Convex Hulls in 3D

Outline:

I. Algebraic definition

II. Complexity of a convex hull

III. Visible facets

IV. Conflict sets

V. Algorithm
I. Convex Sets

A set $S \subseteq \mathbb{R}^n$ is *convex* if the line segment $\overline{pq} \subset S$ for any pair of points $p, q \in S$. 
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It is **concave** if the set does not contain all the line segments.
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A set $S \subseteq \mathbb{R}^n$ is \textit{convex} if the line segment $pq \subset S$ for any pair of points $p, q \in S$.

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Convex Hulls

The *convex hull* of a set of points $S \subseteq \mathbb{R}^n$ is the *intersection* of all convex sets containing $S$. 
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Every $x \in [x_1, x_2]$ satisfies

$$x = \lambda_1 x_1 + \lambda_2 x_2$$

where $\lambda_1, \lambda_2 \geq 0$

$$\lambda_1 + \lambda_2 = 1$$
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\( \lambda_1, \lambda_2 \): **barycentric coordinates**

\[
\lambda_1 = \frac{x_2 - x}{x_2 - x_1} \quad \lambda_2 = \frac{x - x_1}{x_2 - x_1}
\]
Line Segment

\[ S = \{p_1, p_2\} \]

A point \( p \) on the segment \( \overline{p_1p_2} \)
A point $p$ on the segment $p_1p_2$

$p = \lambda_1 p_1 + \lambda_2 p_2$

where $\lambda_1, \lambda_2 \geq 0$

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A point \( p \) on the segment \( p_1p_2 \)

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\[
\lambda_1 = \frac{||p - p_2||}{||p_2 - p_1||}
\]

\[
\lambda_2 = \frac{||p - p_1||}{||p_2 - p_1||}
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\( S = \{p_1, p_2\} \)
Three Non-Collinear Points in 2D

A point \( p \) in the convex hull (bounded by triangle \( \Delta p_1p_2p_3 \)):

\[
p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3
\]

where \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \)

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\( p_1 = (x_1, y_1) \)
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In fact, let $A = area(\Delta p_1p_2p_3)$
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\lambda_1 = \frac{area(\Delta p_2p_3q)}{A}
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$n$ Points in the Plane

$n$ points $p_1, p_2, \ldots, p_n$
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A point $p$ in the convex hull has

$$p = \sum_{i=1}^{n} \lambda_i p_i$$

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$\lambda_1, \lambda_2, ..., \lambda_n$ are not uniquely determined when $n > 3$. 

$n$ points $p_1, p_2, ..., p_n$
Vertices of the Convex Hull

$k$ vertices: $p_{i1}, p_{i2}, ..., p_{ik}$

\[ p = \sum_{j=1}^{k} \mu_j p_{ij} \]

where \( \mu_1, \mu_2, ..., \mu_k \geq 0 \)

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Non-Coplanar Points in 3D

Tetrahedron (for 4 points)

\[ p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4 \]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \)

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II. Faces vs Facets

*Faces* are features of all dimensions on a polyhedron.

- **0-faces**: vertices
- **1-faces**: edges
- **2-faces** (*facets*): polygonal faces

A dodecahedron has

- 20 vertices
- 30 edges
- 12 facets
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- A \(d\)-D polytope \(P\) has 0-faces, 1-faces, ..., \((d - 1)\)-faces.

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- The generalization of a polyhedron in the $d$-dimensional ($d$-D) space is called a polytope.
- A $d$-D polytope $P$ has 0-faces, 1-faces, ..., $(d - 1)$-faces.
- The facets of $P$ are its $(d - 1)$-faces.
Complexity of a Convex Hull in 3D

$S$: a set of $n$ points  
$P$: convex hull of $S$ (a convex polyhedron)

**Theorem**  
$\#\text{edges} \leq 3n - 6$ and $\#\text{facets} \leq 2n - 4$
Complexity of a Convex Hull in 3D

$S$: a set of $n$ points  \hspace{1cm} P$: convex hull of $S$ (a convex polyhedron)

**Theorem** \hspace{0.5cm} \#edges $\leq 3n - 6$ and \#facets $\leq 2n - 4$

**Proof** The surface of a convex polyhedron can be seen as a planar graph.
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**Theorem** \ ['#edges \leq 3n - 6 \text{ and } #facets \leq 2n - 4']

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- facet $\mapsto$ face
- top facet $\mapsto$ unbounded face
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Apply Euler’s formula:

\[ n_v - n_e + n_f = 2 \]
Proof (cont’d)

Every facet of the polyhedron has $\geq 3$ edges.
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\[\downarrow\]

Every face of the planar graph has $\geq 3$ edges.
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Every edge is adjacent to two faces.
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Every facet of the polyhedron has $\geq 3$ edges.

$\Downarrow$

Every face of the planar graph has $\geq 3$ edges.

$\Rightarrow$  

Every edge is adjacent to two faces.

$2n_e \geq 3n_f$
Proof (cont’d)

Every facet of the polyhedron has \( \geq 3 \) edges.

\[ \Downarrow \]

Every face of the planar graph has \( \geq 3 \) edges.

Every edge is adjacent to two faces.

\[
\begin{align*}
2e & \geq 3f \\
n_v - n_e + n_f & = 2 \\
n_v + n_f - 2 & = n_e \geq \frac{3}{2} n_f
\end{align*}
\]
Proof (cont’d)

Every facet of the polyhedron has \( \geq 3 \) edges.

\[ \Rightarrow \]

Every face of the planar graph has \( \geq 3 \) edges.

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\[ n_v - n_e + n_f = 2 \]
\[ n_v + n_f - 2 = n_e \geq \frac{3}{2} n_f \]
\[ n \geq n_v \]
\[ n + n_f - 2 \geq \frac{3}{2} n_f \]
Proof (cont’d)

Every facet of the polyhedron has $\geq 3$ edges.

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\[
2n_e \geq 3n_f
\]

\[
n_v - n_e + n_f = 2
\]

\[
n_v + n_f - 2 = n_e \geq \frac{3}{2} n_f
\]

\[
n \geq n_v
\]

\[
n + n_f - 2 \geq \frac{3}{2} n_f
\]

\[
n_f \leq 2n - 4
\]
Proof (cont’d)

Every facet of the polyhedron has $\geq 3$ edges.

Every face of the planar graph has $\geq 3$ edges.

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\[
\begin{align*}
2n_e & \geq 3n_f \\
_n_v - n_e + n_f &= 2 \\
n_v + n_f - 2 &= n_e \geq \frac{3}{2} n_f \\
n \geq n_v \\
n + n_f - 2 &\geq \frac{3}{2} n_f \\
n_f &\leq 2n - 4 \\
n_e &\leq n + n_f - 2 \\
n_e &\leq 3n - 6
\end{align*}
\]
Simplicial Polytope

**Corollary** The complexity of the convex hull of $n$ points in 3D is $O(n)$. 
Simplicial Polytope

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A simplicial polytope has every facet as a triangle.
Simplicial Polytope

**Corollary** The complexity of the convex hull of \( n \) points in 3D is \( O(n) \).

A *simplicial polytope* has every facet as a triangle.

\[
2n_e = 3n_f \\
\quad n_v = n
\]
Simplicial Polytope

**Corollary**  The complexity of the convex hull of $n$ points in 3D is $O(n)$.

A *simplicial polytope* has every facet as a triangle.

\[
\begin{align*}
2n_e &= 3n_f \\
n_v &= n
\end{align*}
\]

Proof of the theorem

\[
\begin{align*}
n_e &= 3n - 6 \\
n_f &= 2n - 4
\end{align*}
\]
III. Computing a Convex Hull

Randomized incremental construction
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Randomized incremental construction

- Choose four points $p_1, p_2, p_3, p_4 \in S$ that are not co-planar.

  Their convex hull is a tetrahedron.
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Randomized incremental construction

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III. Computing a Convex Hull

Randomized incremental construction

- Choose four points $p_1, p_2, p_3, p_4 \in S$ that are not co-planar. $O(n)$
  
  Their convex hull is a tetrahedron.

- Compute a random permutation $p_5, p_6, \ldots, p_n$.
  
  $$P_r = \{p_1, p_2, \ldots, p_r\} \quad r \geq 1$$
III. Computing a Convex Hull

Randomized incremental construction

- Choose four points $p_1, p_2, p_3, p_4 \in S$ that are not co-planar. $O(n)$
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  $P_r = \{p_1, p_2, \ldots, p_r\}$ \quad $r \geq 1$

- For $r \geq 5$, add $p_r$ to the convex hull $CH(P_{r-1})$. 
III. Computing a Convex Hull

Randomized incremental construction

- Choose four points $p_1, p_2, p_3, p_4 \in S$ that are not co-planar. $O(n)$
  
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- For $r \geq 5$, add $p_r$ to the convex hull $CH(P_{r-1})$.
  
  $p_r$ inside $CH(P_{r-1})$ or on its boundary.
III. Computing a Convex Hull

Randomized incremental construction

♦ Choose four points \( p_1, p_2, p_3, p_4 \in S \) that are not co-planar. \( O(n) \)

Their convex hull is a tetrahedron.

♦ Compute a random permutation \( p_5, p_6, ..., p_n \).

\[
P_r = \{p_1, p_2, ..., p_r\} \quad r \geq 1
\]

♦ For \( r \geq 5 \), add \( p_r \) to the convex hull \( CH(P_{r-1}) \).

• \( p_r \) inside \( CH(P_{r-1}) \) or on its boundary.

\[
CH(P_r) = CH(P_{r-1})
\]
• $p_r$ outside $CH(P_{r-1})$. 

\[
p_r
\]
Visible Facets

- $p_r$ outside $CH(P_{r-1})$. 
Visible Facets

- $p_r$ outside $CH(P_{r-1})$.

- Visible facets form a connected region on the surface of $CH(P_{r-1})$. 

$p_r$
Visible Facets

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- Boundary of this visible region is called the *horizon* of $CH(P_{r-1})$. 
Visible Facets

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- Visible facets form a connected region on the surface of $CH(P_{r-1})$.

- Boundary of this visible region is called the **horizon** of $CH(P_{r-1})$. 

![Diagram showing visible facets and horizon](image-url)
Visible Facets

• \( p_r \) outside \( CH(P_{r-1}) \).

Visible facets form a connected region on the surface of \( CH(P_{r-1}) \).

• Boundary of this visible region is called the \textit{horizon} of \( CH(P_{r-1}) \).

Observation

A facet \( f \) is visible from \( p_r \) if \( p_r \) and \( CH(P_{r-1}) \) lie on opposite sides of the half-plane containing \( f \).
Hull Update

Strategy:

- Keep all invisible facets.
Hull Update

Strategy:

- Keep all invisible facets.
- Replace visible facets with facets connecting $p_r$ to its horizon.
Hull Update

Strategy:

- Keep all invisible facets.
- Replace visible facets with facets connecting $p_r$ to its horizon.
Degeneracy Handling

- Check if \( p_r \) lies in the plane of a facet of \( CH(P_{r-1}) \).

\[ p_r \text{ coplanar with } \Delta p_i p_j p_k \]
**Degeneracy Handling**

- Check if $p_r$ lies in the plane of a facet of $CH(P_{r-1})$.

![Diagram showing coplanar points and merge process]
Degeneracy Handling

✦ Check if $p_r$ lies in the plane of a facet of $CH(P_{r-1})$.

$p_r$ coplanar with $\Delta p_ip_jp_k$
Degeneracy Handling

- Check if $p_r$ lies in the plane of a facet of $CH(P_{r-1})$. 

$p_r$ coplanar with $\Delta p_i p_j p_k$
Doubly-connected edge list (DCEL)

because convex hull can be interpreted as a planar graph.

- Every vertex represents a point in space.
- Every edge represents an edge on the convex hull.
- Transforming $\text{DCEL}_{r-1}$ for CH$(P_{r-1})$ to $\text{DCEL}_r$ for CH$(P_r)$ takes time \textit{linear} in the total complexity of the visible facets.
IV. Finding Visible Facets

Which faces of $CH(P_{r-1})$ are visible to $p_r$?
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Which faces of \( CH(P_{r-1}) \) are visible to \( p_r \)?

Slow strategy

Test every facet \( f \) whether \( p_r \) and \( CH(P_{r-1}) \) are on the opposite sides of the plane \( \Pi \) containing \( f \).
IV. Finding Visible Facets

Which faces of $CH(P_{r-1})$ are visible to $p_r$?

Slow strategy

Test every facet $f$ whether $p_r$ and $CH(P_{r-1})$ are on the opposite sides of the plane $\Pi$ containing $f$.

- $O(1)$ for each facet.
- $O(n)$ for all facets.
IV. Finding Visible Facets

Which faces of $CH(P_{r-1})$ are visible to $p_r$?

Slow strategy

Test every facet $f$ whether $p_r$ and $CH(P_{r-1})$ are on the opposite sides of the plane $\Pi$ containing $f$.

- $O(1)$ for each facet.
- $O(n)$ for all facets.

Algorithm runs in $O(n^2)$ time.
Faster Testing

**Heuristic** Maintain additional information related to $CH(P_{r-1})$. 

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- for every facet $f$ of $CH(P_{r-1})$
  
  $$P_{conflict}(f) = \{ p_t \mid r \leq t \leq n \text{ and } f \text{ visible from } p_t \}$$
Faster Testing

**Heuristic** Maintain additional information related to $CH(P_{r-1})$.

- for every facet $f$ of $CH(P_{r-1})$
  \[ P_{\text{conflict}}(f) = \{p_t \mid r \leq t \leq n \text{ and } f \text{ visible from } p_t \} \]

- for every point $p_t$, $t \geq r$
  \[ F_{\text{conflict}}(p_t) = \{ \text{ facets of } CH(P_{r-1}) \text{ visible from } p_t \} \]
Faster Testing

**Heuristic** Maintain additional information related to $CH(P_{r-1})$.

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visibility $\iff$ conflict
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  \[ F_{\text{conflict}}(p_t) = \{ \text{facets of } CH(P_{r-1}) \text{ visible from } p_t \} \]

Visibility $\iff$ conflict

- $p_t \in P_{\text{conflict}}(f)$ is in conflict with $f$.
- $f \in F_{\text{conflict}}(p_t)$ is in conflict with $p_t$. 
Faster Testing

Heuristic  Maintain additional information related to $CH(P_{r-1})$.

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  \[ P_{conflict}(f) = \{ p_t \mid r \leq t \leq n \text{ and } f \text{ visible from } p_t \} \]

- for every point $p_t$, $t \geq r$
  \[ F_{conflict}(p_t) = \{ \text{ facets of } CH(P_{r-1}) \text{ visible from } p_t \} \]

visibility $\iff$ conflict

- $p_t \in P_{conflict}(f)$ is in conflict with $f$.
  \[ f \in F_{conflict}(p_t) \text{ is in conflict with } p_t. \]

- Once we add $p_t, f$ must be deleted.
Conflict Graph

- Bipartite graph $G$
Conflict Graph

- **Bipartite graph** $G$

  - Vertices are decomposed into two sets:
    - $\{p_r, \ldots, p_n\}$  // those to be added
    - facets of $CH(P_{r-1})$
Conflict Graph

- Bipartite graph $G$

  - Vertices are decomposed into two sets:
    - $\{p_r, \ldots, p_n\}$  // those to be added
    - facets of $CH(P_{r-1})$

  - Every edge connects a point and a facet.

    An edge $\langle p_t, f \rangle$ exists if $f$ is visible from $p_t$. 
Conflict Graph Illustration

\[ G: \]

\[ p_r \]

\[ p_{r+1} \]

\[ p_t \]

\[ p_n \]

\[ F_{\text{conflict}}(p_t) \]

\[ F_{\text{conflict}}(p_t) \]

\[ P_{\text{conflict}}(f) \]

facets in \( CH(P_{r-1}) \)
When inserting $p_r$ into $CH(P_{r-1})$, look up $F_{\text{conflict}}(p_r)$ to get the visible facets.
Graph Initialization & Updates

Initialize $G$:

- $P_4 = \{p_1, p_2, p_3, p_4\}$ is a tetrahedron.
- Check every point $p_i, 5 \leq i \leq n$, which of the four facets are visible.
Graph Initialization & Updates

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Update $G$ after adding $p_r$:
Graph Initialization & Updates

Initialize $G$:

- $P_4 = \{p_1, p_2, p_3, p_4\}$ is a tetrahedron.
- Check every point $p_i, 5 \leq i \leq n$, which of the four facets are visible.

$O(n)$

Update $G$ after adding $p_r$:

- Discard neighbors (all visible facets) of $p_r$ in $G$.
- Delete the node representing $p_r$.
- Add nodes for the new facets (which connect $p_r$ to the horizon).
Updating the Conflict Sets of New Faces

- Construct $P_{conflict}(f)$ for every new facet $f$.
  \[
  \{ p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t \}\]
Updating the Conflict Sets of New Faces

- Construct $P_{conflict}(f)$ for every new facet $f$.
- $\{p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t\}$

Suppose a point $p_t$ can see $f$. 

![Diagram showing new faces and points]
Updating the Conflict Sets of New Faces

- Construct $P_{conflict}(f)$ for every new facet $f$.

$$\{p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t\}$$

Suppose a point $p_t$ can see $f$.

- $p_t$

$e$: edge of $f$ on the horizon and opposite to $p_r$

$f_1, f_2$: facets in $CH(P_{r-1})$ that are incident on $e$. 
Updating the Conflict Sets of New Faces

- Construct $P_{\text{conflict}}(f)$ for every new facet $f$.

$\{p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t\}$

Suppose a point $p_t$ can see $f$.

Then it can see $e$.

- $p_t$

$e$: edge of $f$ on the horizon and opposite to $p_r$

$f_1, f_2$: facets in $CH(P_{r-1})$ that are incident on $e$. 
Updating the Conflict Sets of New Faces

• Construct $P_{conflict}(f)$ for every new facet $f$.

\[ \{p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t\} \]

Suppose a point $p_t$ can see $f$.

Then it can see $e$, which is an edge in $CH(P_{r-1})$ that bounds $f$.

\[ e: \text{ edge of } f \text{ on the horizon and opposite to } p_r \]

\[ f_1, f_2: \text{ facets in } CH(P_{r-1}) \text{ that are incident on } e. \]
Updating the Conflict Sets of New Faces

- Construct $P_{conflict}(f)$ for every new facet $f$.

\[ \{ p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t \} \]

Suppose a point $p_t$ can see $f$.

Then it can see $e$, which is an edge in $CH(P_{r-1})$ that bounds $f$.

\[ e \text{ must have been visible from } p_t \text{ in } CH(P_{r-1}) \subseteq CH(P_r) \]
Updating the Conflict Sets of New Faces

- Construct $P_{\text{conflict}}(f)$ for every new facet $f$.

\[ \{p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t \} \]

Suppose a point $p_t$ can see $f$.

Then it can see $e$, which is an edge in $CH(P_{r-1})$ that bounds $f$.

\[ e \text{ must have been visible from } p_t \text{ in } CH(P_{r-1}) \subseteq CH(P_r) \]

$f_1$ or $f_2$ is visible from $p_t$. 

- $e$: edge of $f$ on the horizon and opposite to $p_r$.
- $f_1, f_2$: facets in $CH(P_{r-1})$ that are incident on $e$. 
- $p_t$: $P_{\text{conflict}}(f)$
Updating the Conflict Sets of New Faces

- Construct $P_{\text{conflict}}(f)$ for every new facet $f$.
  
  \[ \{ p_t \mid r < t \leq n \text{ and } f \text{ visible from } p_t \} \]

Suppose a point $p_t$ can see $f$.

Then it can see $e$, which is an edge in $CH(P_{r-1})$ that bounds $f$.

\[ e \text{ must have been visible from } p_t \text{ in } CH(P_{r-1}) \subseteq CH(P_r) \]

\[ f_1 \text{ or } f_2 \text{ is visible from } p_t. \]

Test all points $p_t \in P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$, where $r < t \leq n$, add $\langle p_t, f \rangle$ to $G$ if $f$ is visible from $p_t$. 

\[ e: \text{ edge of } f \text{ on the horizon and opposite to } p_r \]

\[ f_1, f_2: \text{ facets in } CH(P_{r-1}) \text{ that are incident on } e. \]
V. Algorithm

ConvexHull(P)

1. find $p_1, p_2, p_3, p_4 \in P$ that form a tetrahedron
2. $C \leftarrow CH(\{p_1, p_2, p_3, p_4\})$
3. compute a random permutation $p_5, p_6, ..., p_n$
4. initialize the conflict graph $G$ over all facets of $C$ and $p_5, p_6, ..., p_n$
5. for $r \leftarrow 5$ to $n$
6. do // insert $p_r$ to $C$
7. if $F_{\text{conflict}}(p_r) \neq \emptyset$ // $p_r$ lies outside $C$
8. then
9. delete all facets in $F_{\text{conflict}}(p_r)$ from $C$
10. find the horizon by walking along the boundary of the visible region of $p_r$
10. for each edge $e$ on the horizon
11. do connect $e$ to $p_r$ to form a triangle $f$
Algorithm (cont’d)

12. if \( f \) is coplanar with its neighbor facet \( f' \) along \( e \)
13. then merge \( f \) with \( f' \) and the merged facet inherits the latter’s conflict set
14. else // determine conflicts for \( f \)
15. create a node for \( f \) in \( G \)
16. \( f_1, f_2 \): facets incident to \( e \)
17. for all \( p \in P_{conflict}(f_1) \cup P_{conflict}(f_2) \)
18. do
19. if \( f \) is visible from \( p \)
20. add \( \langle p, f \rangle \) to \( G \) // update \( P_{conflict}(f) \) and \( F_{conflict}(p) \)
21. delete the node corresponding to \( p_r \) and
22. the nodes corresponding to the facets in \( F_{conflict}(p_r) \) from \( G \), along with incident arcs
23. return \( C \)
Analysis

Theorem  The randomized incremental algorithm computes the convex hull of $n$ points in 3D in $O(n \log n)$ expected time.

Proof  (omitted)