Yan-Bin Jia, Iowa State University

# Line Segment Intersection

## Outline

I. Intersecting two segments

II. Intersecting $n$ segment via a plane sweep

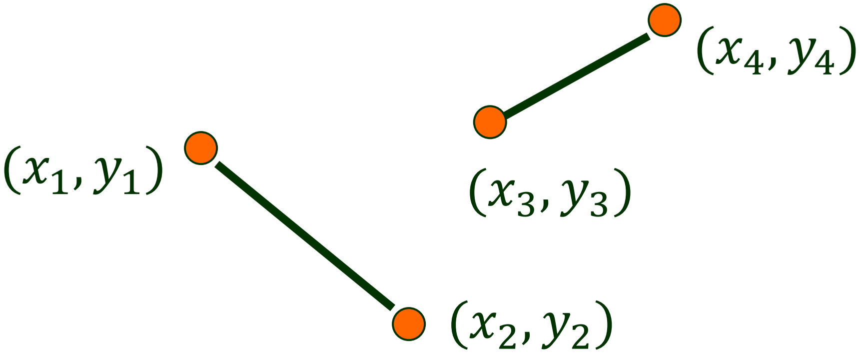
III. Data structures for the algorithm

IV. Running time and storage

Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

# I. Two Segments Intersect?

Straightforward approach:



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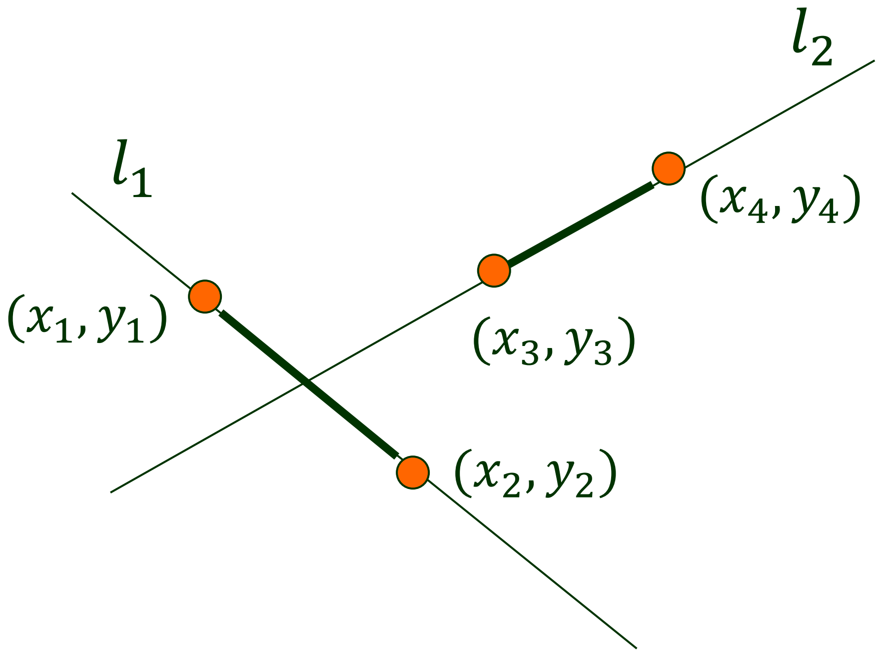

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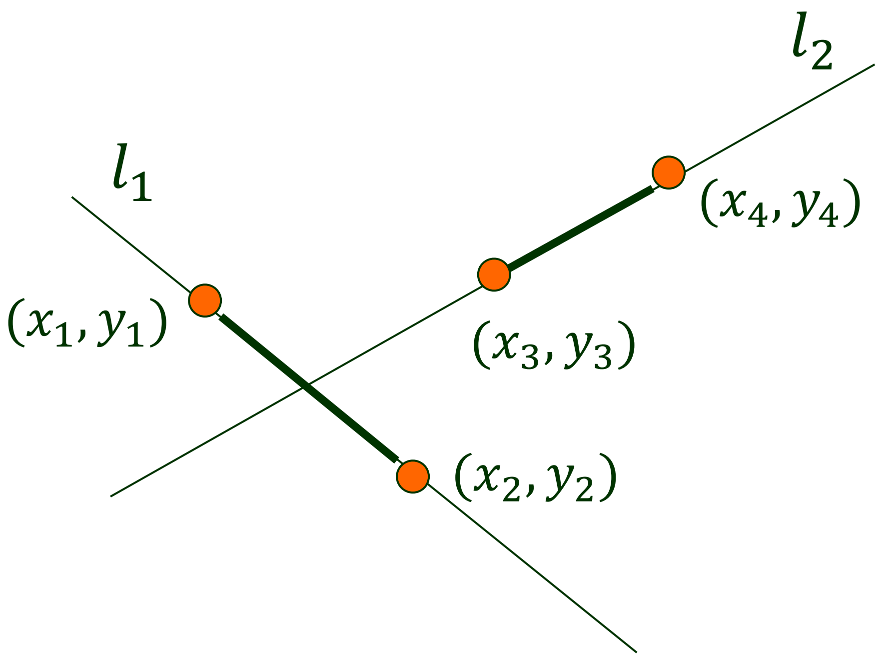

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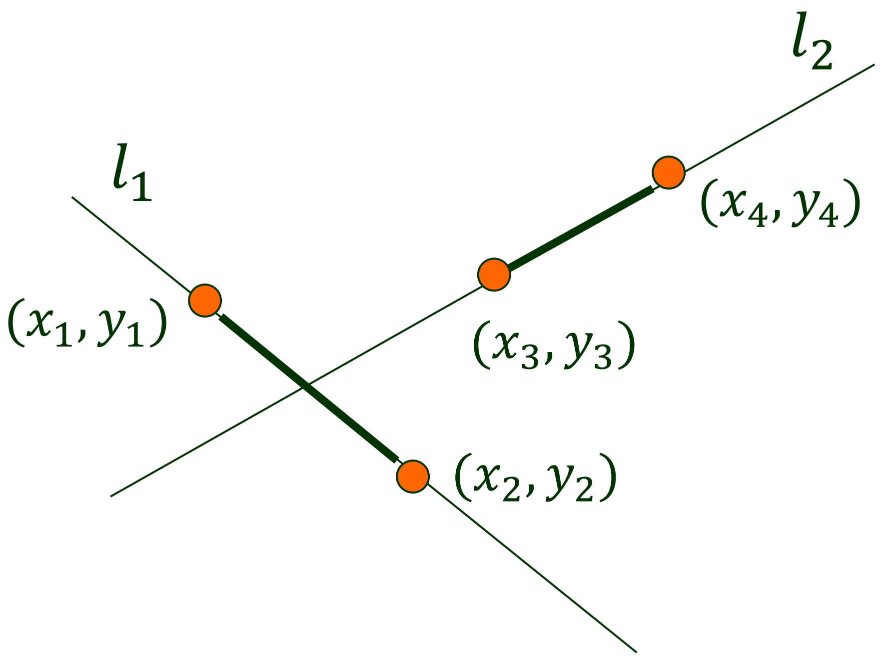

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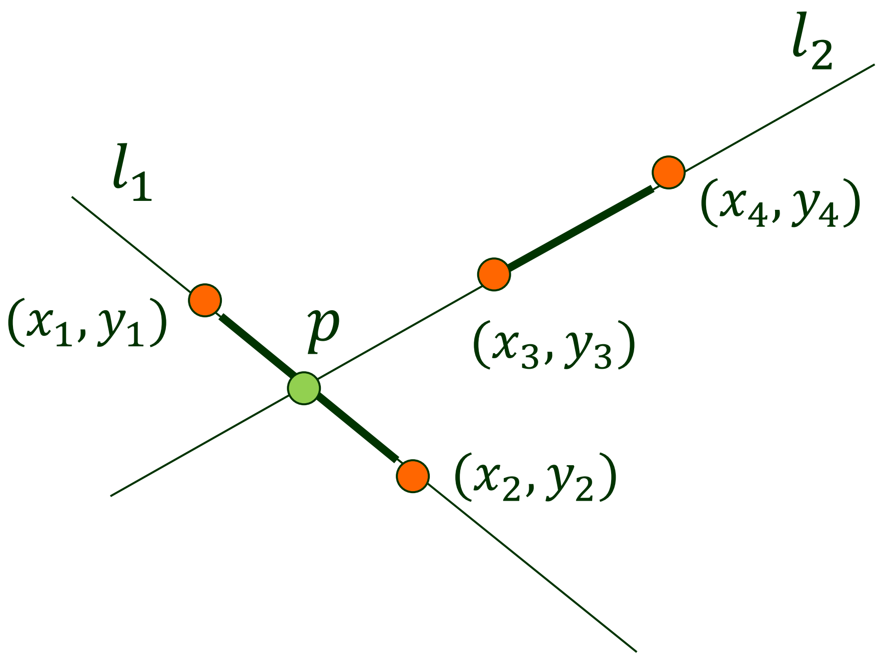

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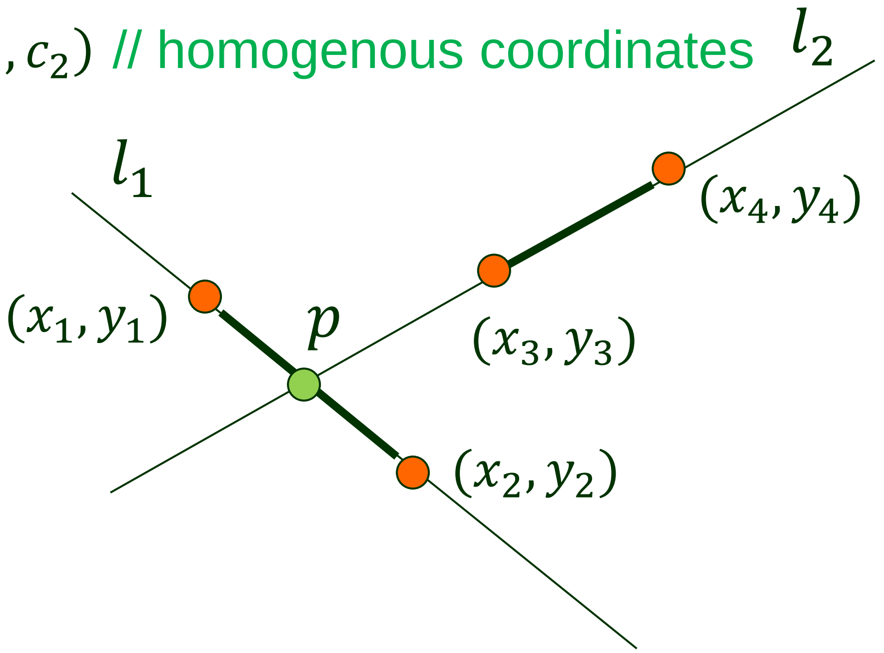

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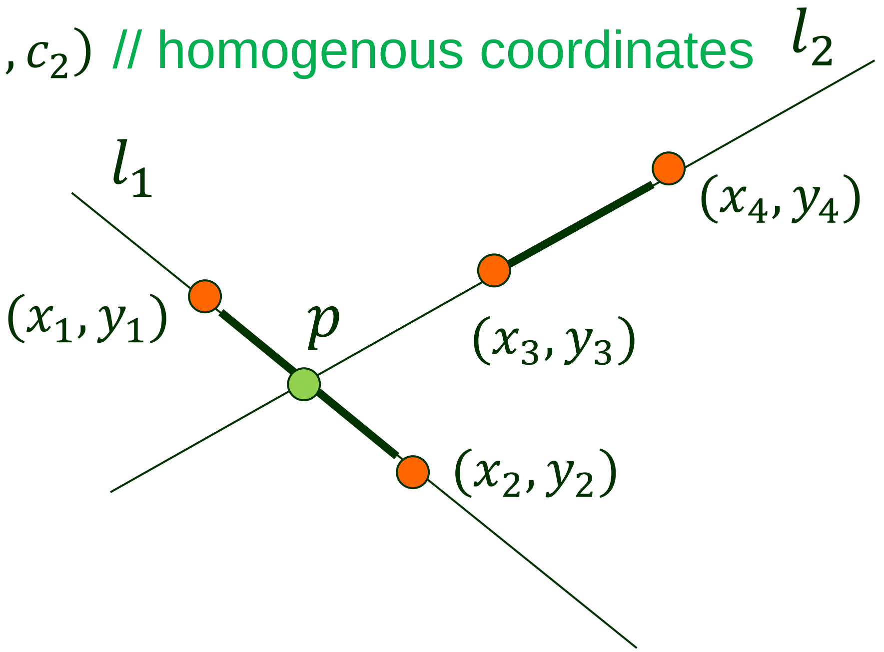

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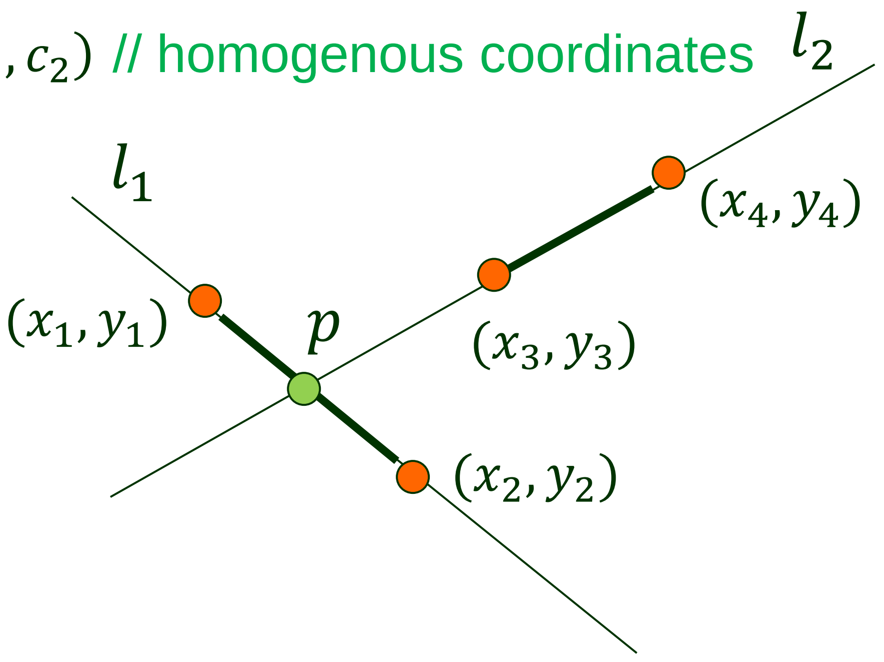

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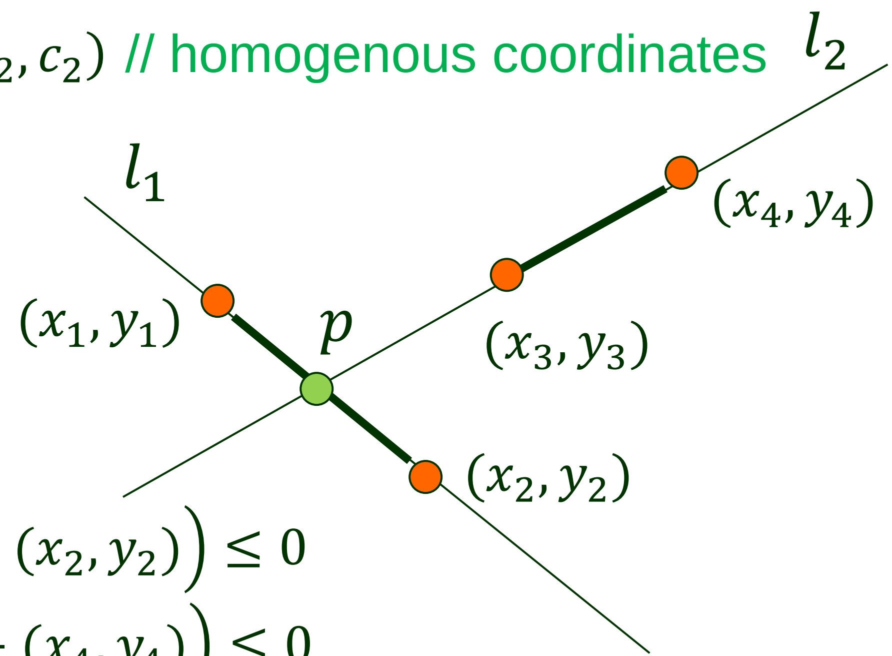


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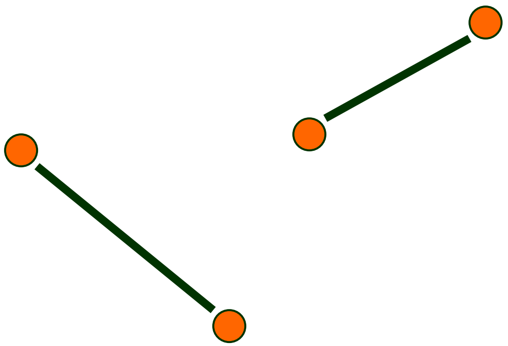

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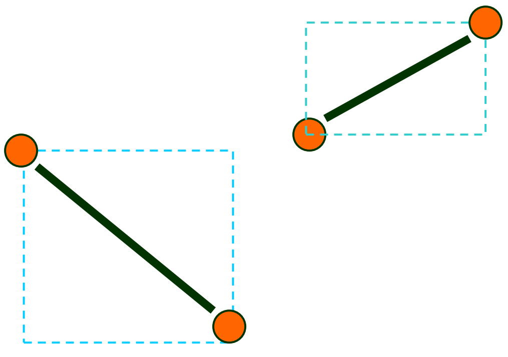

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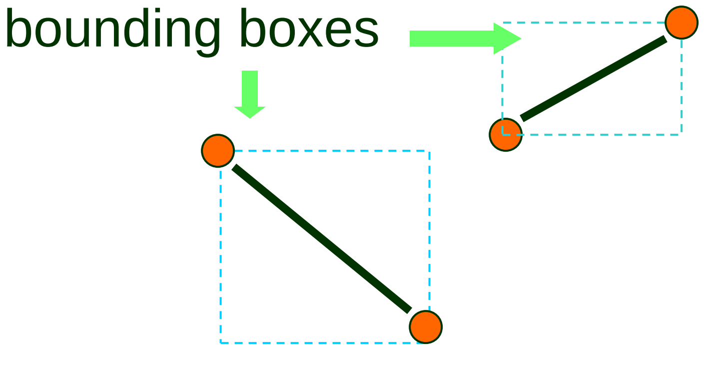

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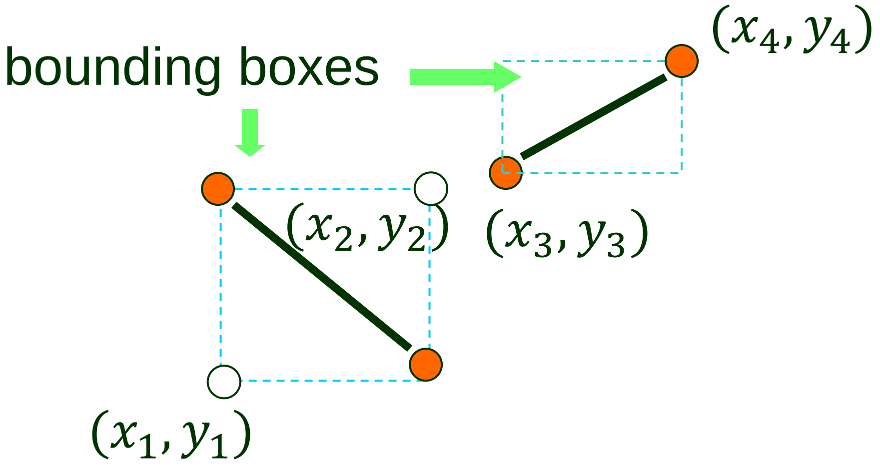

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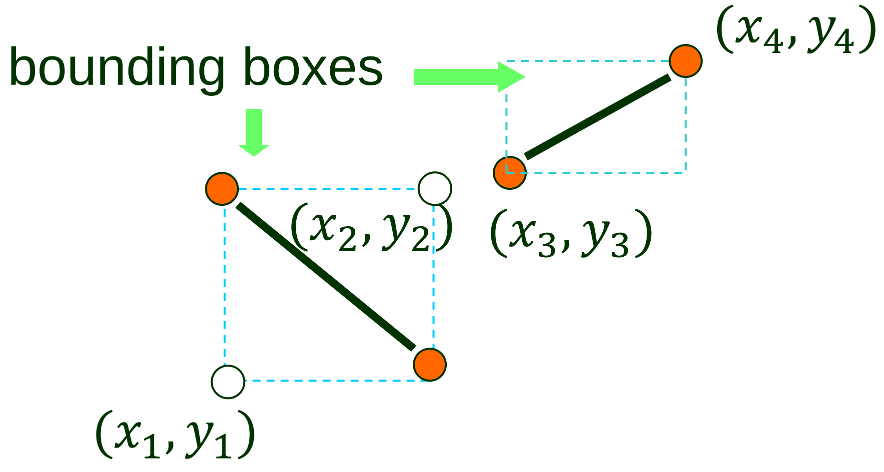

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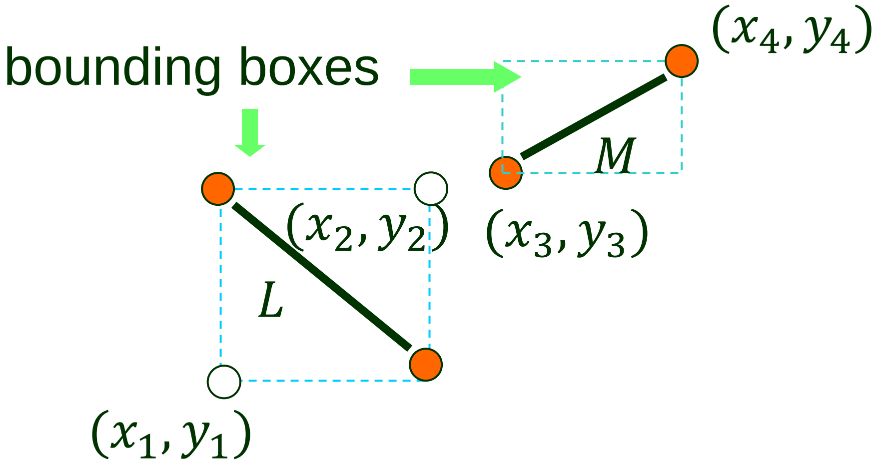

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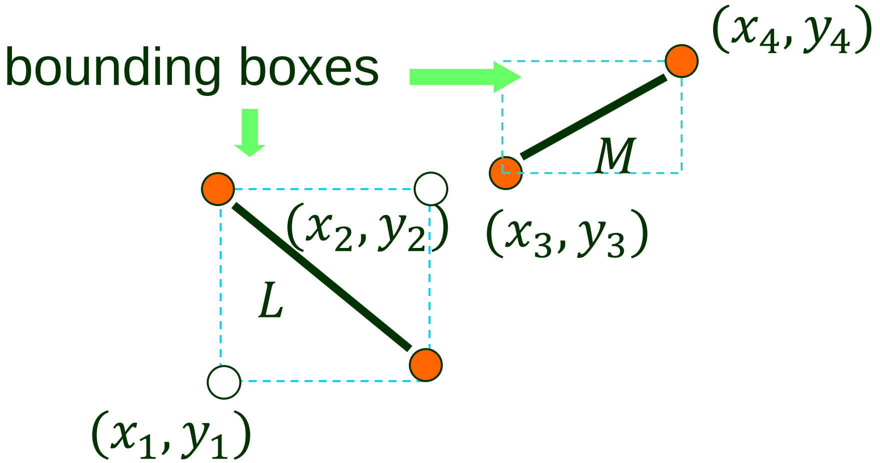

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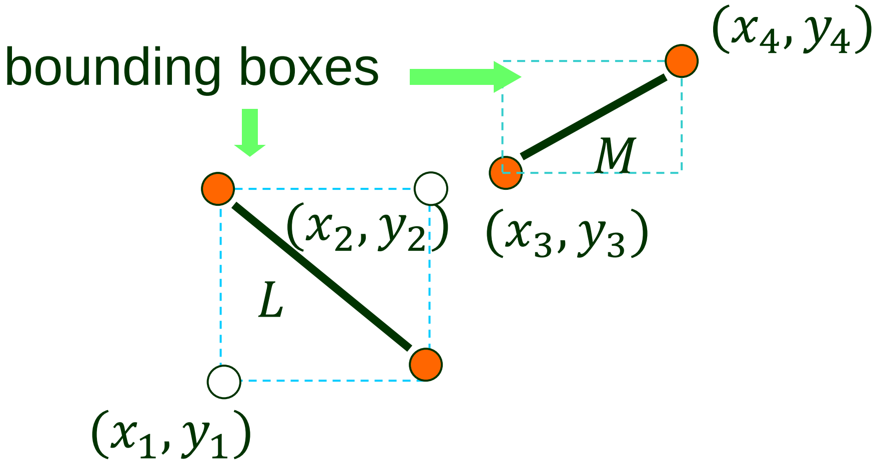
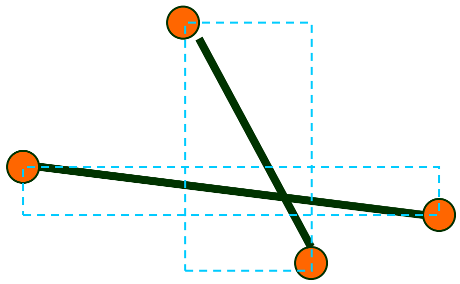
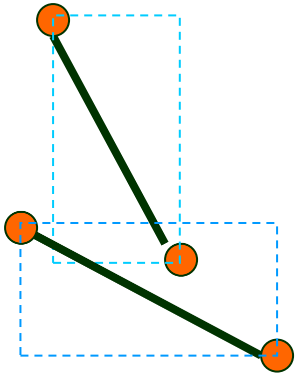
**Case 1**: bounding boxes do not intersect; neither will the segments**.**



# Bounding Box

**Case 2**: Bounding boxes intersect; the segments may or may not intersect. Needs to be further checked in Stage 2.

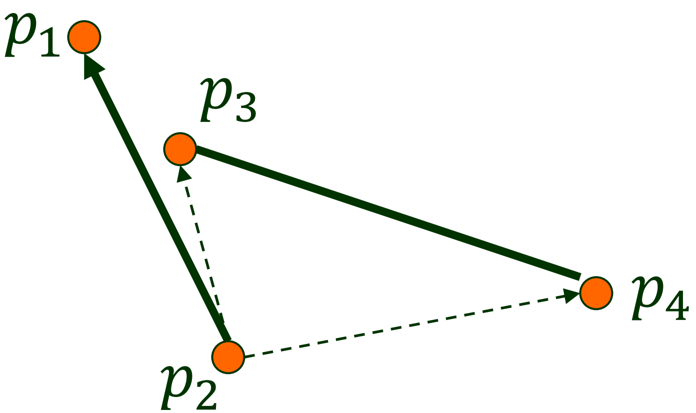




# Necessary and Sufficient Condition for No Intersection

Two line segments do ***not*** intersect if and only if

*one of them lies entirely to one side of the line containing the other one.*




The above condition is equivalent to that

*at least one of the two pairs of cross products below has the same sign:*

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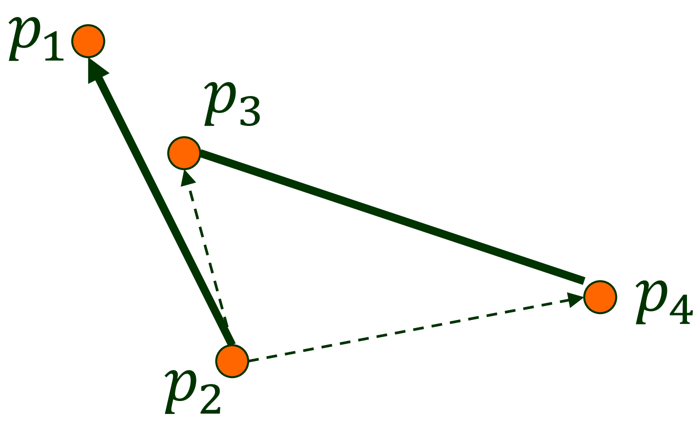


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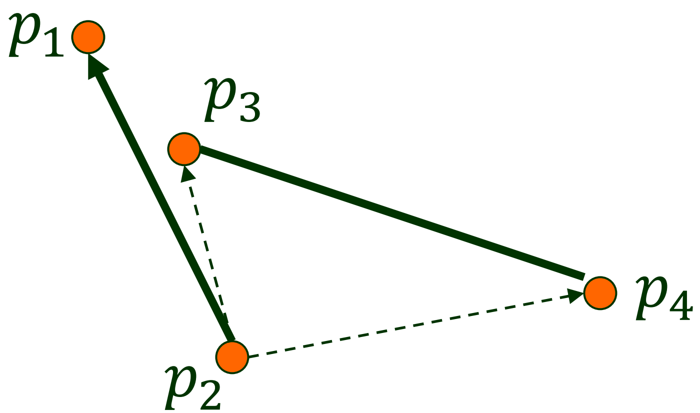


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Two line segments intersect if and only if the condition is false.

# When the Necessary & Sufficient Condition is False

The cross products in either pair have different signs (or at least one cross product in the pair is 0).

$(p_3 - p_2) \times (p_1 - p_2)$ and $(p_4 - p_2) \times (p_1 - p_2)$ // the line through $p_3, p_4$ // intersects $\overline{p_1 p_2}$.

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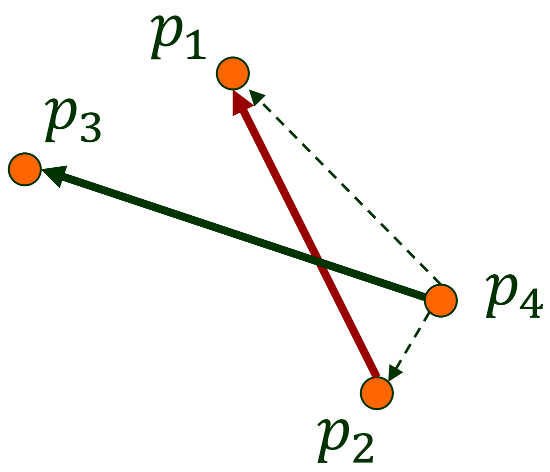

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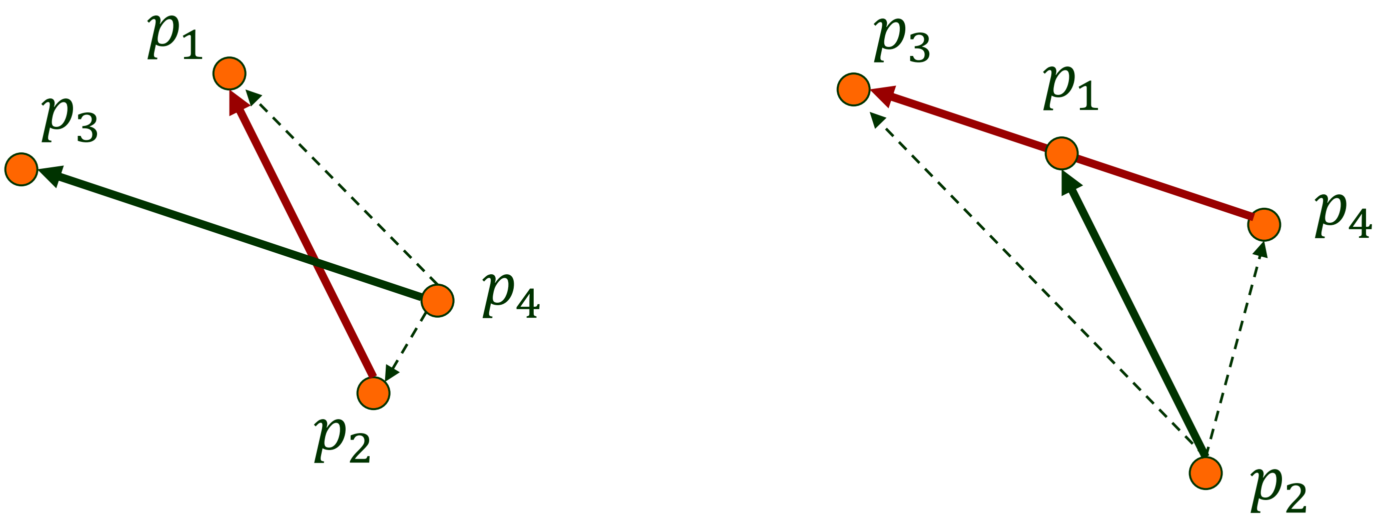

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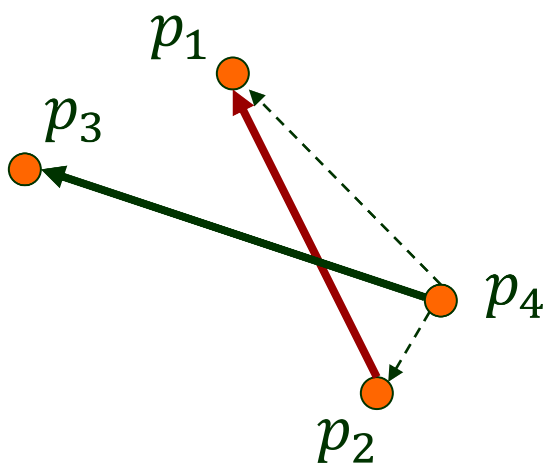
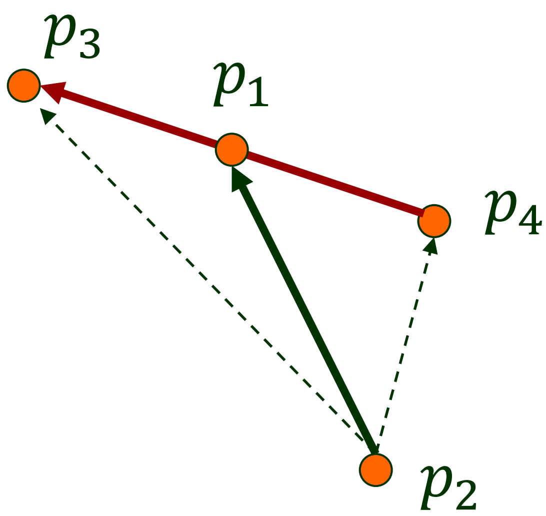


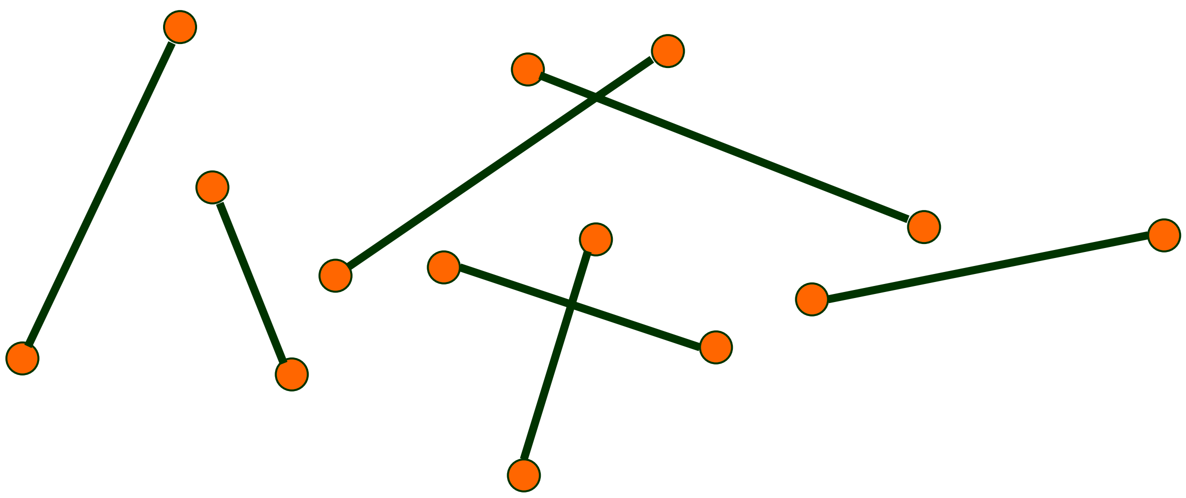




Not as convenient as testing the falsity of non-intersecting of the two segments.

# II. $n$ Line Segments

**Input:** a set of $n$ line segments in the plane.

**Output**: all intersections and for each intersection the involved segments.



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Need a faster algorithm to deal with such situations!

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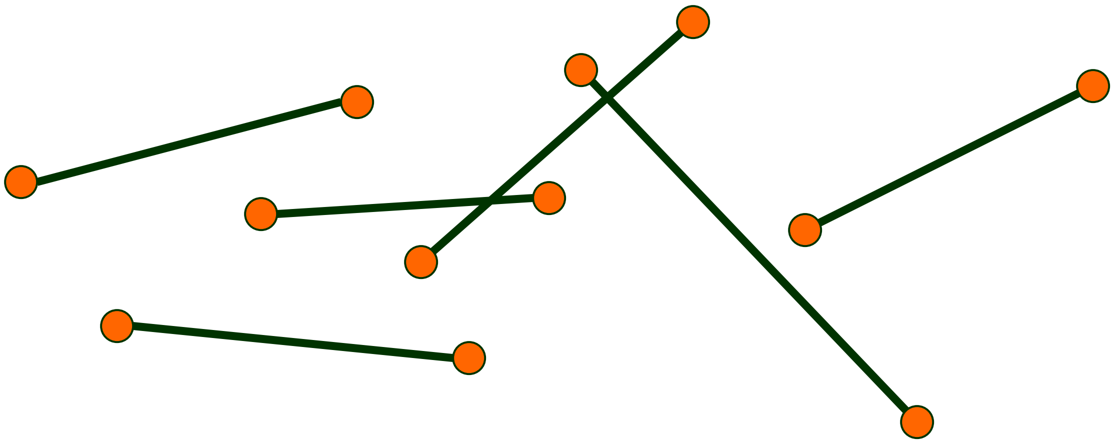

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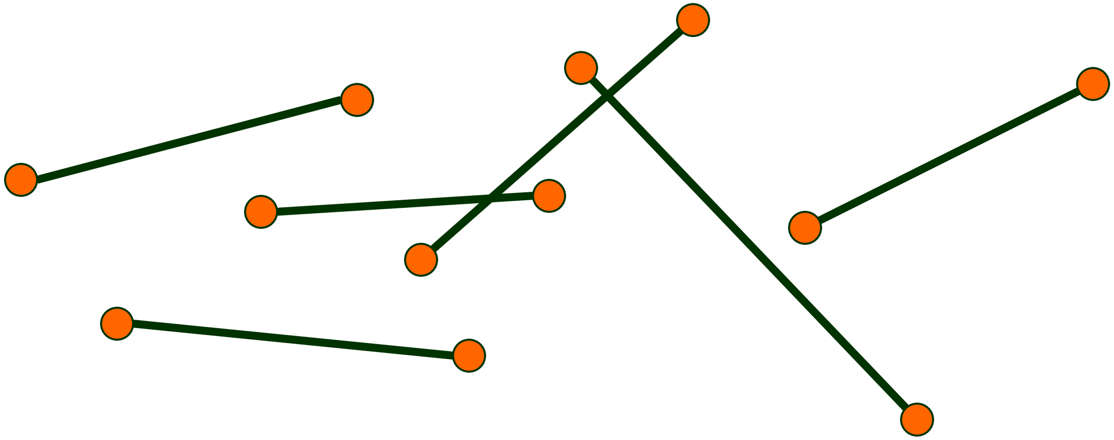
**Idea**: *Imagine* a vertical sweep line passes through the given set of line segments, from left to right.



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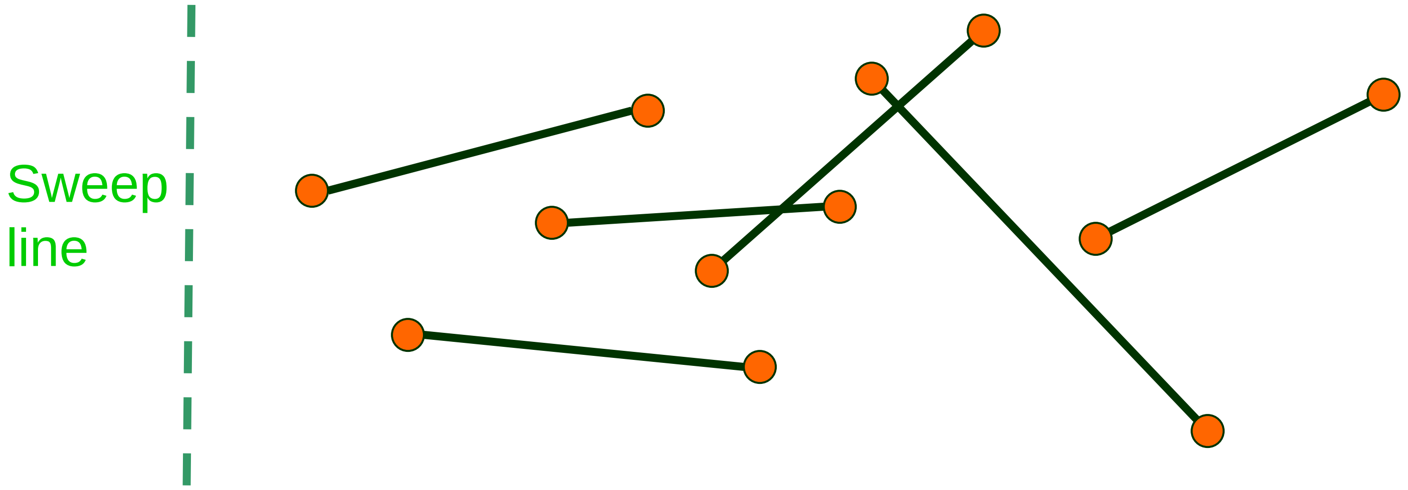

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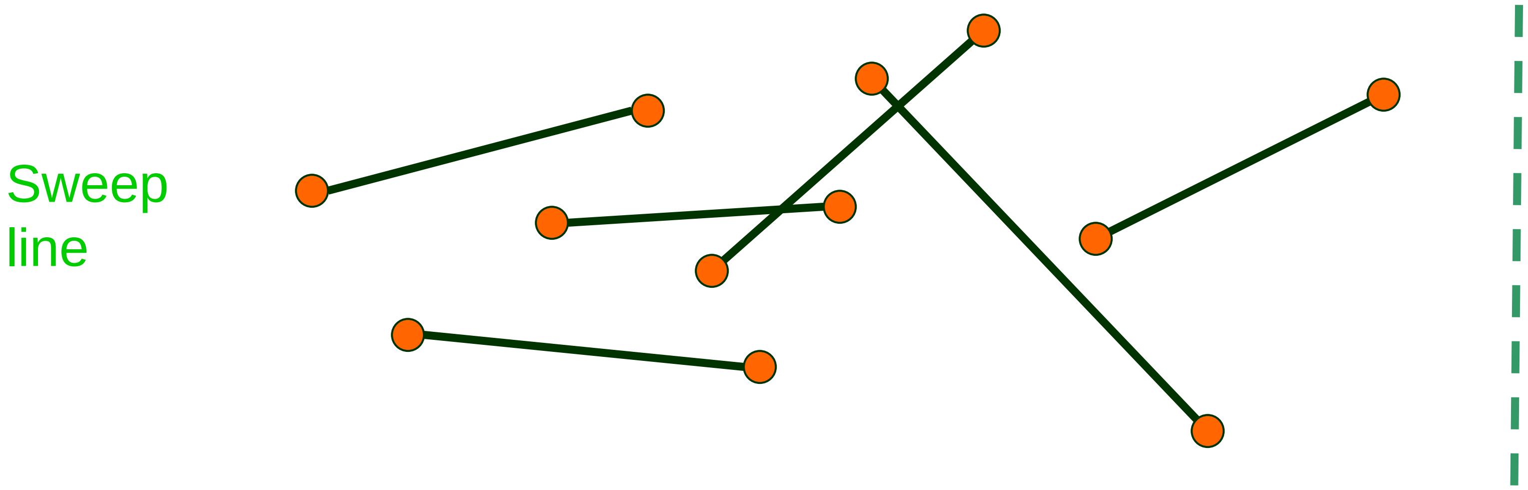

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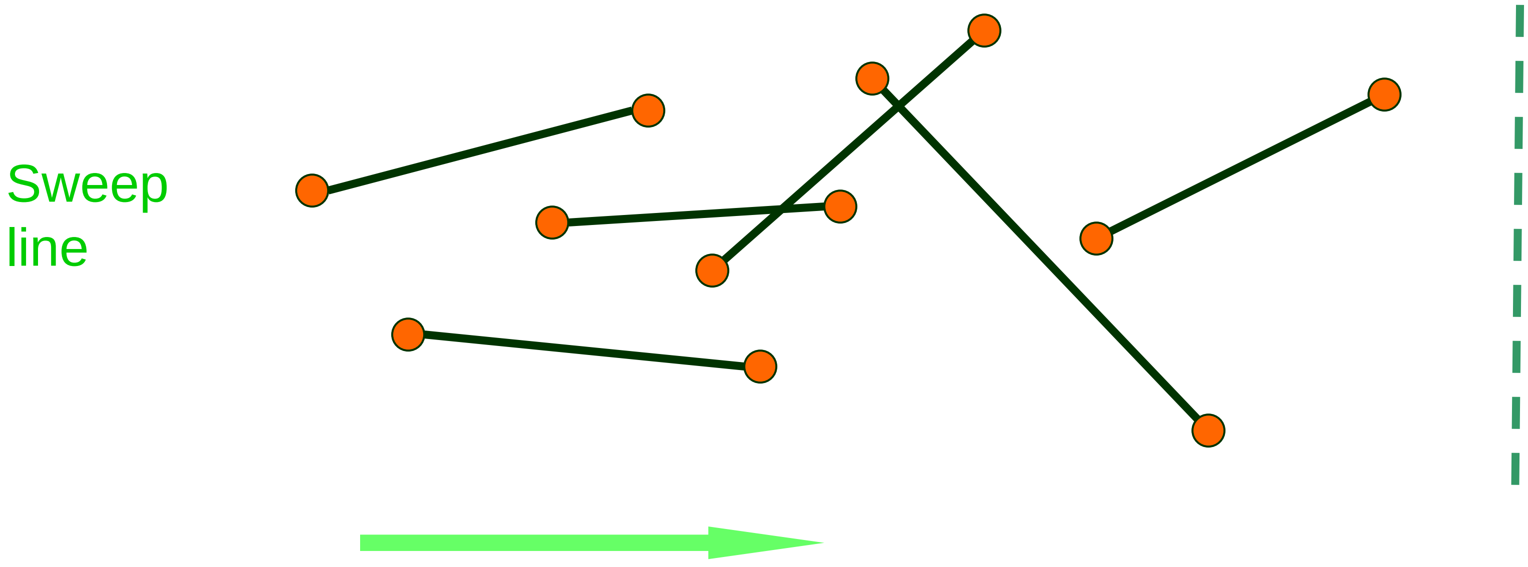
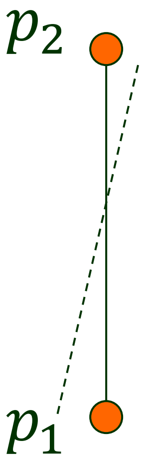


# Handling Non-degeneracy

If $\geq 1$ vertical segment, imagine all segments are rotated clockwise by a tiny angle and then test for intersection.

For each vertical segment, the sweep line will hit its lower endpoint before upper point.



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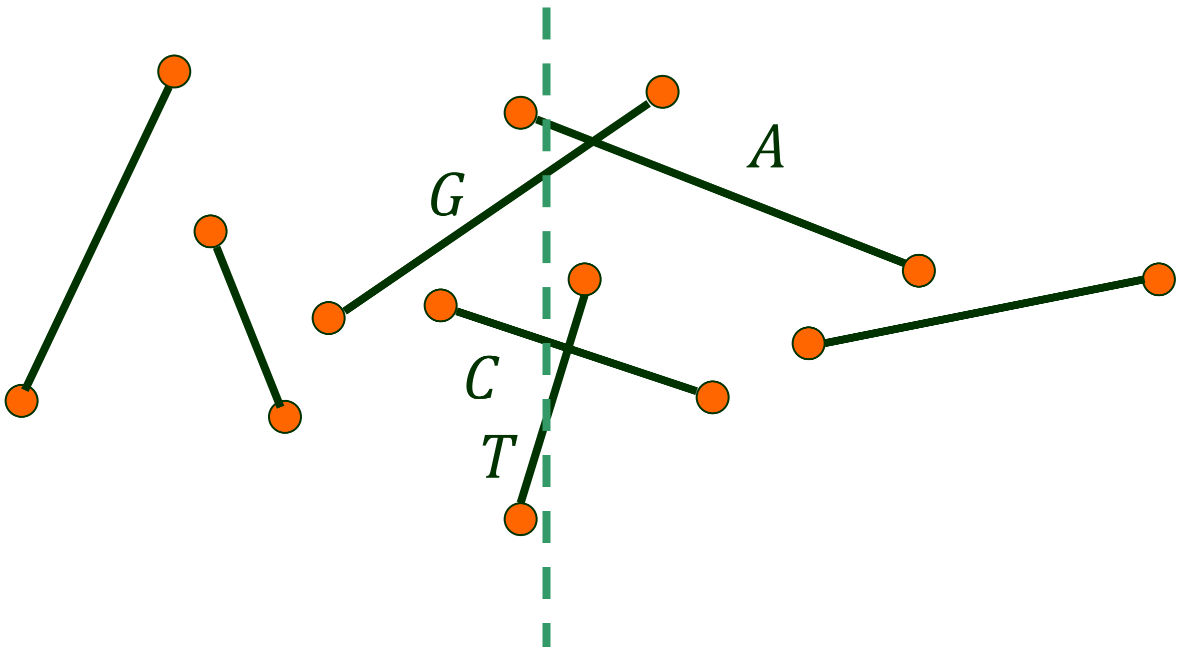

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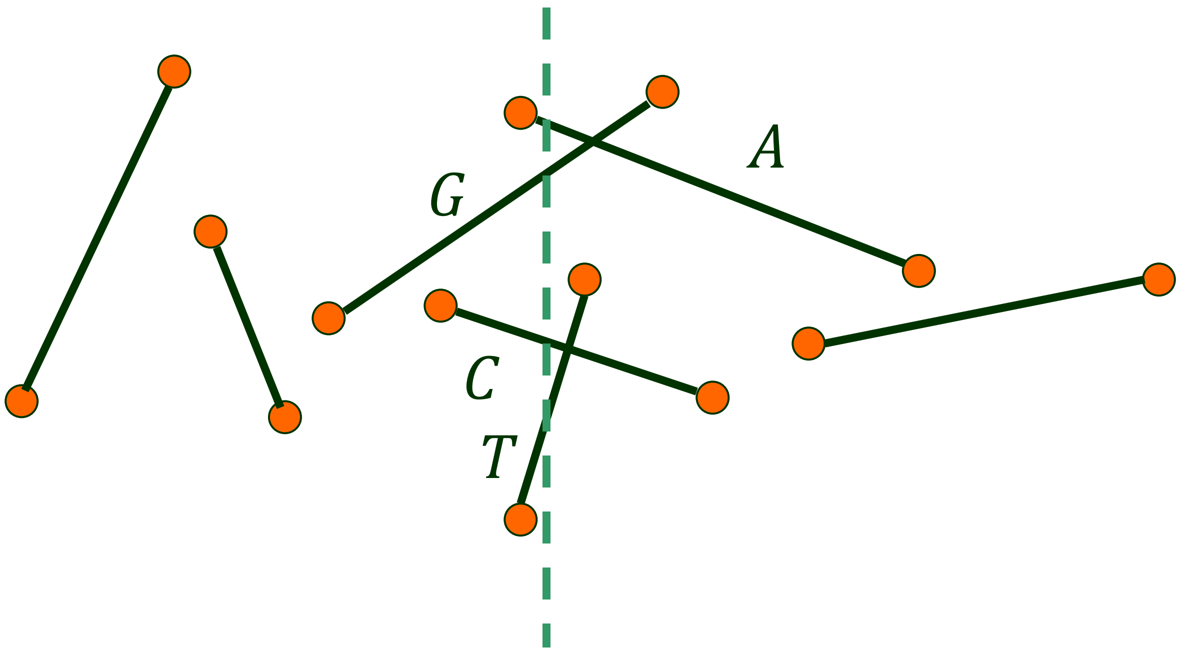

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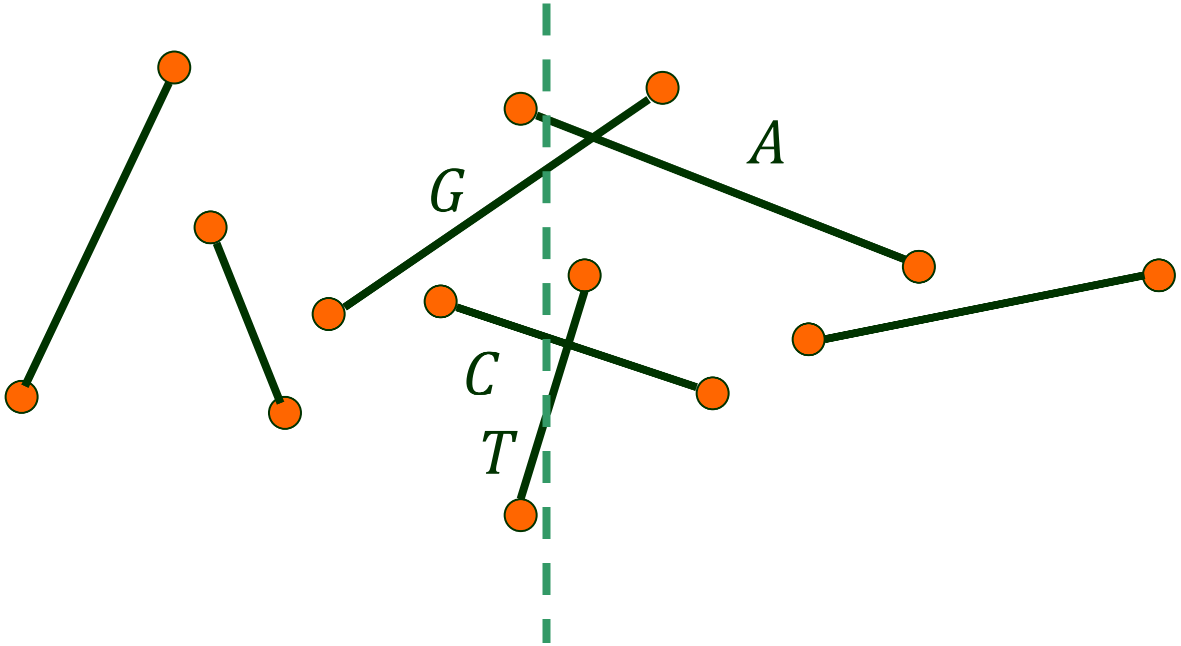
Updates of status happen only at *event points*.



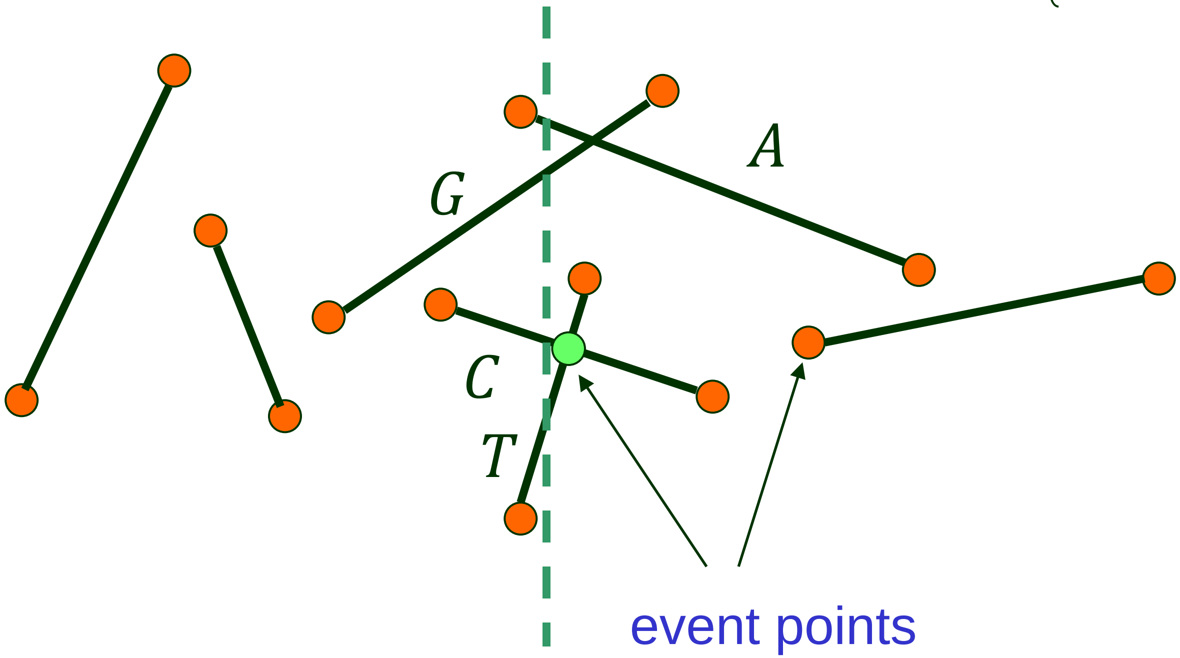
# Sweep Line Status

*The set of segments intersecting the sweep line.*

It changes as the sweep line moves, but *not continuously*.

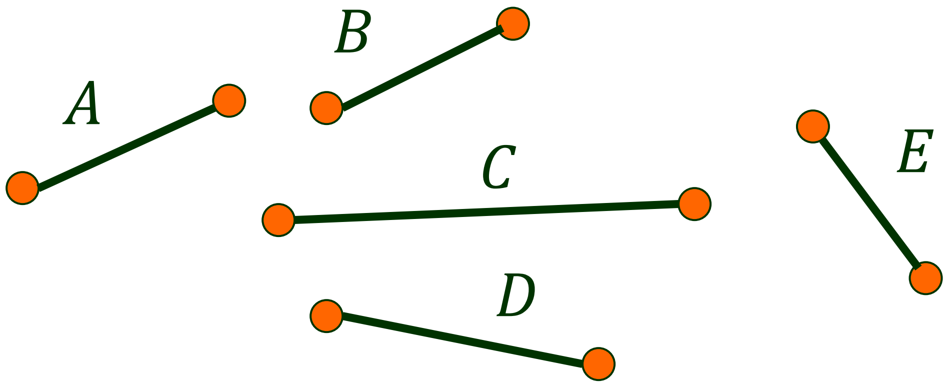
Updates of status happen only at *event points*.

- endpoints
- intersections



# Ordering Segments

A *total order* over the segments that intersect the current position of the sweep line:



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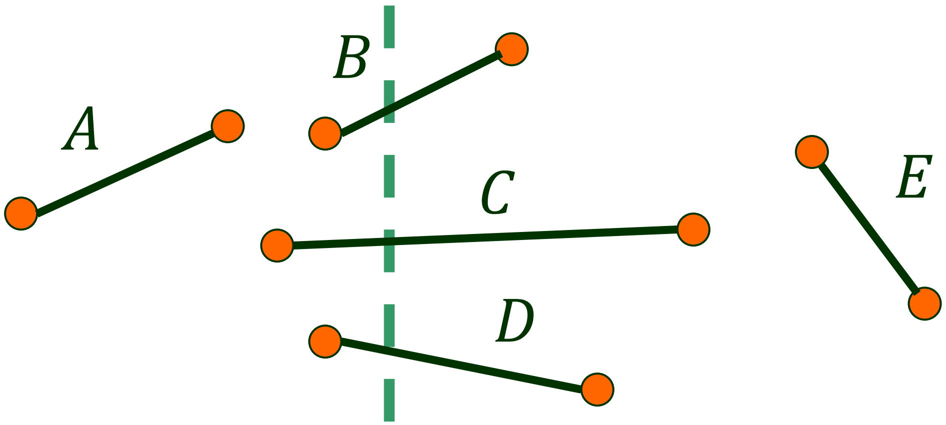


$D < C < B$
($A$ and $E$ not in the ordering)

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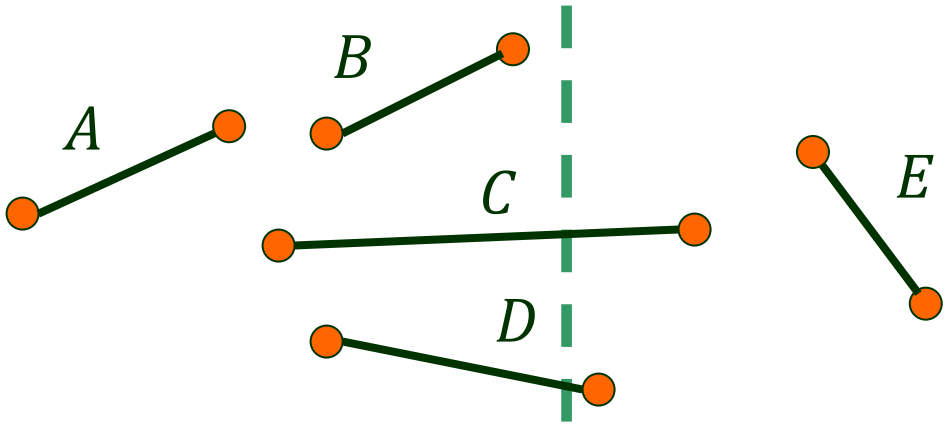


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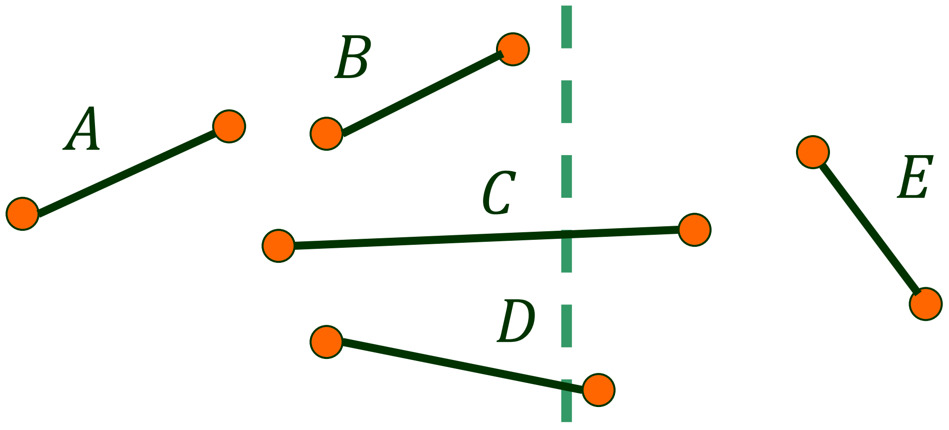


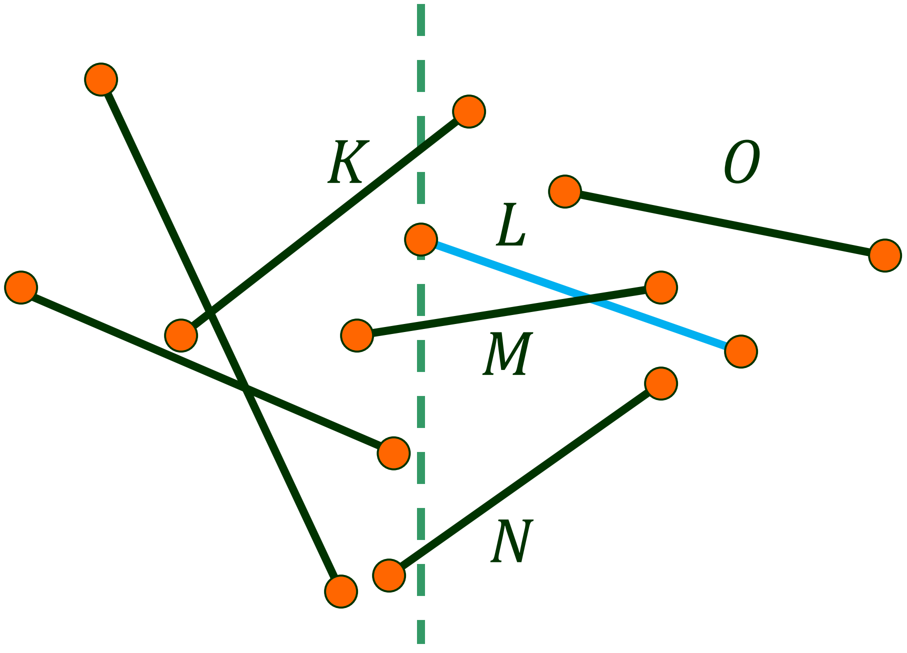

$D < C$
($B$ drops out of the ordering)

At an event point, the sequence of segments changes:

- Update the status.
- Detect the intersections.

# Status Update (1)

Event point is the left endpoint of a segment.




*N*, *M*, *K*

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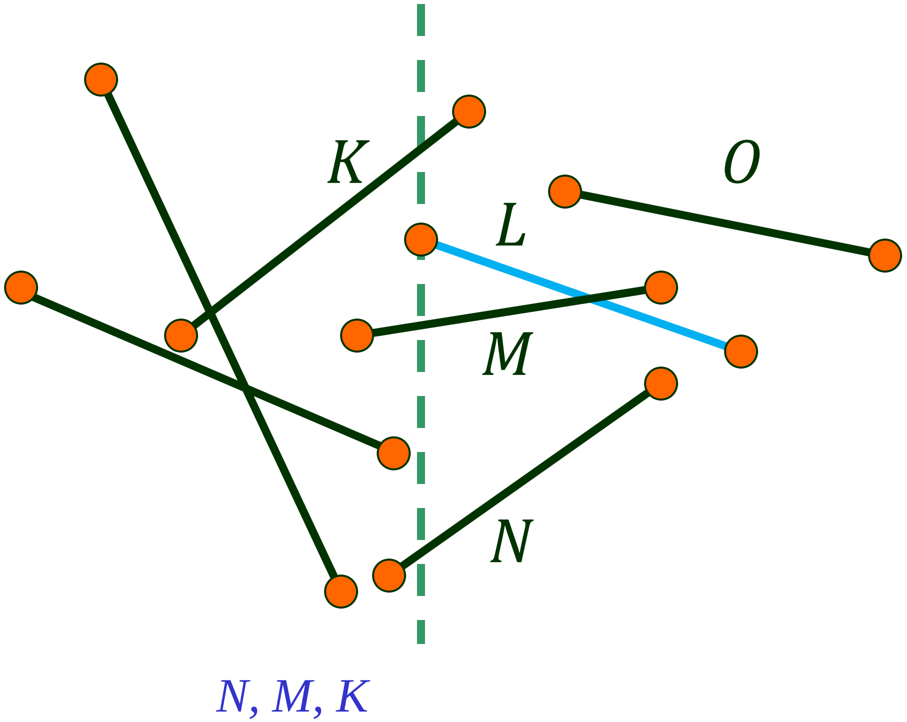


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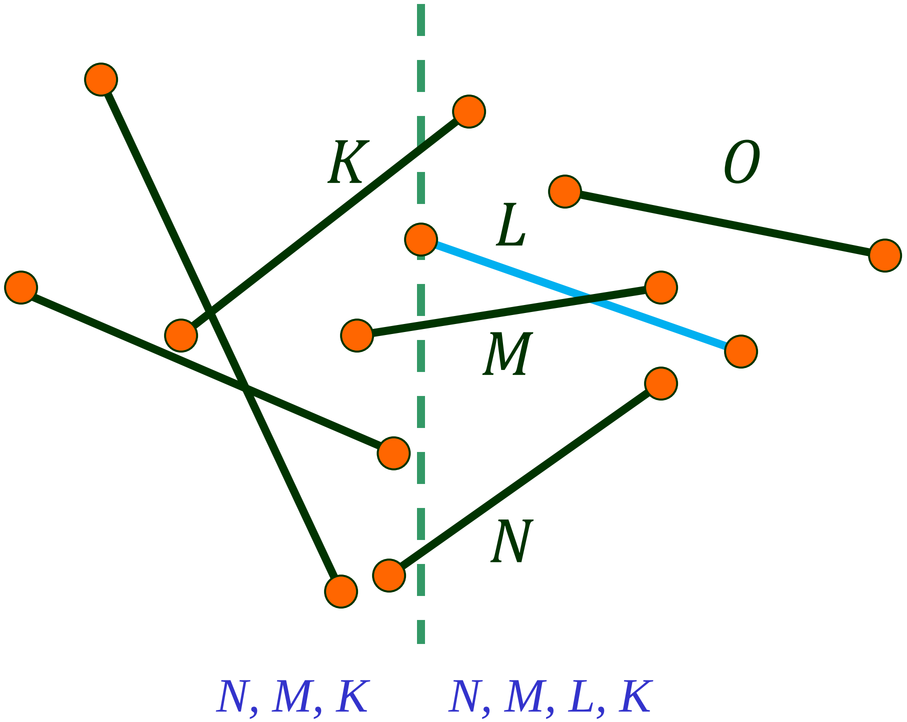


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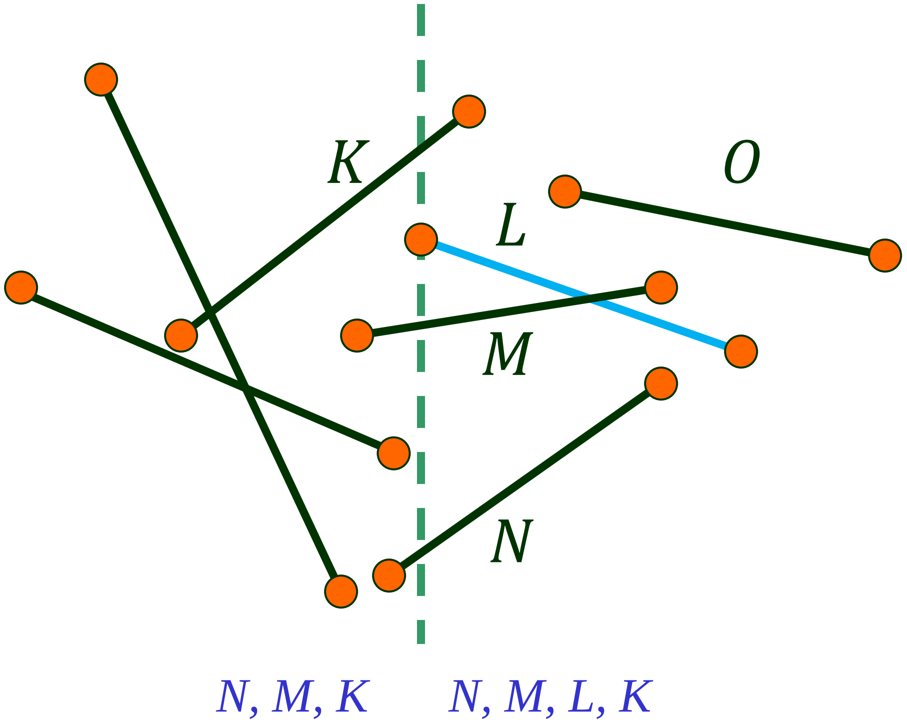


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- Check if $L$ intersects with the segment above ($K$) and the segment below ($M$).

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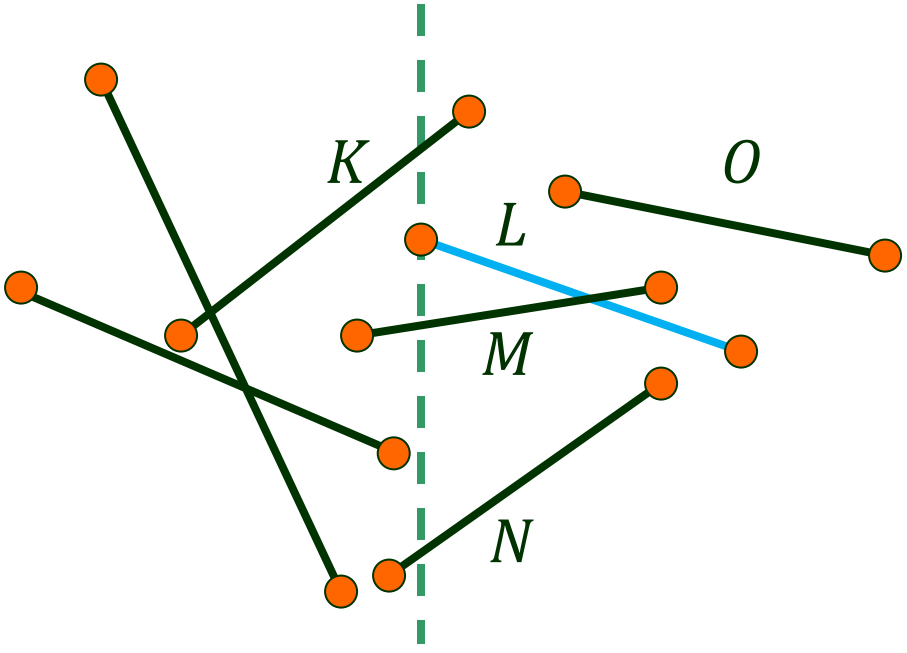


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Two segments must have become neighbors before they can intersect as the sweep line moves.

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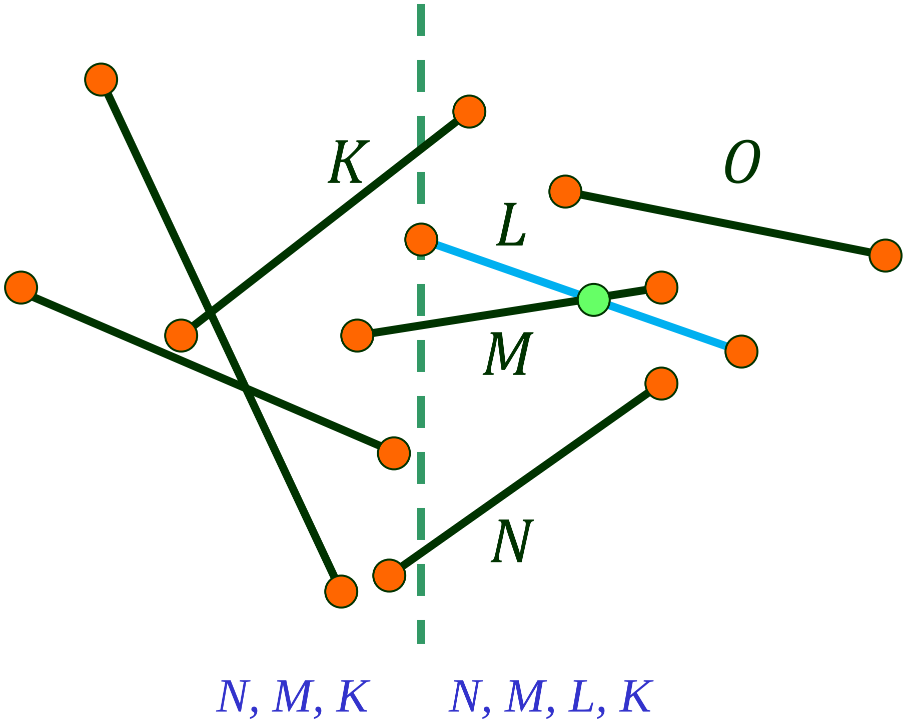


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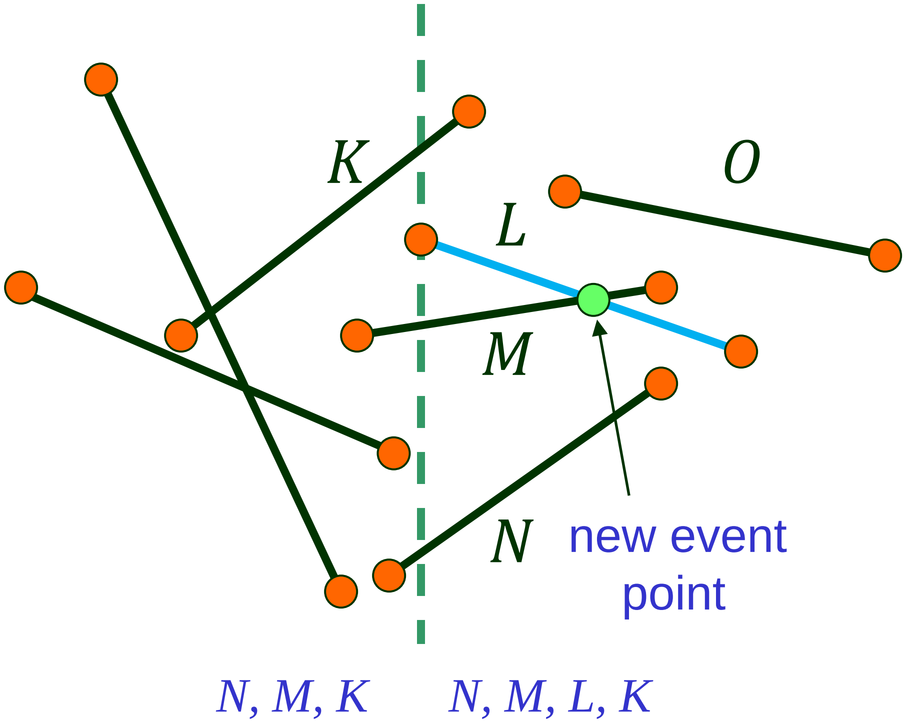


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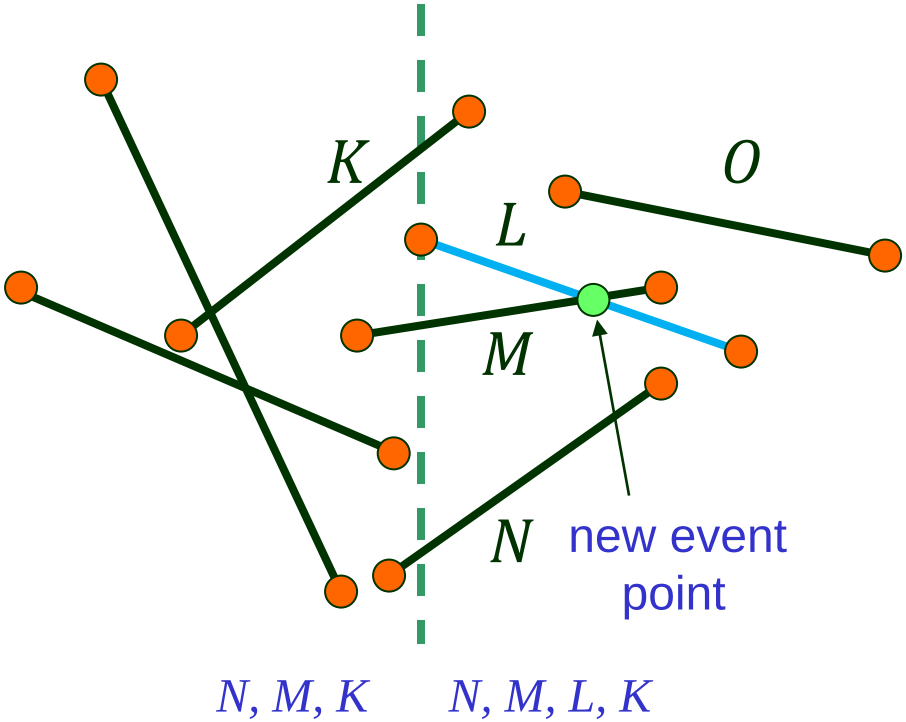


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- Intersection(s) are new event points.

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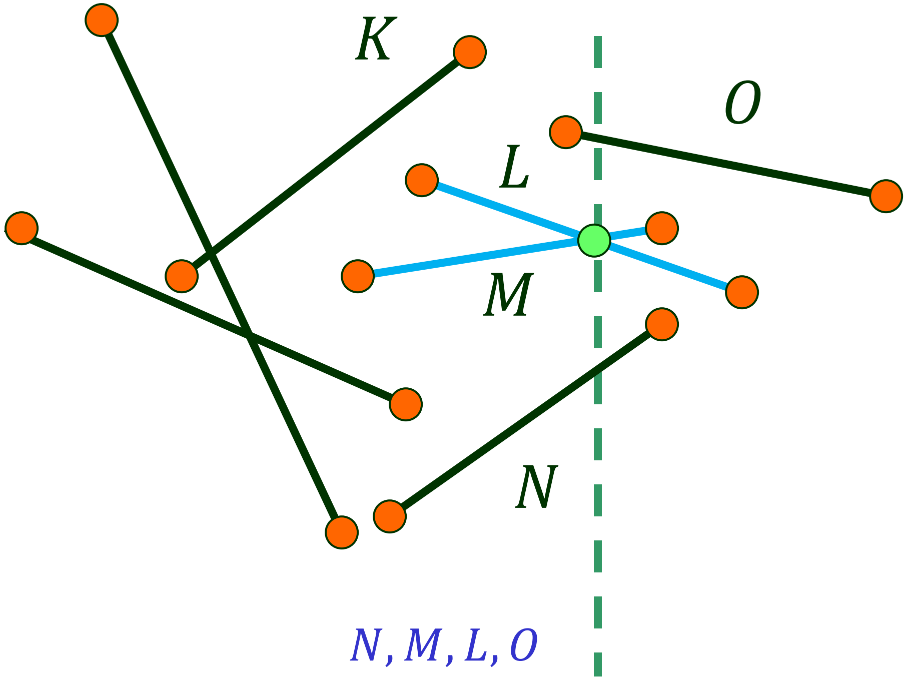

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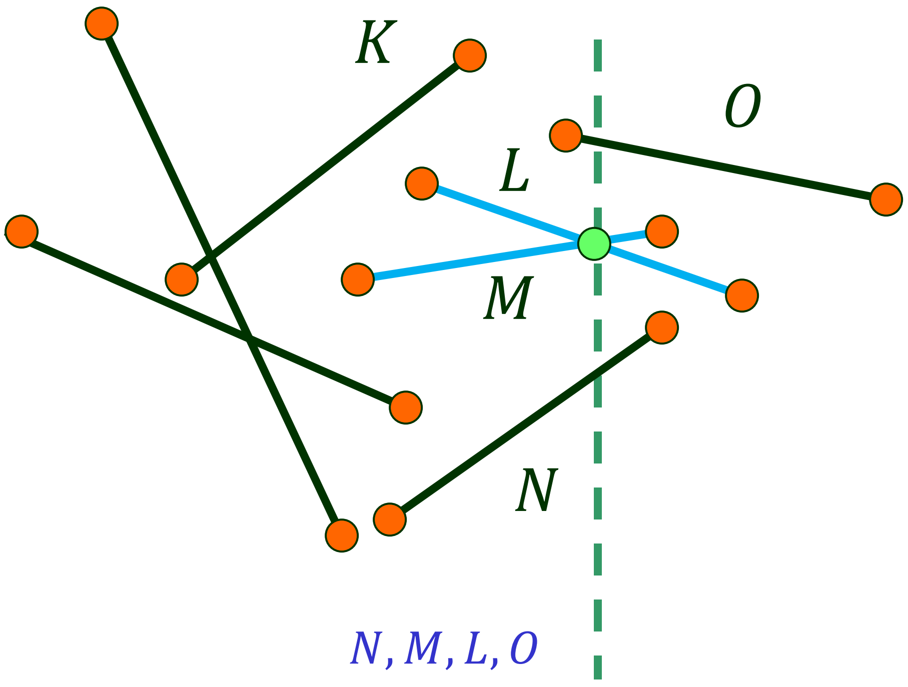


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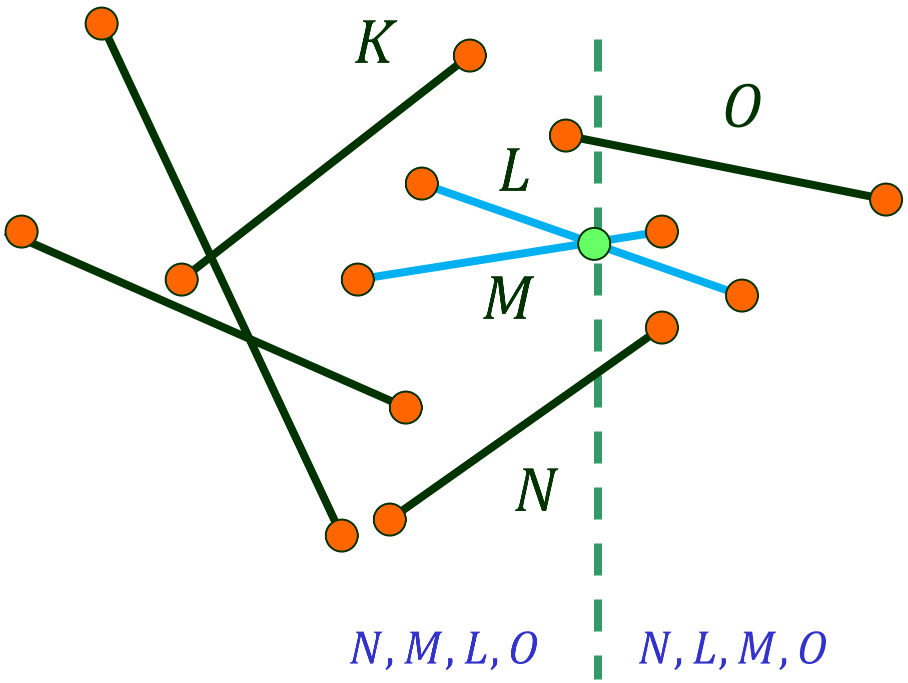


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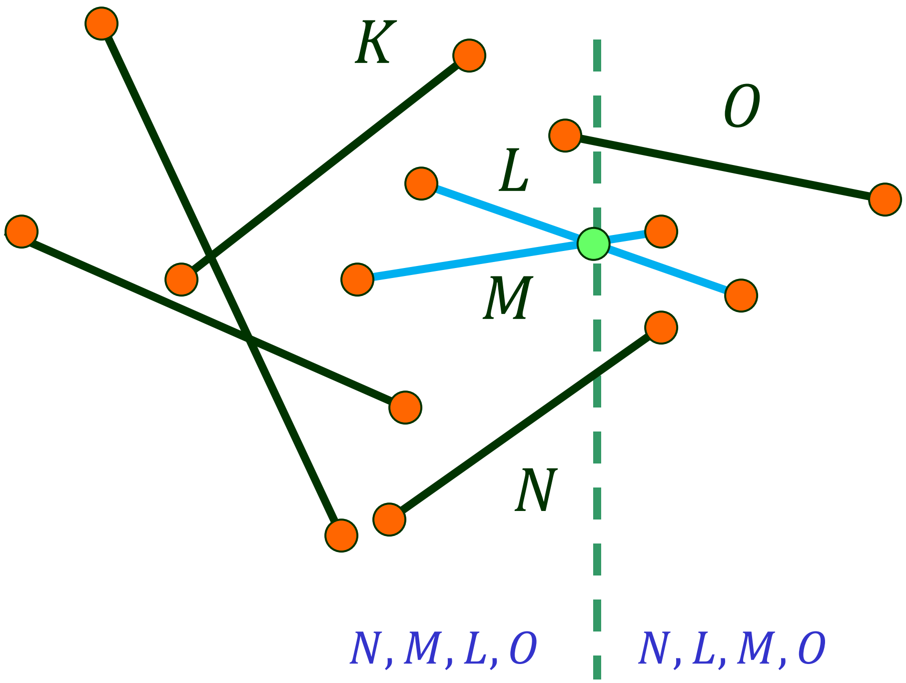


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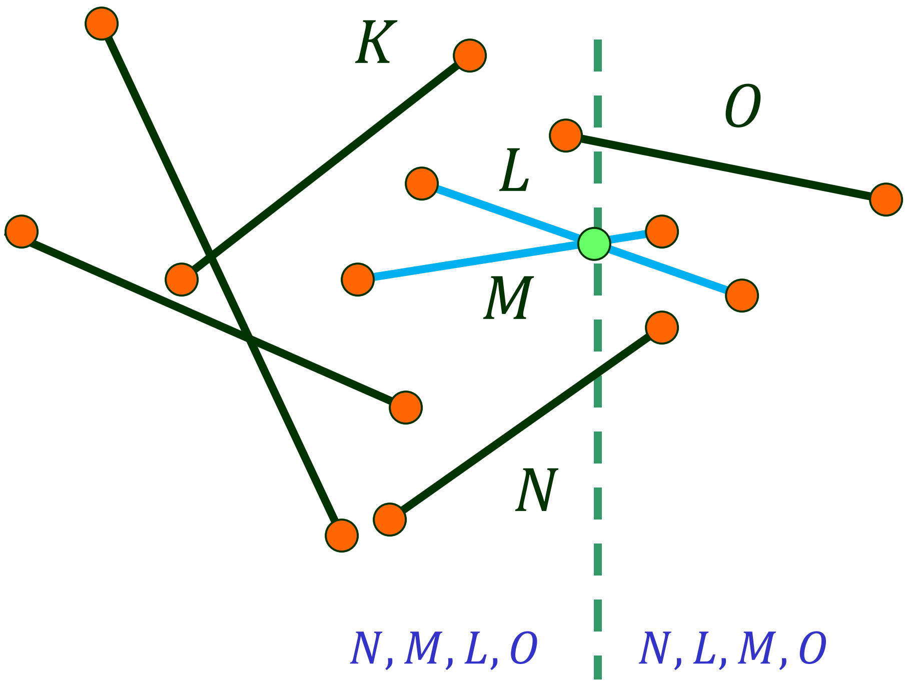


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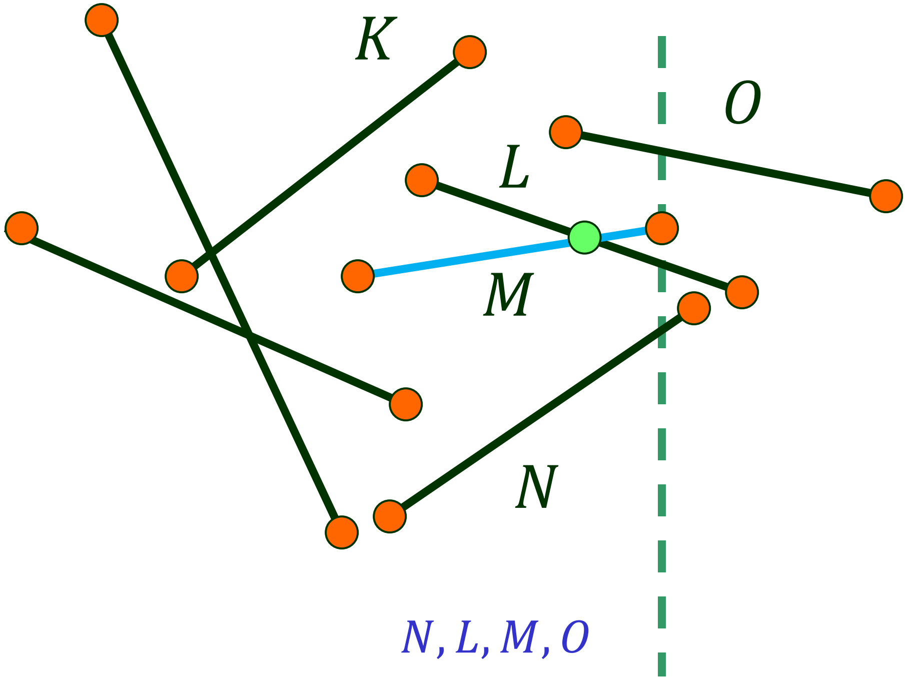

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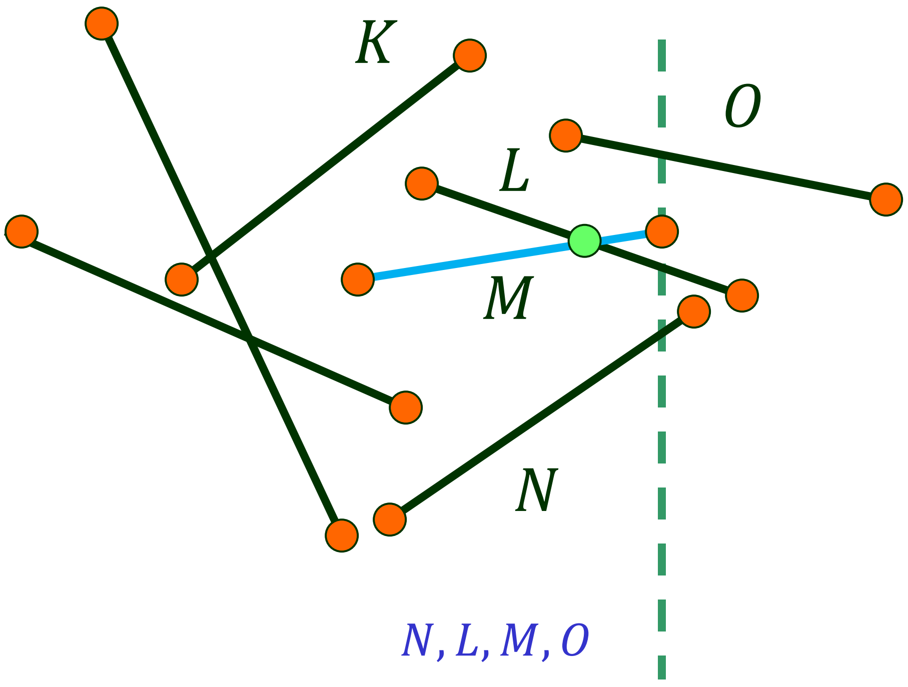


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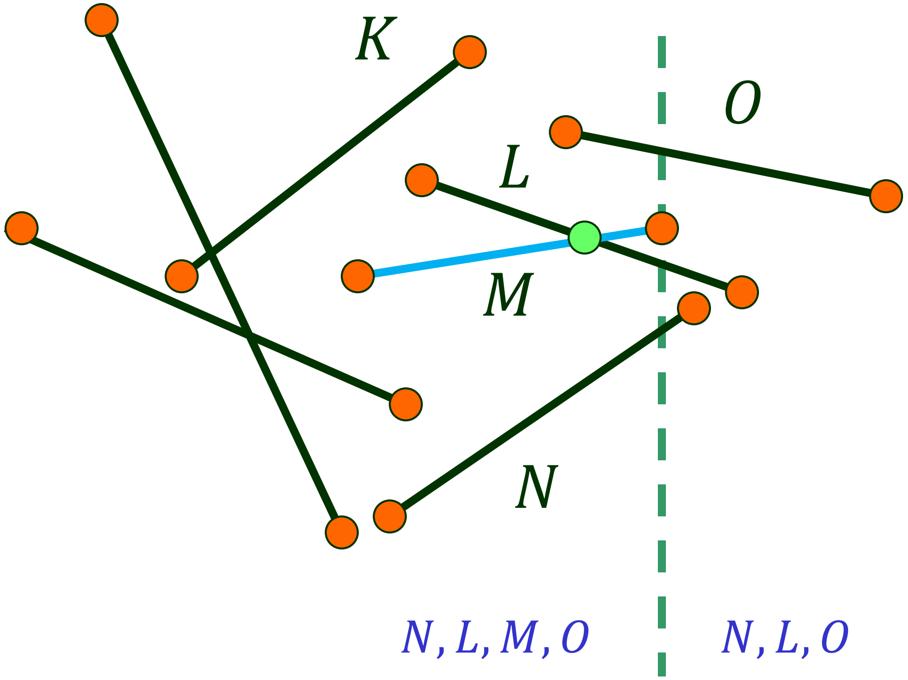


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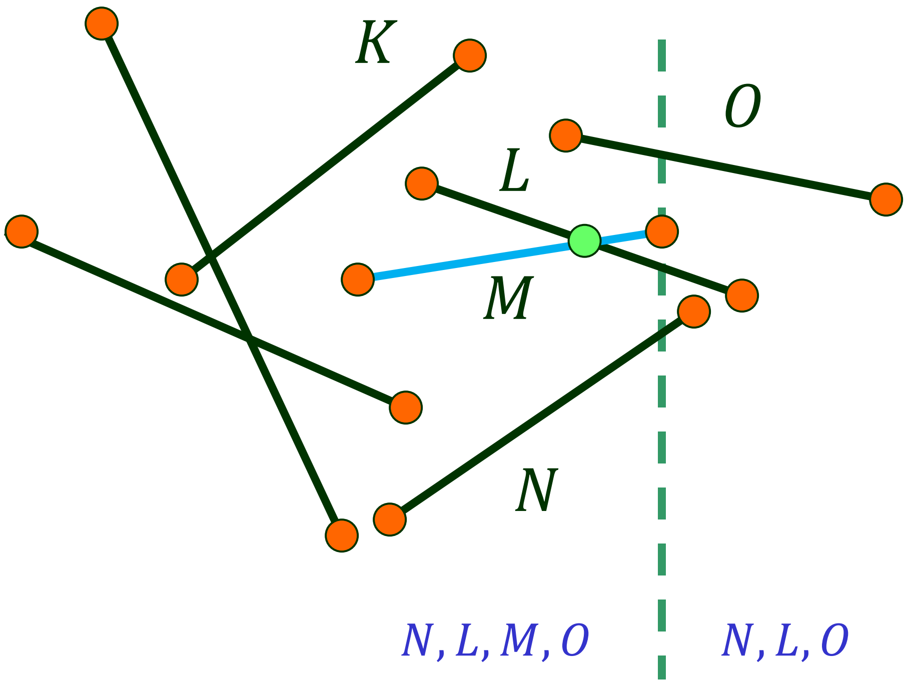


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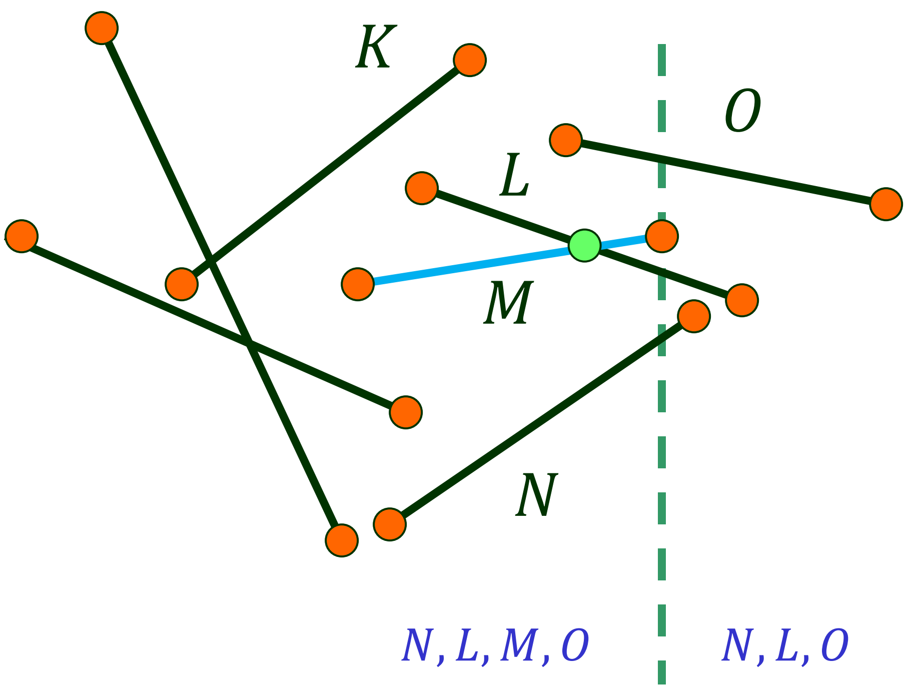


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- Check if they ($O$ and $L$) intersect.
- Intersection is new event point.

# Correctness

***Invariant*** at any time during the plane sweep:

All intersection points to the left of the sweep line have been computed correctly.

The correctness of the algorithm thus follows.

# III. Data Structure for Event Queue

Ordering of event points:

- by $x$-coordinates
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- inserting an event // $O(\log m)$

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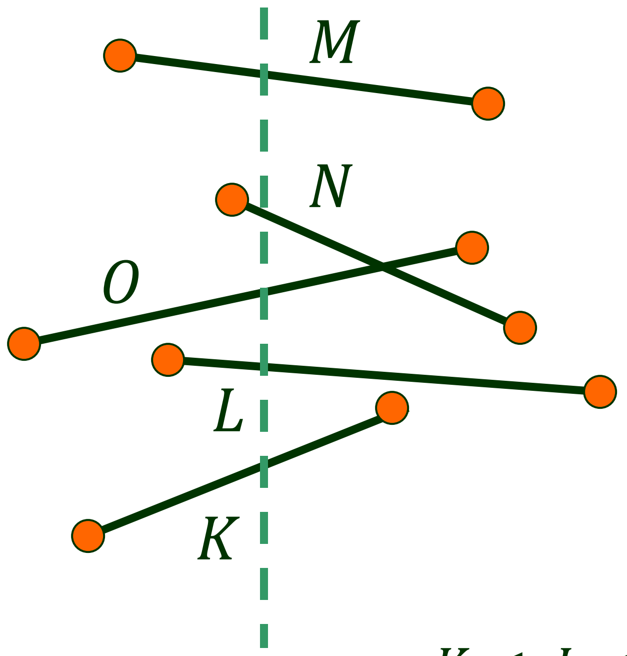
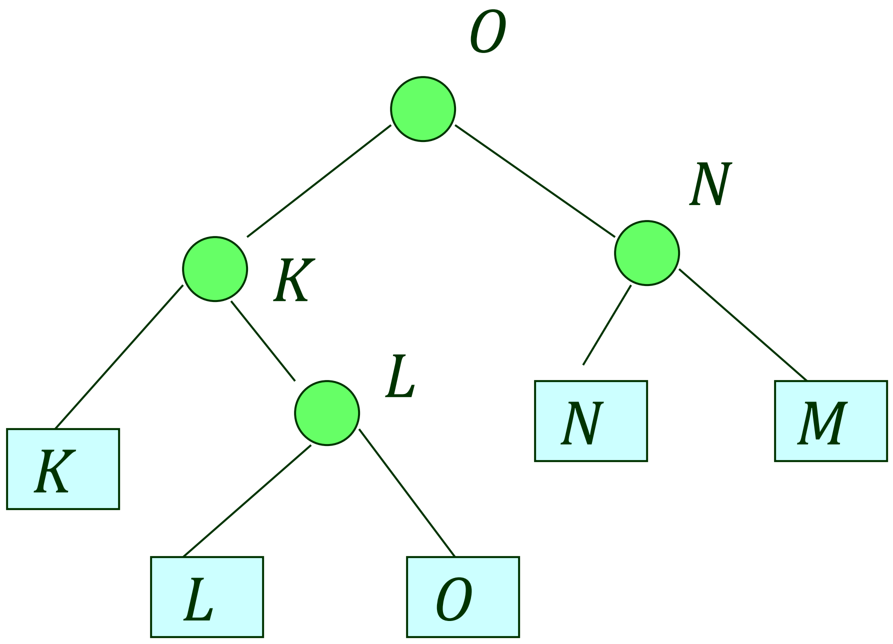
- $O(\log n)$ for each operation.

# Example




$K < L < O < N < M$

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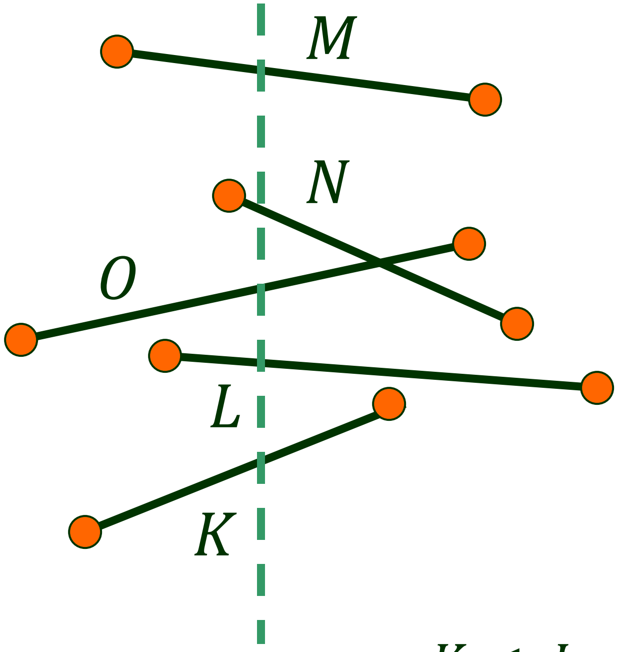


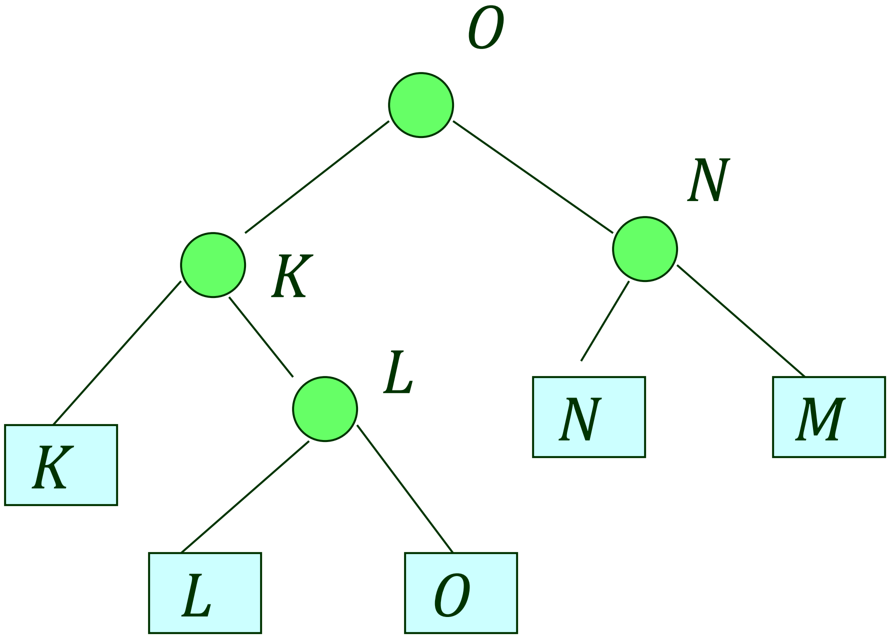


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- The bottom-up order of the segments corresponds to the *left-to-right* order of the leaves in the tree $T$.

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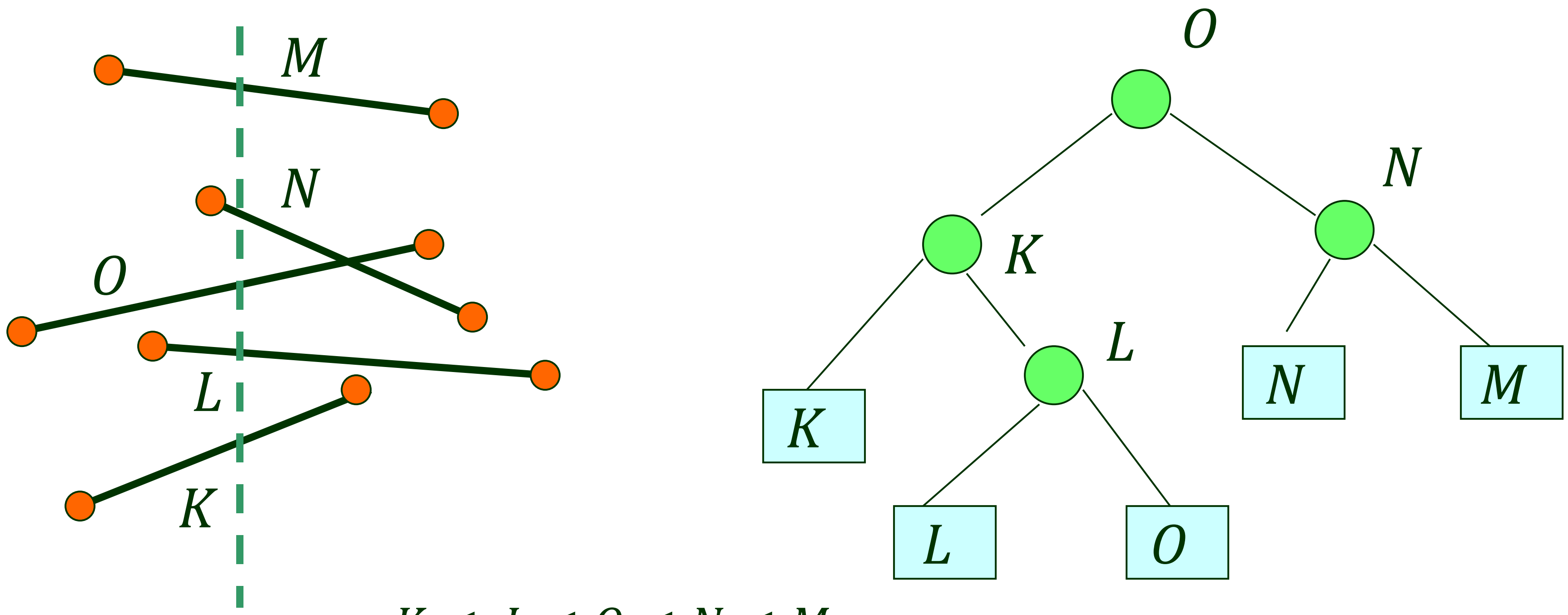


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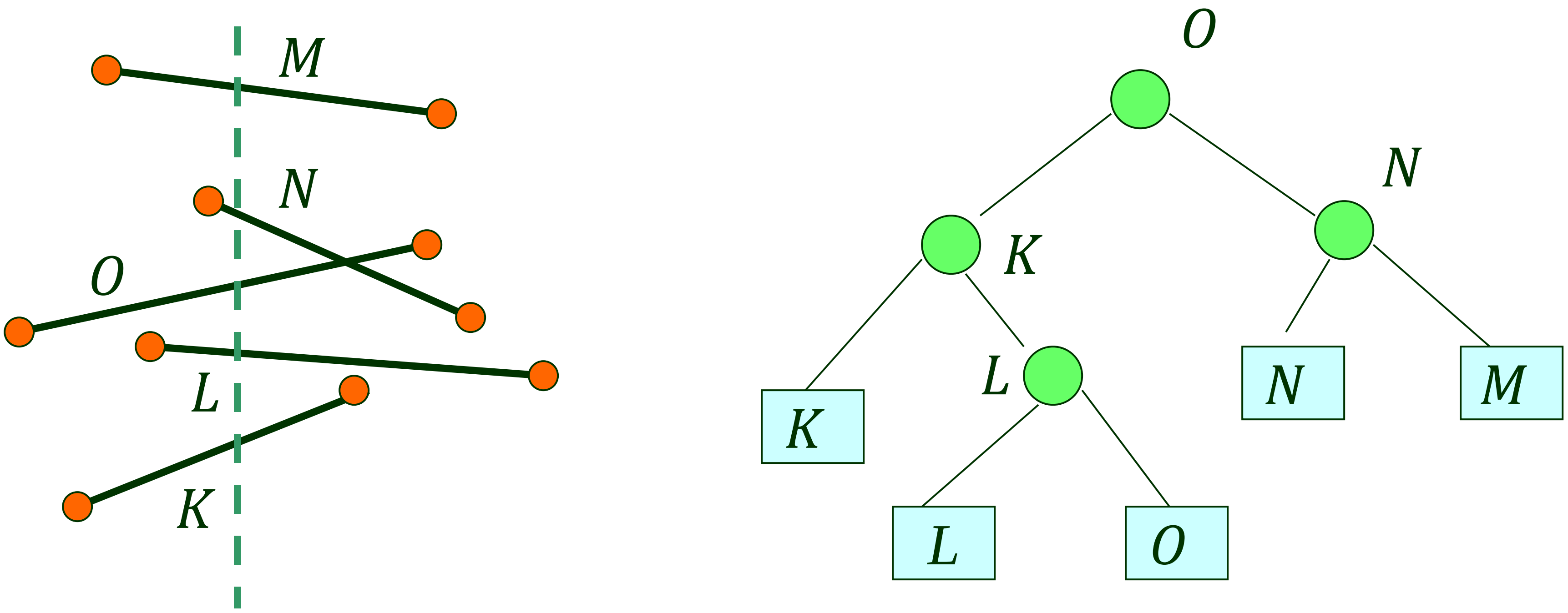
- The bottom-up order of the segments corresponds to the *left-to-right* order of the leaves in the tree $T$.
- Each internal node stores the segment from the *rightmost leaf* in its left subtree.

# Additional Operation




Searching for the segment immediately below some point $p$ on the sweep line.

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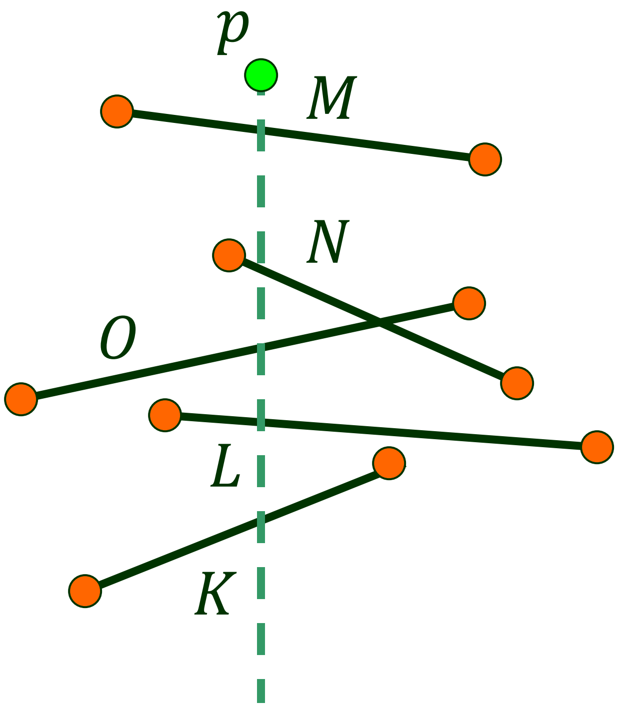


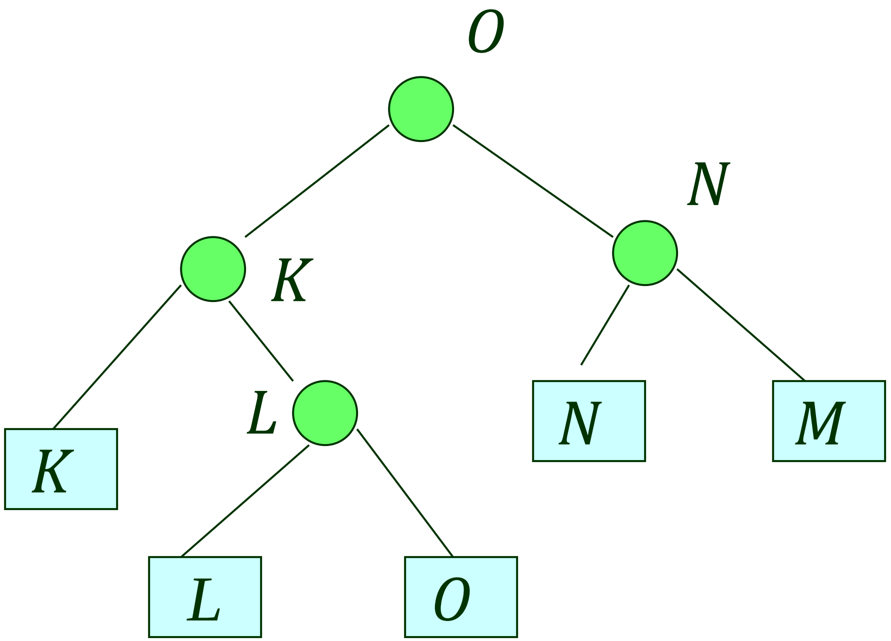


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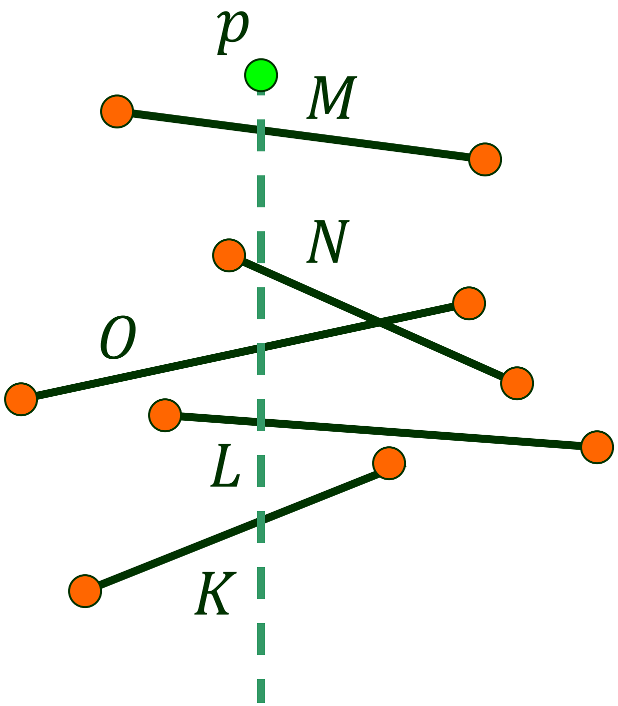


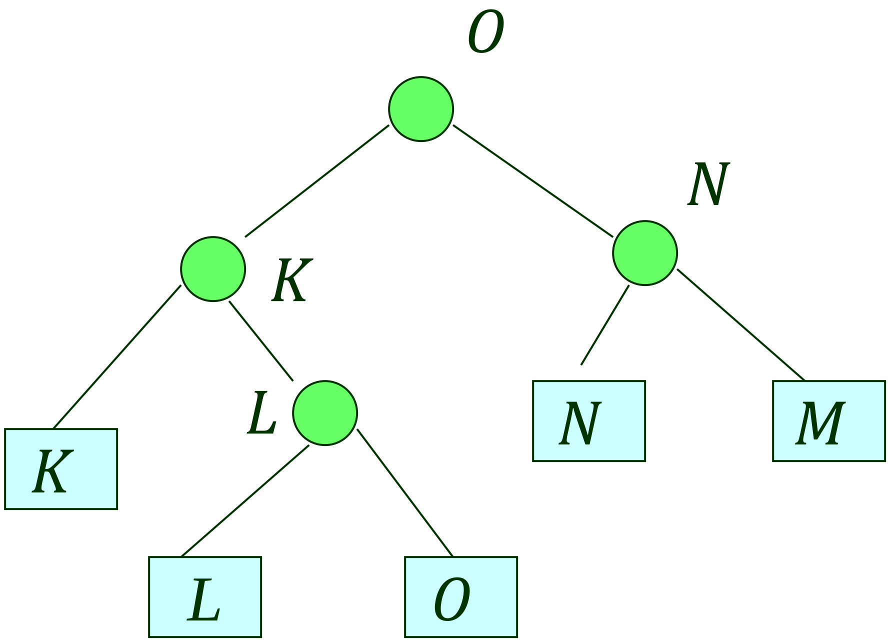


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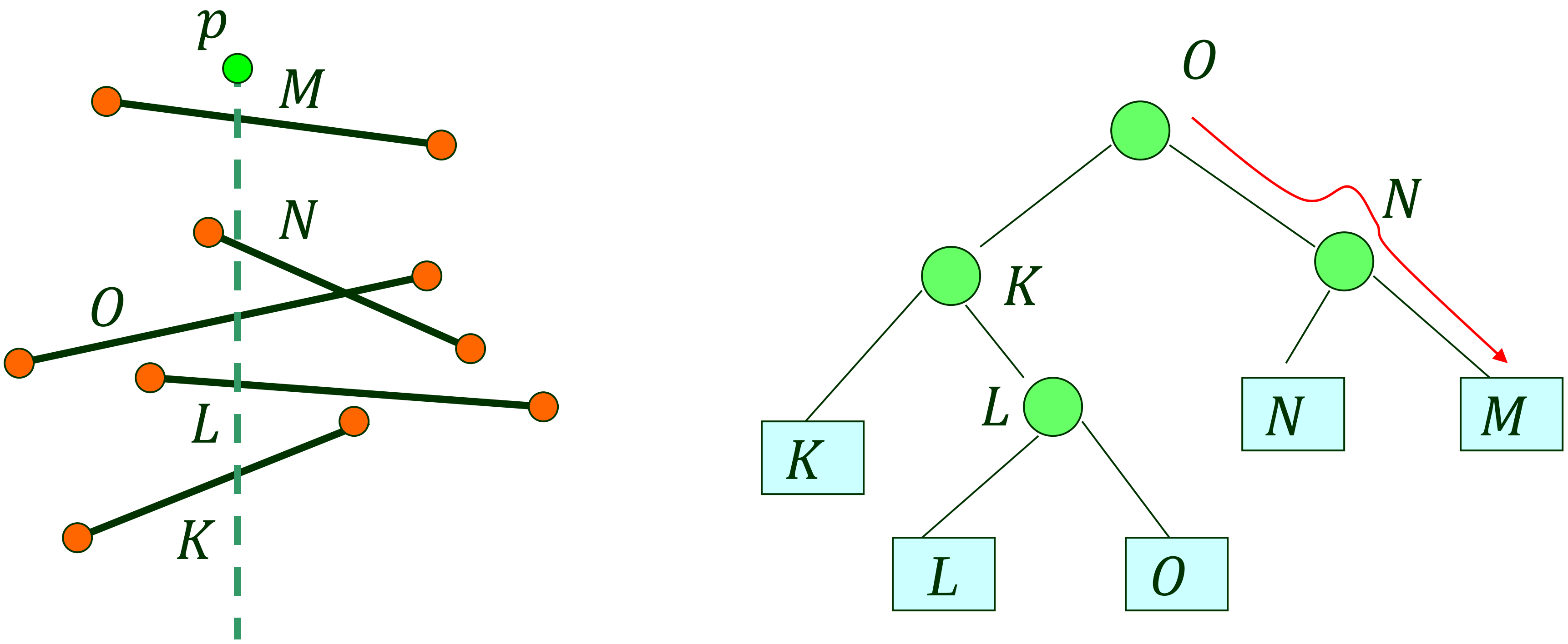


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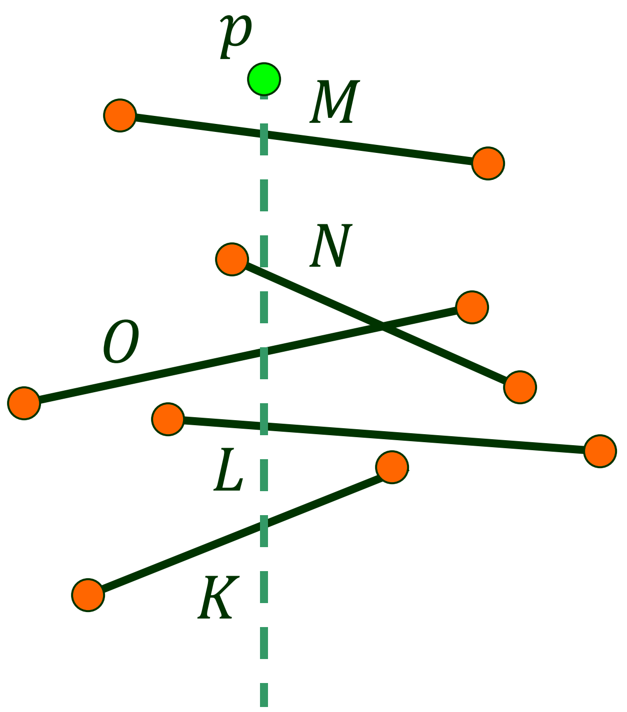


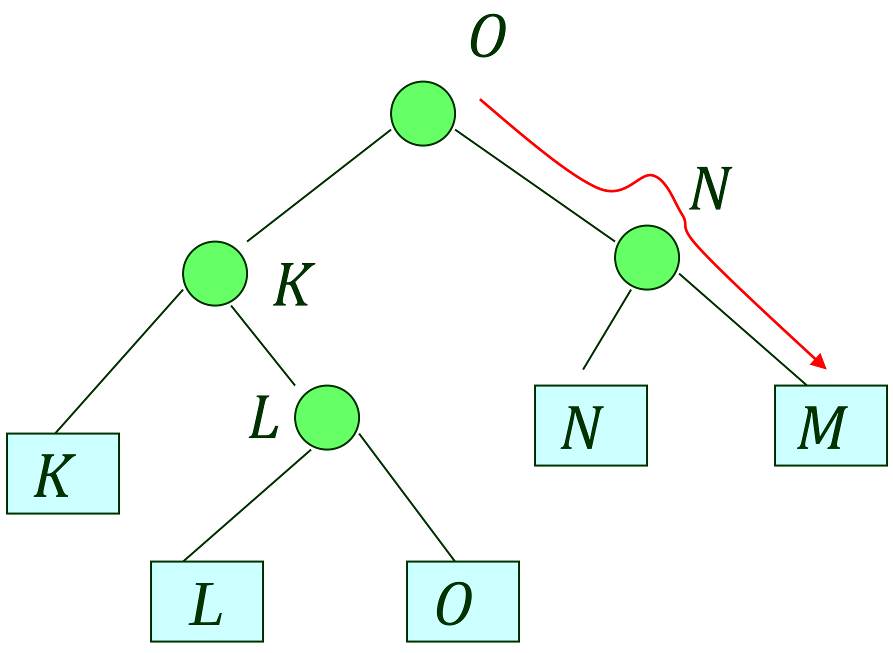


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- Descend binary search tree all the way down to a leaf.
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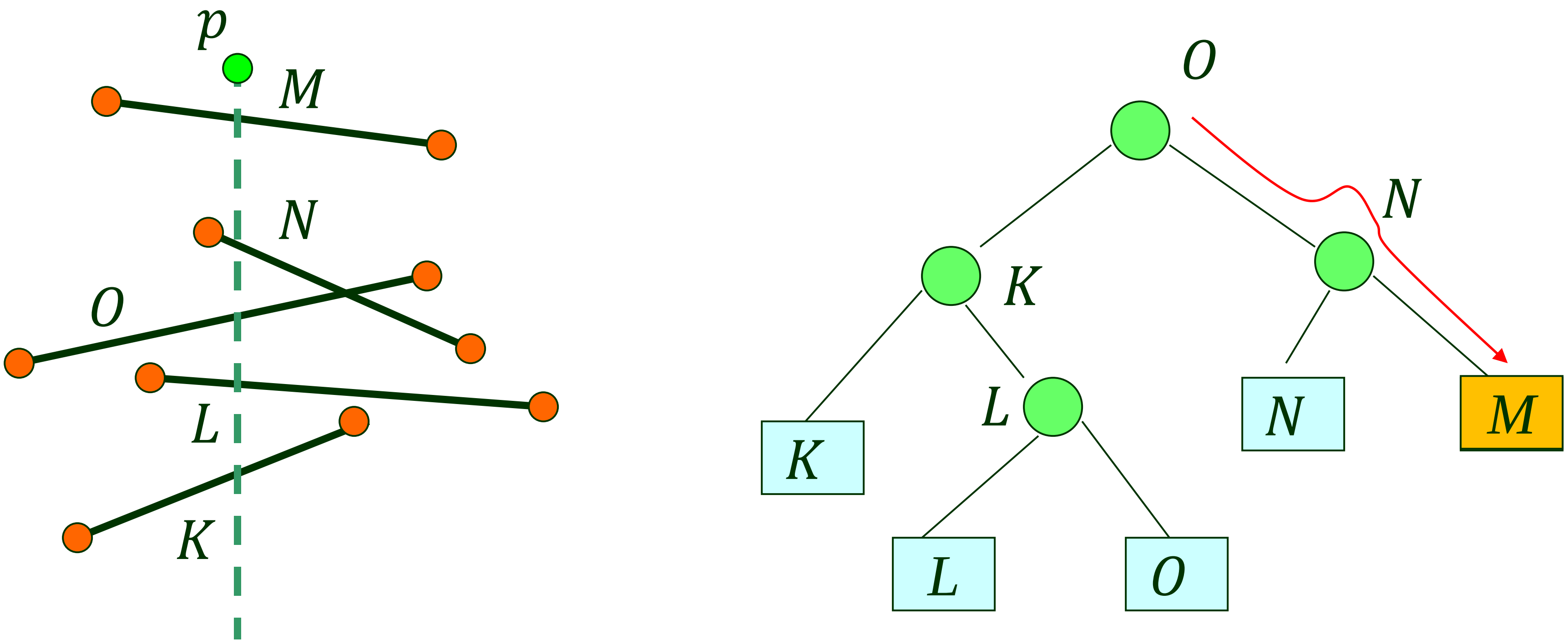


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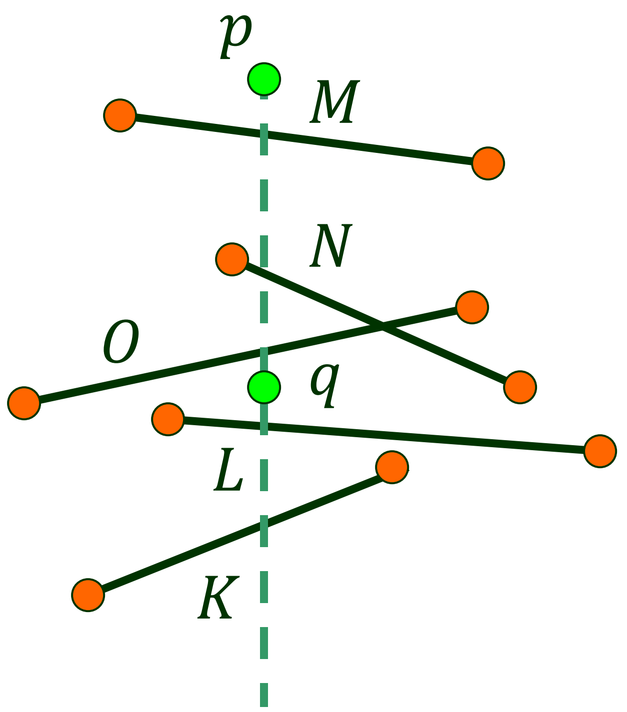


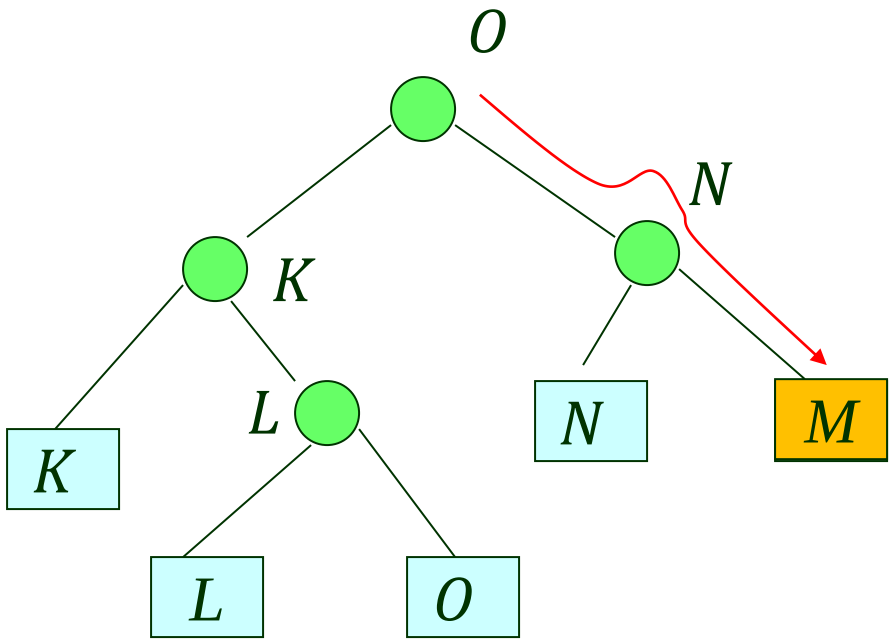


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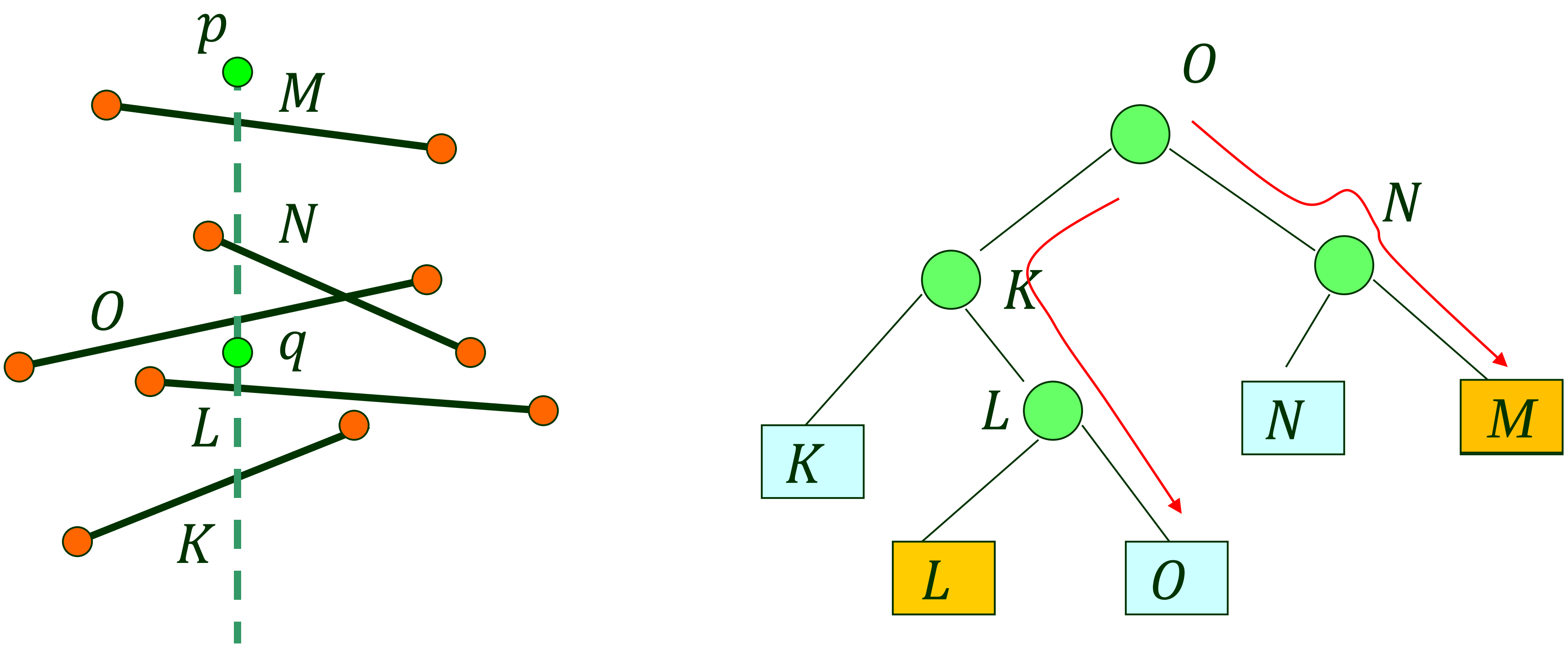


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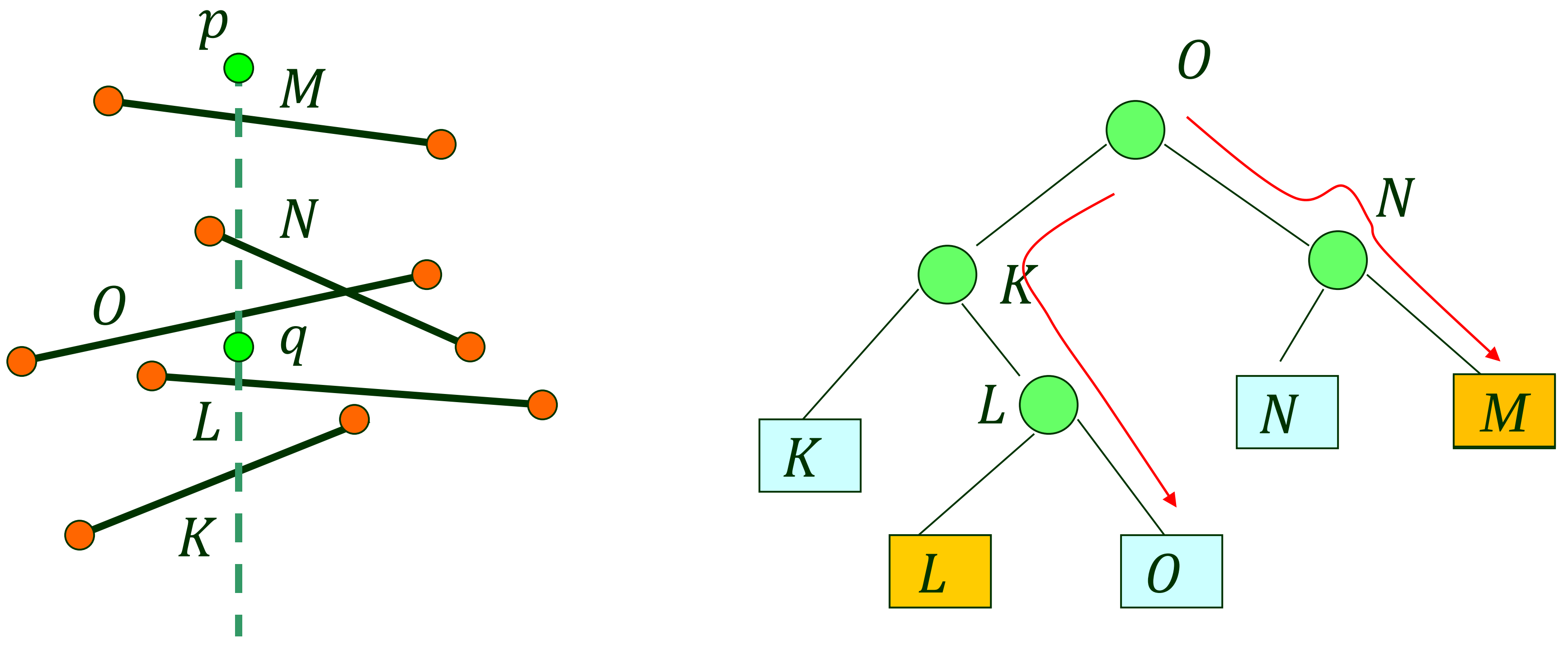


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$O(\log n)$ time

# The Algorithm

FindIntersections($S$)

**Input**: a set $S$ of line segments

**Ouput**: all intersection points and for each intersection the segment containing it.

1. $Q \leftarrow \emptyset$ // initialize an empty event queue
2. Insert the segment endpoints into $Q$ // store with every left // endpoint the **corresponding segments**
3. $T \leftarrow \emptyset$ // initialize an empty status structure
4. while $Q \neq \emptyset$
5. do extract the next event point $p$
6. $Q \leftarrow Q - \{p\}$
7. HandleEventPoint($p$)

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Status updates (1) – (3) presented earlier.

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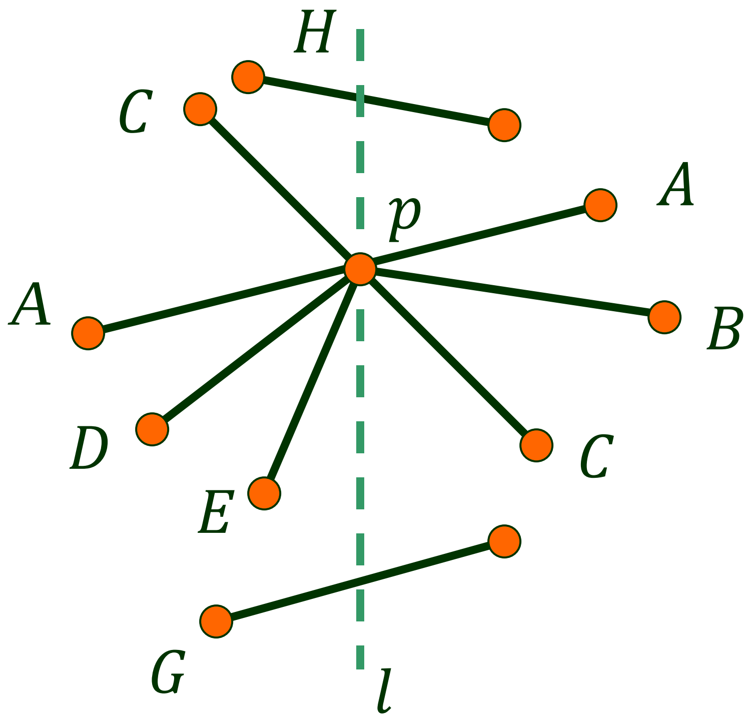

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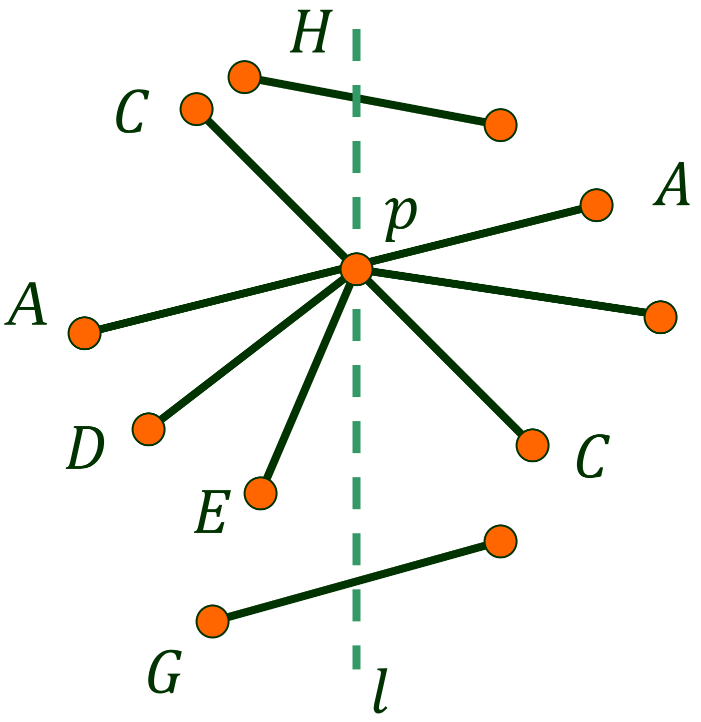


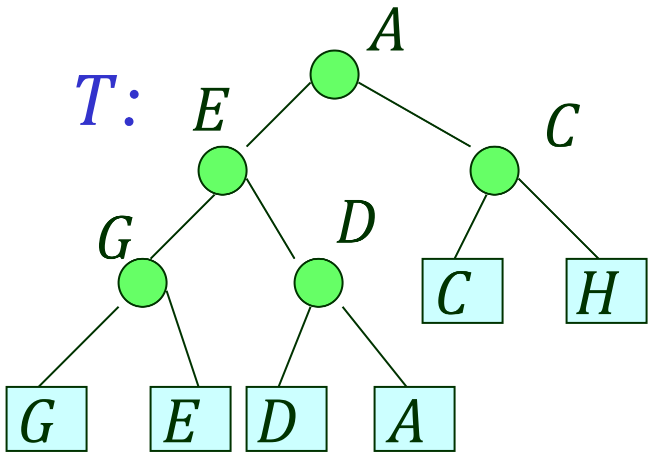

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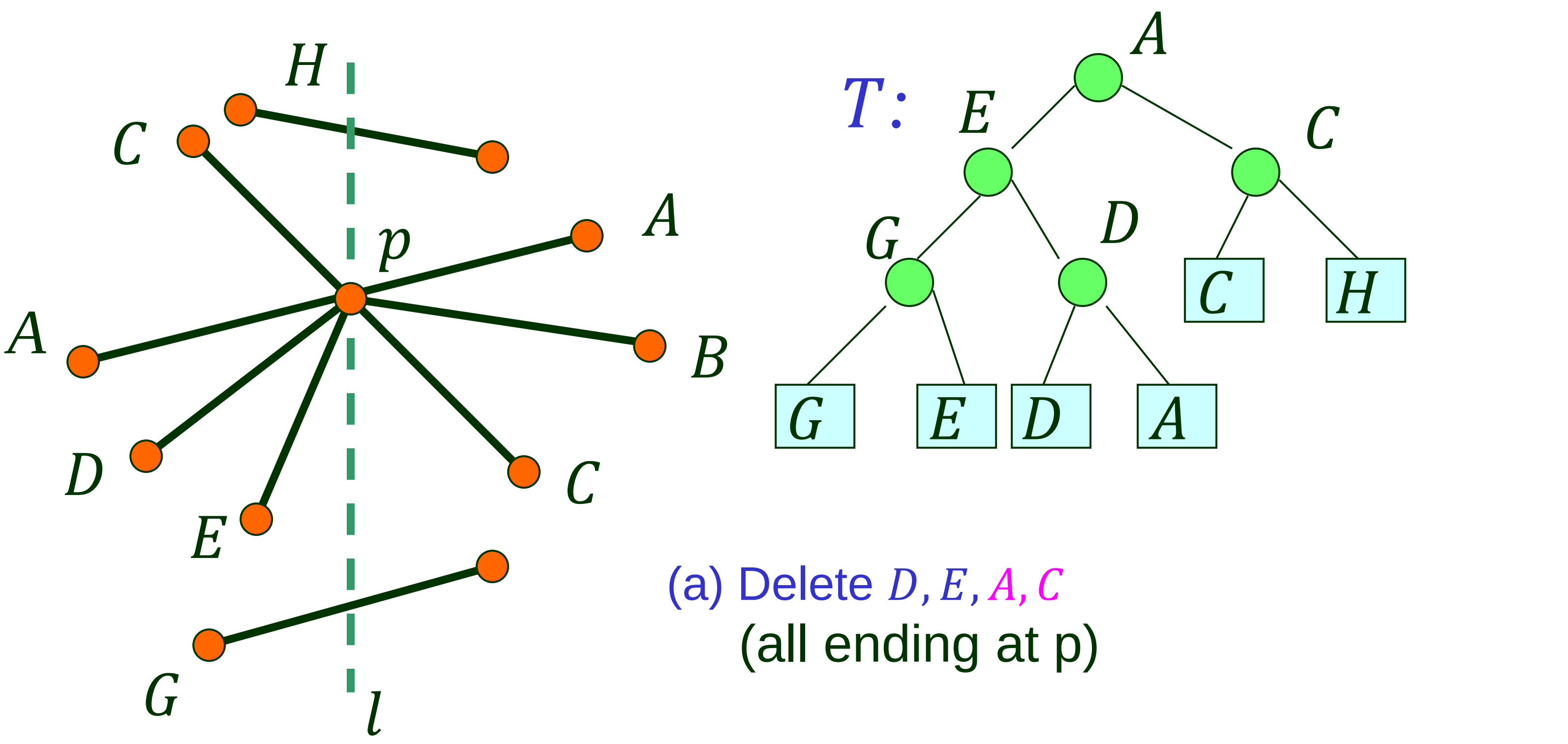


(a) Delete $D, E, A, C$
(all ending at p)

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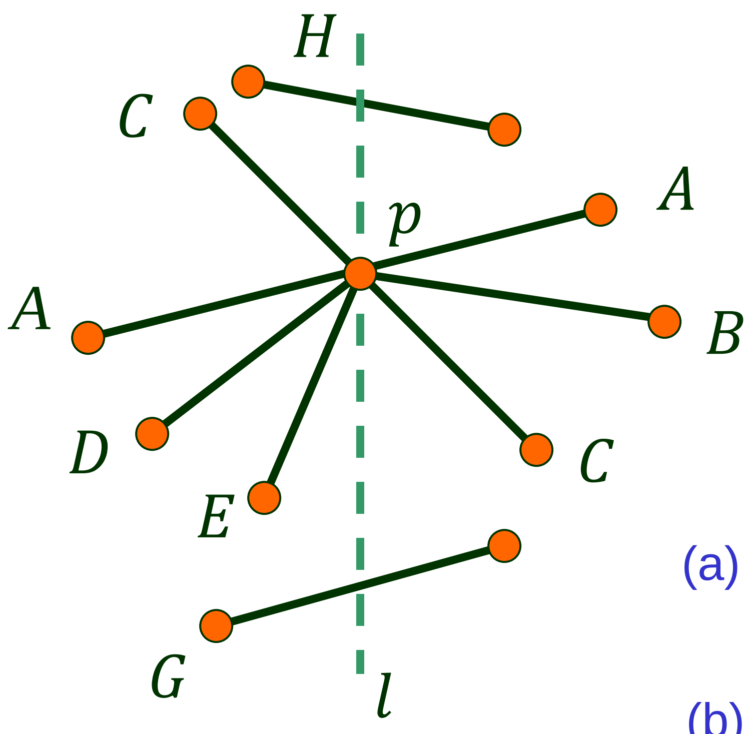


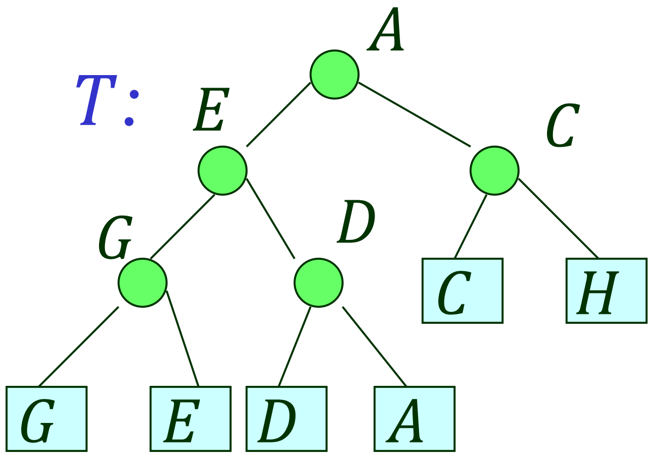


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(all ending at p)

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# Handling Event Points

Status updates (1) – (3) presented earlier.

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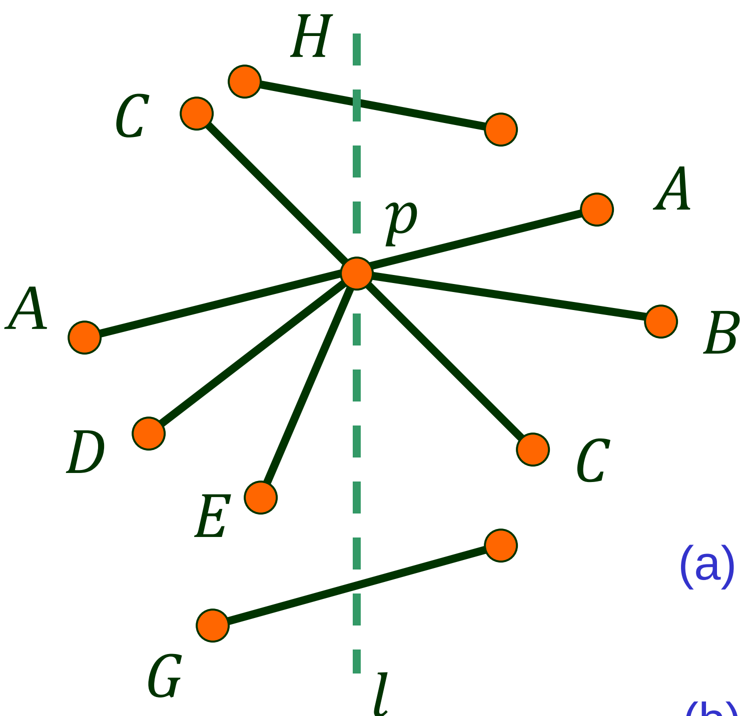
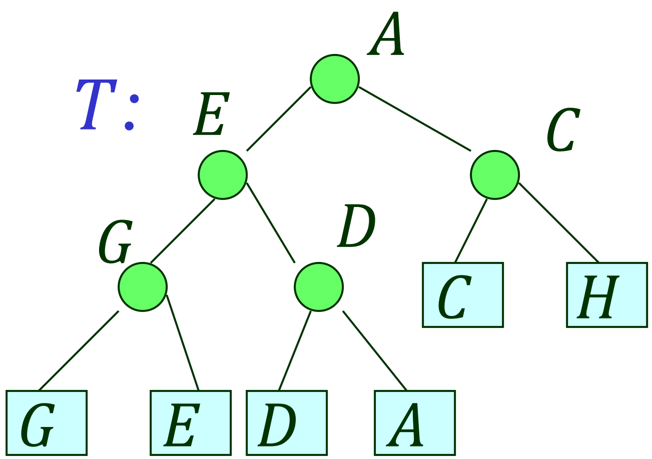



$T$:

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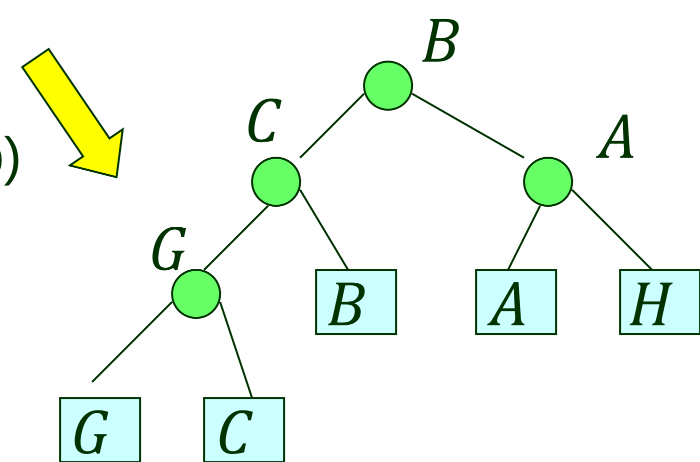

# The Code for Event Handling

HandleEventPoint($p$)

$L(p) \leftarrow \{$ segments with left endpoint $p$ $\}$ // they are stored with $p$

$R(p) \leftarrow \{$ segments with right endpoint $p$ $\}$

$C(p) \leftarrow \{$ segments with $p$ in interior $\}$

**if** $|L \cup R \cup C| > 1$

  **then** report $p$ as an intersection along with $L, R, C$

$T \leftarrow T - (R \cup C)$

$T \leftarrow T \cup (L \cup C)$ // order as intersected by sweep line just to the right of $p$.
// segments in $C(p)$ have order reversed.

**if** $L \cup C = \emptyset$ // right endpoint only

  **then** let $s_a$ and $s_b$ be the neighbors right above and below $p$ in $T$

    FindNewEvent($s_b, s_a, p$)

  **else** $s' \leftarrow$ lowest segment in $L \cup C$

    $s_b \leftarrow$ segment right below $s'$

    FindNewEvent($s_b, s', p$)

    $s'' \leftarrow$ highest segment in $L \cup C$

    $s_a \leftarrow$ segment right above $s''$

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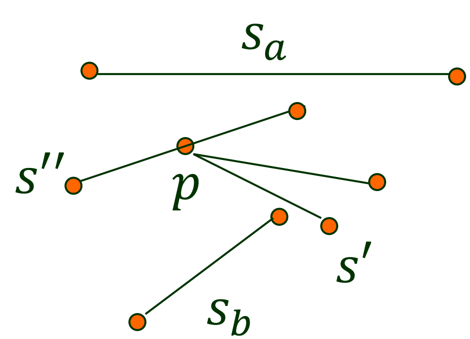

# Finding New Event

FindNewEvent($s_l, s_r, p$)

1. if $s_l$ and $s_r$ intersect to the right of $p$ // sweep line position
2. then insert the intersection point as an event in $Q$

# IV. Time & Storage

The tree $T$ stores every segment once. $O(n)$

The size of the event queue $Q$: $O(n+I)$.

↑ # intersections

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**Theorem** All $I$ intersections of $n$ line segments in the plane can be reported in $O((n+I)\log n)$ time and $O(n)$ space.

# Correctness

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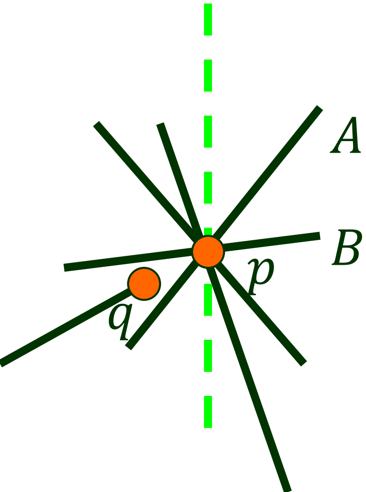

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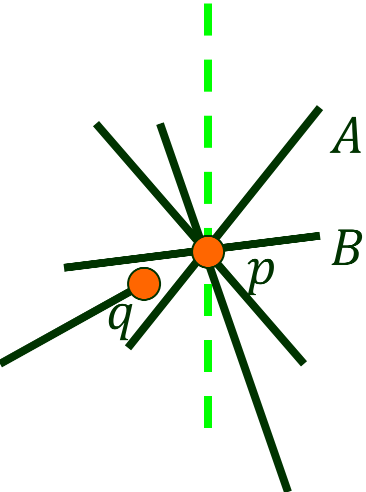

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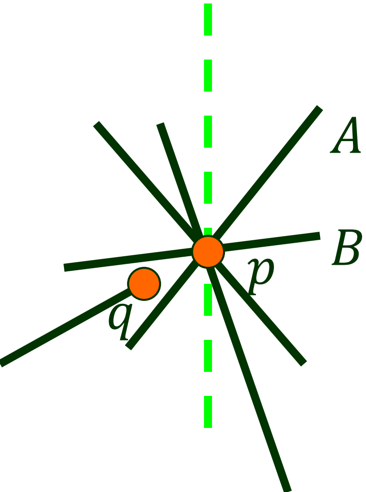


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$\Rightarrow$ By induction, $q$ was handled correctly and $p$ is detected and stored in $Q$

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Not tight! – the total number of intersections may still be $\Theta(n)$.

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                      // #operations on T is Θ(|L(p) ∪ R(p) ∪ C(p)|)
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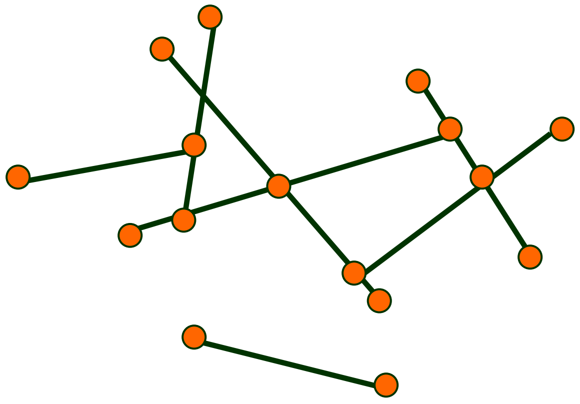

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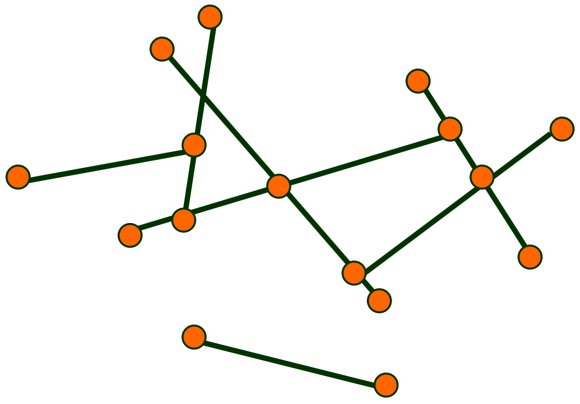

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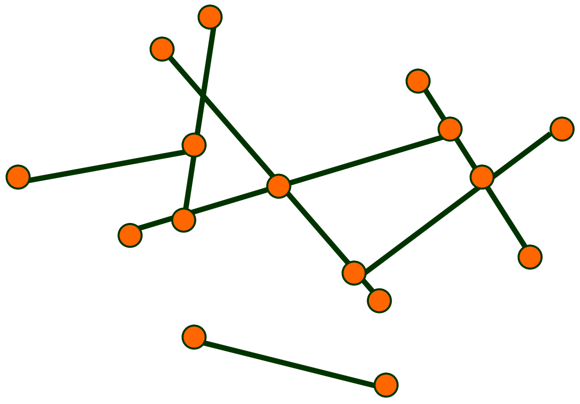

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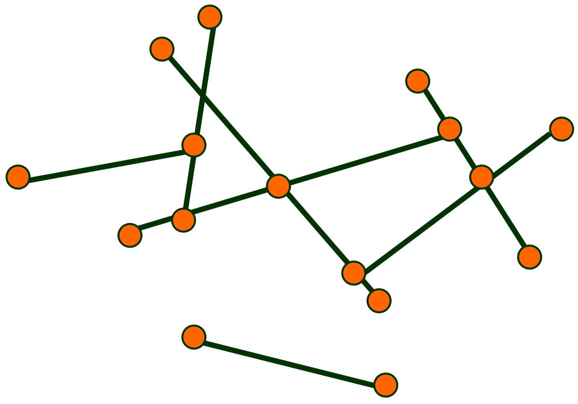

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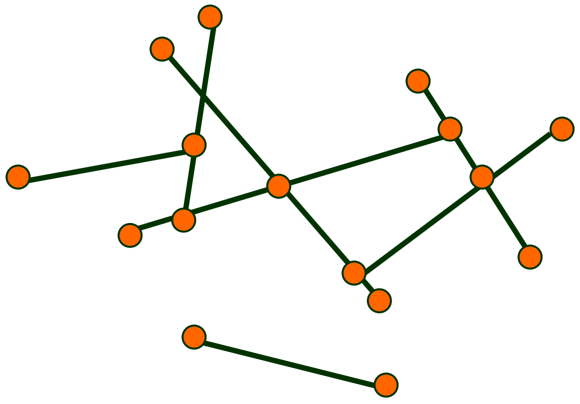

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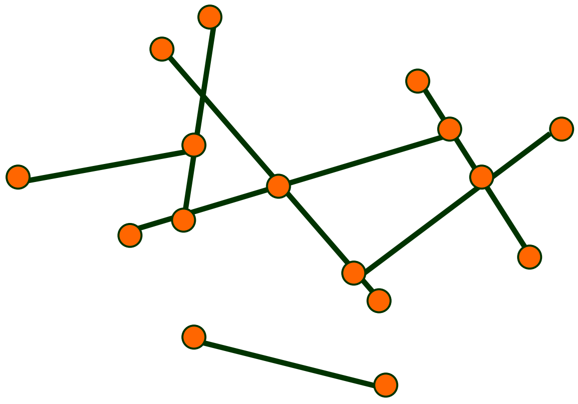

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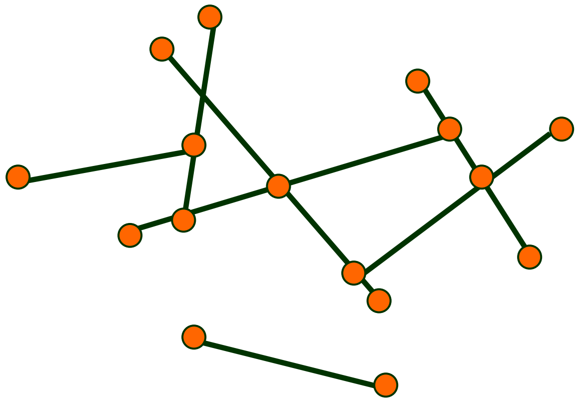

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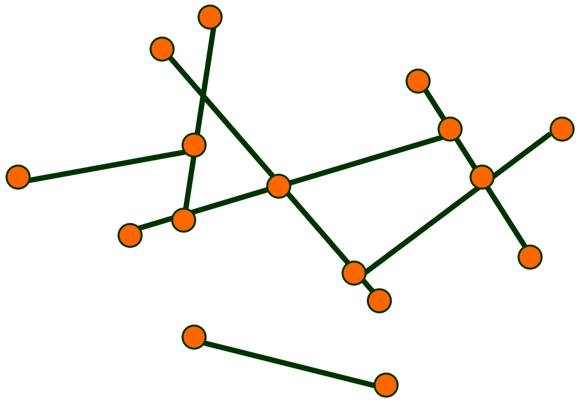

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Euler's equation:

# connected components

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$$n_e \leq 6n + 3I - 6$$

⇓

$$m \leq 2n_e \leq 12n + 6I - 12 = O(n + I)$$

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\# connected components

$$n_v - n_e + n_f = l + 1 \geq 2$$

$$\Downarrow \quad \frac{2}{3}n_e \geq n_f$$

$$n_v - n_e + \frac{2}{3}n_e \geq n_v - n_e + n_f \geq 2$$

$$\Downarrow$$

$$n_e \leq 3n_v - 6$$

$$\Downarrow \quad n_v \leq 2n + I$$

$$n_e \leq 6n + 3I - 6$$

$$\Downarrow$$

$$m \leq 2n_e \leq 12n + 6I - 12 = O(n + I)$$