Lecture Notes for *Introduction to Computational Geometry* (Com S 418/518)
Yan-Bin Jia, Iowa State University

Point Set Triangulation

Outline:

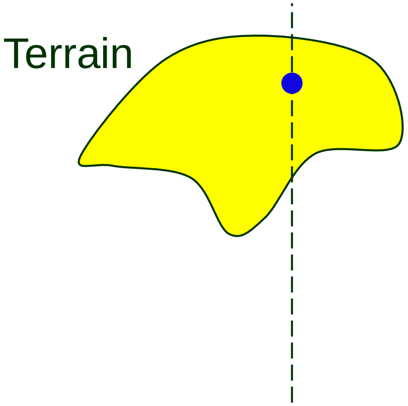
I. Height interpolation

II. Existence and complexity of a triangulation

III. Legal Triangulation

Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

I. Terrain



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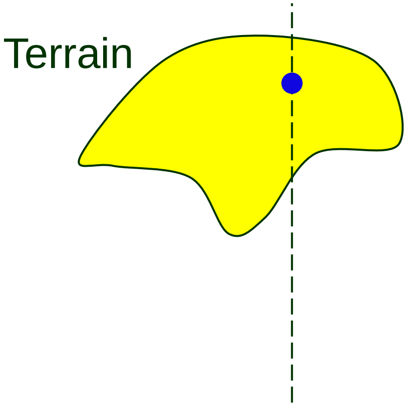

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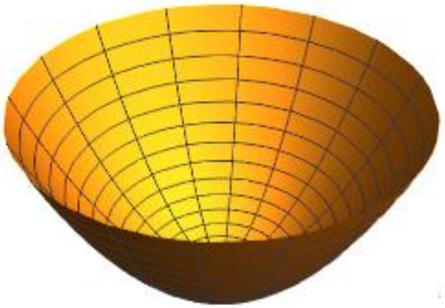

$z = x^2 + y^2$

Paraboloid
(terrain)

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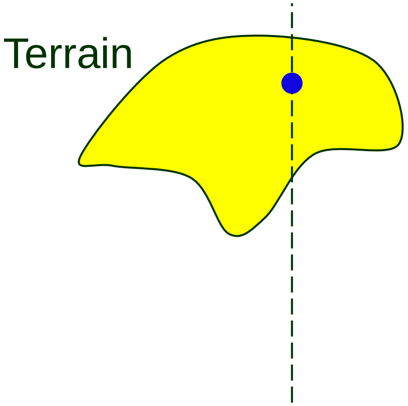

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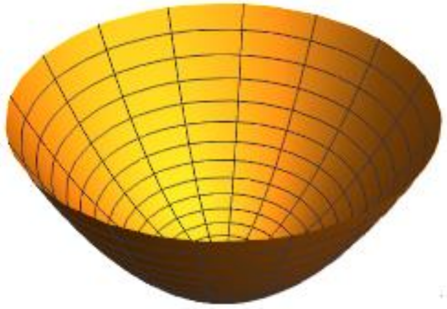

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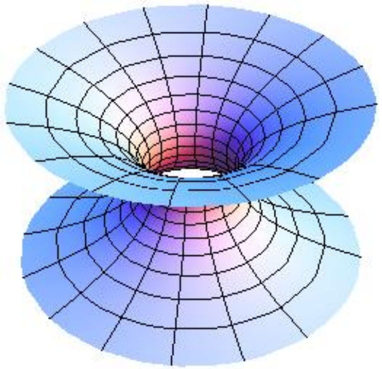

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Catenoid (Com S 477/577)
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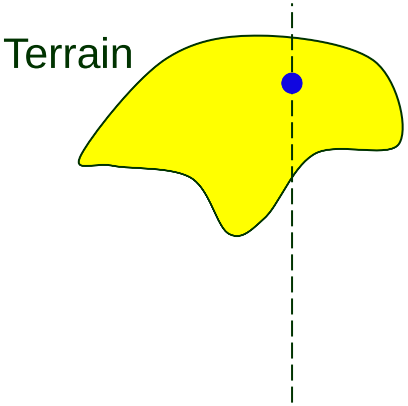

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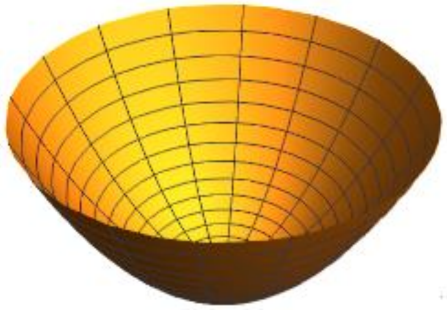

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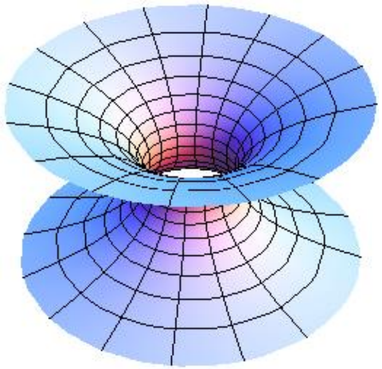

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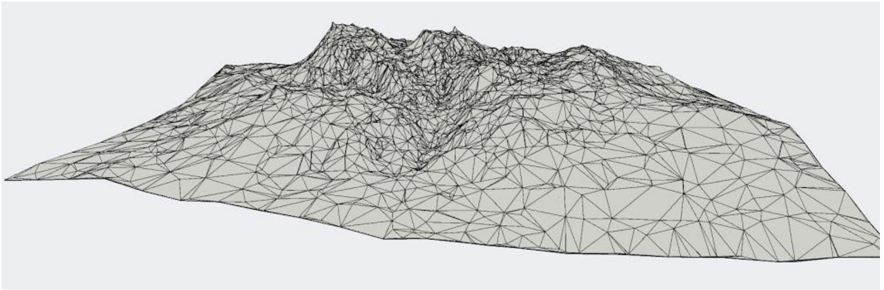

Landscape
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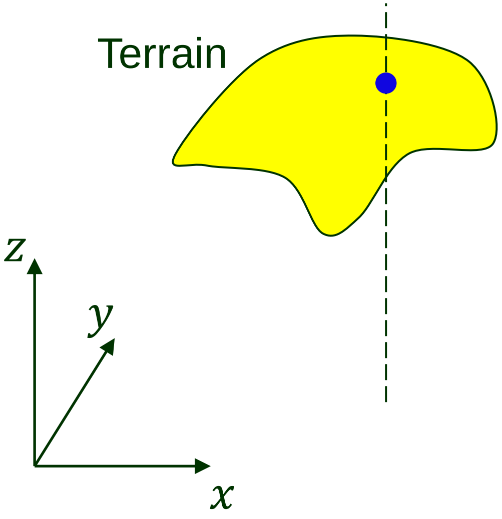

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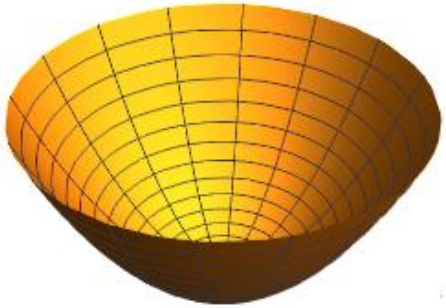

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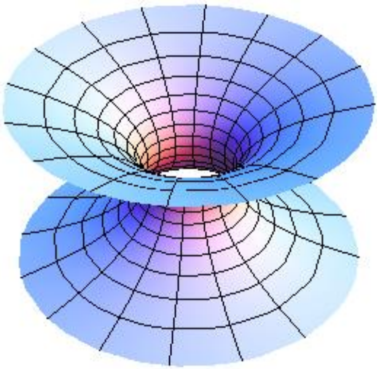

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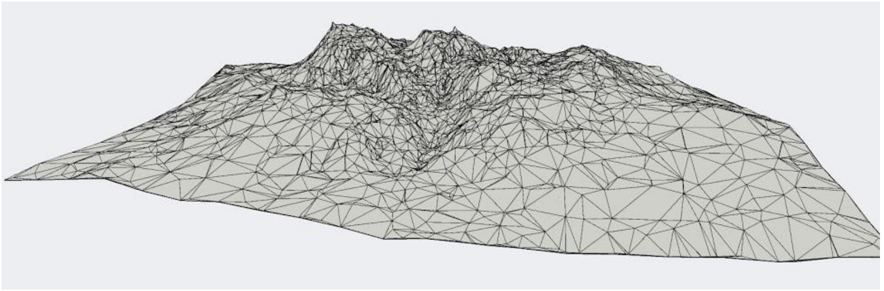

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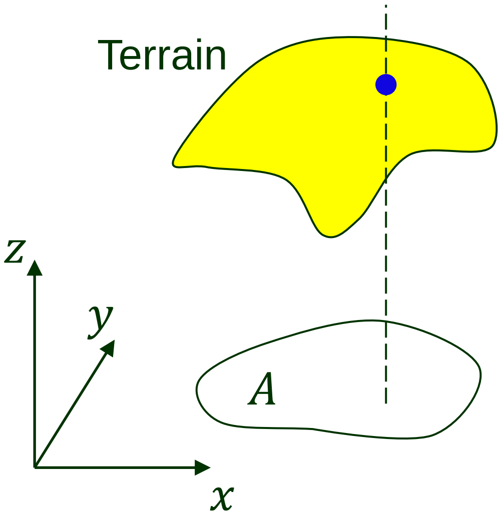

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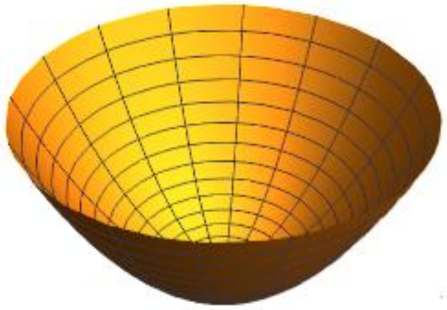

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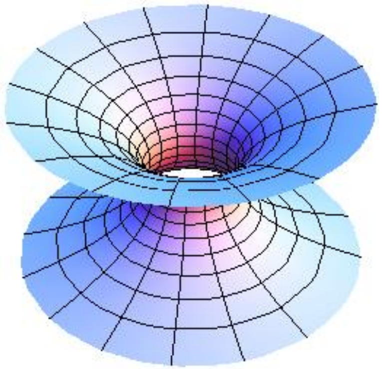

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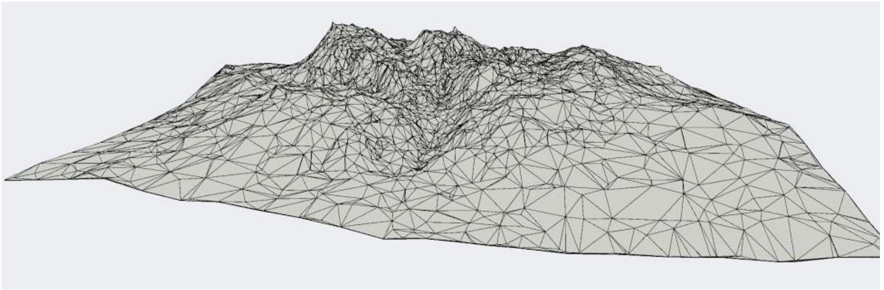

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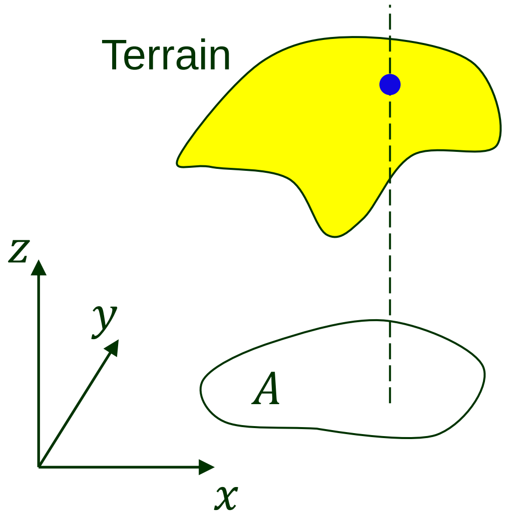

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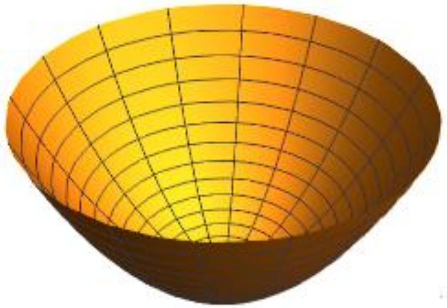

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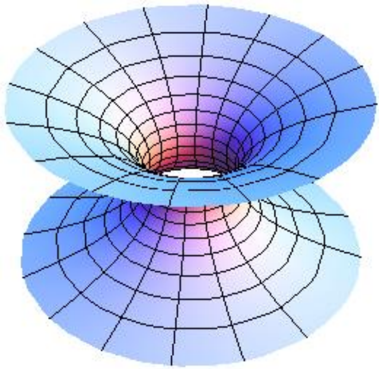

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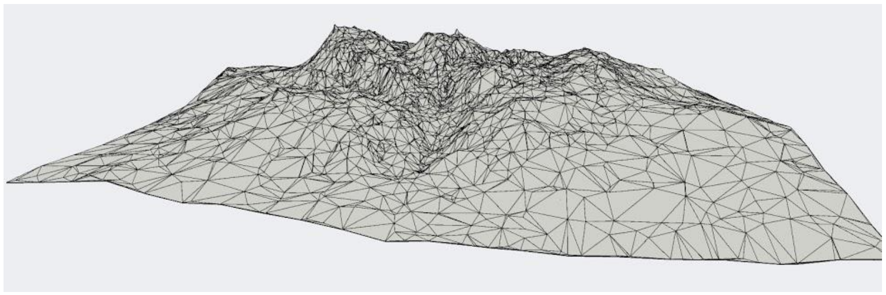

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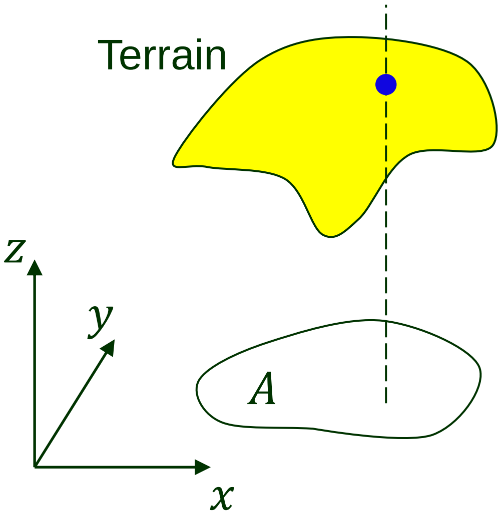

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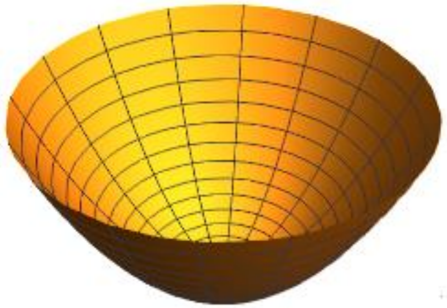

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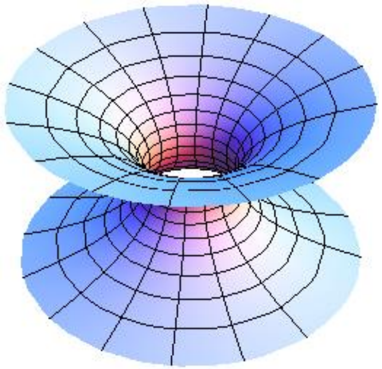

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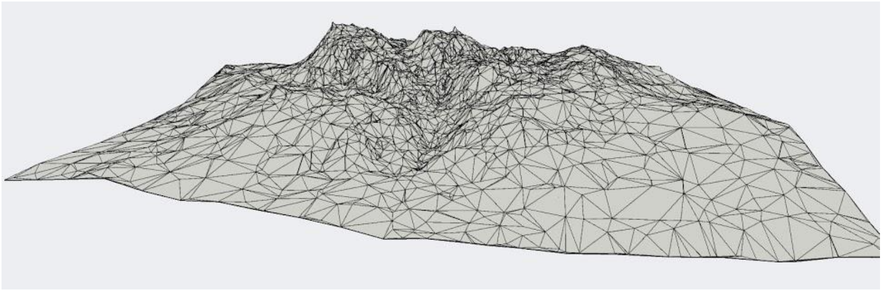

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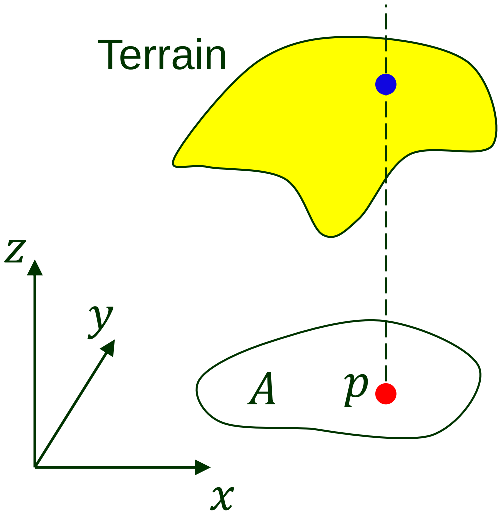

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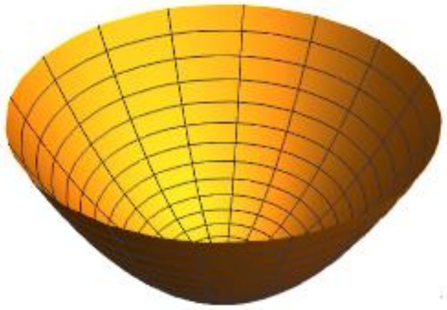

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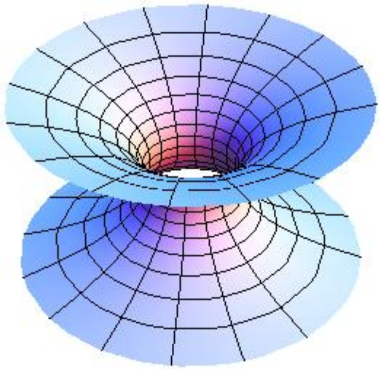

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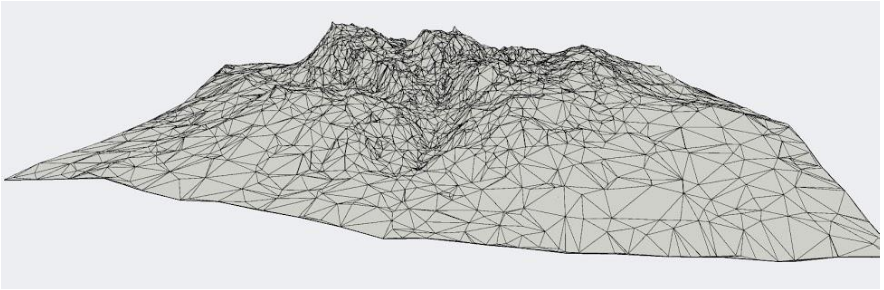

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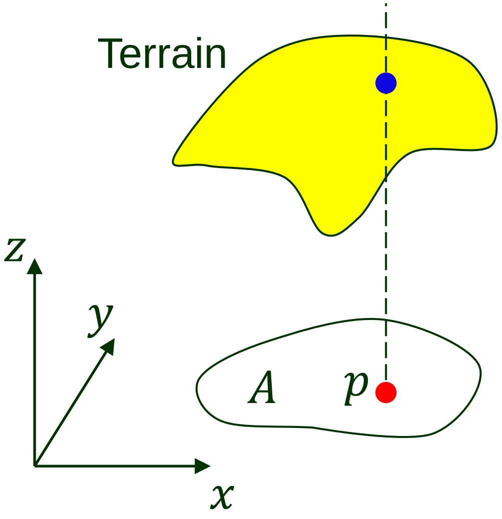

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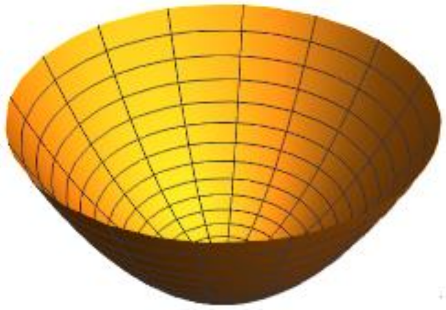

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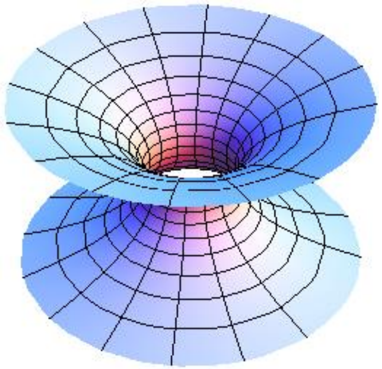

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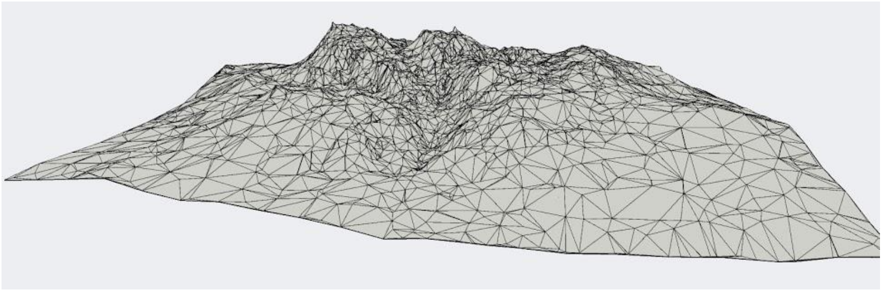

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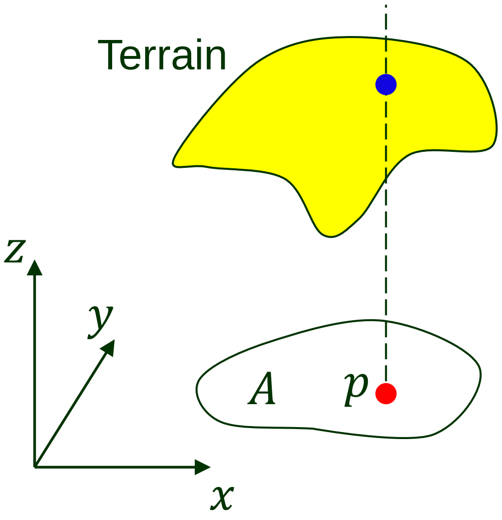

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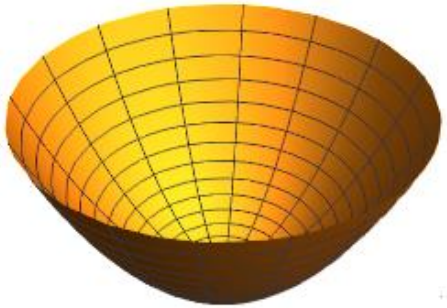

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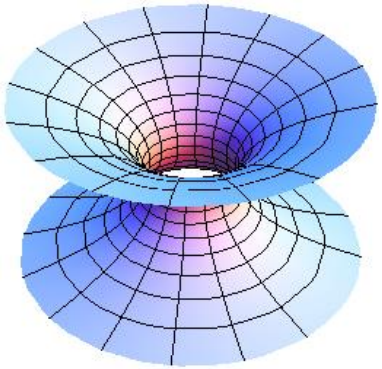

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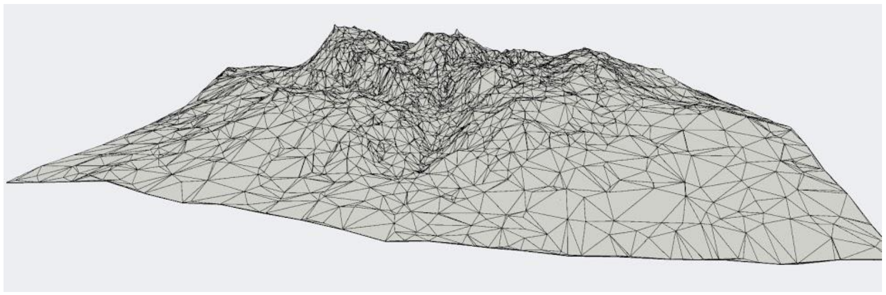

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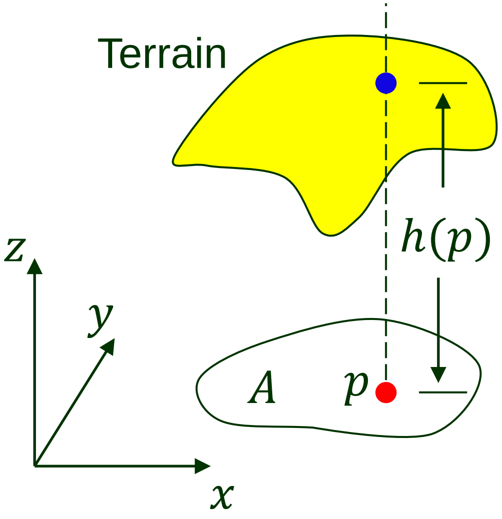

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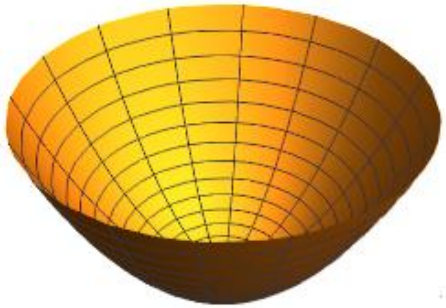

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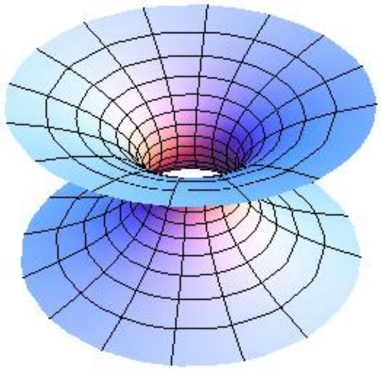

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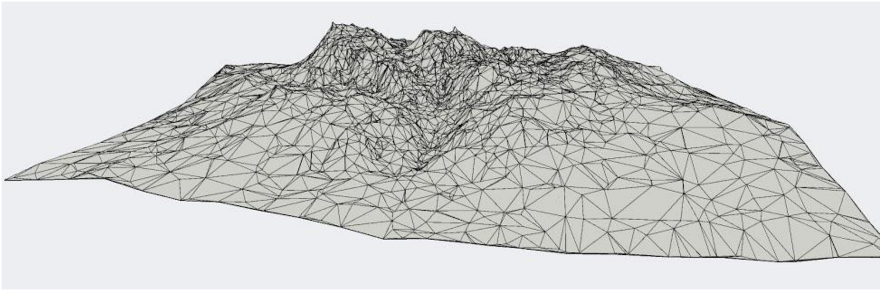

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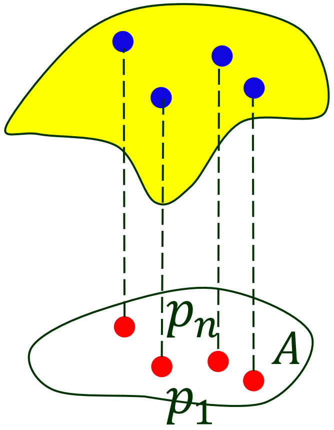

Sample the value of h at a finite set

$$P = \{p_1, p_2, \dots, p_n\} \subseteq A$$

of points.

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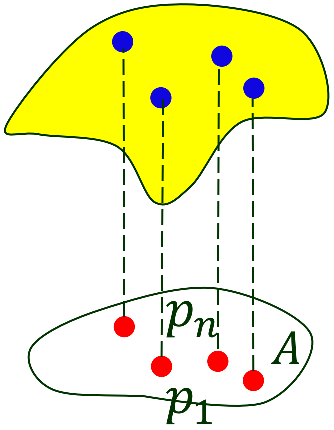

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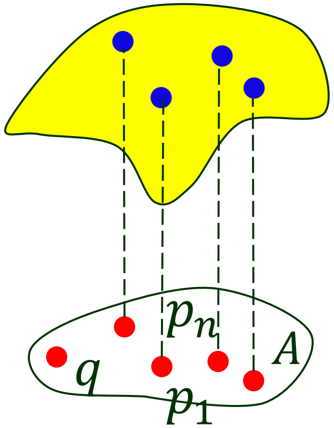

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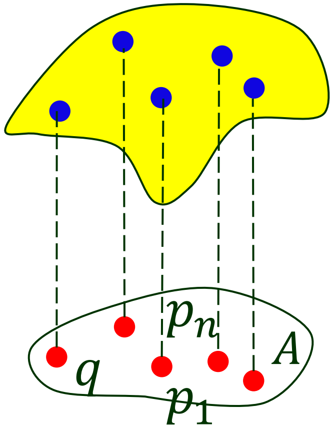

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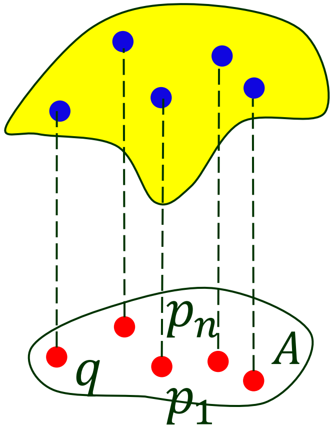

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Naïve approach:

$$h(q) = h(p_i)$$

where $|q - p_i| \le |q - p_j|$ for all $p_j \in A$ (i.e., p_i is the *closest* to q among all points in A).

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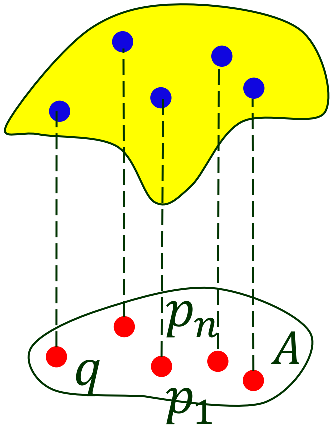

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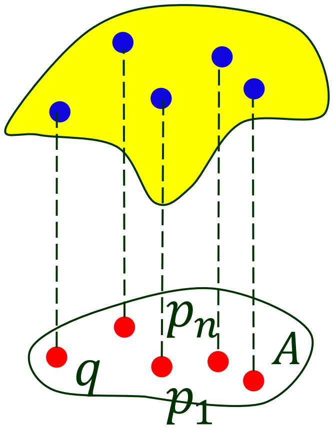

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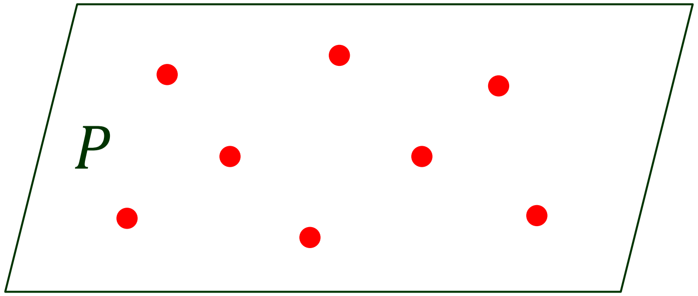
♦ $O(\log n)$ – using the Voronoi diagram of P for fast location of p_i.

- ♣ Not natural (stairlike)
- ♣ Not continuous

Better Interpolation

- Determine a triangulation of P.



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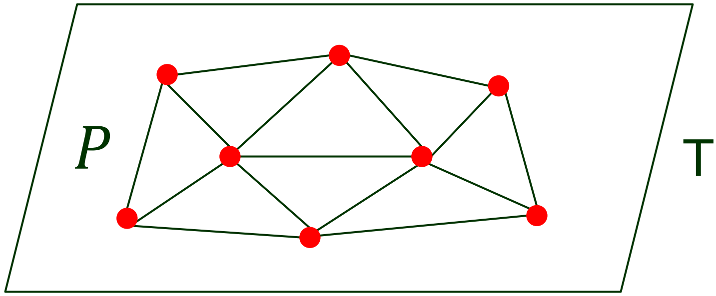
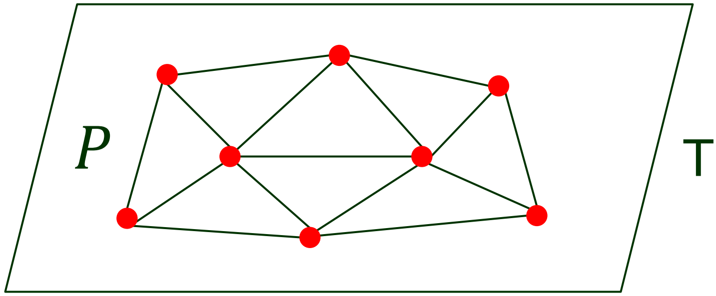


Triangulation

Better Interpolation

- Determine a triangulation of P.
- Lift every sample point to its height.



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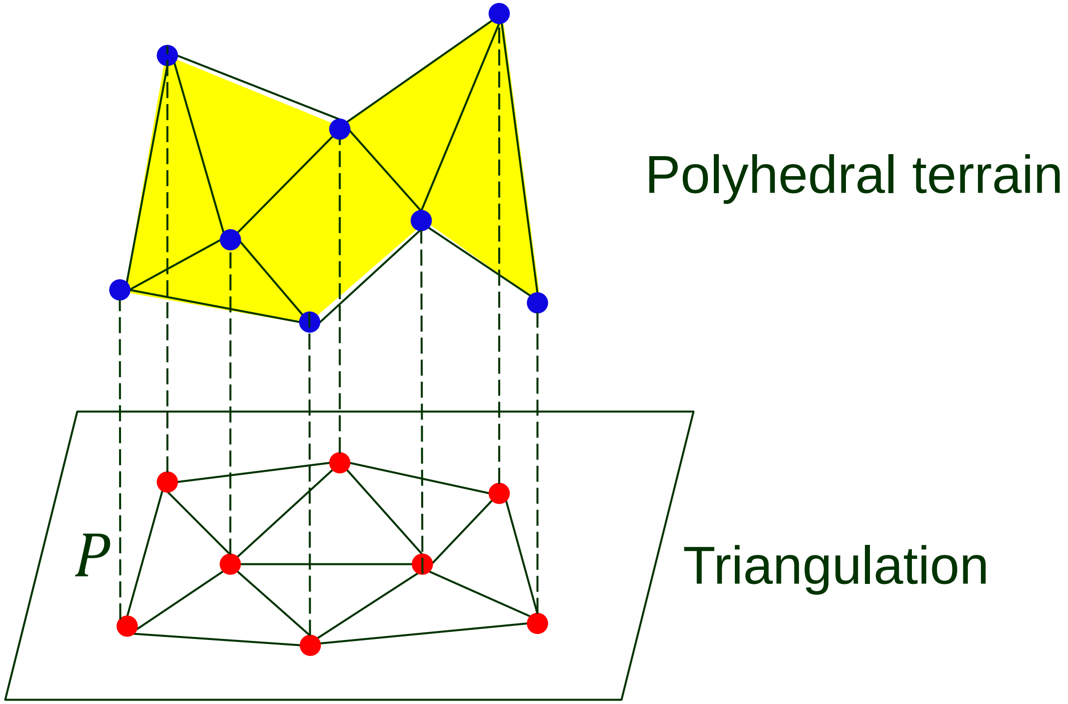

Better Interpolation

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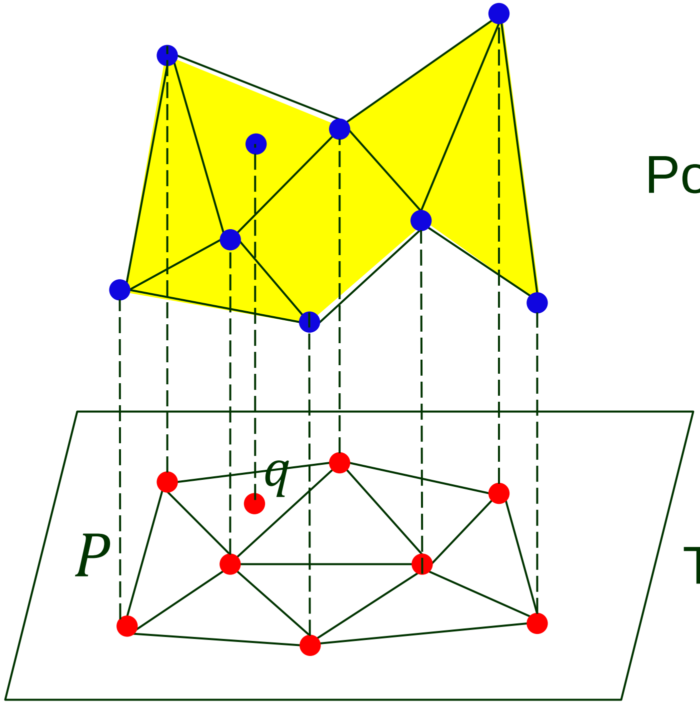

Polyhedral terrain

Triangulation

Better Interpolation

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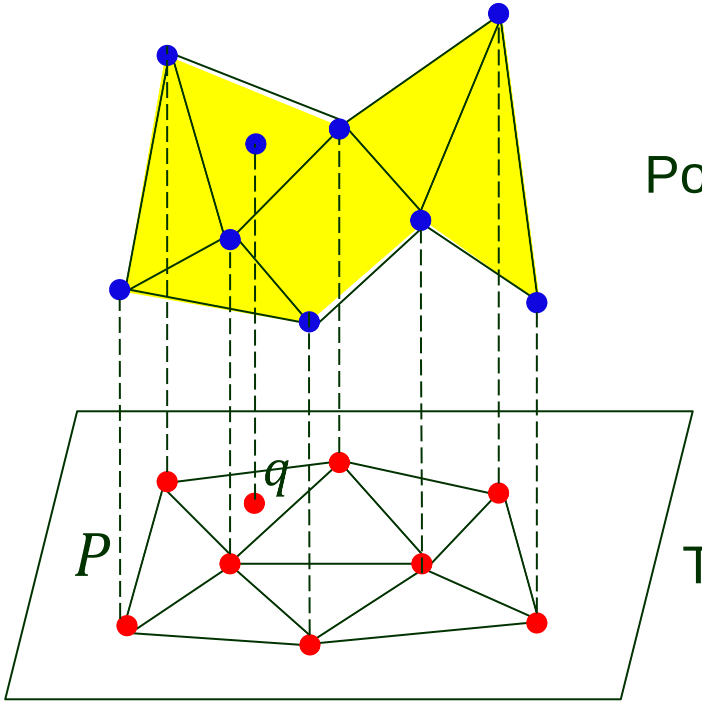
- Lift every sample point to its height.

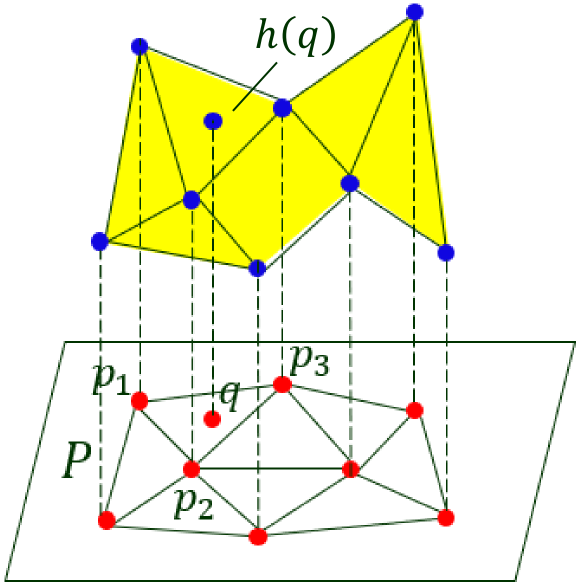


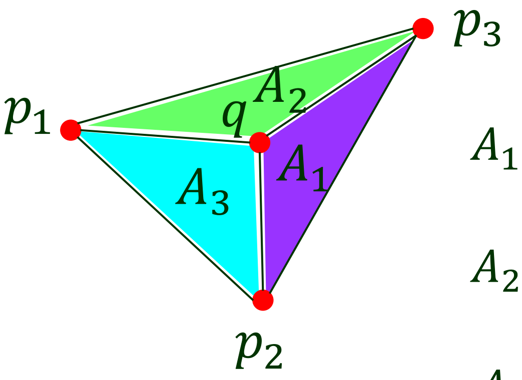
Polyhedral terrain

Triangulation

- What is the height at q?
- How to triangulate P?
- Which triangulation is the most appropriate for approximating a terrain?

Barycentric Interpolation





A_1: area of $\triangle\, qp_2p_3$
(facing p_1)

A_2: area of $\triangle\, qp_3p_1$
(facing p_2)

A_3: area of $\triangle\, qp_1p_2$
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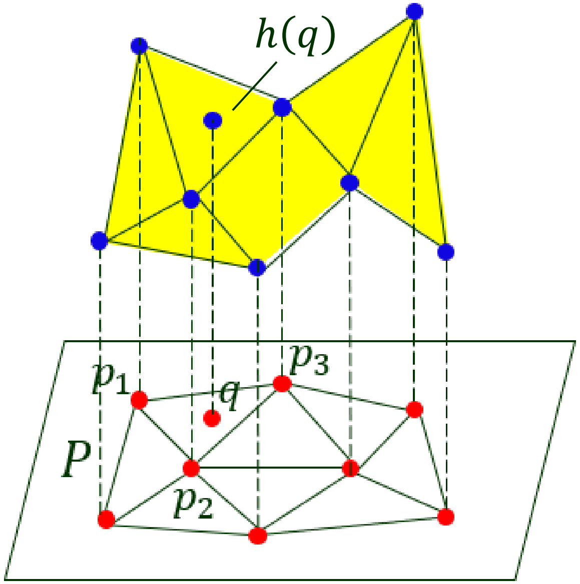
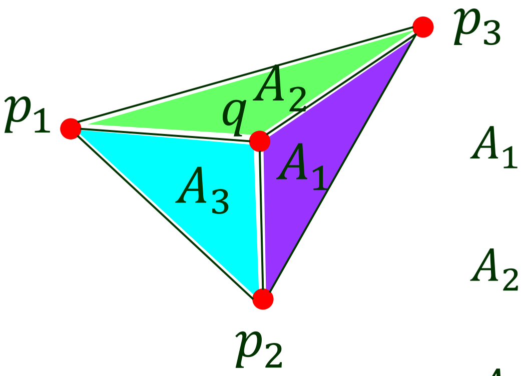
$A = A_1 + A_2 + A_3$: area of $\triangle\, p_1p_2p_3$

Barycentric Interpolation





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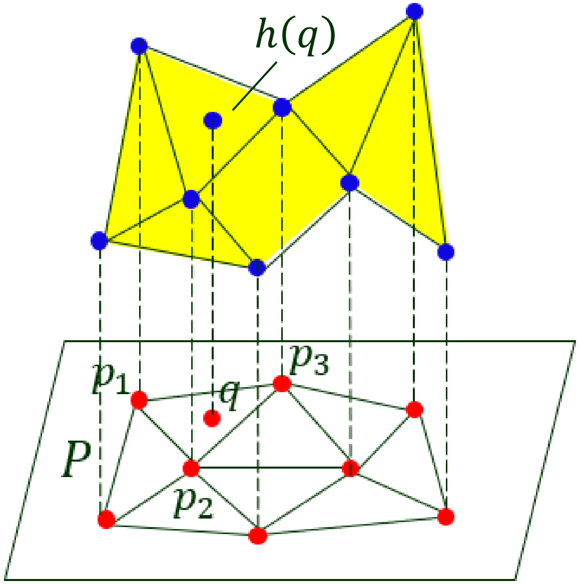
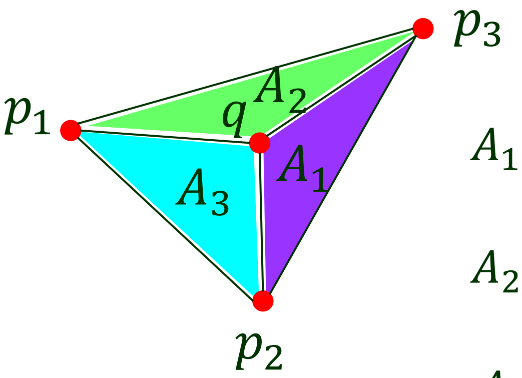
$A = A_1 + A_2 + A_3$: area of $\triangle\, p_1p_2p_3$

$$q = \frac{A_1}{A}p_1 + \frac{A_2}{A}p_2 + \frac{A_3}{A}p_3$$

Barycentric Interpolation





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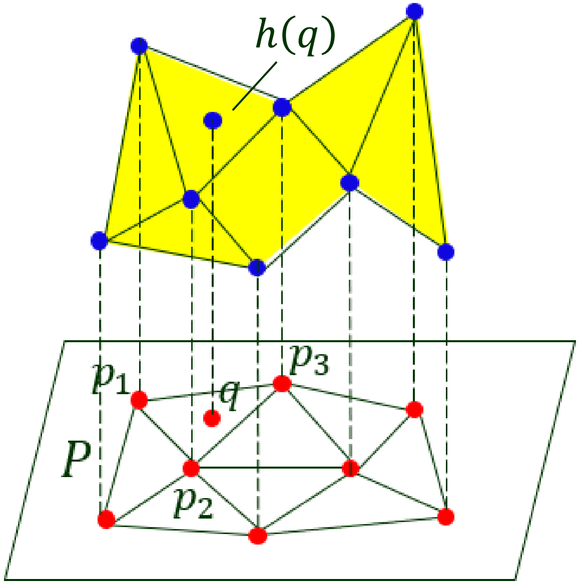
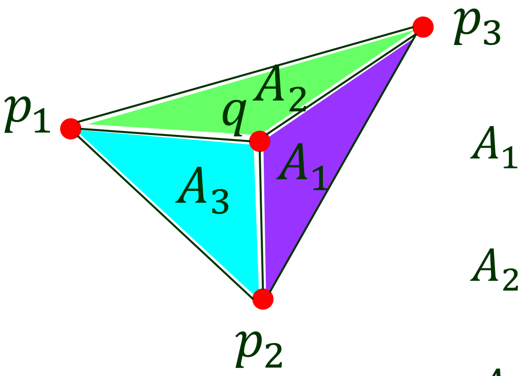
Barycentric coordinates of q: $\left(\frac{A_1}{A}, \frac{A_2}{A}, \frac{A_3}{A}\right)$

Barycentric Interpolation





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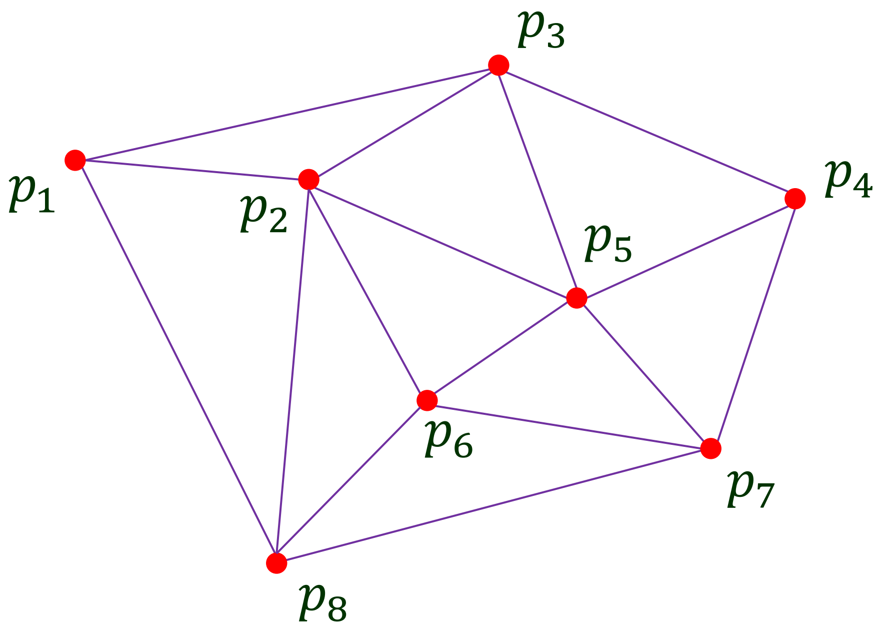
$A = A_1 + A_2 + A_3$: area of $\triangle\, p_1p_2p_3$

$$h(q) = \frac{A_1}{A} h(p_1) + \frac{A_2}{A} h(p_2) + \frac{A_3}{A} h(p_3) \quad \Longleftarrow \quad q = \frac{A_1}{A} p_1 + \frac{A_2}{A} p_2 + \frac{A_3}{A} p_3$$

Barycentric coordinates of q: $\left(\frac{A_1}{A}, \frac{A_2}{A}, \frac{A_3}{A}\right)$

II. Triangulation of a Planar Point Set

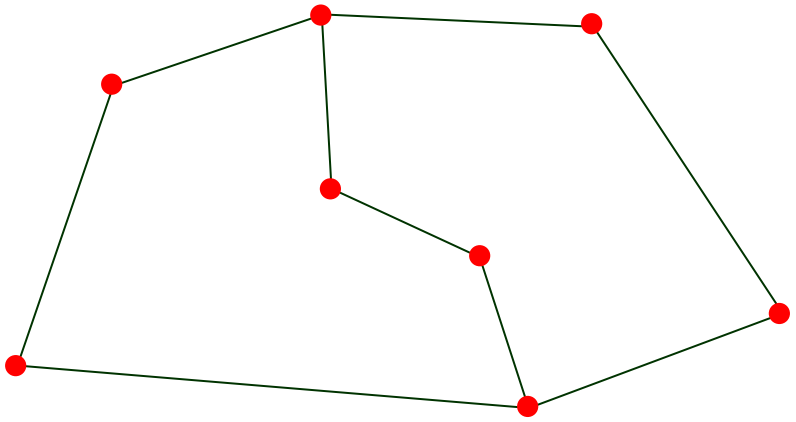
A planar subdivision S is ***maximal*** if no (straight) edge connecting two vertices can be added without destroying its planarity.



A ***triangulation*** of a point set $P = \{p_1, p_2, \ldots, p_n\}$ is the maximal planar subdivision whose vertex set is P.

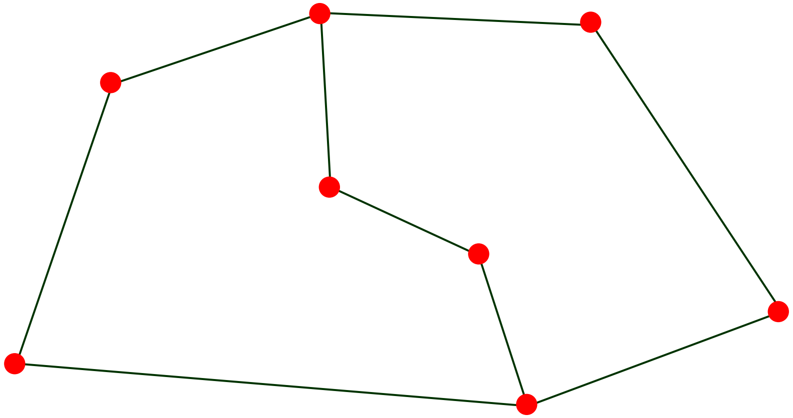
Existence of a Triangulation

- Every bounded face of such a planar subdivision with vertex set P is a polygon.



Existence of a Triangulation

- Every bounded face of such a planar subdivision with vertex set P is a polygon.



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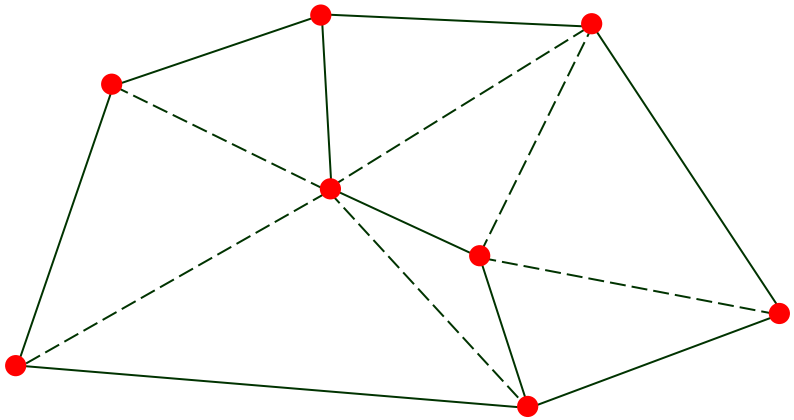

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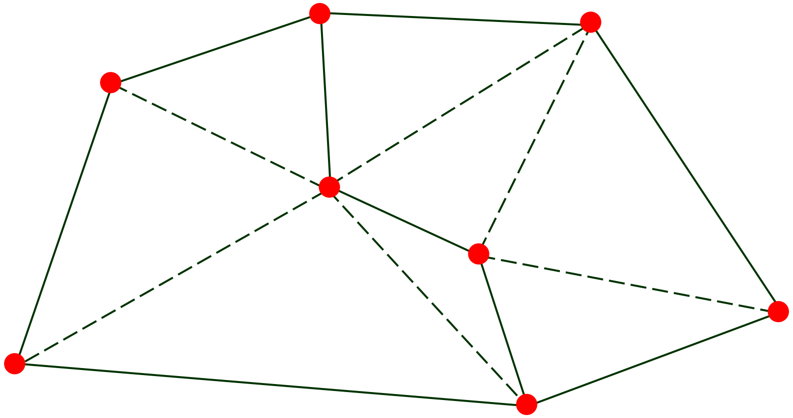

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- Union of bounded faces is the convex hull of P.

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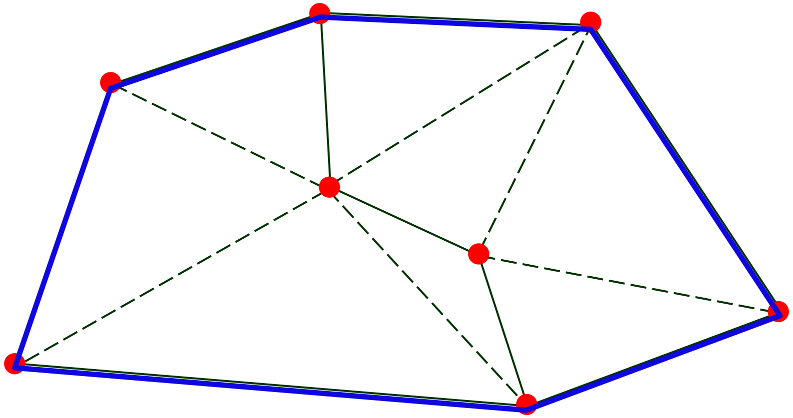

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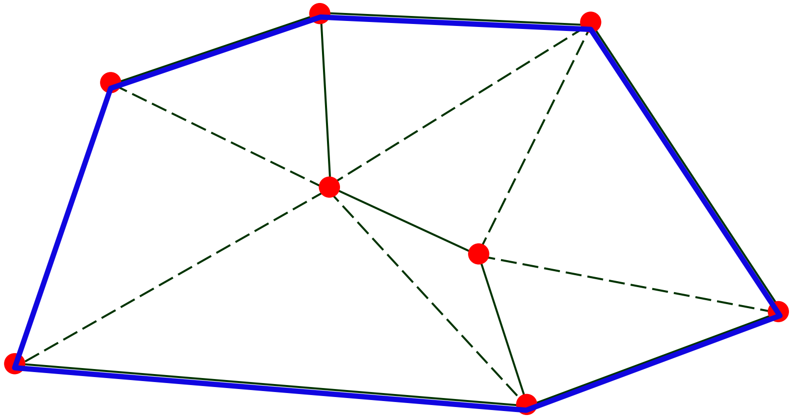

Any polygon can be triangulated!

- Union of bounded faces is the convex hull of P.
- Unbounded face is the complement of the convex hull (i.e., the remaining region in the plane).

Complexity of Triangulation

Assumption The n points in P are not all collinear.

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Theorem Any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.

points from P on its convex hull

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Proof T is a triangulation with m triangles.

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n_f = #faces

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$$n_e = \#\text{edges}$$

$$n_f = \#\text{faces} = m + 1$$

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$$\Rightarrow \quad 3m + k = 2n_e$$

$$\Downarrow$$

$$n_e = (3m + k)/2$$

Complexity (cont'd)

Euler's formula:

$$n - n_e + n_f = 2$$

Complexity (cont'd)

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$$n - n_e + n_f = 2$$

$$n_f = m + 1$$

Complexity (cont'd)

Euler's formula:

$$\left.\begin{aligned} n - n_e + n_f &= 2 \\ n_f &= m + 1 \end{aligned}\right\} \Rightarrow$$

$$m = 2 - n + n_e - 1$$

Complexity (cont’d)

Euler’s formula:

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$$m = 2 - n + n_e - 1$$

$$n_e = (3m + k)/2$$

$$m = 2n - 2 - k$$

$$n_e = (3m + k)/2$$

$$n_e = 3n - 3 - k$$

Complexity (cont'd)

Euler's formula:

$$\left.\begin{aligned} n - n_e + n_f &= 2 \\ n_f &= m + 1 \end{aligned}\right\} \Rightarrow$$

$$m = 2 - n + n_e - 1$$

$$\Big\Downarrow \quad n_e = (3m + k)/2$$

$$m = 2n - 2 - k$$

$$\Big\Downarrow \quad n_e = (3m + k)/2$$

$$n_e = 3n - 3 - k$$

$\square$

III. Angle Vector

T: triangulation with m triangles which have $3m$ angles.

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- Order these angles into an *angle vector*:

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- T' is another triangulation with angle vector:

$$A(T') = \langle \alpha'_1, \alpha'_2, \ldots, \alpha'_{3m} \rangle$$

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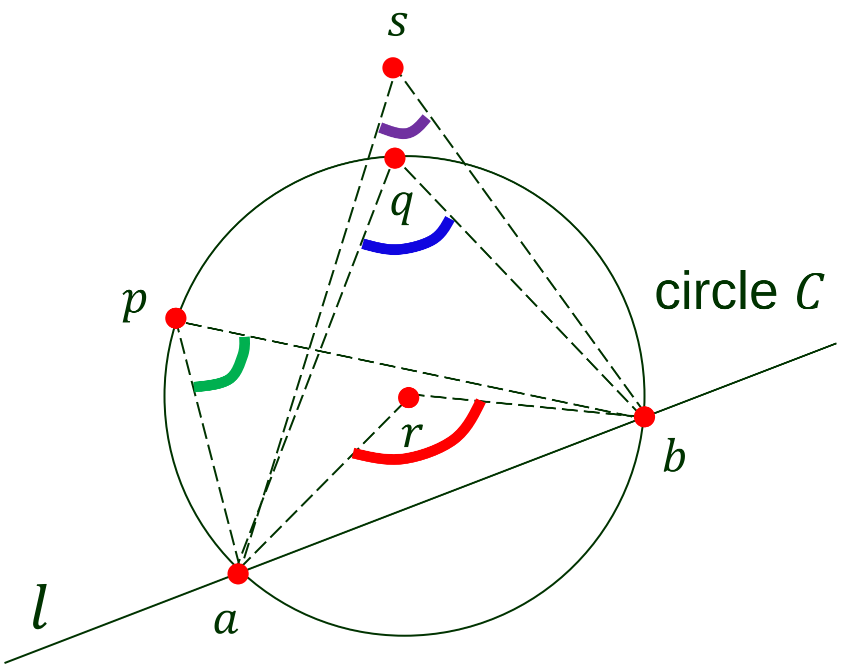
- T is ***angle optimal*** if $A(T) \geq A(T')$ for all triangulations T'.

Thales' Theorem

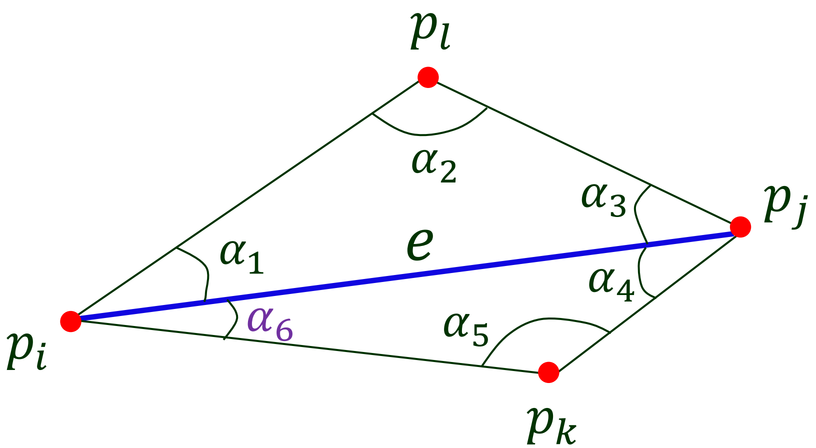


Four angles subtended by the arc $\widehat{ab}$:

$$\angle arb > \angle apb = \angle aqb > \angle asb$$

Illegal Edge

An edge $e = \overline{p_i p_j}$ is incident to two triangles.



$\langle \alpha_6, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_5 \rangle$

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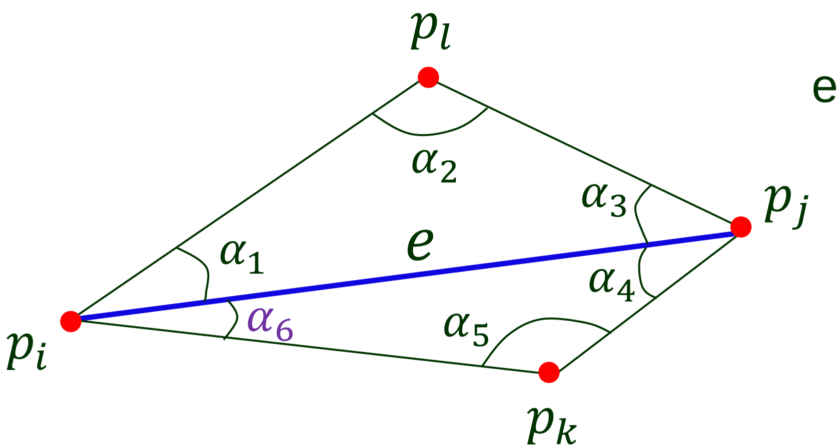

edge flip

$\langle \alpha_6, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_5 \rangle$

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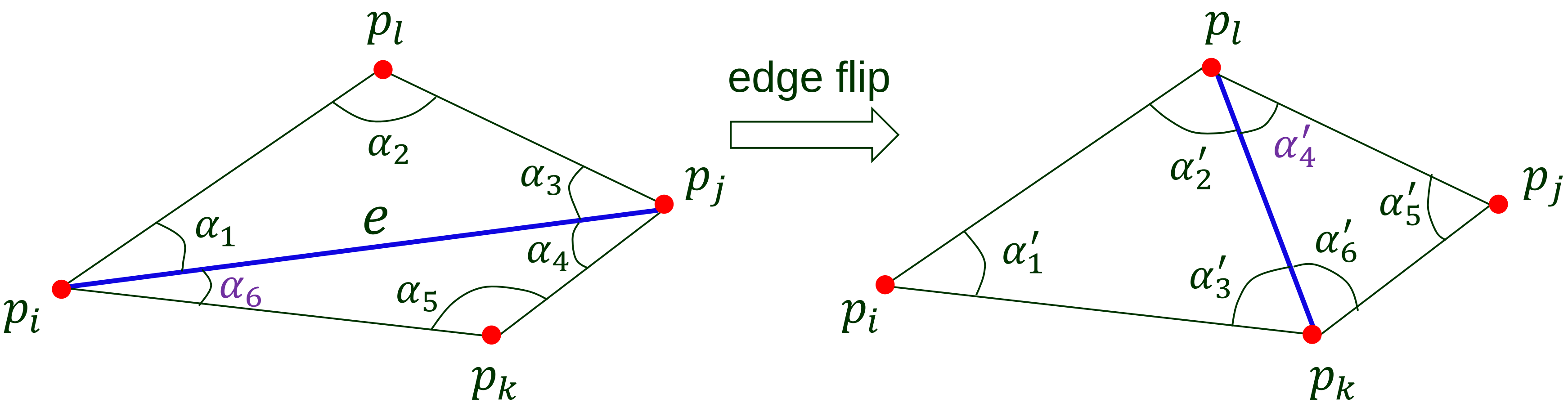
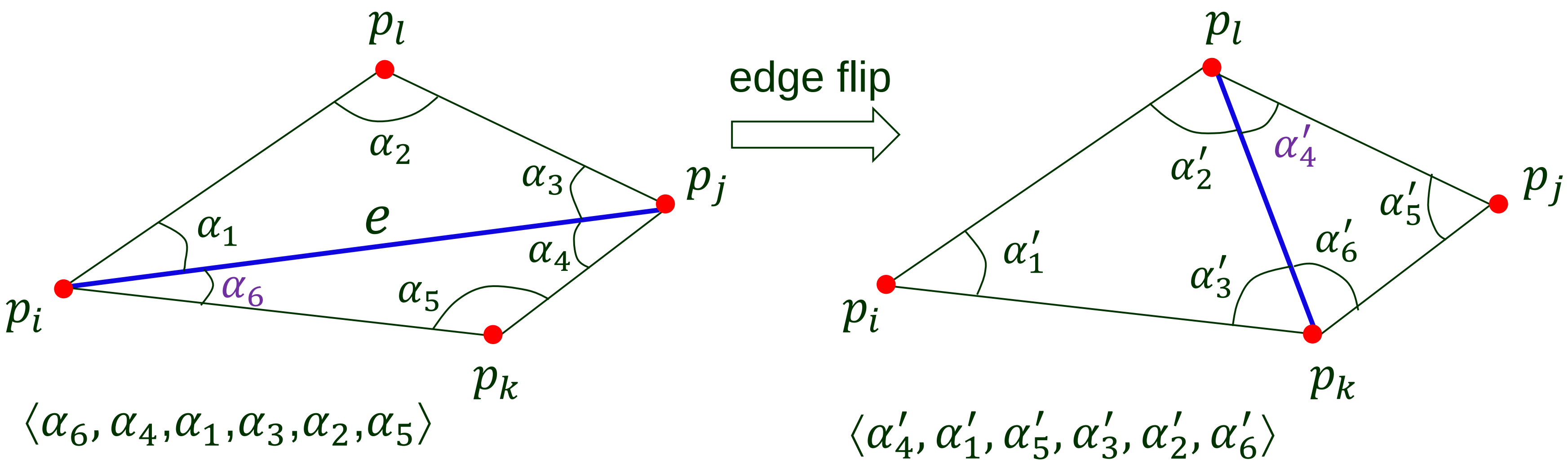


$\langle \alpha_6, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_5 \rangle$

$\langle \alpha'_4, \alpha'_1, \alpha'_5, \alpha'_3, \alpha'_2, \alpha'_6 \rangle$

Illegal Edge

An edge $e = \overline{p_i p_j}$ is incident to two triangles.



e is an *illegal edge* if $$\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i$$

Illegal Edge

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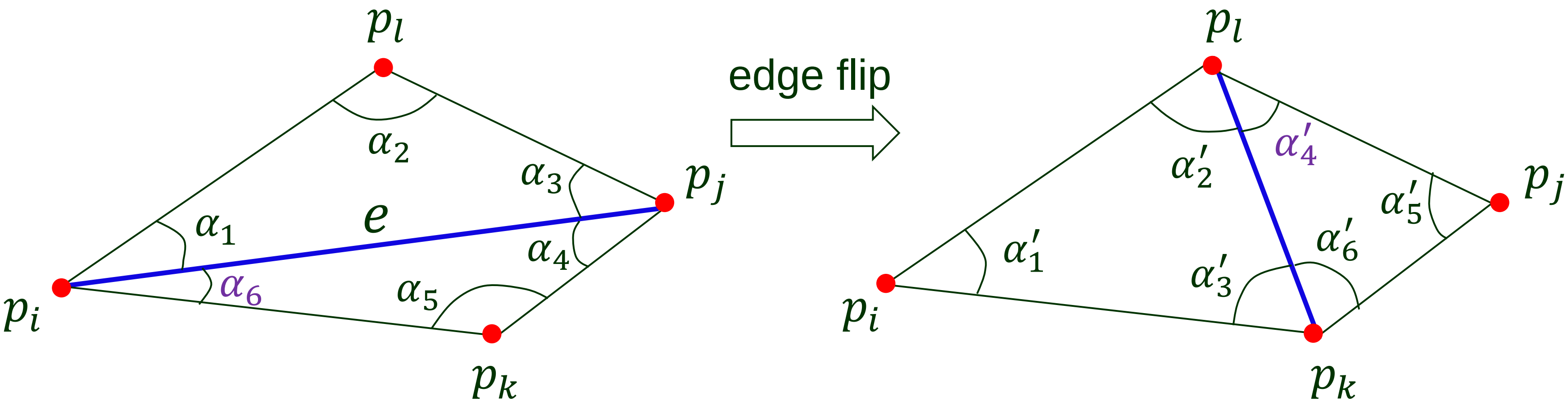

$\langle \alpha_6, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_5 \rangle$

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$= \alpha_6$

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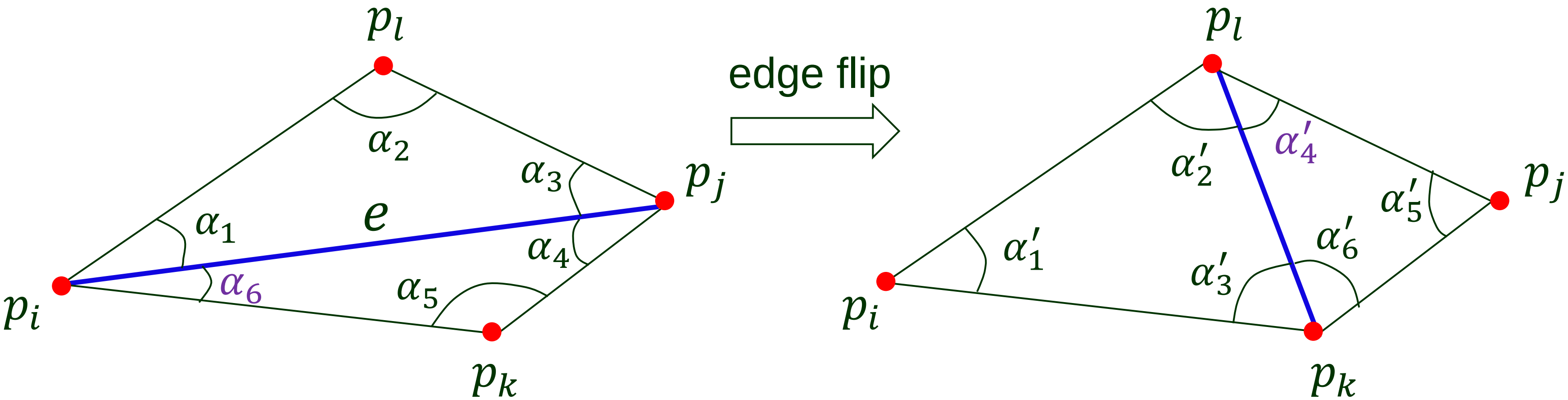

$\langle \alpha_6, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_5 \rangle$

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$= \alpha_6$

$= \alpha'_4$

Illegal Edge

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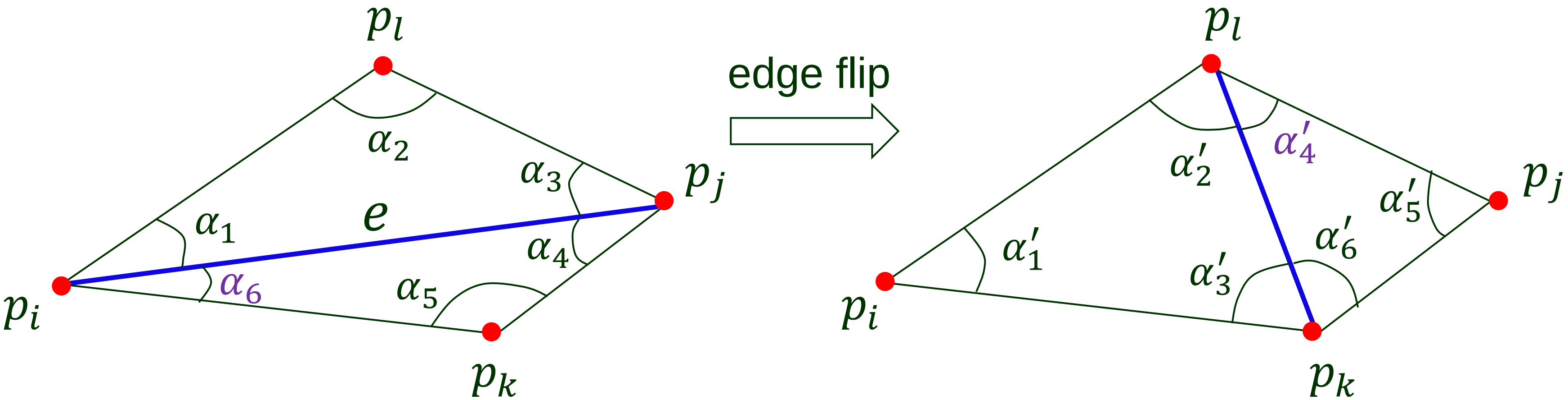


$\langle \alpha_6, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_5 \rangle$

$\langle \alpha'_4, \alpha'_1, \alpha'_5, \alpha'_3, \alpha'_2, \alpha'_6 \rangle$

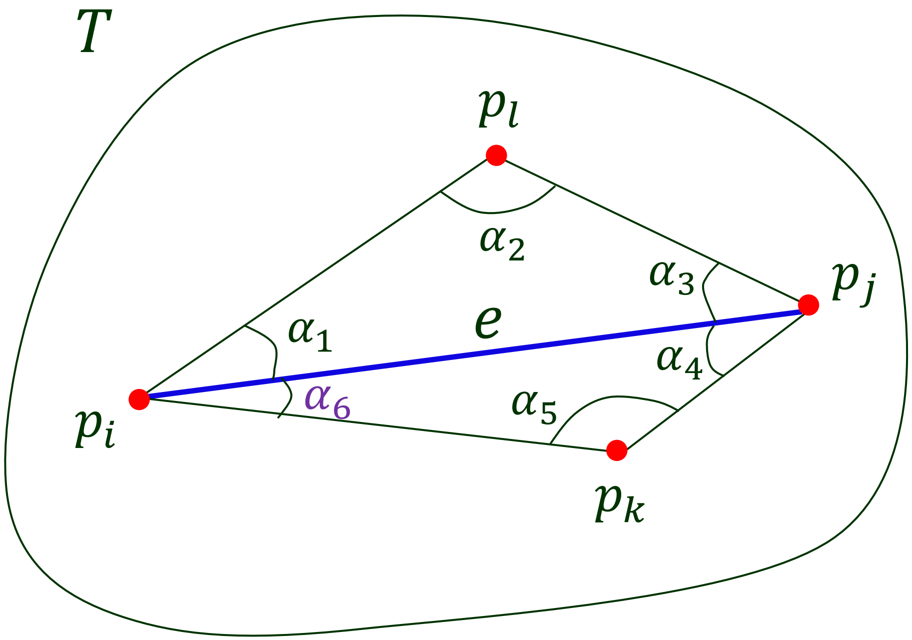
e is an *illegal edge* if $\min\limits_{1 \le i \le 6} \alpha_i < \min\limits_{1 \le i \le 6} \alpha'_i$

$= \alpha_6$

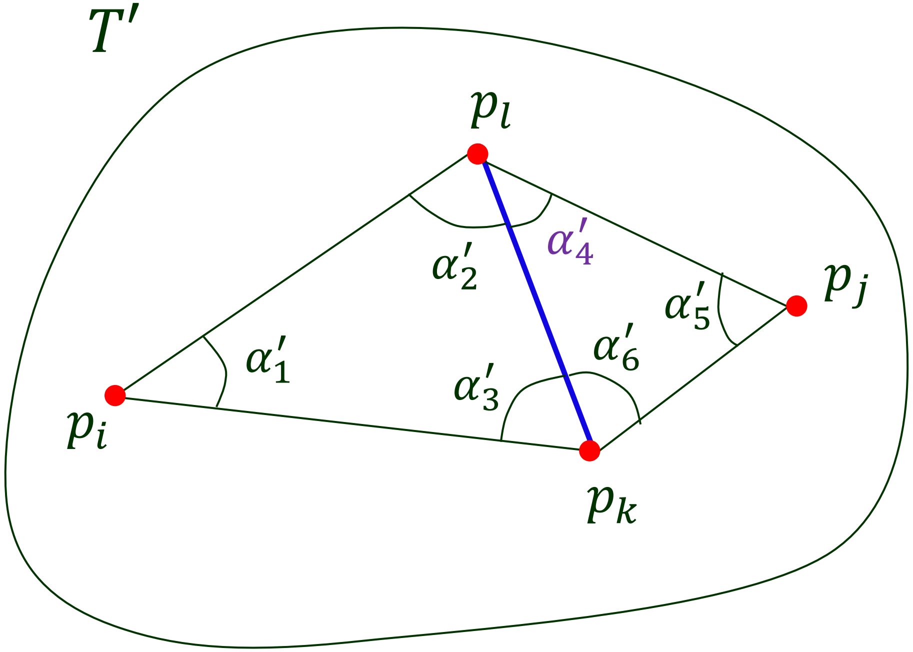
$= \alpha'_4$

We can locally increase the smallest angle by flipping edges.

Edge Flip

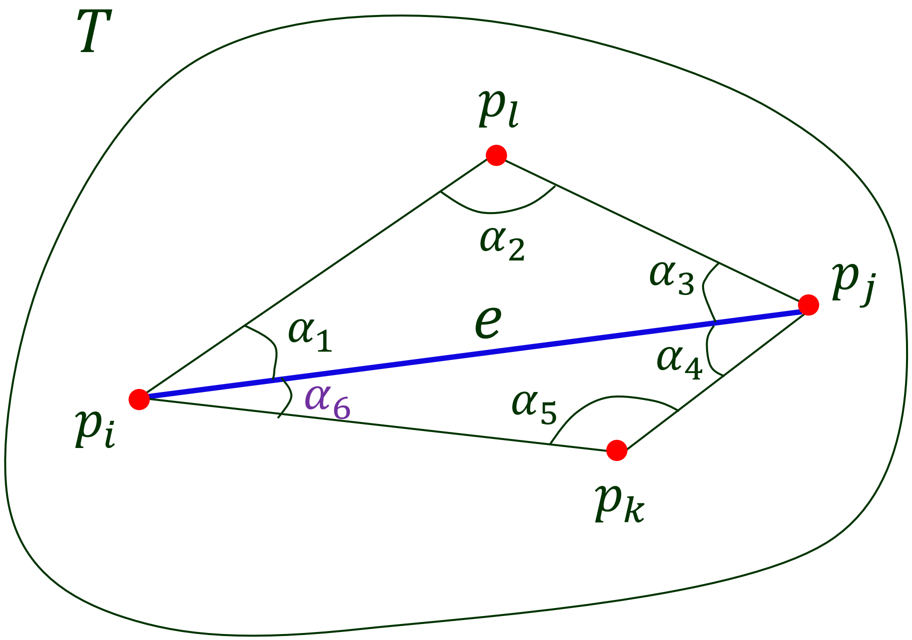


Triangulation T
(before flipping e)



Triangulation T'
(after flipping e)

Edge Flip



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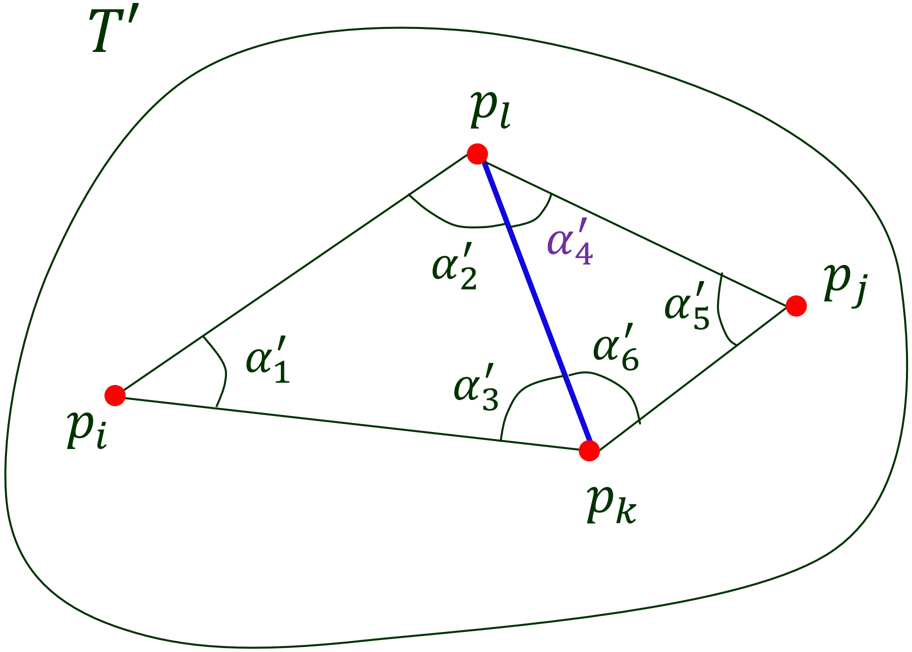
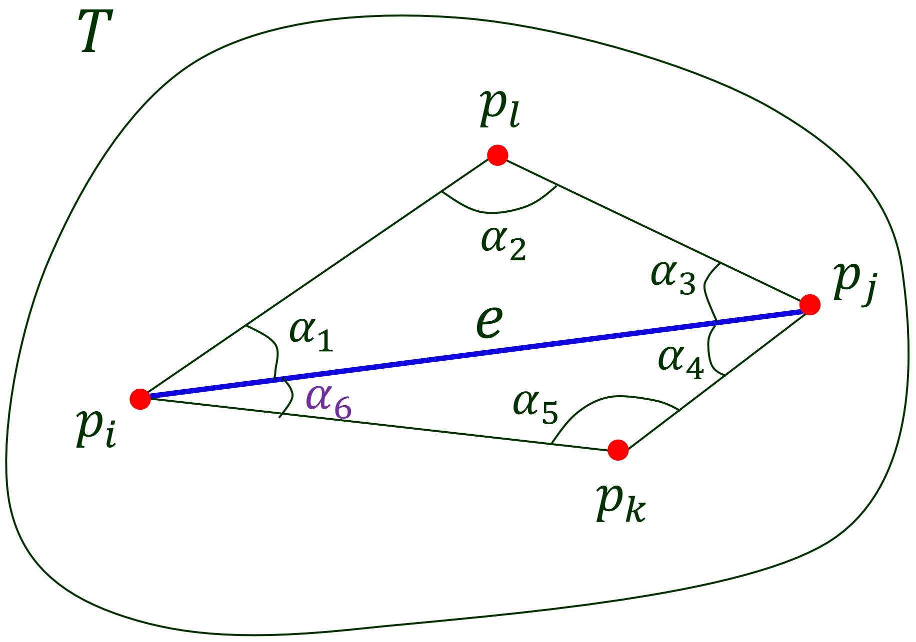


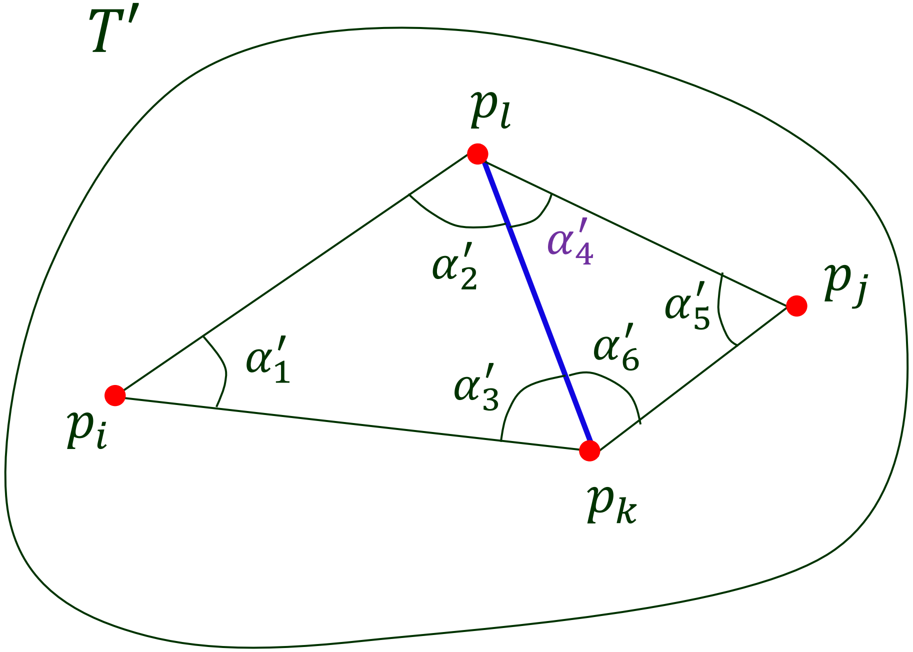
Triangulation T'
(after flipping e)

- It can be shown that $A(T') > A(T)$.

Edge Flip



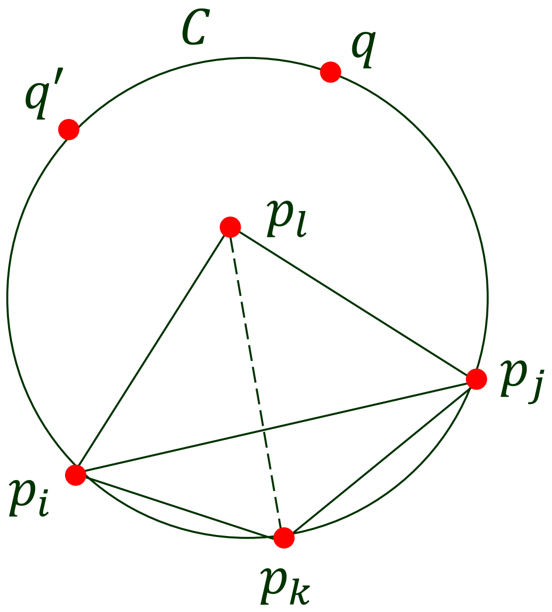
Triangulation T
(before flipping e)



Triangulation T'
(after flipping e)

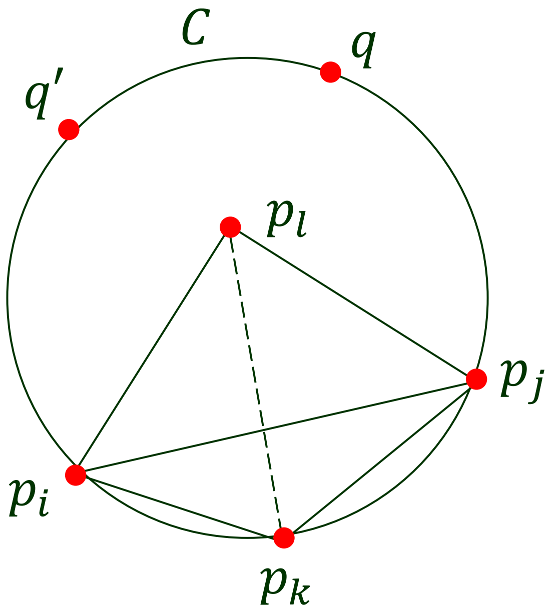
- It can be shown that $A(T') > A(T)$.
- Computing $\alpha_1, \alpha_2, \ldots, \alpha_6$ and $\alpha'_1, \alpha'_2, \ldots, \alpha'_6$ is unnecessary!

Determining an Illegal Edge



Theorem Edge $\overline{p_i p_j}$ is illegal iff p_l lies in the interior of circle C determined by p_i, p_j, p_k.

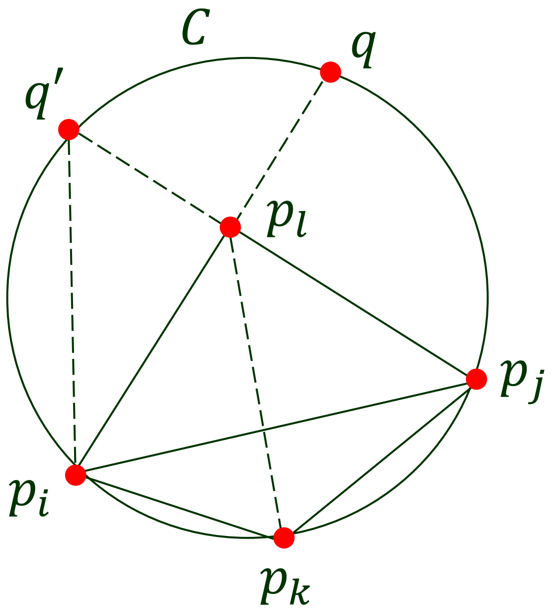
Determining an Illegal Edge



Theorem Edge $\overline{p_i p_j}$ is illegal iff p_l lies in the interior of circle C determined by p_i, p_j, p_k.

Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

Determining an Illegal Edge



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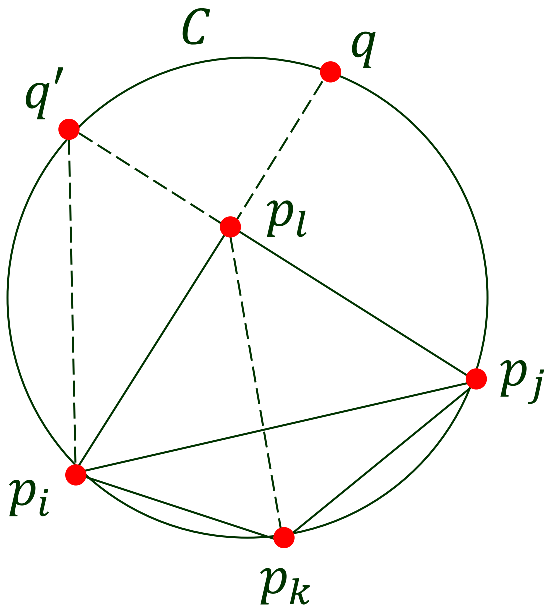
Determining an Illegal Edge



$\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T.

Theorem Edge $\overline{p_i p_j}$ is illegal iff p_l lies in the interior of circle C determined by p_i, p_j, p_k.

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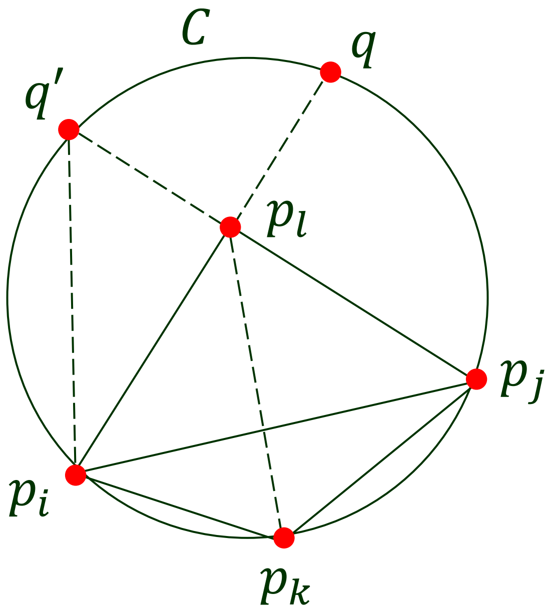
Determining an Illegal Edge



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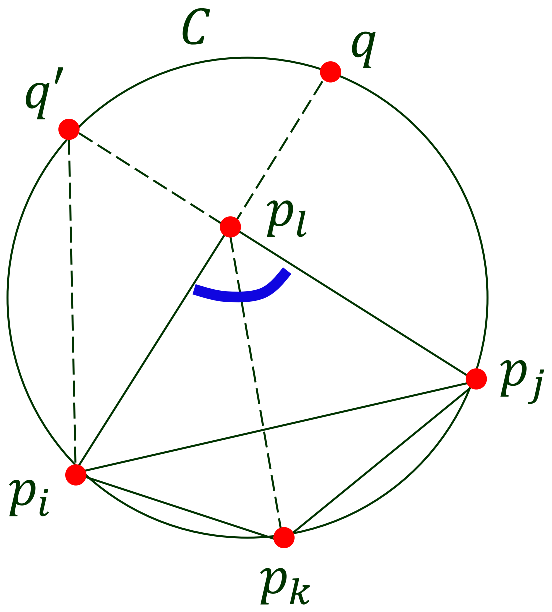
$\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T. ⇐ {

Determining an Illegal Edge



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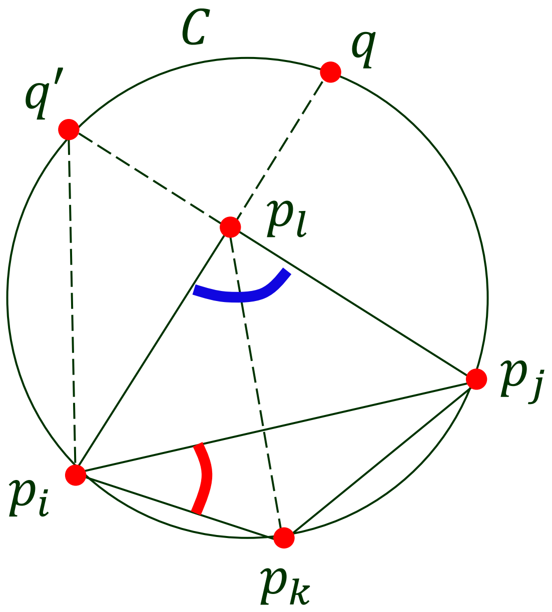
Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

$\angle p_i p_l p_j$

$\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T. ⇐ $\Big\{$

Determining an Illegal Edge



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$\angle p_i p_l p_j$

$\angle p_k p_i p_j$

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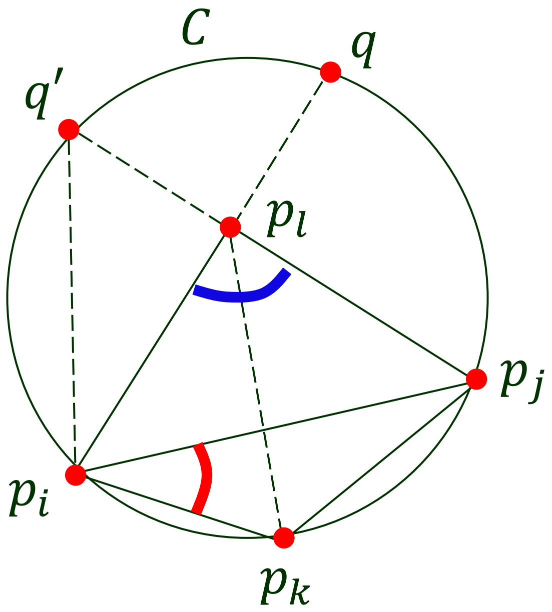
Determining an Illegal Edge



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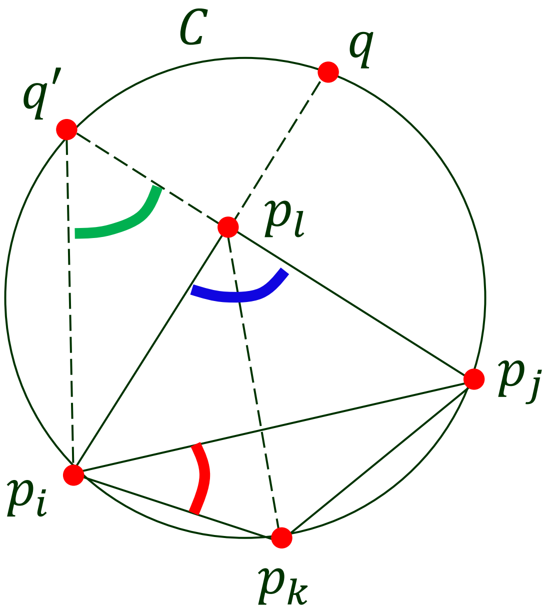
Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

$\angle p_i p_l p_j > \angle p_i q' p_j$ $\angle p_k p_i p_j$

$\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T.

Determining an Illegal Edge



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Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

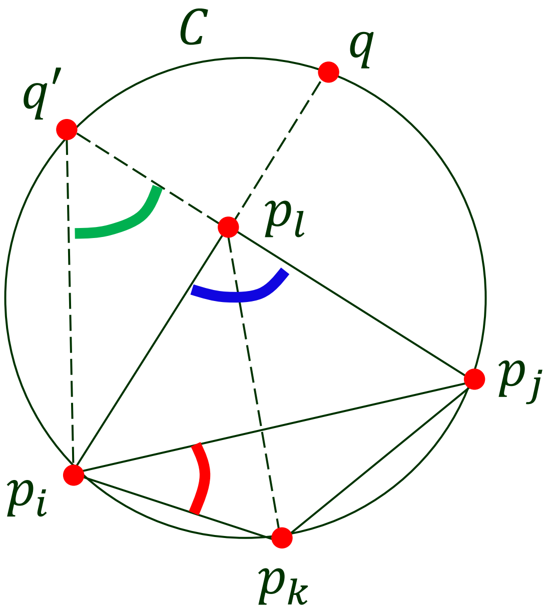
$\angle p_ip_lp_j > \angle p_iq'p_j$ $\angle p_kp_ip_j$

subtended by

arc $\widehat{p_ip_j}$

$\angle p_ip_lp_j$ and $\angle p_ip_kp_j$ are not the smallest angle in T.

Determining an Illegal Edge



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$\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T.

⇐

$\angle p_i p_l p_j > \angle p_i q' p_j$

subtended by ⇓

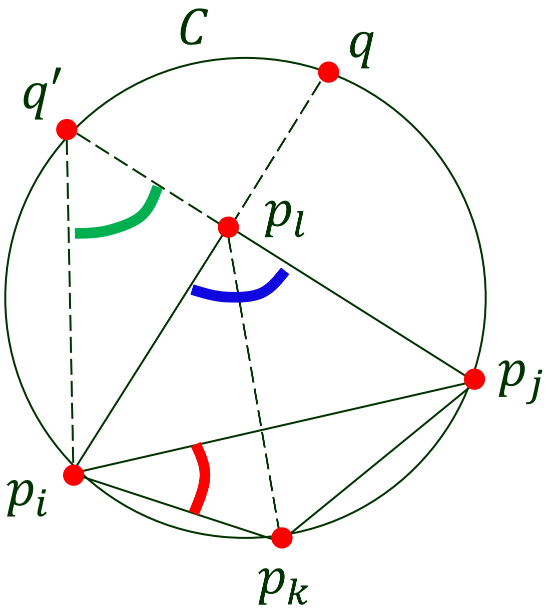
arc $\widehat{p_i p_j}$

$\angle p_k p_i p_j$

⇓ subtended by

arc $\widehat{p_k p_j}$

Determining an Illegal Edge



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Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

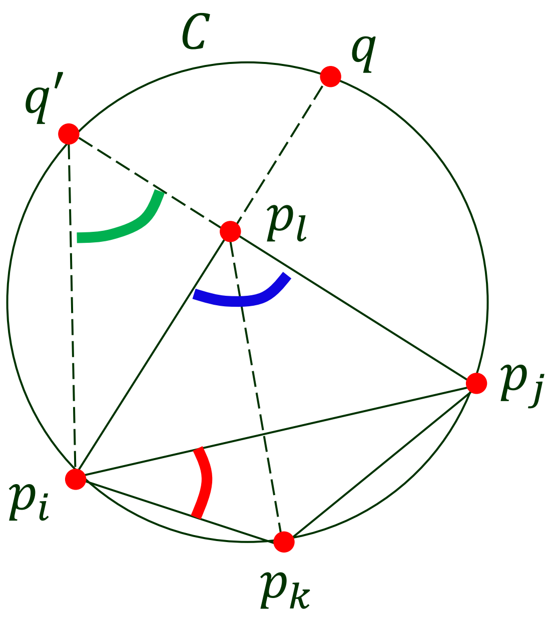
$\angle p_i p_l p_j > \angle p_i q' p_j$ $\quad\quad$ $\angle p_k p_i p_j$

subtended by ⇓ $\quad\quad$ ⇓ subtended by

arc $\widehat{p_i p_j} \supset$ arc $\widehat{p_k p_j}$

⇐ $\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T.

Determining an Illegal Edge



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Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

$$\angle p_ip_lp_j > \angle p_iq'p_j > \angle p_kp_ip_j$$

subtended by ⇓ ⇓ subtended by

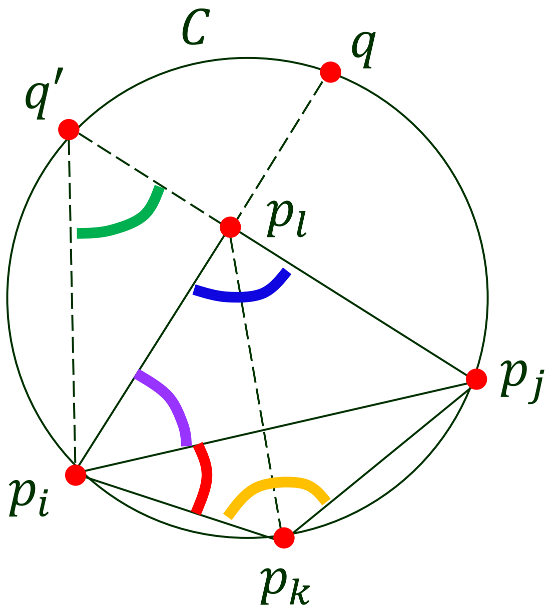
arc $\widehat{p_ip_j}$ $\supset$ arc $\widehat{p_kp_j}$

⇐

$\angle p_ip_lp_j$ and $\angle p_ip_kp_j$ are not the smallest angle in T.

Determining an Illegal Edge

Theorem Edge $\overline{p_ip_j}$ is illegal iff p_l lies in the interior of circle C determined by p_i, p_j, p_k.



Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

$\angle p_ip_lp_j > \angle p_iq'p_j > \angle p_kp_ip_j$

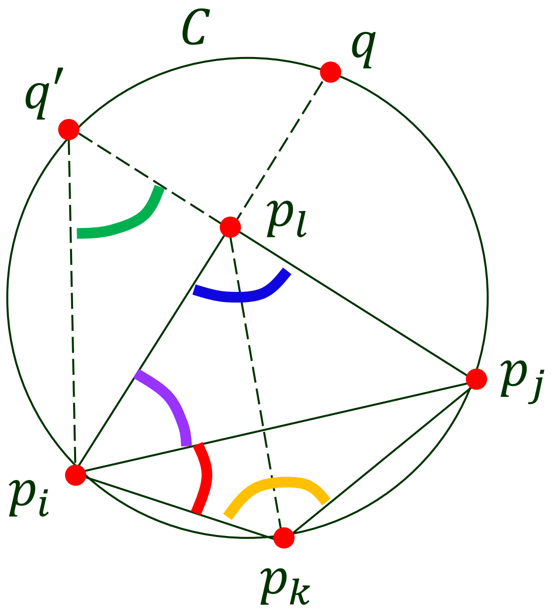
subtended by ⇓ ⇓ subtended by

arc $\widehat{p_ip_j} \supset$ arc $\widehat{p_kp_j}$

$\angle p_ip_kp_j$ $\angle p_lp_ip_j$

⇐ $\angle p_ip_lp_j$ and $\angle p_ip_kp_j$ are not the smallest angle in T.

Determining an Illegal Edge



Theorem Edge $\overline{p_i p_j}$ is illegal iff p_l lies in the interior of circle C determined by p_i, p_j, p_k.

Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

$\angle p_i p_l p_j > \angle p_i q' p_j > \angle p_k p_i p_j$

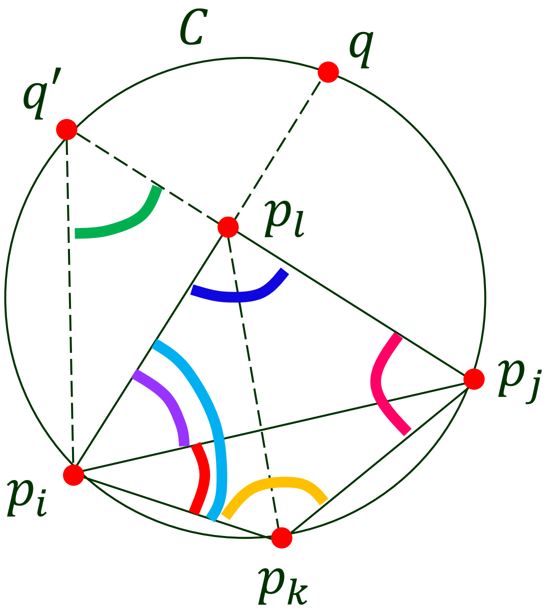
subtended by ⇓ ⇓ subtended by

arc $\widehat{p_i p_j} \supset$ arc $\widehat{p_k p_j}$

$\angle p_i p_k p_j > \angle q p_i p_j = \angle p_l p_i p_j$

⇐ $\angle p_i p_l p_j$ and $\angle p_i p_k p_j$ are not the smallest angle in T.

Determining an Illegal Edge



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Proof (⇐) Let T and T' be the triangulations of the 4 points p_i, p_j, p_k, p_l before and after the flip.

$\angle p_ip_lp_j > \angle p_iq'p_j > \angle p_kp_ip_j$

subtended by ⇓ ⇓ subtended by

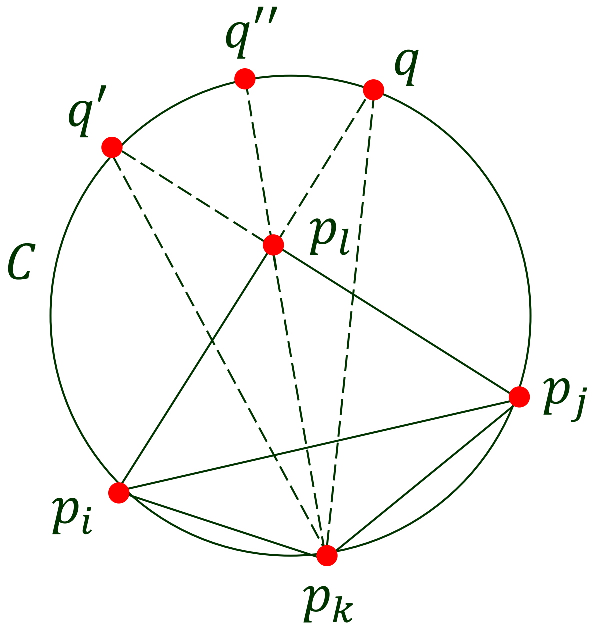
arc $\widehat{p_ip_j} \supset$ arc $\widehat{p_kp_j}$

$\angle p_ip_kp_j > \angle qp_ip_j = \angle p_lp_ip_j$

⇐ $\angle p_ip_lp_j$ and $\angle p_ip_kp_j$ are not the smallest angle in T.

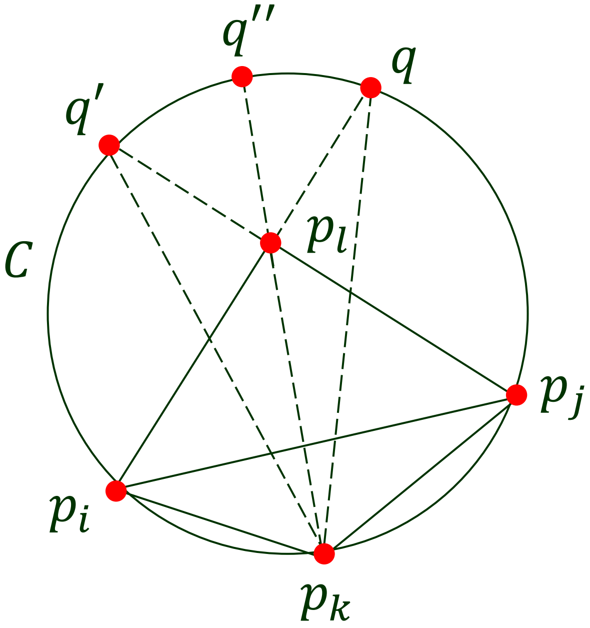
Similarly, $\angle p_lp_ip_k$ and $\angle p_lp_jp_k$ are not the smallest angle in T'.

Cont’d



We need only consider the remaining eight angles, four from each triangulation.

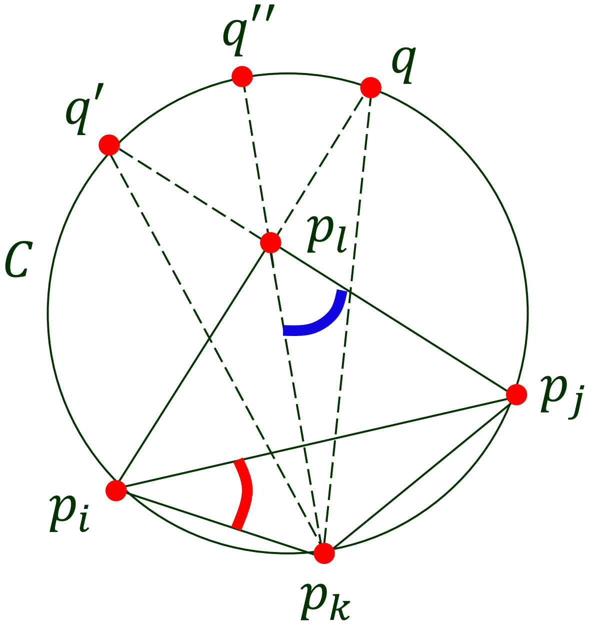
Cont'd



We need only consider the remaining eight angles, four from each triangulation.

T T'

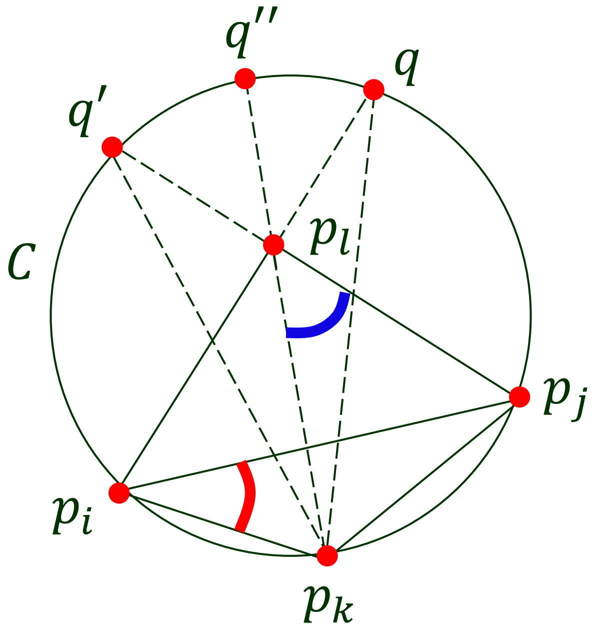
Cont'd



We need only consider the remaining eight angles, four from each triangulation.

T	T'
$\angle p_k p_i p_j$	$\angle p_k p_l p_j$

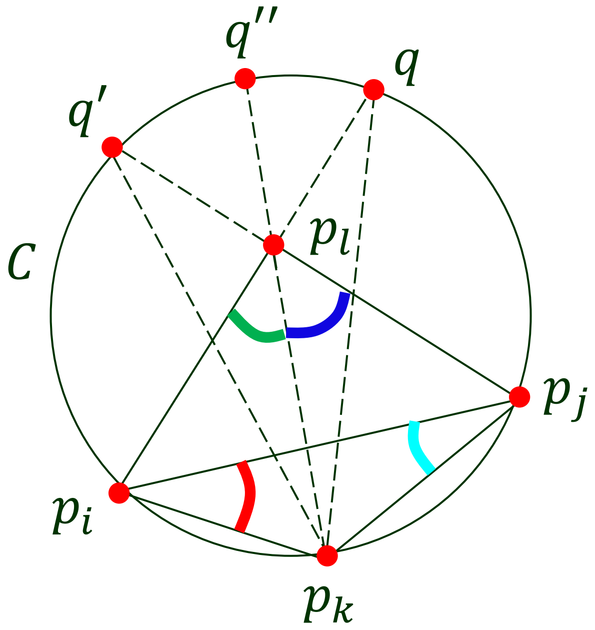
Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$

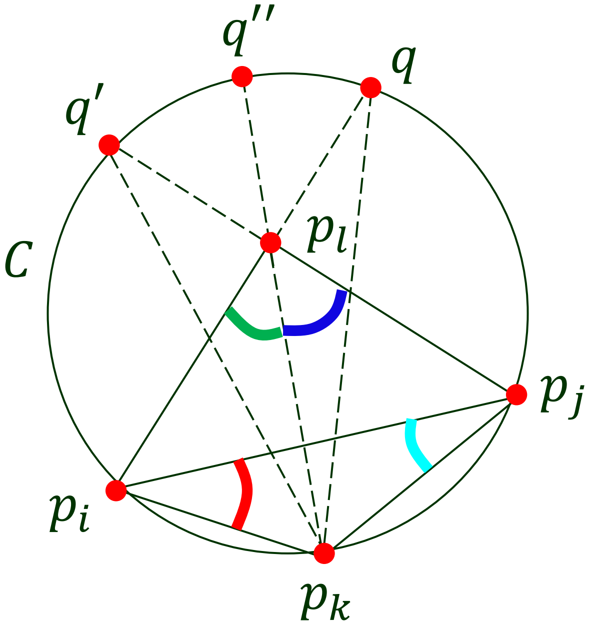
Cont'd



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$		$\angle p_k p_l p_i$

Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$

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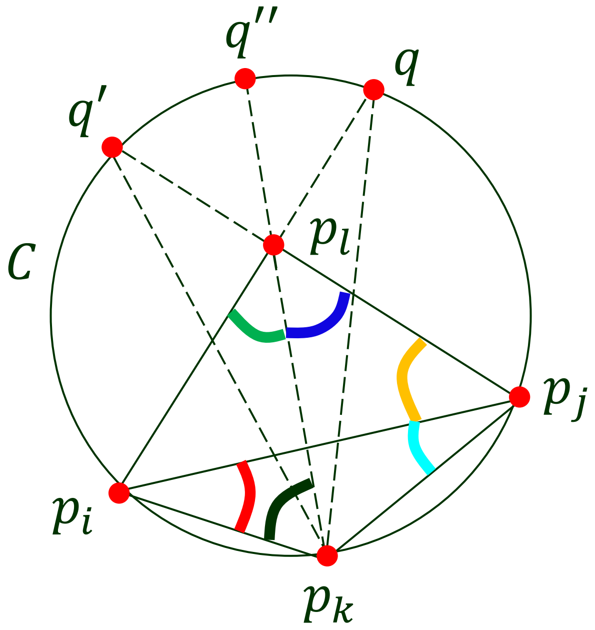
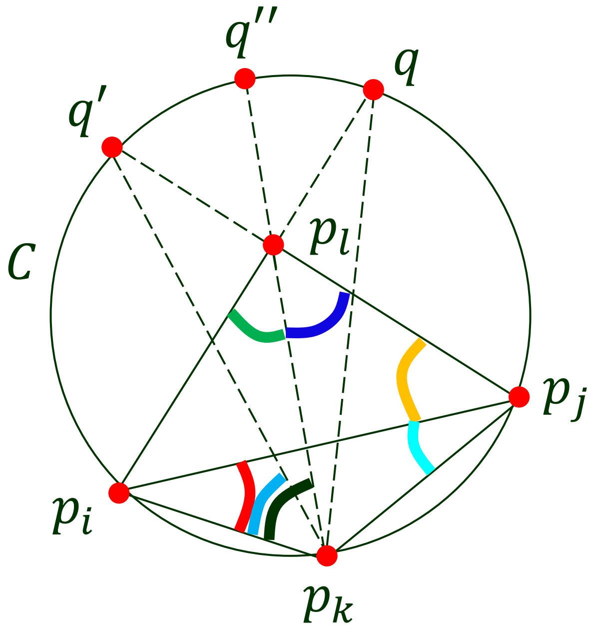


We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$
$\angle p_i p_j p_l$		$\angle p_i p_k p_l$

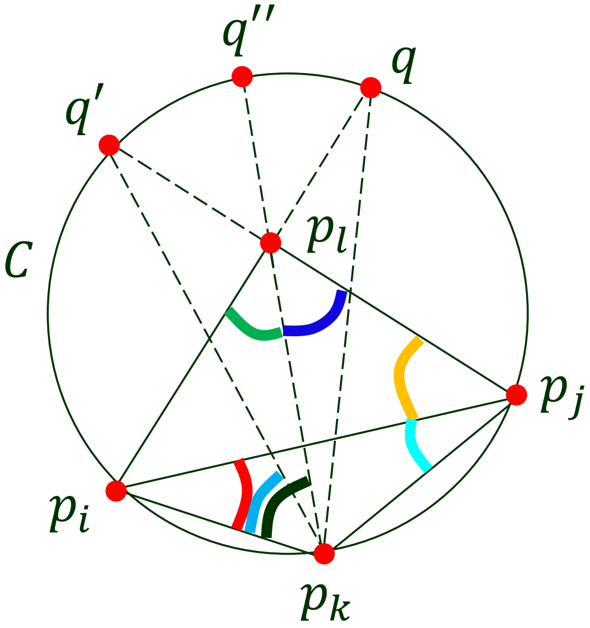
Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$
$\angle p_i p_j p_l = \angle p_i p_k q'$		$\angle p_i p_k p_l$

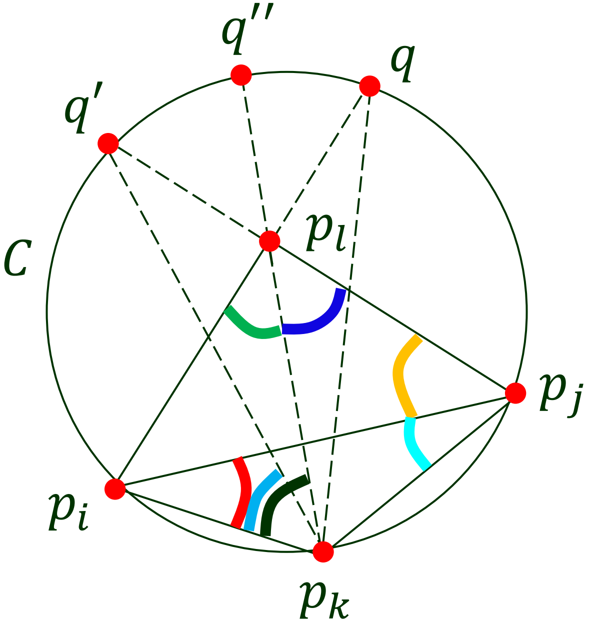
Cont'd



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_kp_ip_j$	$<$	$\angle p_kp_lp_j$
$\angle p_kp_jp_i$	$<$	$\angle p_kp_lp_i$
$\underbrace{\angle p_ip_jp_l = \angle p_ip_kq'}_{\text{two inscribed angles subtended by arc } \overset{\frown}{p_iq'}}$		$\angle p_ip_kp_l$

Cont’d

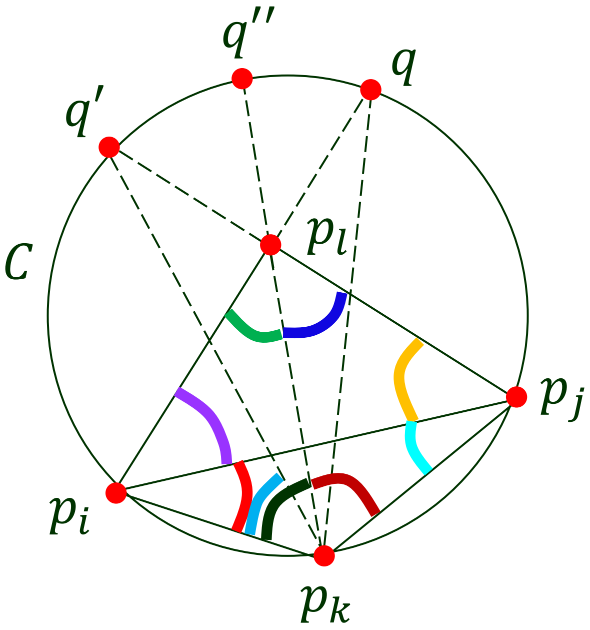


We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$

$$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}} < \angle p_i p_k q'' = \angle p_i p_k p_l$$

Cont’d



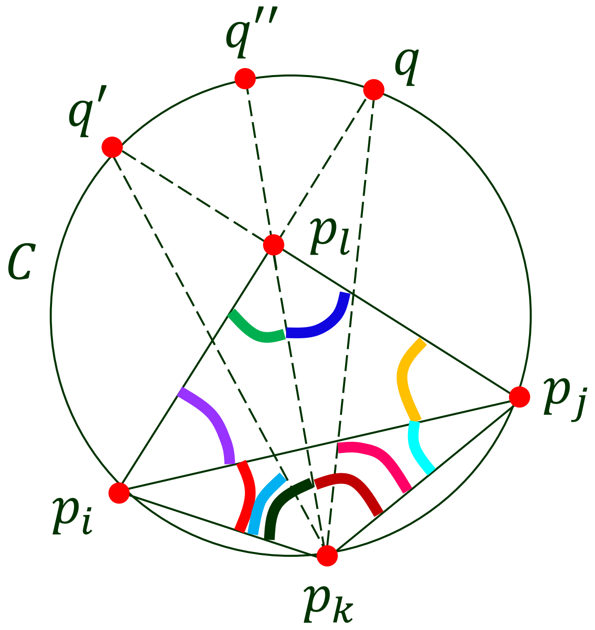
We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$

$$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q}'} < \angle p_i p_k q'' = \angle p_i p_k p_l$$

$\angle p_j p_i p_l$	$\angle p_l p_k p_j$

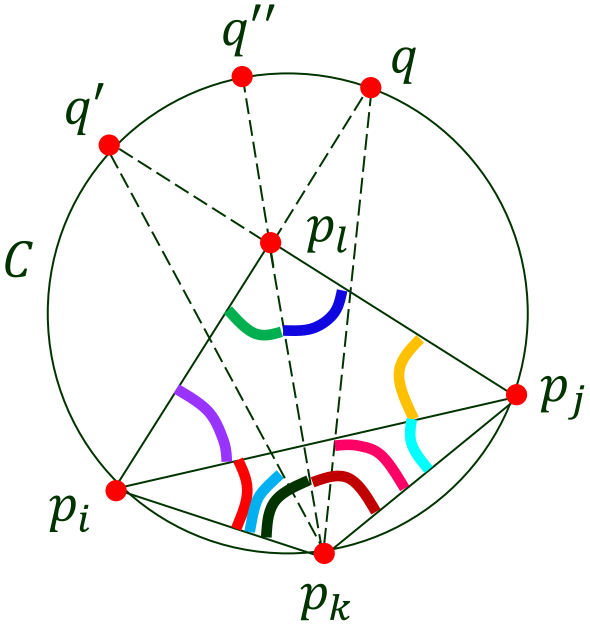
Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$
$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}}$	$<$	$\angle p_i p_k q'' = \angle p_i p_k p_l$
$\angle p_j p_i p_l = \angle p_j p_k q$		$\angle p_l p_k p_j$

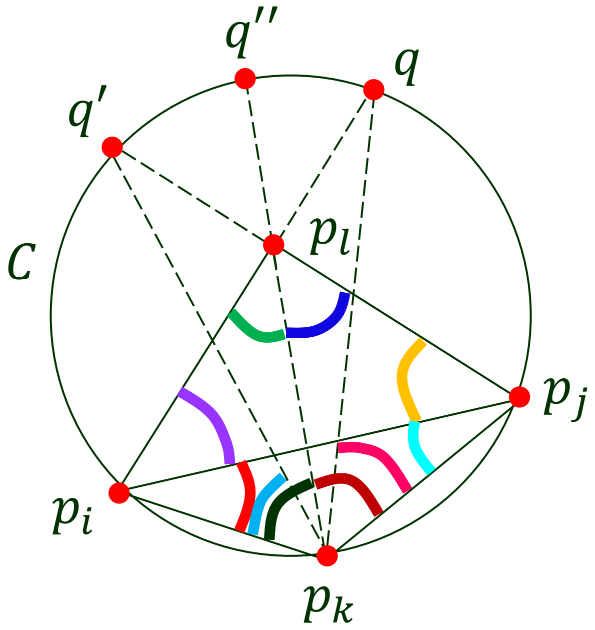
Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$
$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}}$	$<$	$\angle p_i p_k q'' = \angle p_i p_k p_l$
$\underbrace{\angle p_j p_i p_l = \angle p_j p_k q}_{\text{subtended by arc } \widehat{p_j q}}$		$\angle p_l p_k p_j$

Cont'd



We need only consider the remaining eight angles, four from each triangulation.

T T'

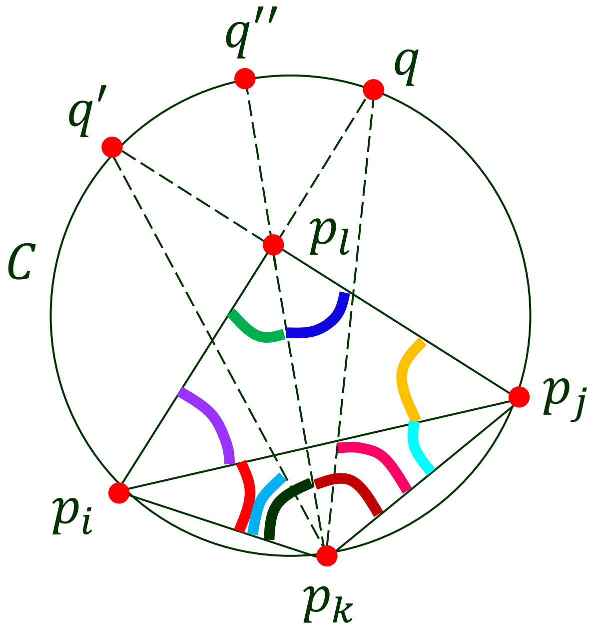
$$\angle p_k p_i p_j < \angle p_k p_l p_j$$

$$\angle p_k p_j p_i < \angle p_k p_l p_i$$

$$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}} < \angle p_i p_k q'' = \angle p_i p_k p_l$$

$$\underbrace{\angle p_j p_i p_l = \angle p_j p_k q}_{\text{subtended by arc } \widehat{p_j q}} < \angle p_l p_k p_j$$

Cont’d



We need only consider the remaining eight angles, four from each triangulation.

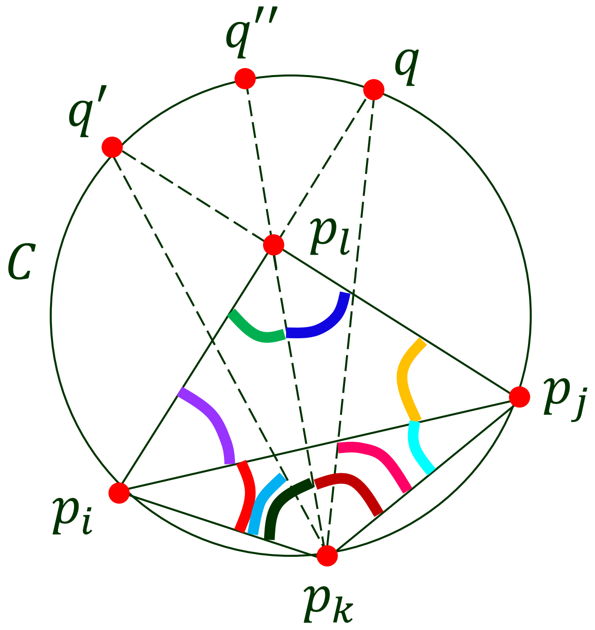
T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$

$$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}} < \angle p_i p_k q'' = \angle p_i p_k p_l$$

$$\underbrace{\angle p_j p_i p_l = \angle p_j p_k q}_{\text{subtended by arc } \widehat{p_j q}} < \angle p_l p_k p_j$$

Thus, the minimum angle in T is less than the minimum angle in T'.

Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$

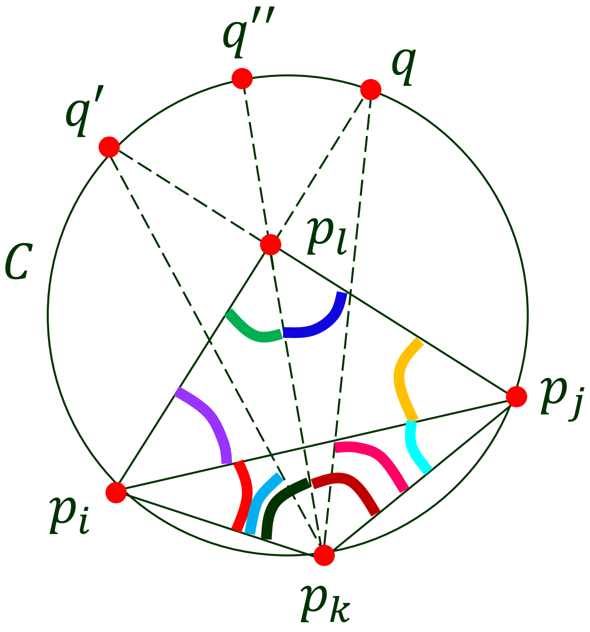
$$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}} < \angle p_i p_k q'' = \angle p_i p_k p_l$$

$$\underbrace{\angle p_j p_i p_l = \angle p_j p_k q}_{\text{subtended by arc } \widehat{p_j q}} < \angle p_l p_k p_j$$

Thus, the minimum angle in T is less than the minimum angle in T'.

(⇒) Omitted.

Cont’d



We need only consider the remaining eight angles, four from each triangulation.

T		T'
$\angle p_k p_i p_j$	$<$	$\angle p_k p_l p_j$
$\angle p_k p_j p_i$	$<$	$\angle p_k p_l p_i$

$$\underbrace{\angle p_i p_j p_l = \angle p_i p_k q'}_{\text{two inscribed angles subtended by arc } \widehat{p_i q'}} < \angle p_i p_k q'' = \angle p_i p_k p_l$$

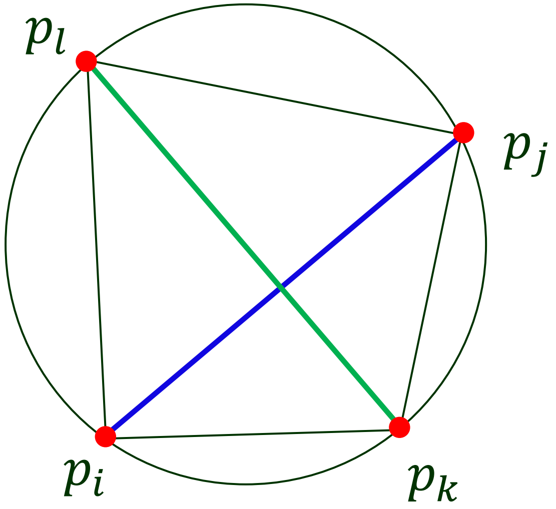
$$\underbrace{\angle p_j p_i p_l = \angle p_j p_k q}_{\text{subtended by arc } \widehat{p_j q}} < \angle p_l p_k p_j$$

Thus, the minimum angle in T is less than the minimum angle in T'.

(⇒) Omitted.

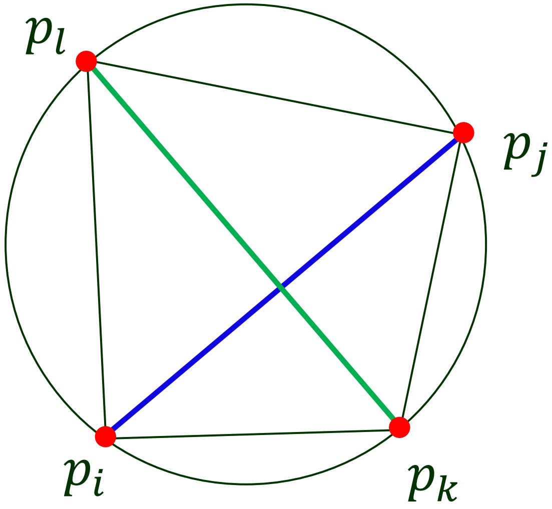
□

Special Case and Legal Triangulation



p_i, p_j, p_k, p_l are cocircular.

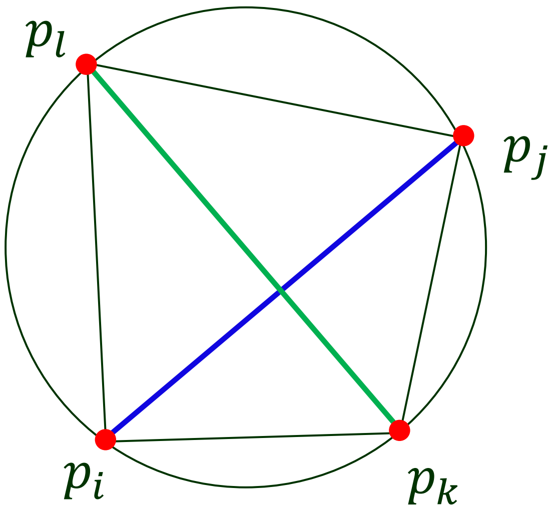
Special Case and Legal Triangulation



p_i, p_j, p_k, p_l are cocircular.

Both $\overline{p_i p_j}$ and $\overline{p_k p_l}$ are legal!

Special Case and Legal Triangulation



p_i, p_j, p_k, p_l are cocircular.

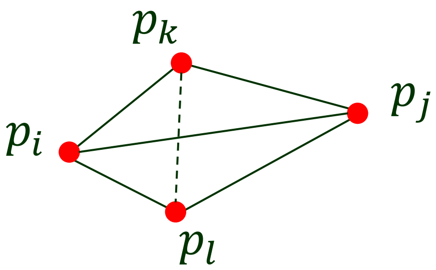
Both $\overline{p_i p_j}$ and $\overline{p_k p_l}$ are legal!

A *legal triangulation* has no illegal edge.

Generating a Legal Triangulation

```
LegalTriangulation($T$)

while $T$ contains an illegal edge $\overline{p_ip_j}$
    do let $p_ip_jp_k$ and $p_ip_jp_l$ be the two triangles adjacent to $\overline{p_ip_j}$
        remove $\overline{p_ip_j}$
        add $\overline{p_kp_l}$
return $T$
```



Generating a Legal Triangulation

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LegalTriangulation($T$)

while $T$ contains an illegal edge $\overline{p_i p_j}$
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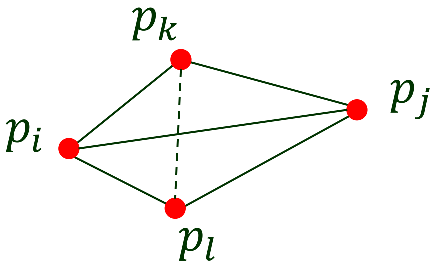
        remove $\overline{p_i p_j}$
        add $\overline{p_k p_l}$
return $T$
```



- Angle vector $A(T)$ increases in every iteration.

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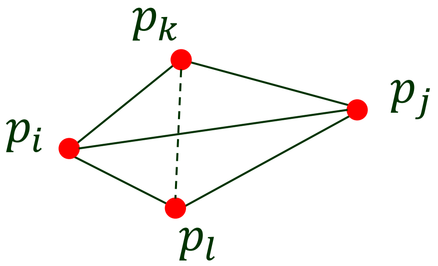
return $T$
```



- Angle vector $A(T)$ increases in every iteration.
- #triangulations is finite.

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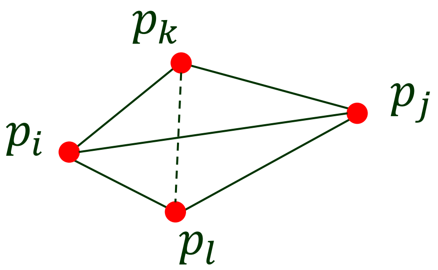

- Angle vector $A(T)$ increases in every iteration.
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The while loop will terminate with a legal triangulation.

Generating a Legal Triangulation

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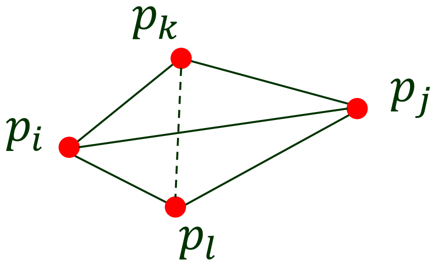
return $T$
```



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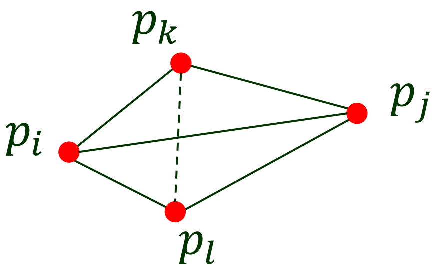
The while loop will terminate with a legal triangulation.

♣ Too slow!

Generating a Legal Triangulation

```
LegalTriangulation($T$)

while $T$ contains an illegal edge $\overline{p_i p_j}$
    do let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$
        remove $\overline{p_i p_j}$
        add $\overline{p_k p_l}$
return $T$
```



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