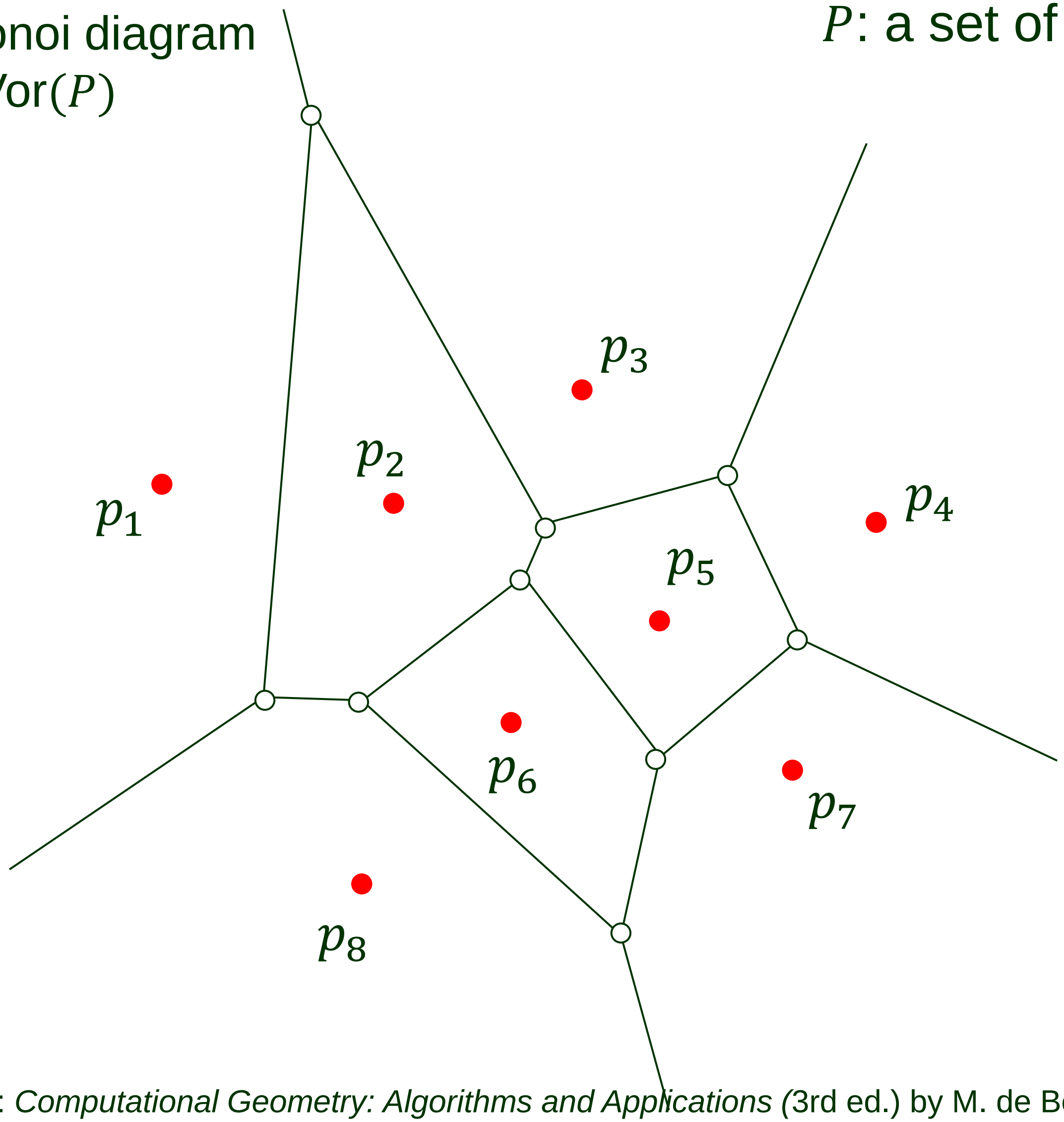
Dual Graph of Voronoi Diagram

P: a set of n points (sites) in the plane.

Voronoi diagram
Vor(P)



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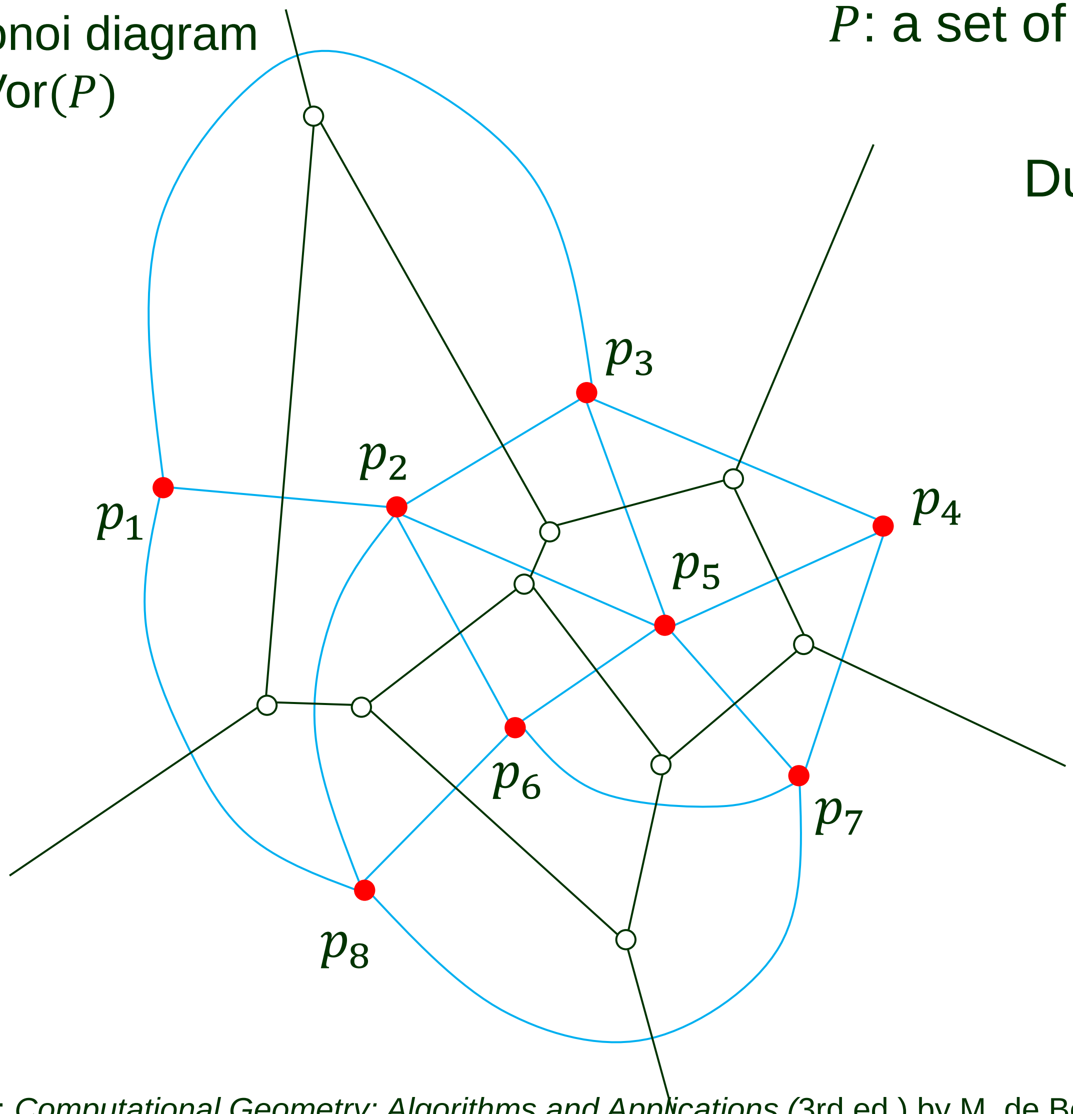

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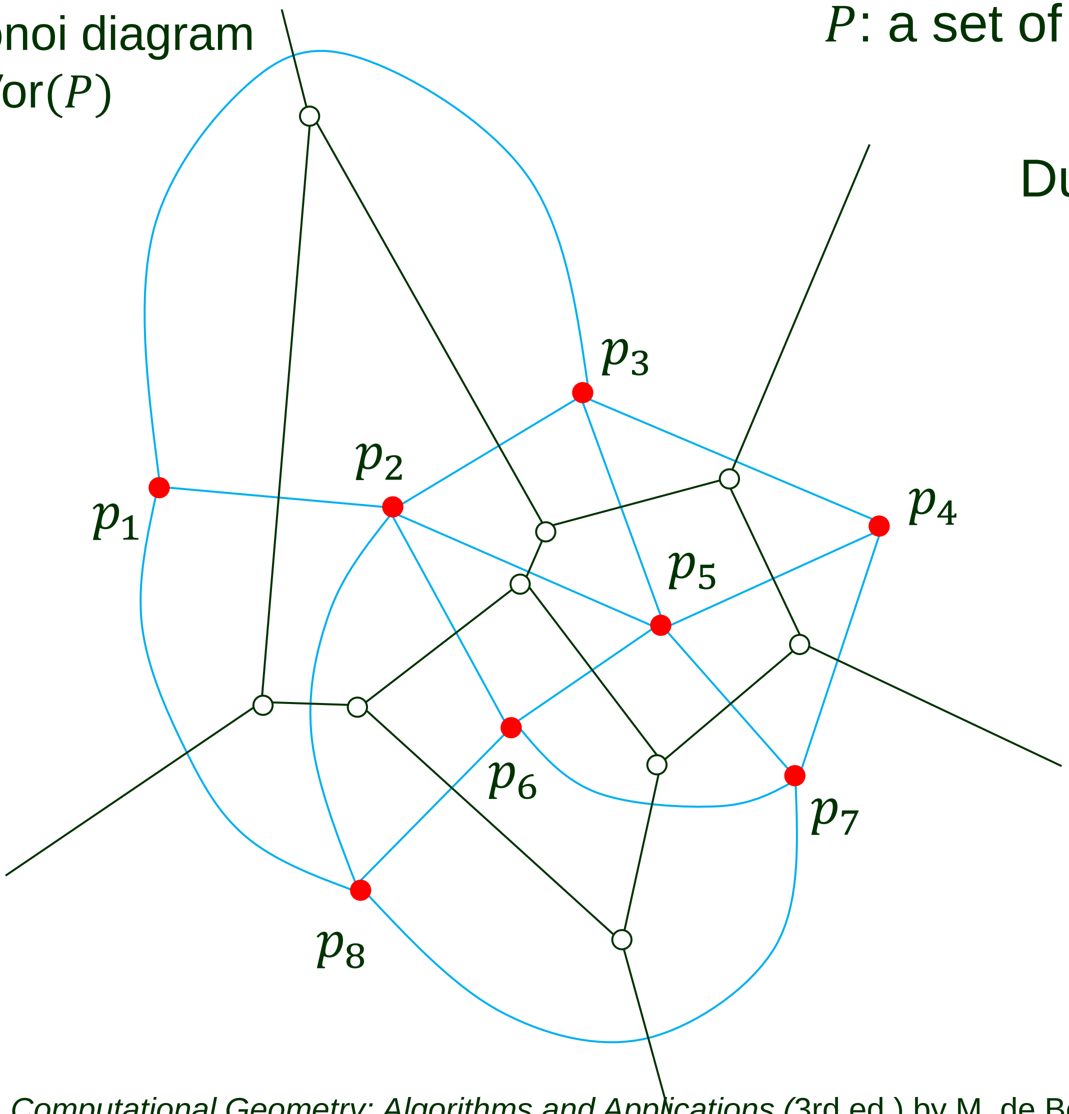

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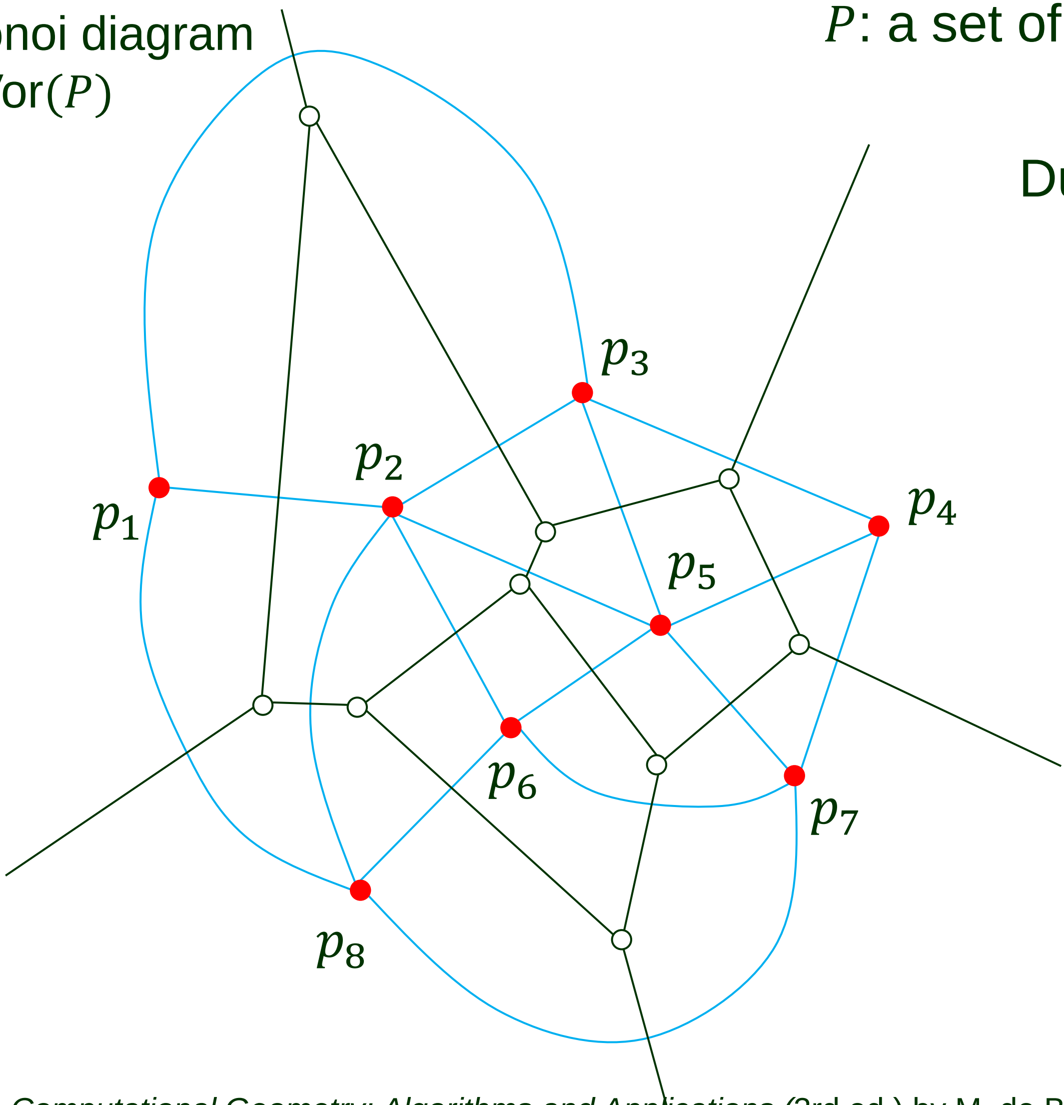

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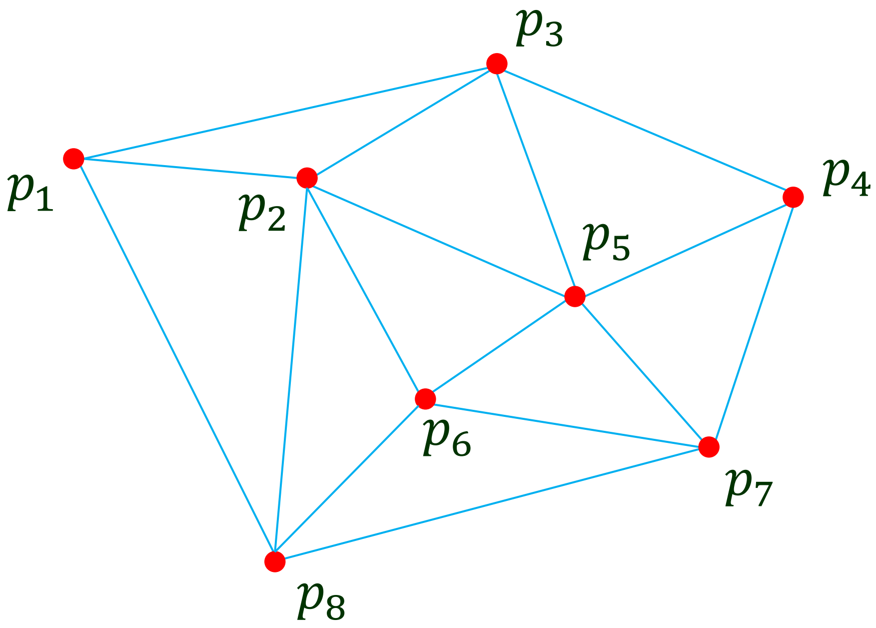
- one node at every site
- an arc between two nodes p and q if $V(p)$ and $V(q)$ share an edge.

Delaunay Graph

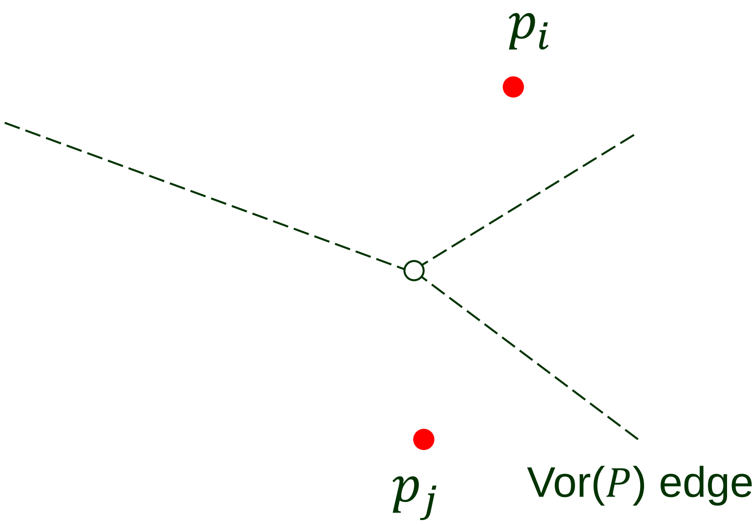
$DG(P)$



Planarity

Theorem 1 $DG(P)$ is a planar graph.

Proof Use a property of Vor(P):



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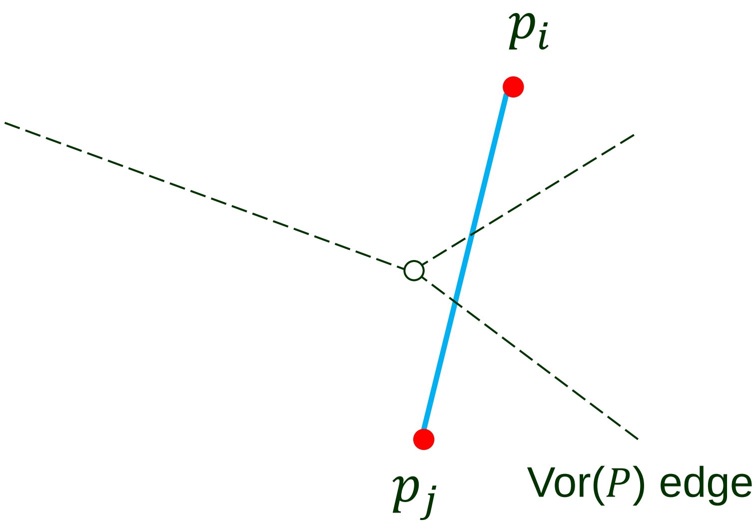

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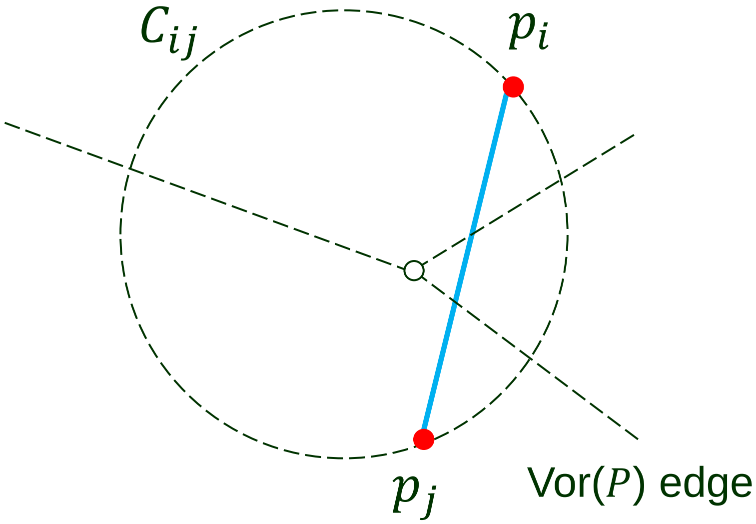

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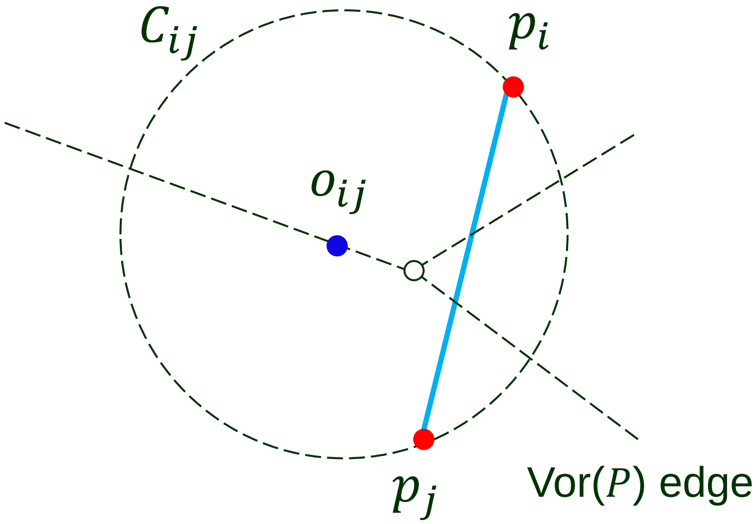

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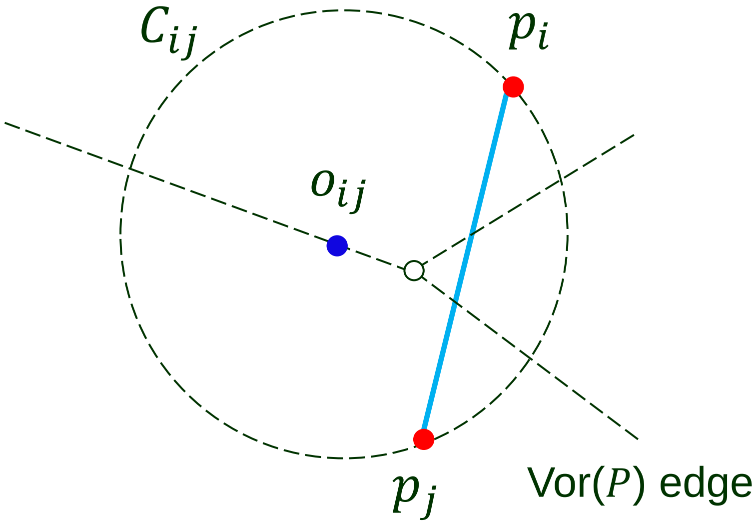

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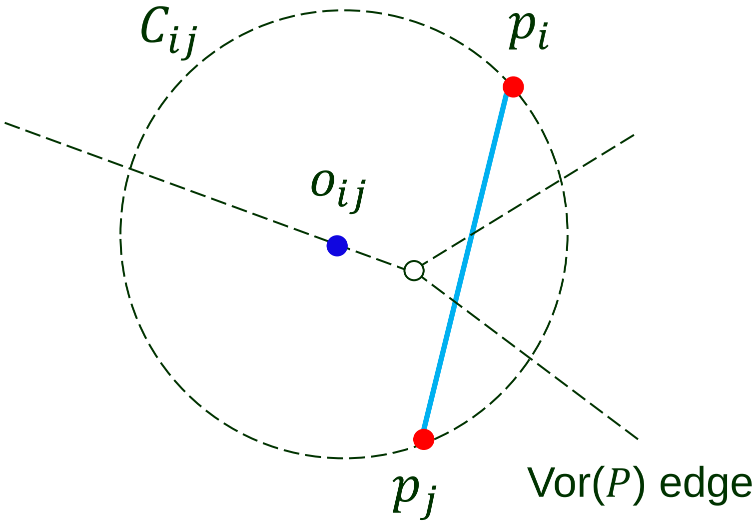

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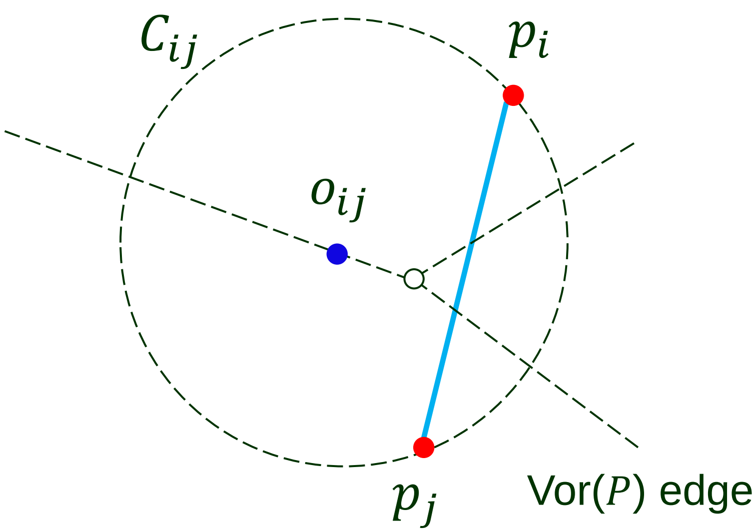

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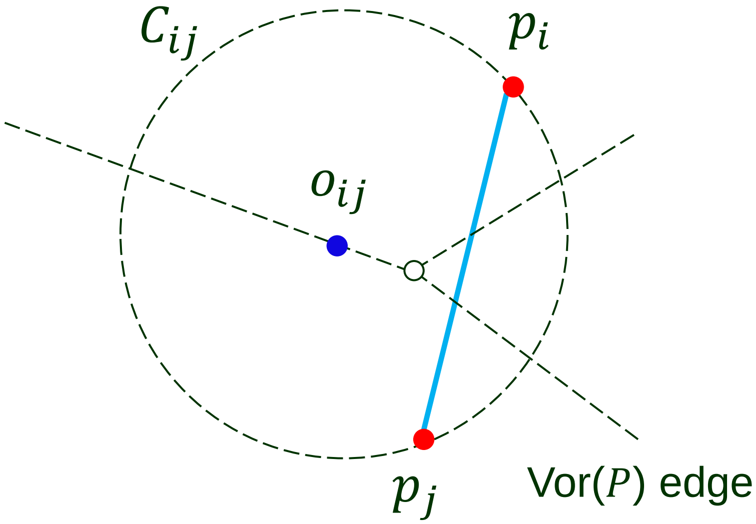

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By contradiction next.

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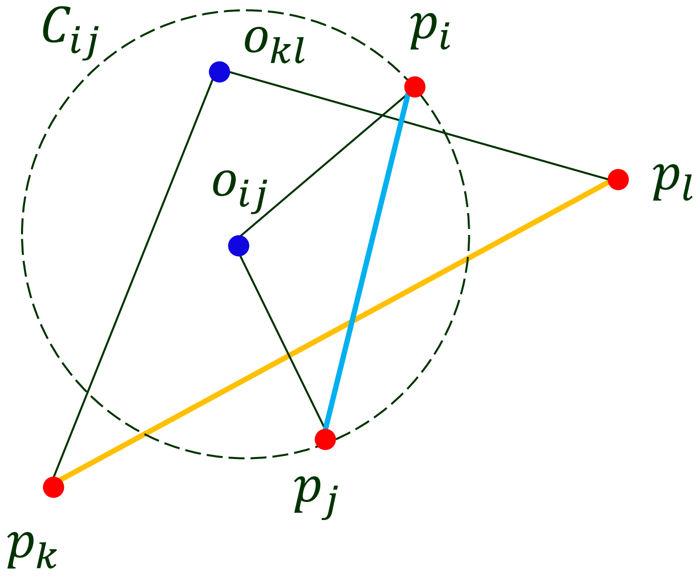

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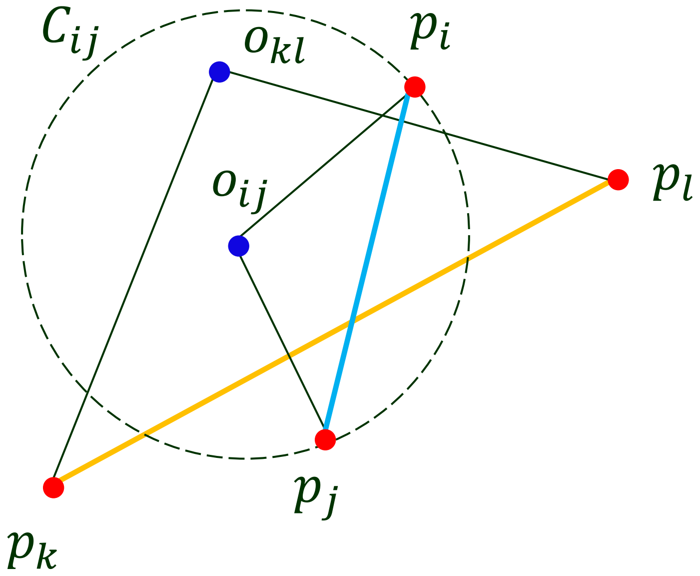

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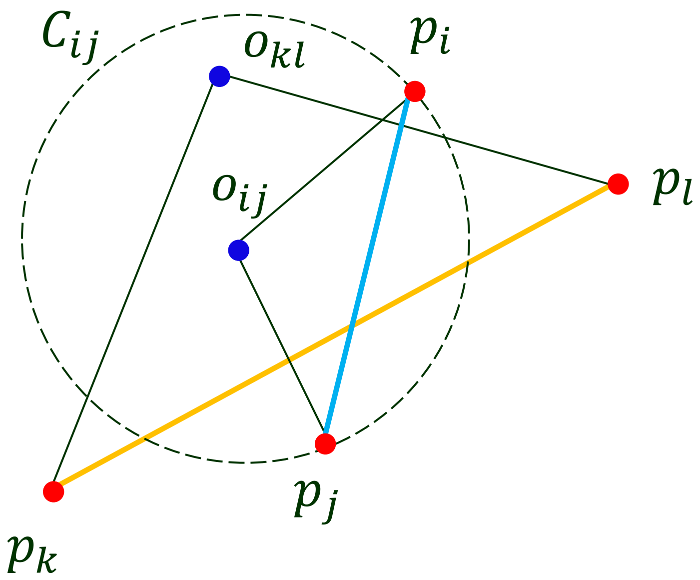

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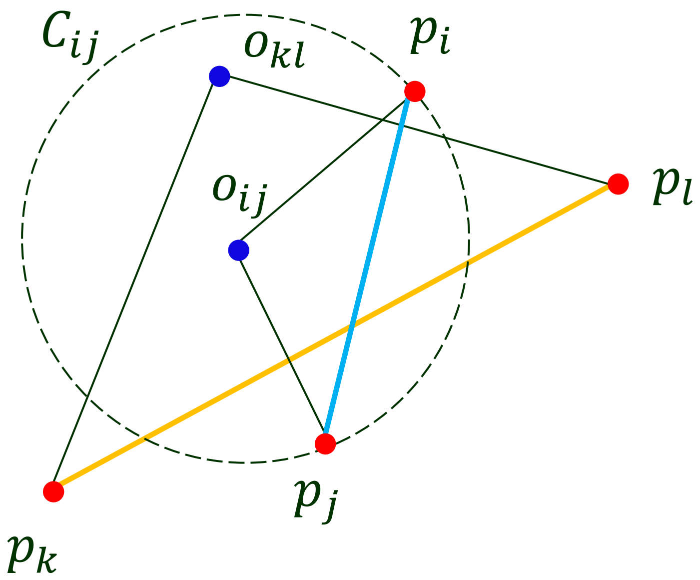

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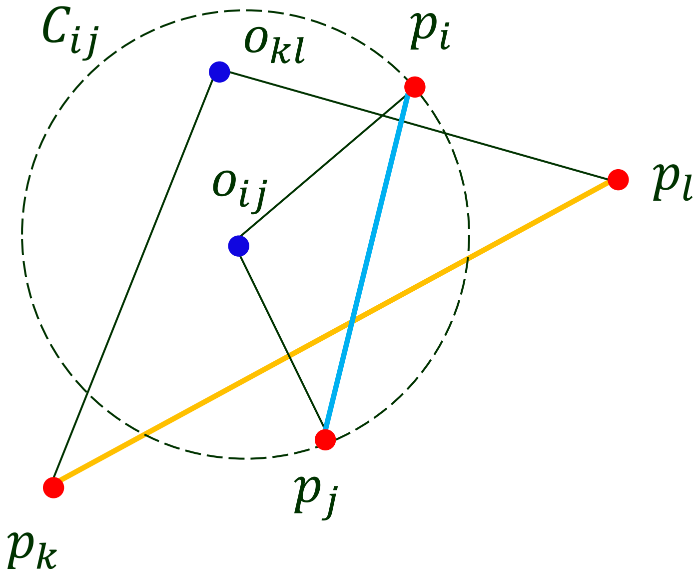

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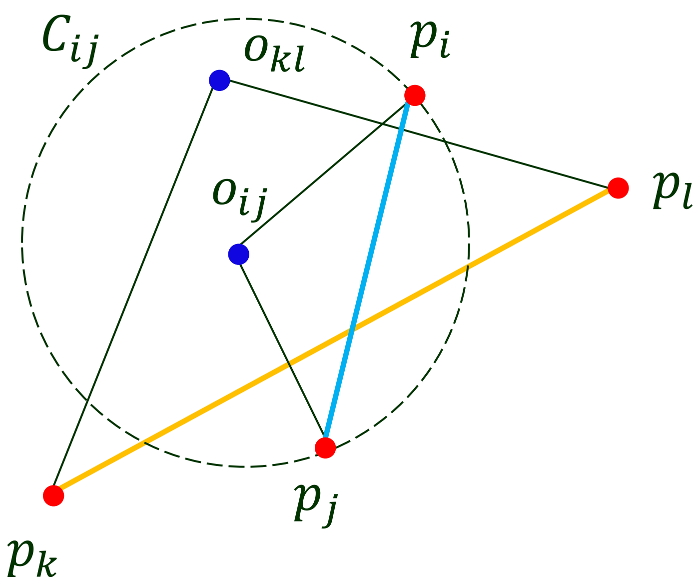

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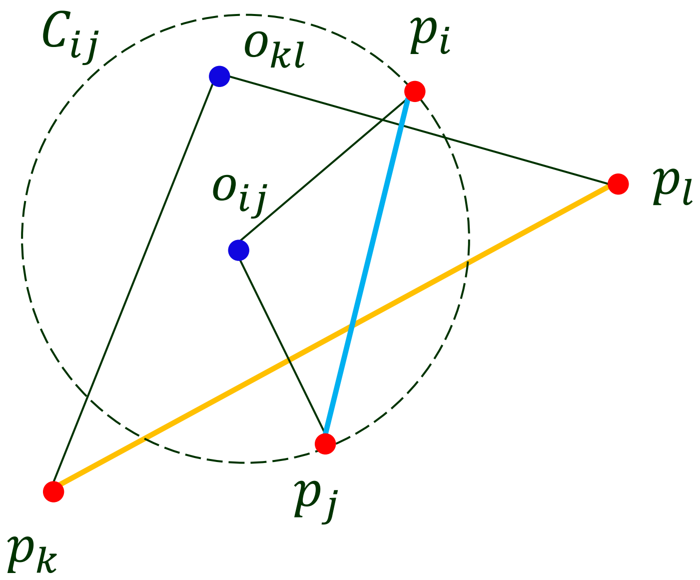

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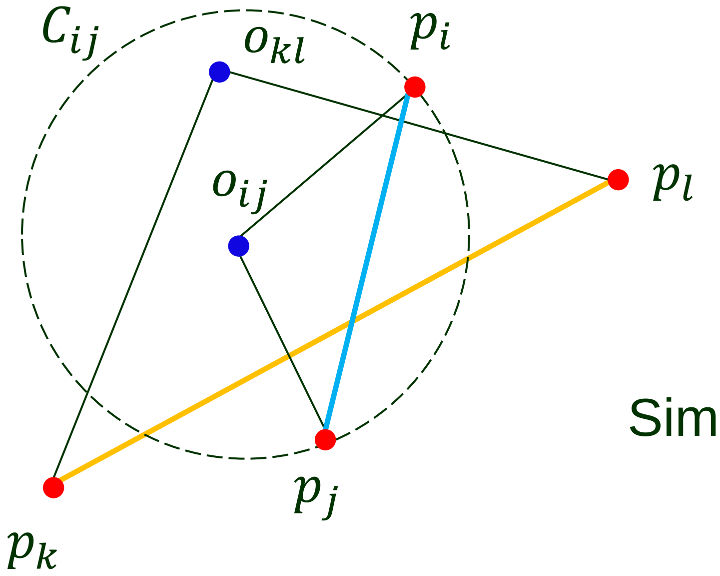

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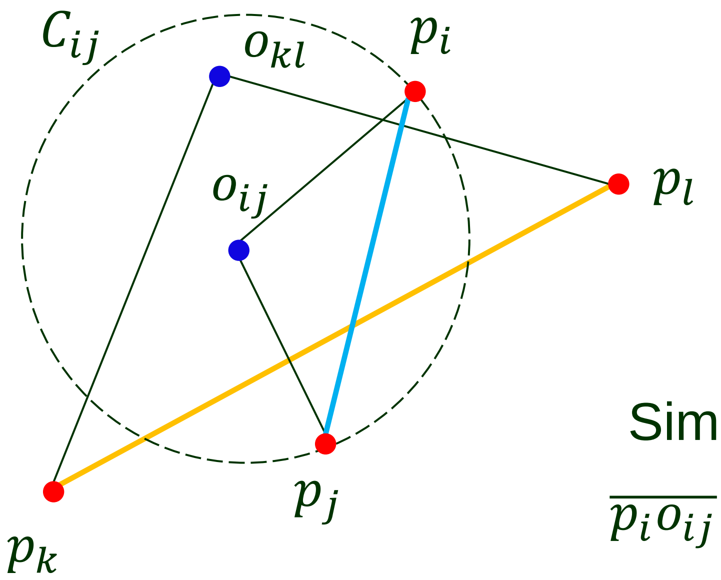

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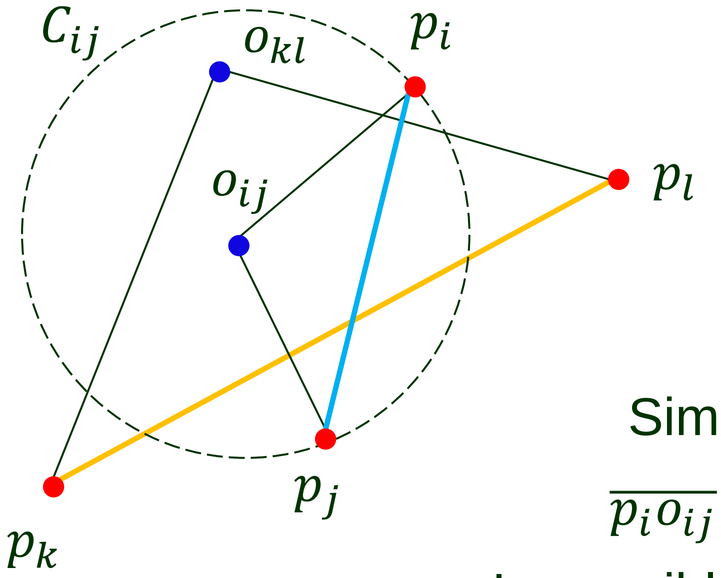

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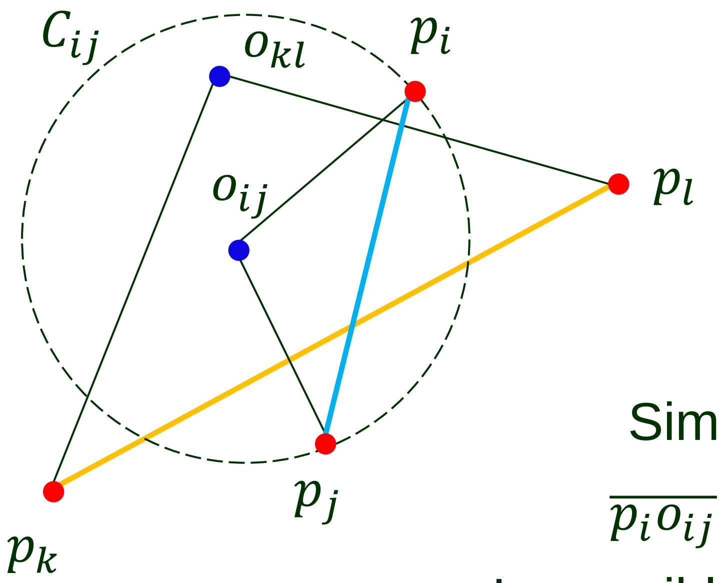

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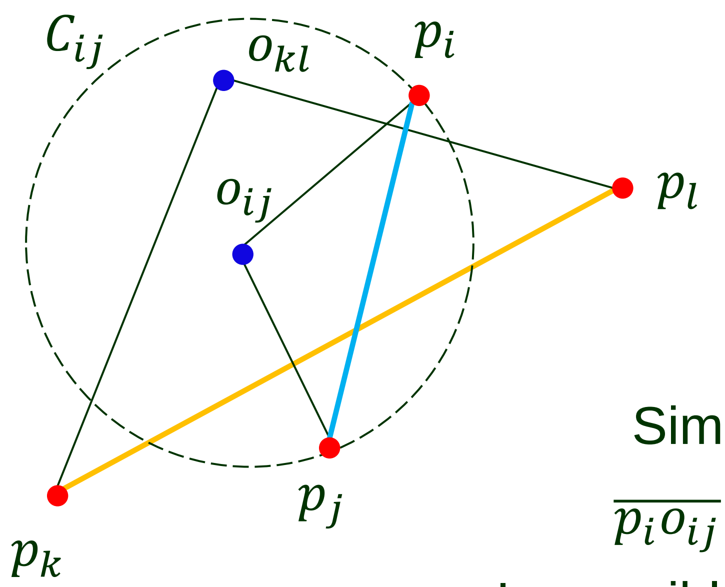

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□

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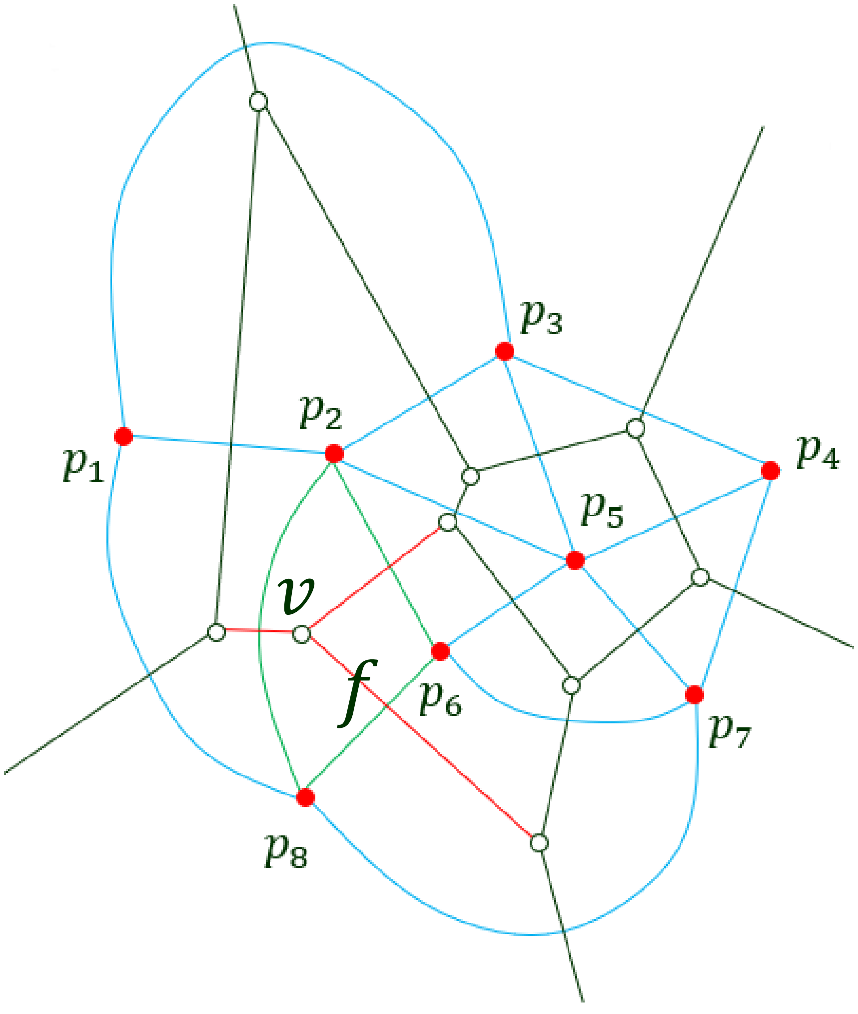
A vertex v in Vor(P) ↔ A face f in $DG(P)$



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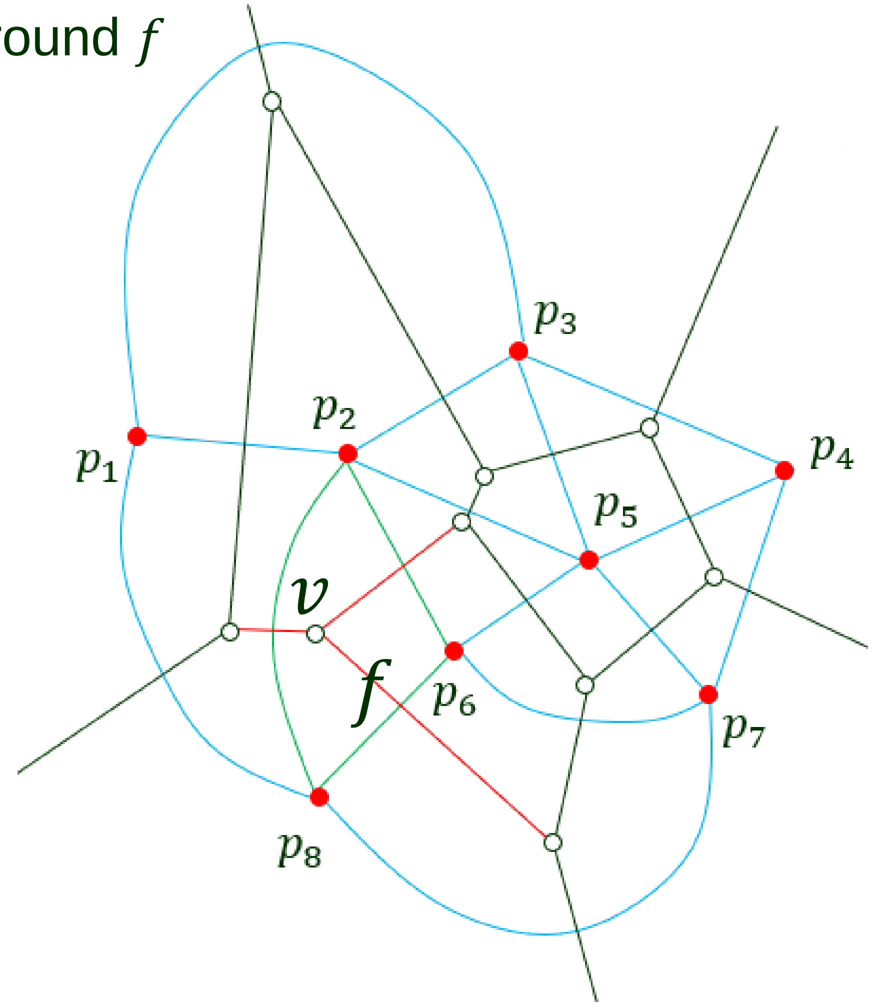
Edges incident to v $\leftrightarrow$ Edges around f



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General position: No four points are cocircular.

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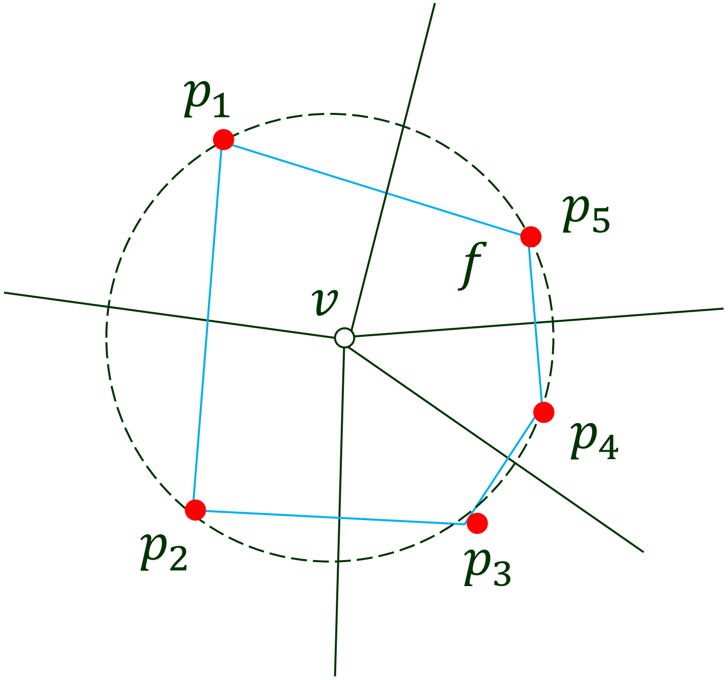

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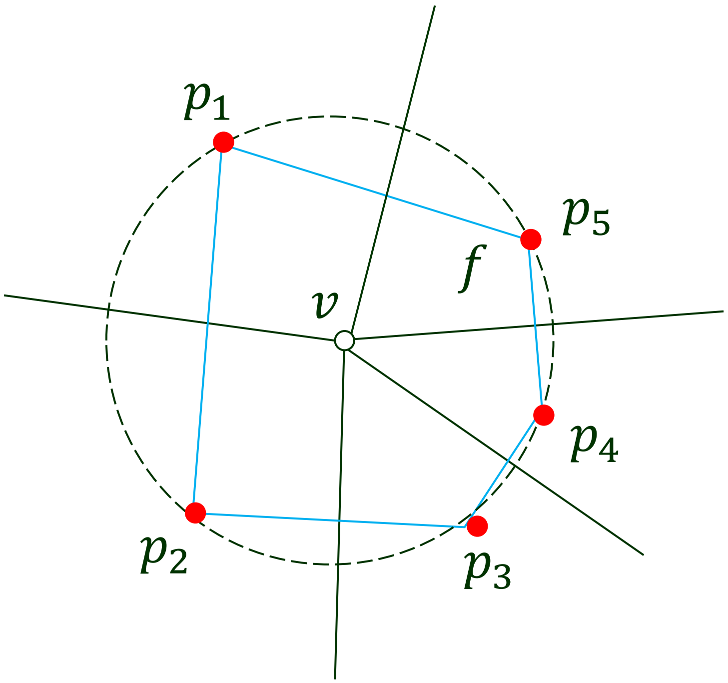

- Otherwise, DG contains some face with > 3 edges.
- Further triangulate every one of such faces.

Properties of Delaunay Graph

Theorem 2 (i) p_i, p_j, p_k are vertices of the same face of $DG(P)$ iff the circle through them contains no other site in its interior.

(ii) $\overline{p_i p_j} \in DG(P)$ iff there is a closed disk C that contains p_i, p_j on its boundary but no other site.

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By Theorem 2(i), we have the following:

Corollary A triangulation T of P is a Delaunay triangulation iff the circumcircle of any triangle contains no point of P in its interior.

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Theorem 3 A triangulation T of P is legal iff T is a Delaunay triangulation of P.

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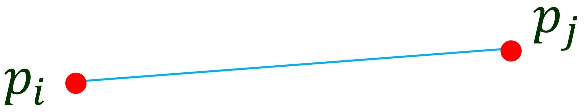

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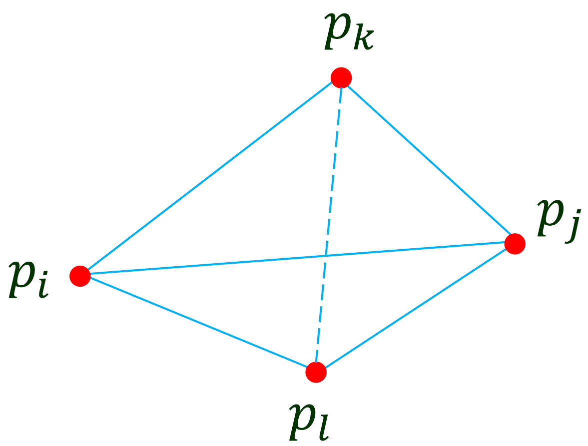

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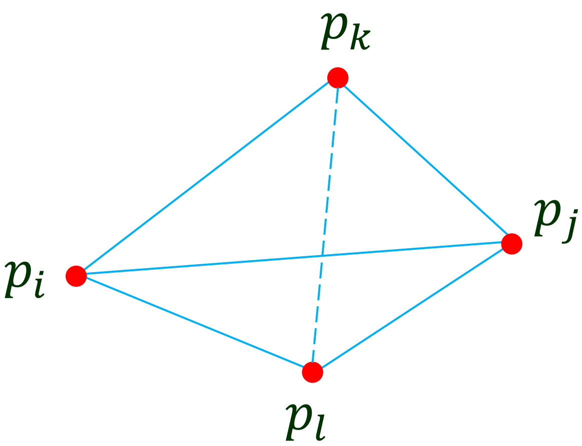

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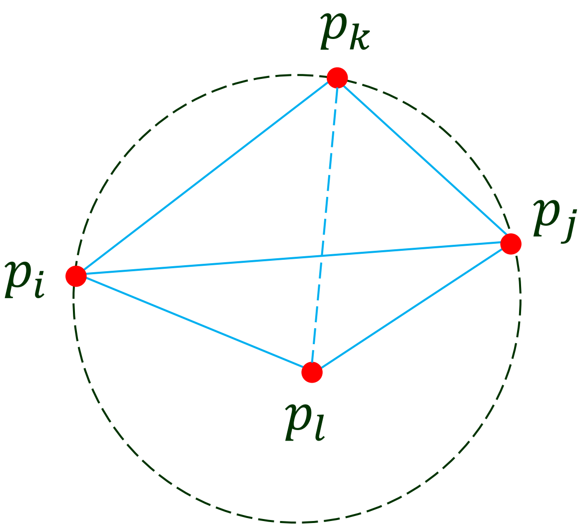

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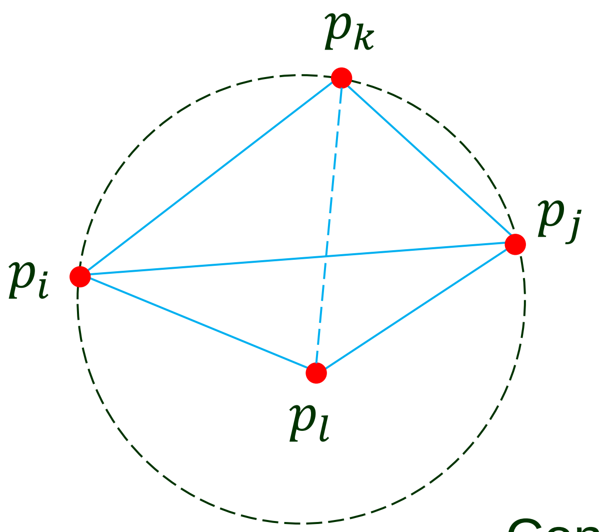

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$\Downarrow$

Contraction to Theorem 2(i), which states that C contains no other sites in its interior because T is Delaunay.

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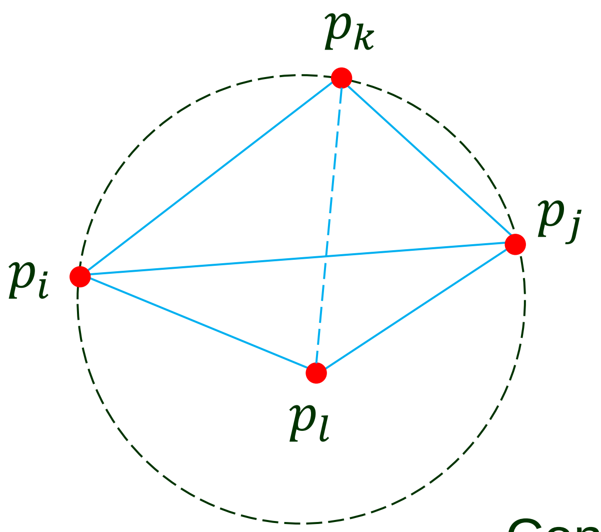


Suppose edge $\overline{p_i p_j}$ is illegal.

Let p_k and p_l be the two sites that are adjacent to both p_i and p_j.

Edge $\overline{p_i p_j}$ is illegal. $\Rightarrow$ The circle C through p_i, p_j, p_k contains p_l in its interior.

$\Downarrow$

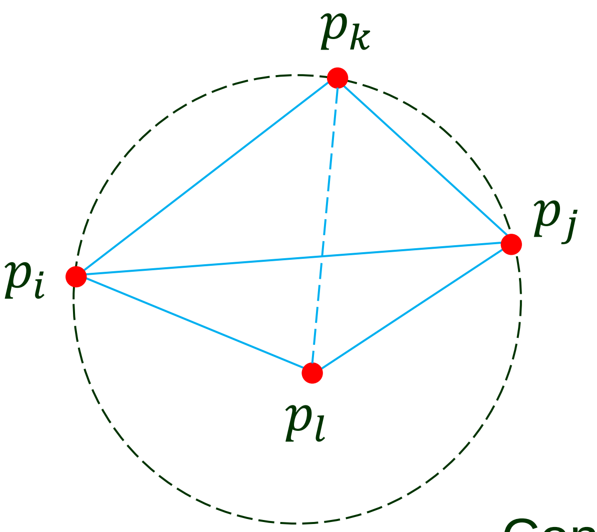
Contraction to Theorem 2(i), which states that C contains no other sites in its interior because T is Delaunay.

($\Longrightarrow$) Prove every legal triangulation is Delaunay by contradiction.

Equivalence Between Legal and Delaunay Triangulations

Theorem 3 A triangulation T of P is legal iff T is a Delaunay triangulation of P.

Proof ($\Longleftarrow$) Let T be a Delaunay triangulation.



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($\Longrightarrow$) Prove every legal triangulation is Delaunay by contradiction. $\square$