Convex Hulls

Outline

I. Convex sets

II. Convex hull properties

III. Graham scan

IV. Correctness and running time

V. Javis' march

I. Convex Sets & Concave Sets

A planar region R is called *convex* if and only if for any pair of points p, q in R, the line segment $\overline{pq}$ lies *completely* in R.

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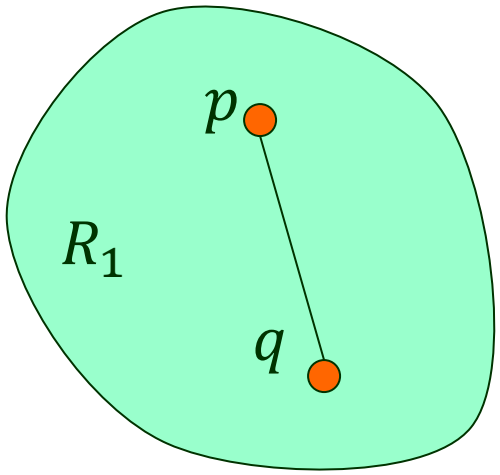

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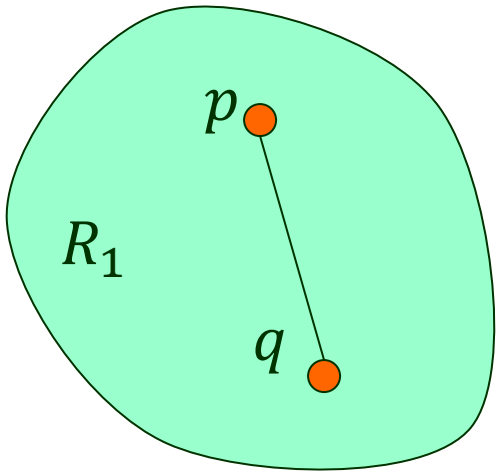

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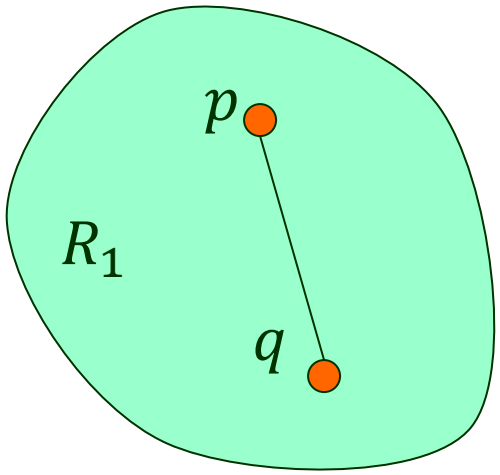

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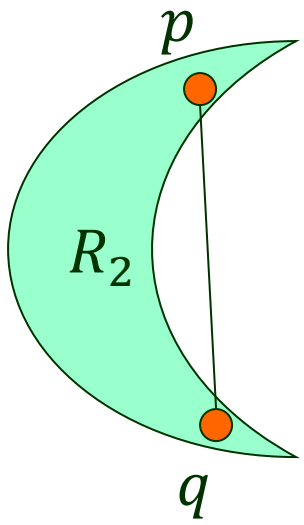
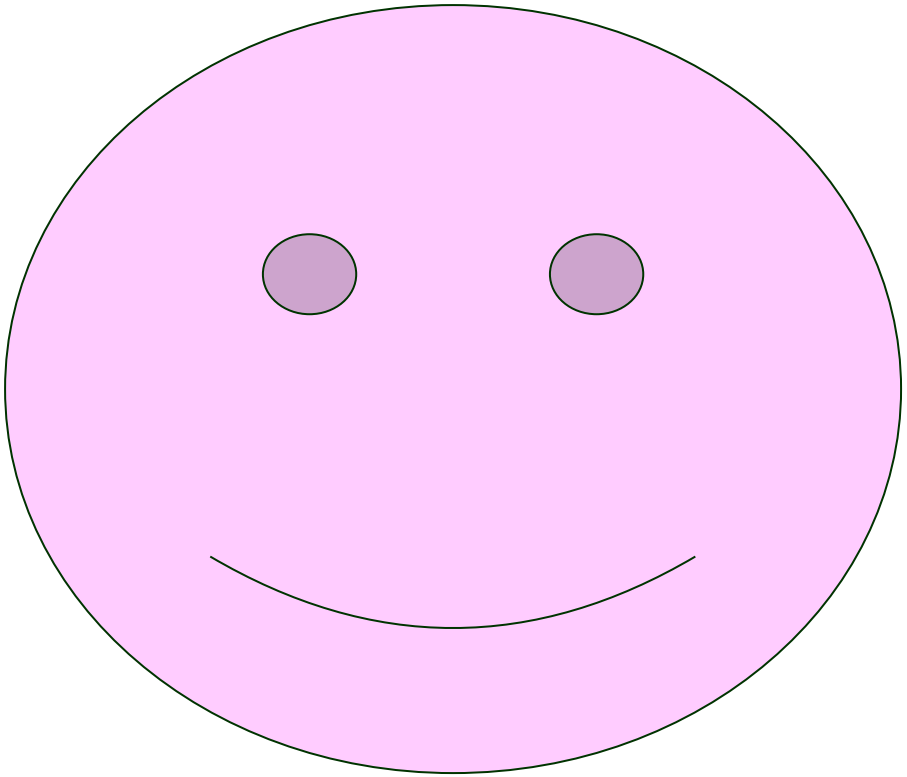


Concave

An Example



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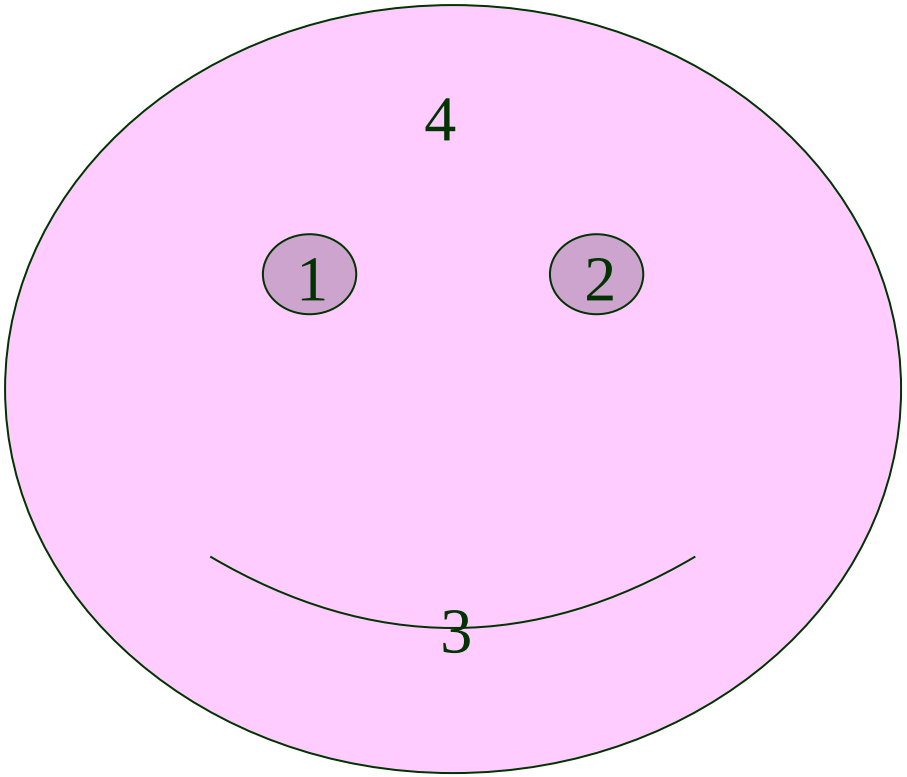

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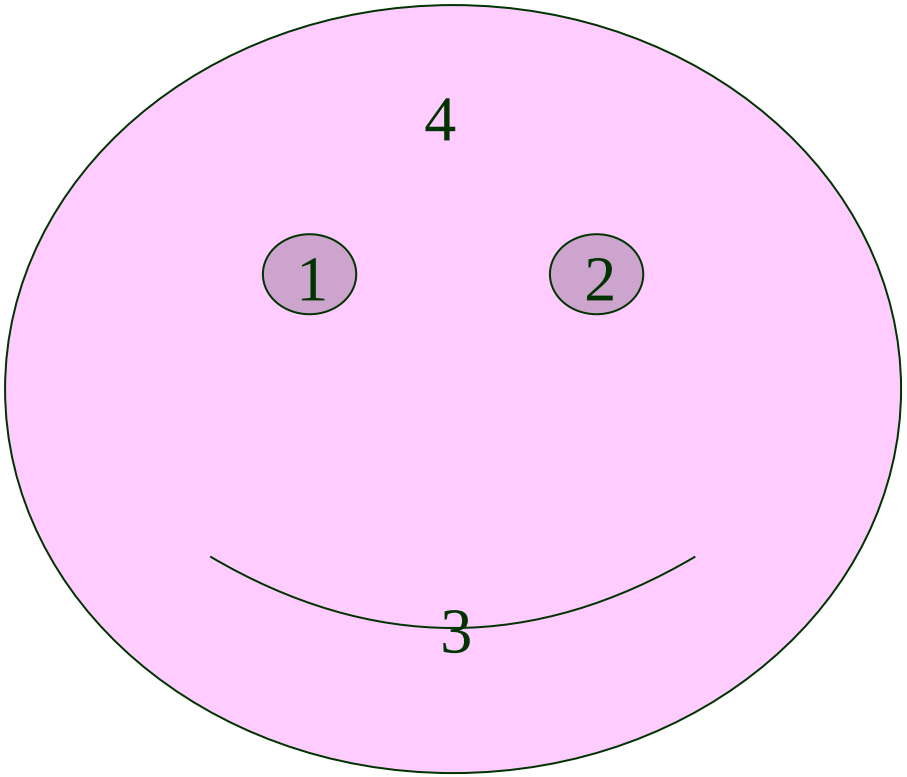
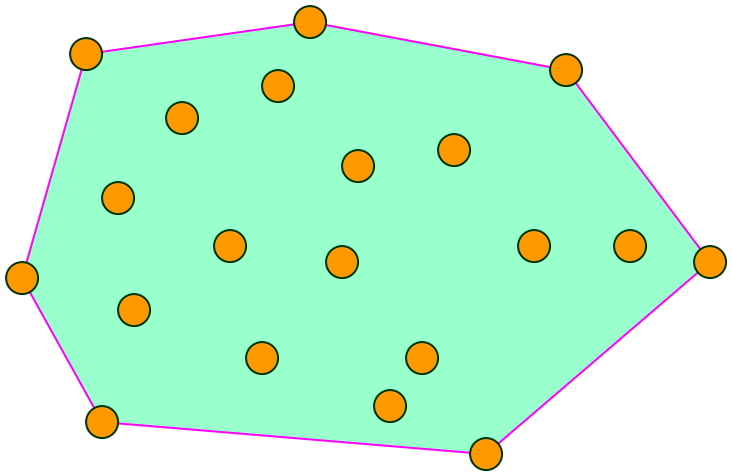


Regions 1 & 2: convex
Regions 3 & 4: concave

Convex Hull

The *convex hull* CH(Q) of a set Q is the *smallest* convex region that contains Q.



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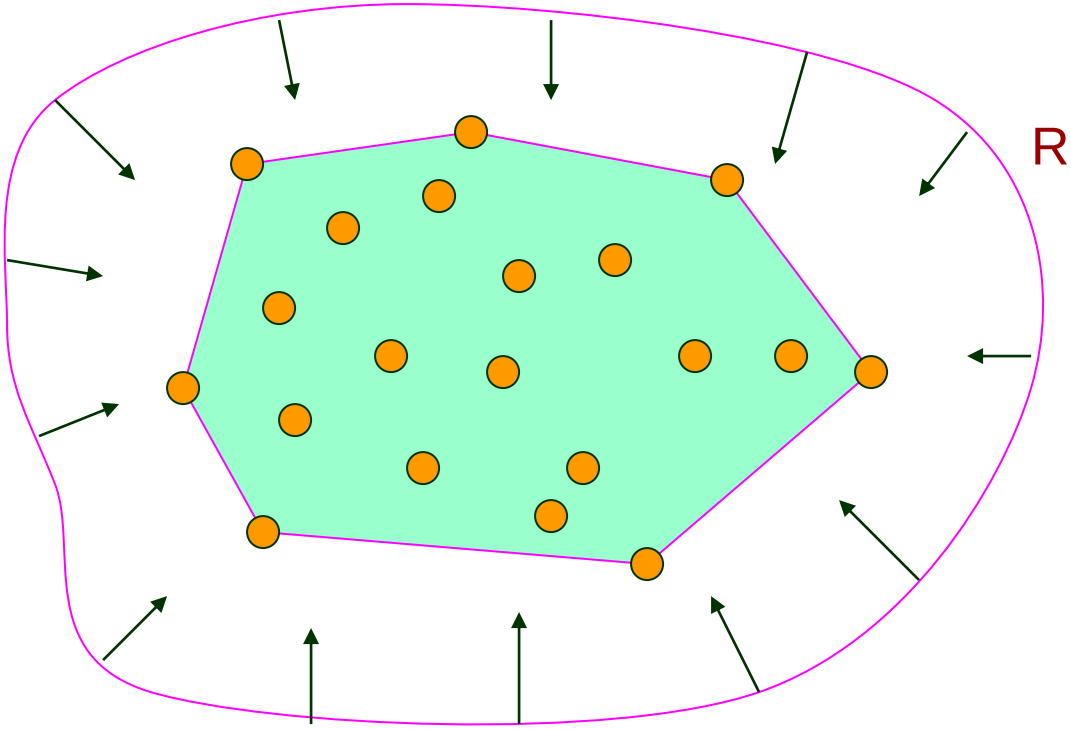

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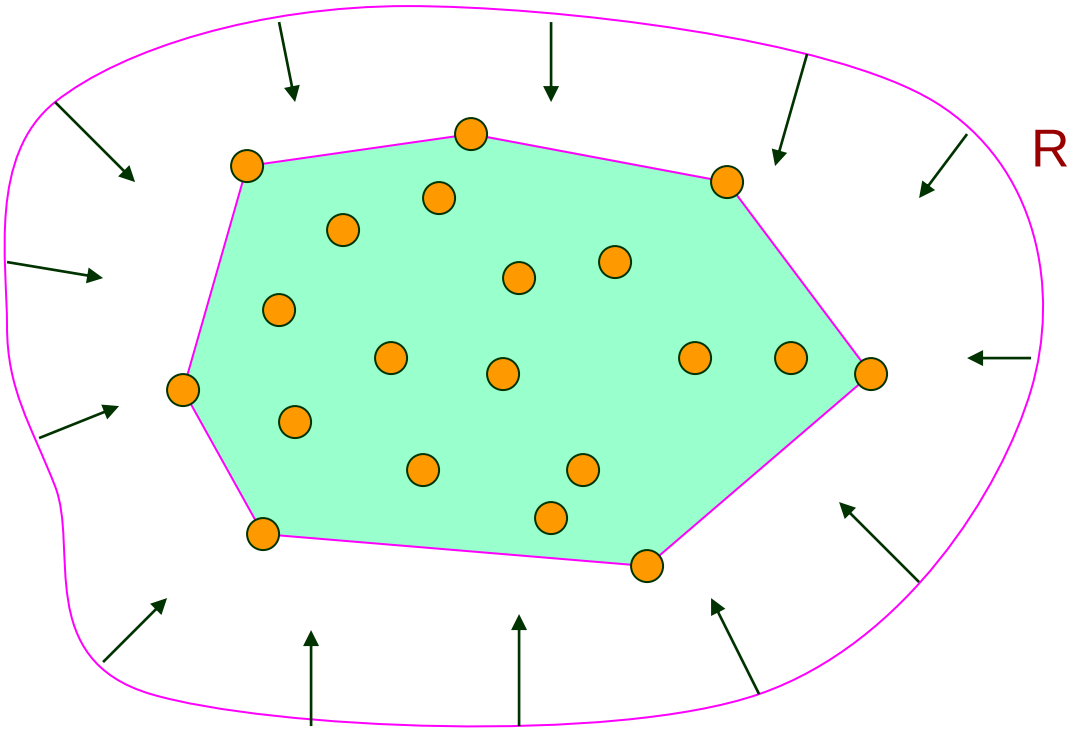

When Q is finite, its convex hull is the unique *convex polygon* whose vertices are from Q and that contains all points of Q.

Degenerate Cases

- The convex hull of a single point is itself.

- The convex hull of several collinear points is the line segment joining the two extreme points.

II. The Convex Hull Problem

Input: a set $P = \{ p_1 , p_2 , \ldots , p_n \}$ of points

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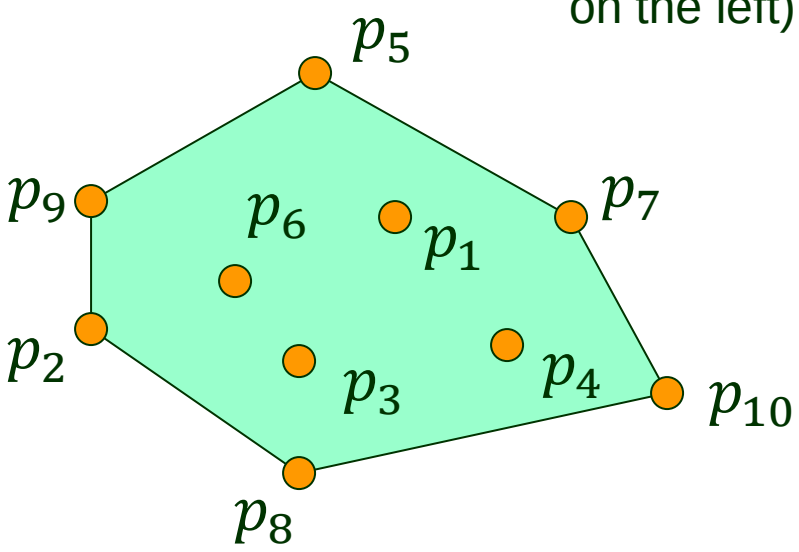

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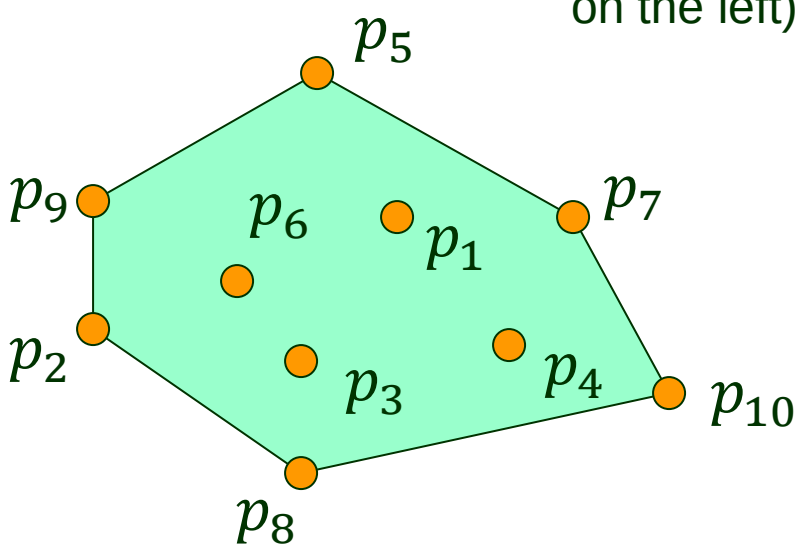
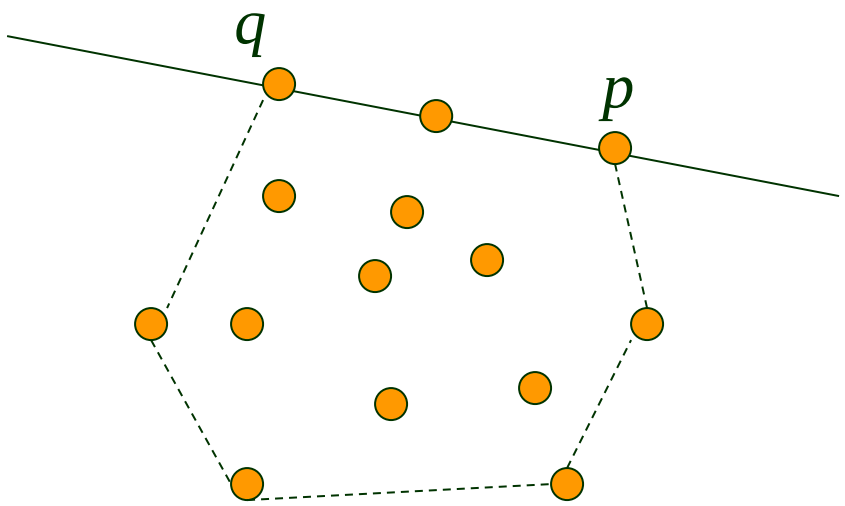


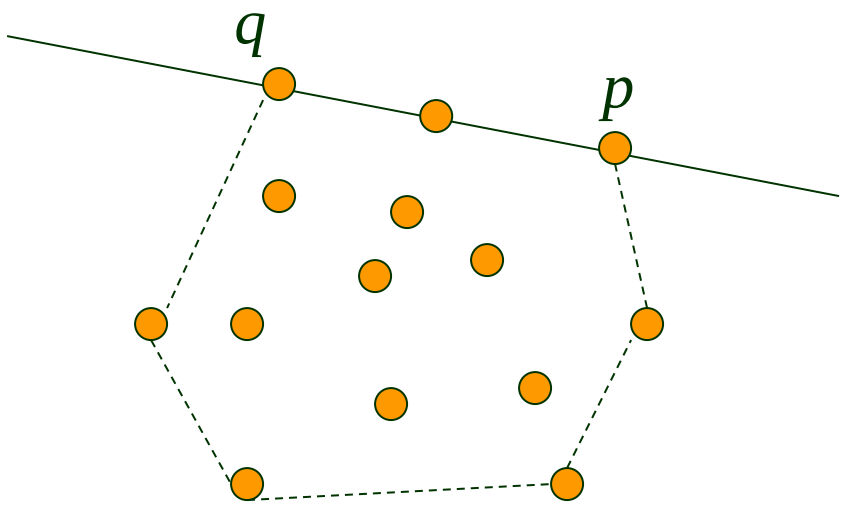
Output: $p_5, p_9, p_2, p_8, p_{10}, p_7.$

Edges of a Convex Hull



Edges of a Convex Hull

- For every edge with endpoints $p, q \in P$.



Edges of a Convex Hull

- For every edge with endpoints $p, q \in P$.
- All other points in P lie
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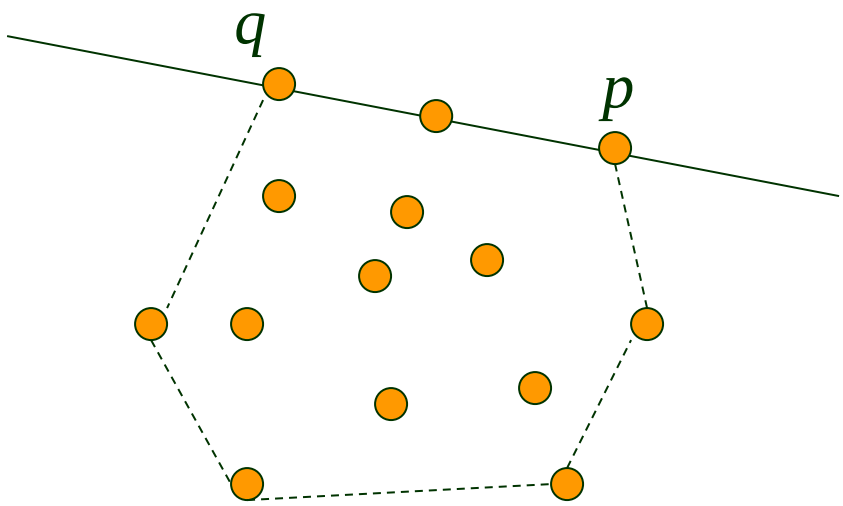

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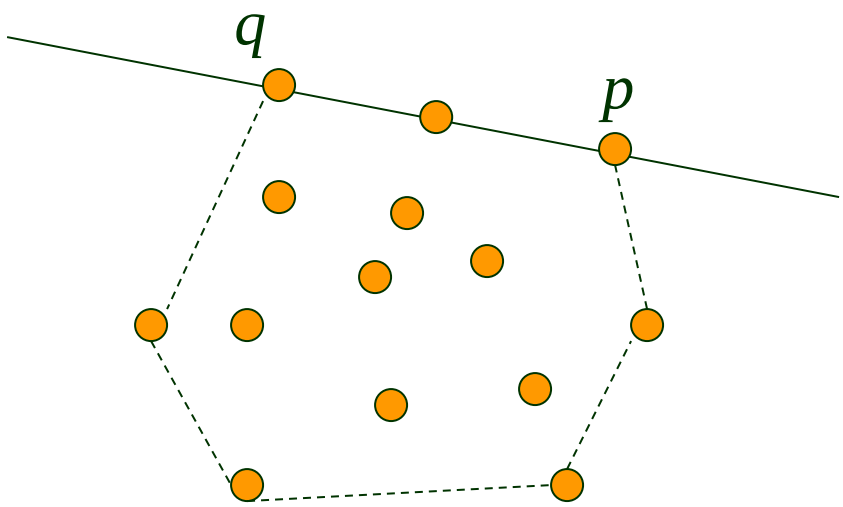

A Slow Convex Hull Algorithm

```
Slow-Convex-Hull(P)
      E ≠ {} // set of directed edges of CH(P) that bounds the
                  // points of P on the right.
      for every ordered pair (p,q), where p,q ∈ P and p ≠ q // Θ(n²) pairs
        do valid ← true
           for every point r ≠ p or q                    // n − 2 such points
              do if r lies to the right of  pq⃗ or
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              then E ≠ E ∪ {pq⃗}    // pq⃗ and qp⃗ cannot be both in E
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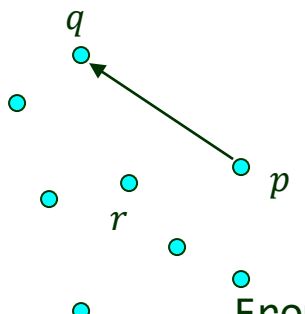


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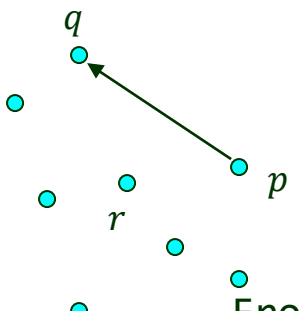

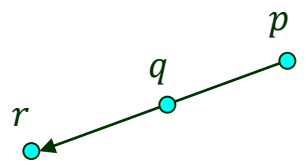

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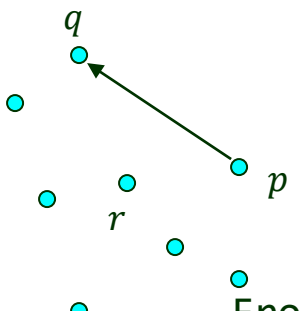
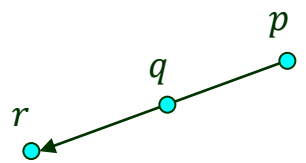
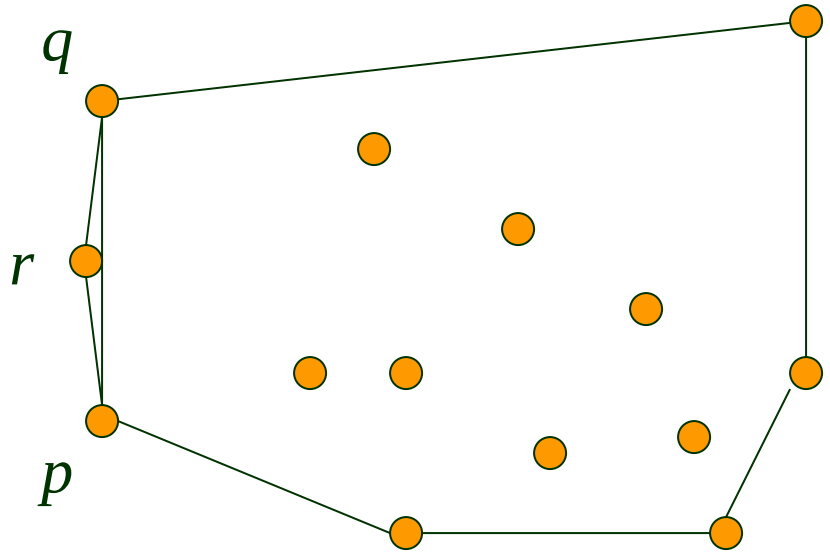




Running time $\Theta(n^3)$

Floating Arithmetic is not Exact!

Nearly colinear points p, q, r.



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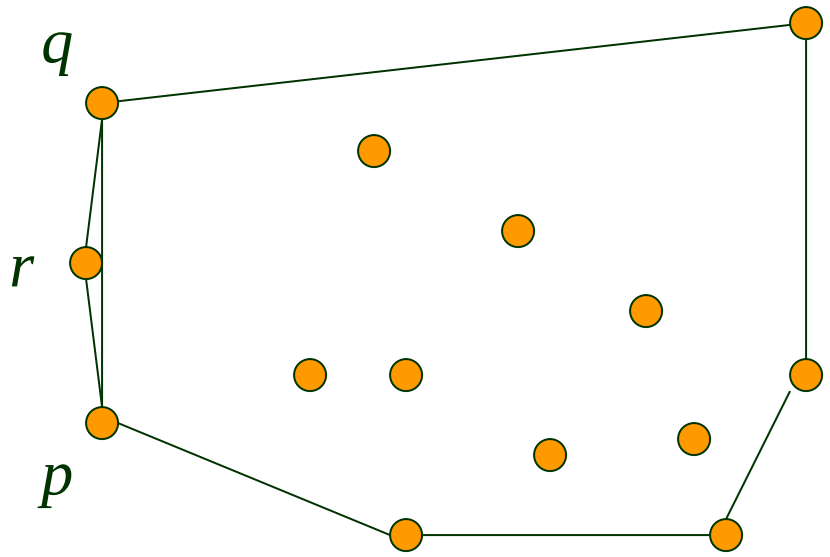

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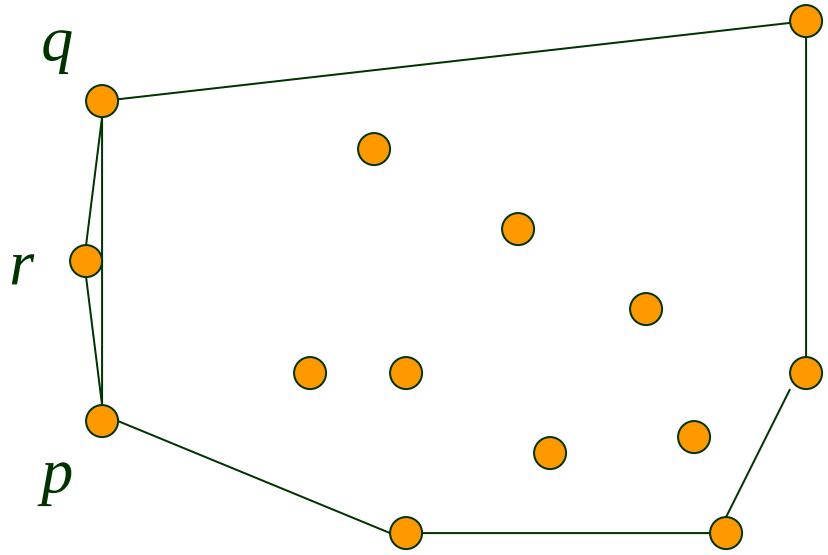

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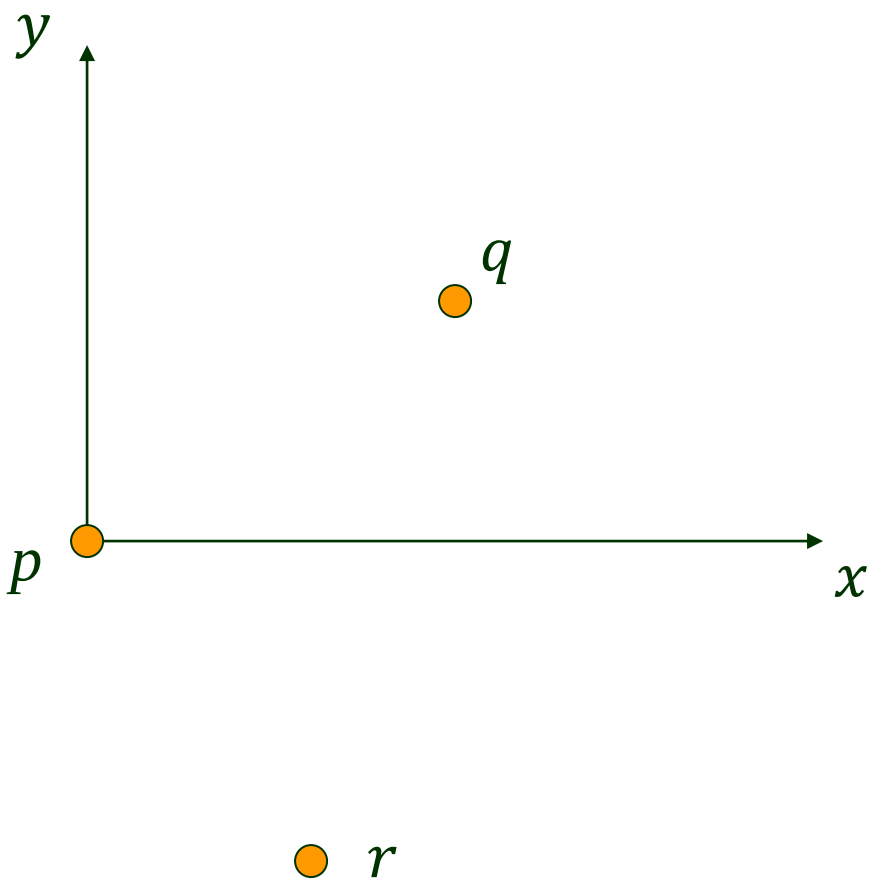
→ All three are accepted as edges!

Not *robust* – the algorithm could fail with small numerical error.

Polar Angle



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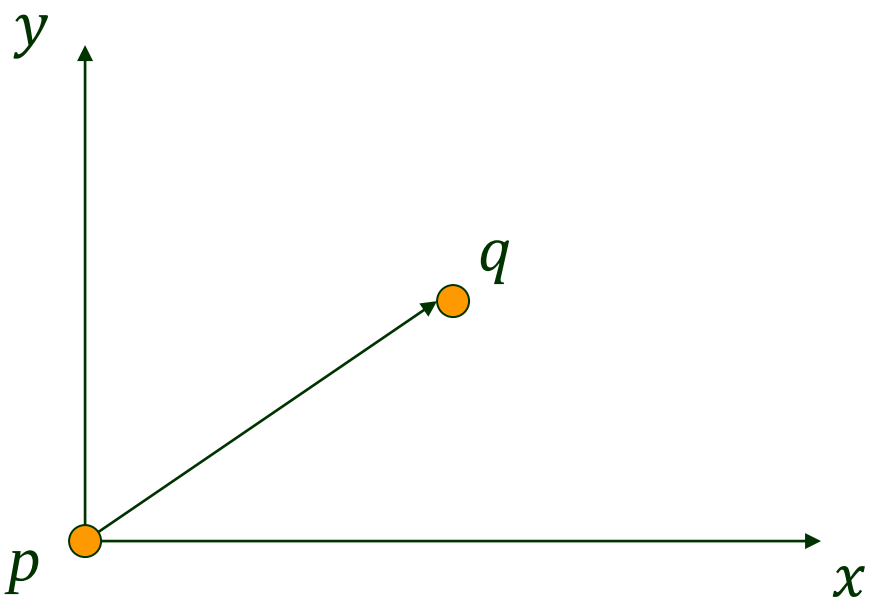

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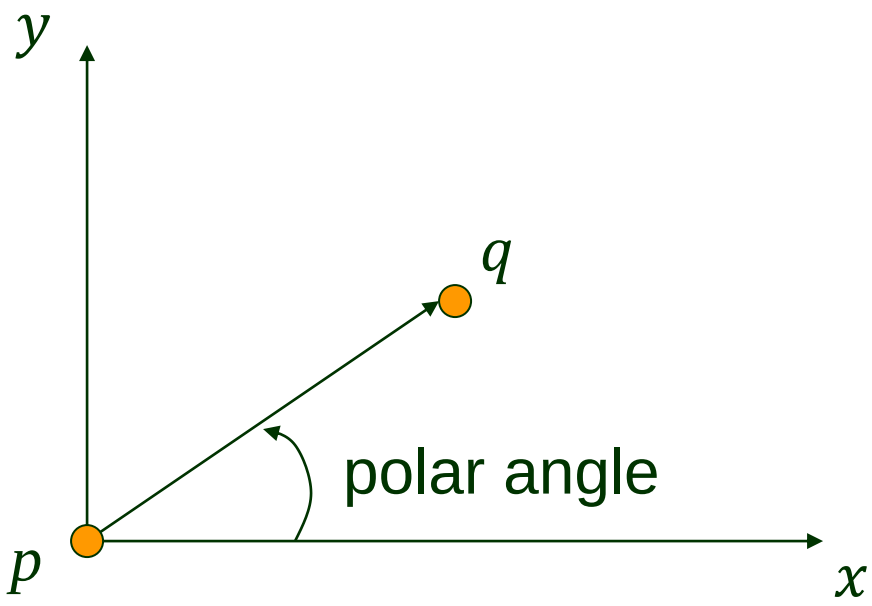

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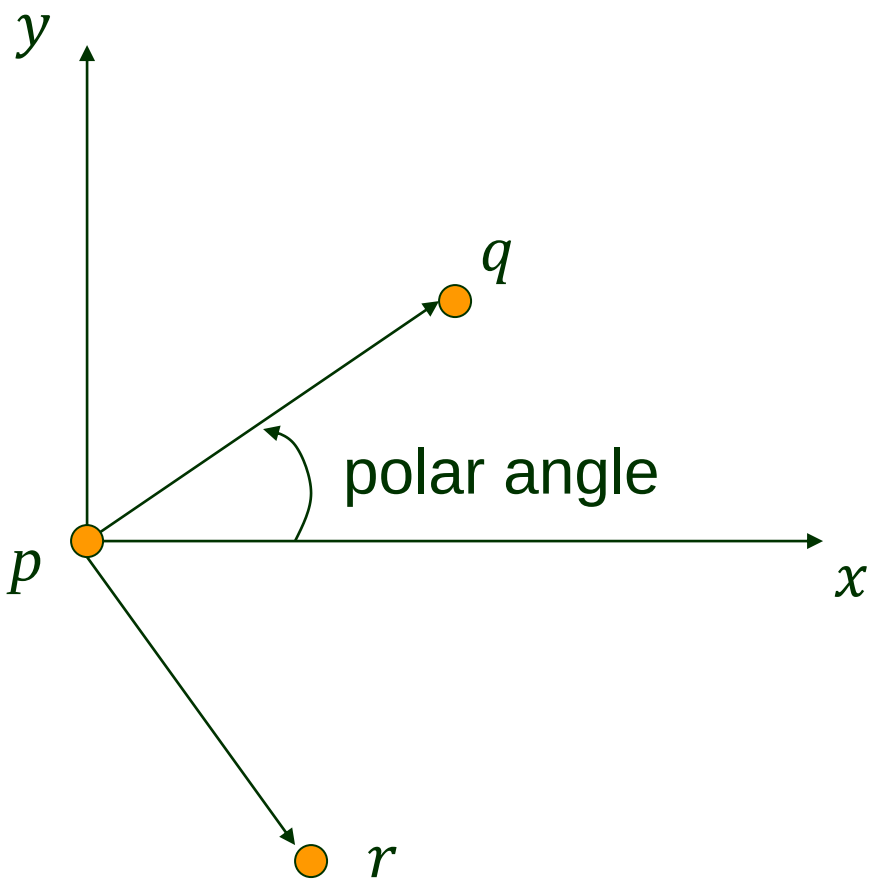

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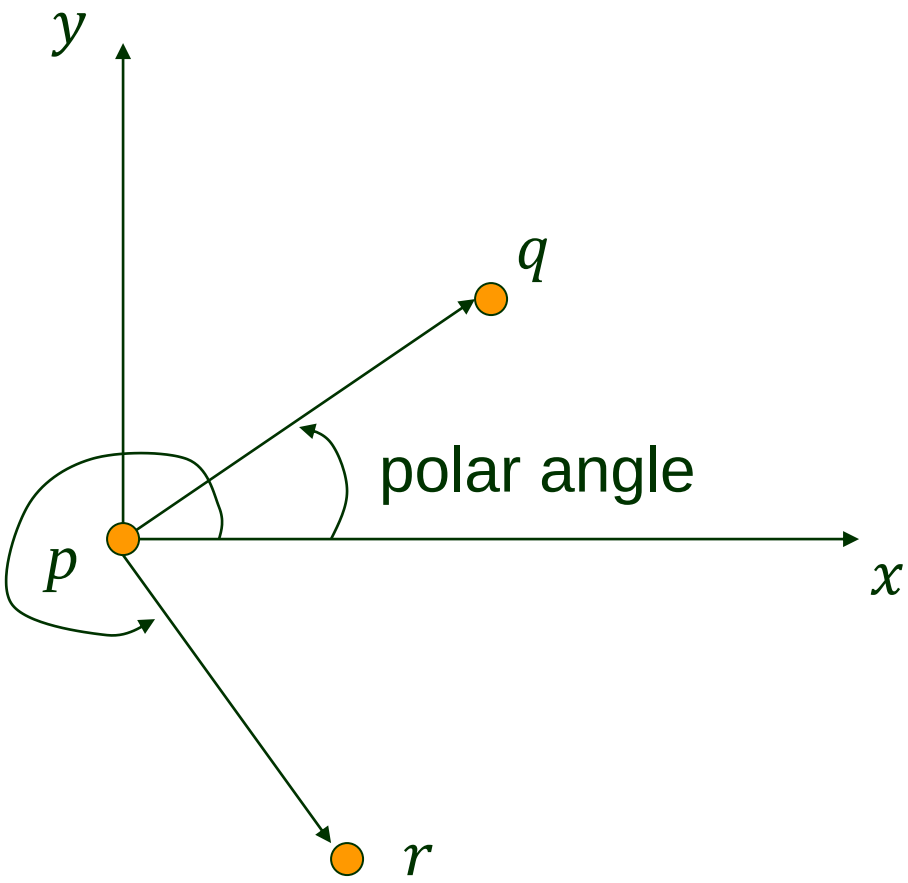
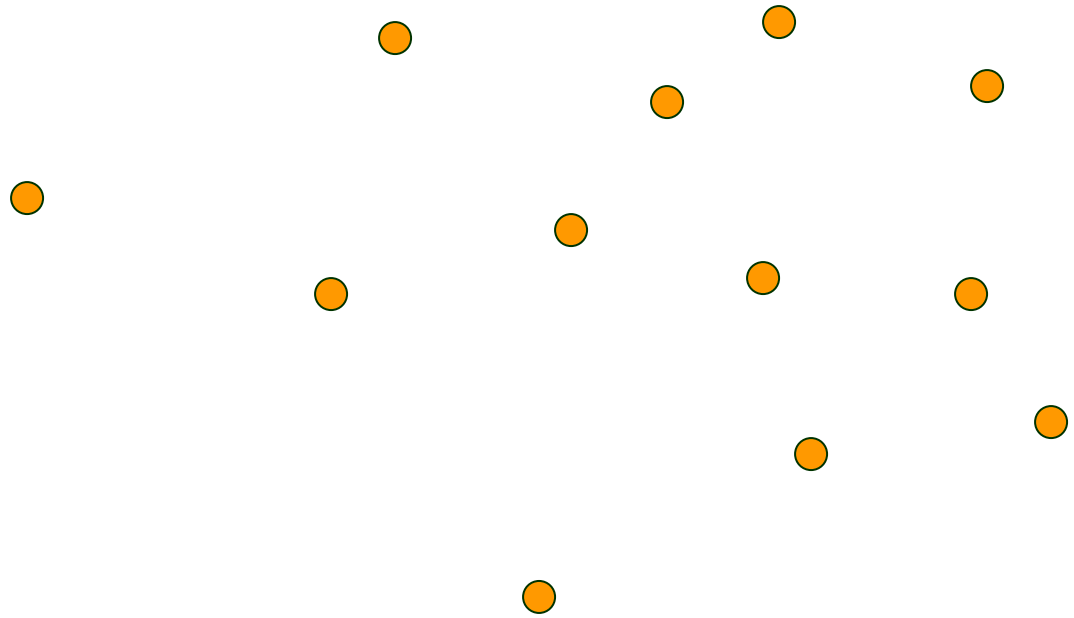


III. Graham Scan



III. Graham Scan

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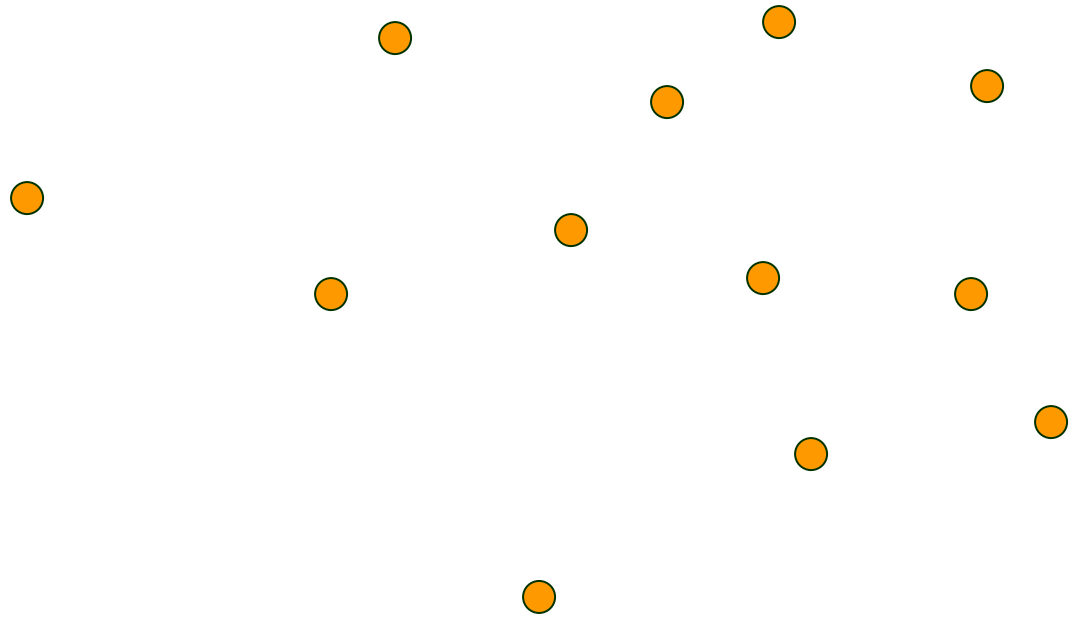

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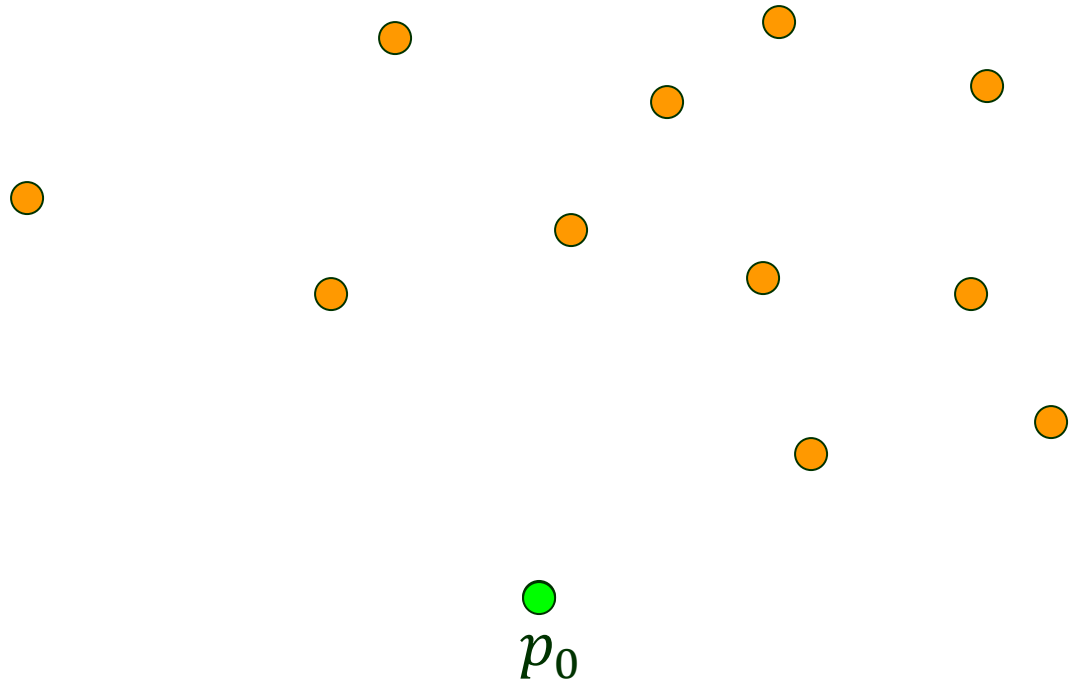

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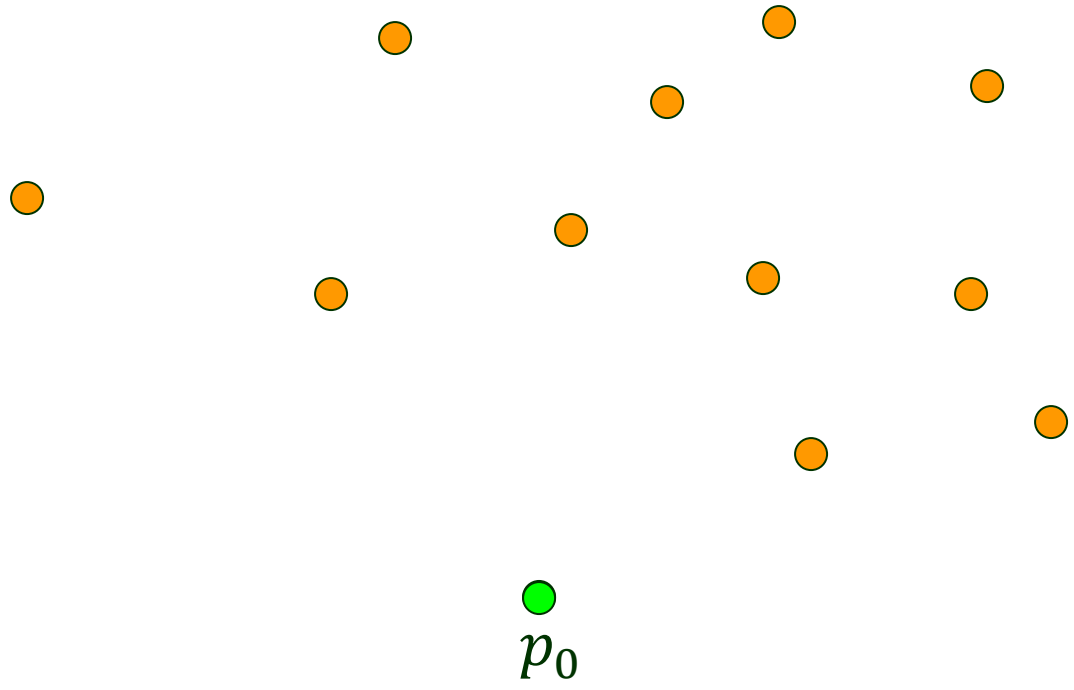
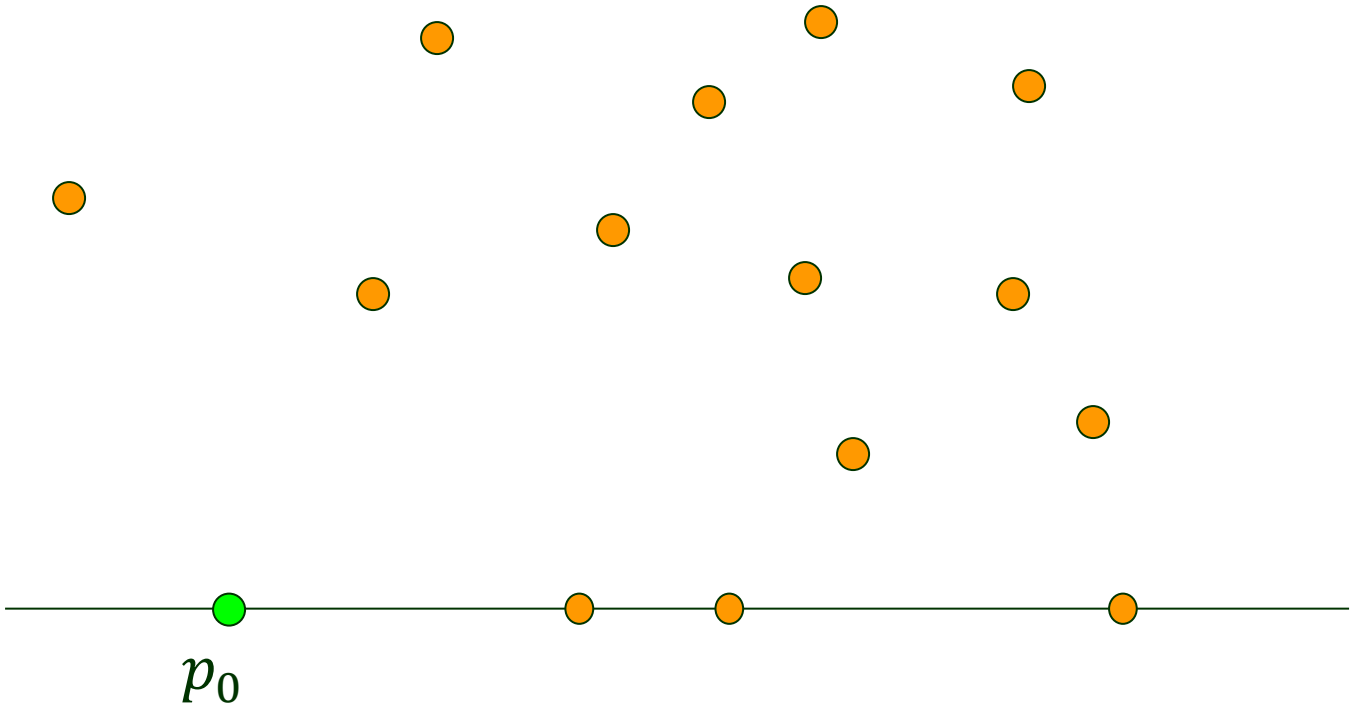


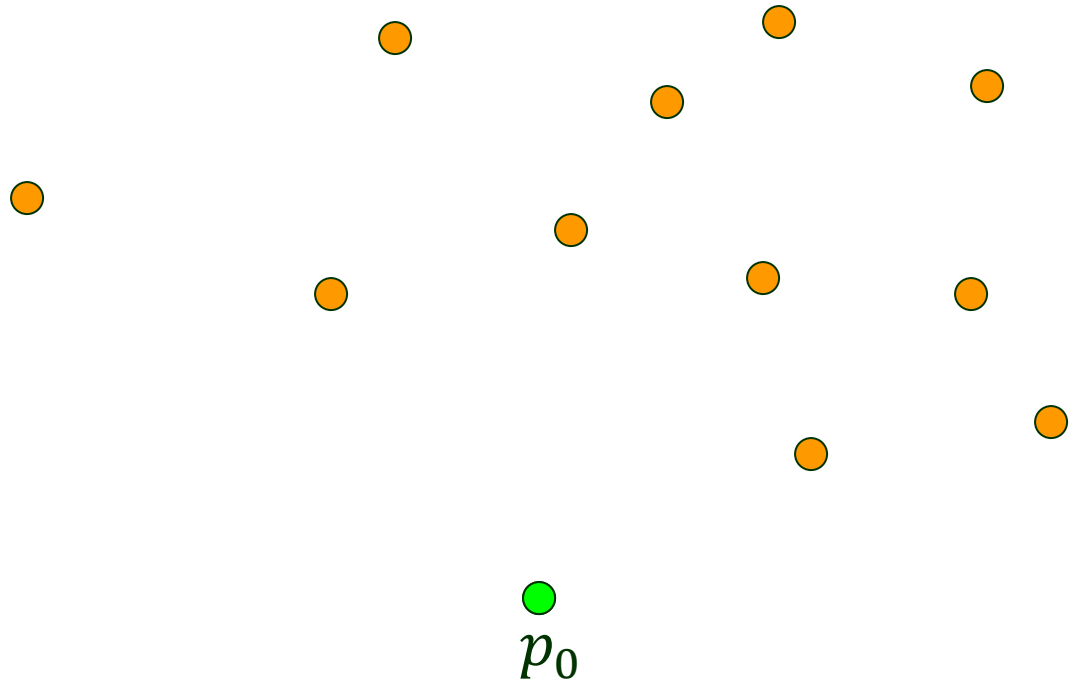
This node will be a vertex of the convex hull.

Tie Breaking (1)

When more than one point has the smallest y coordinate, pick the *leftmost* (or rightmost) one.

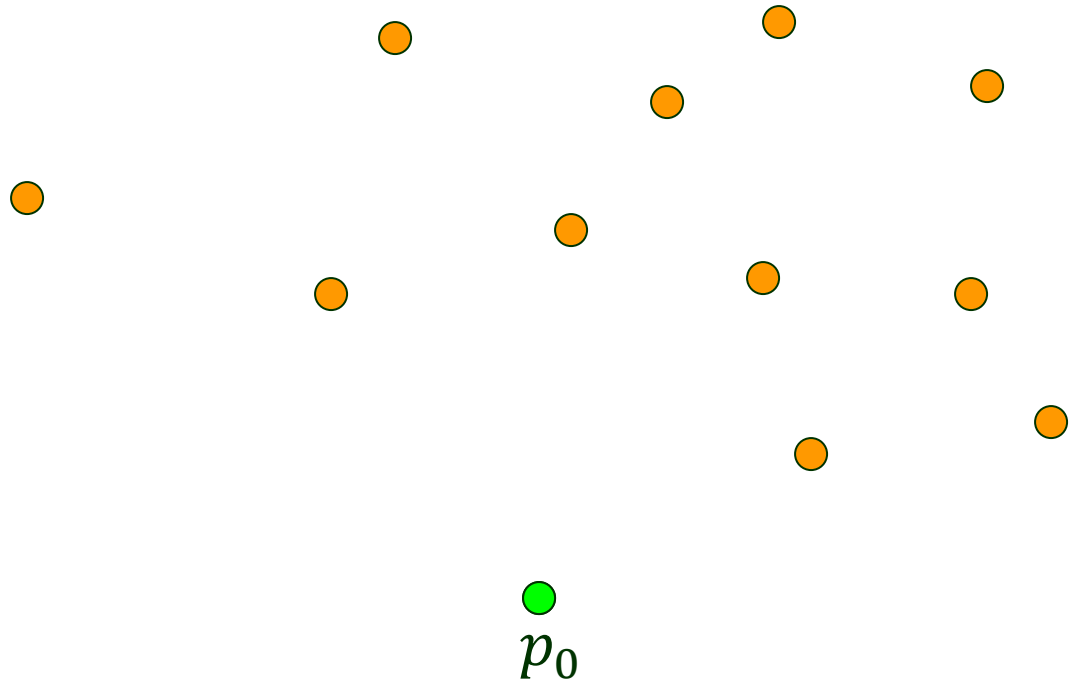


Sorting by Polar Angle



Sorting by Polar Angle

2) Sort by polar angle with respect to p_0.



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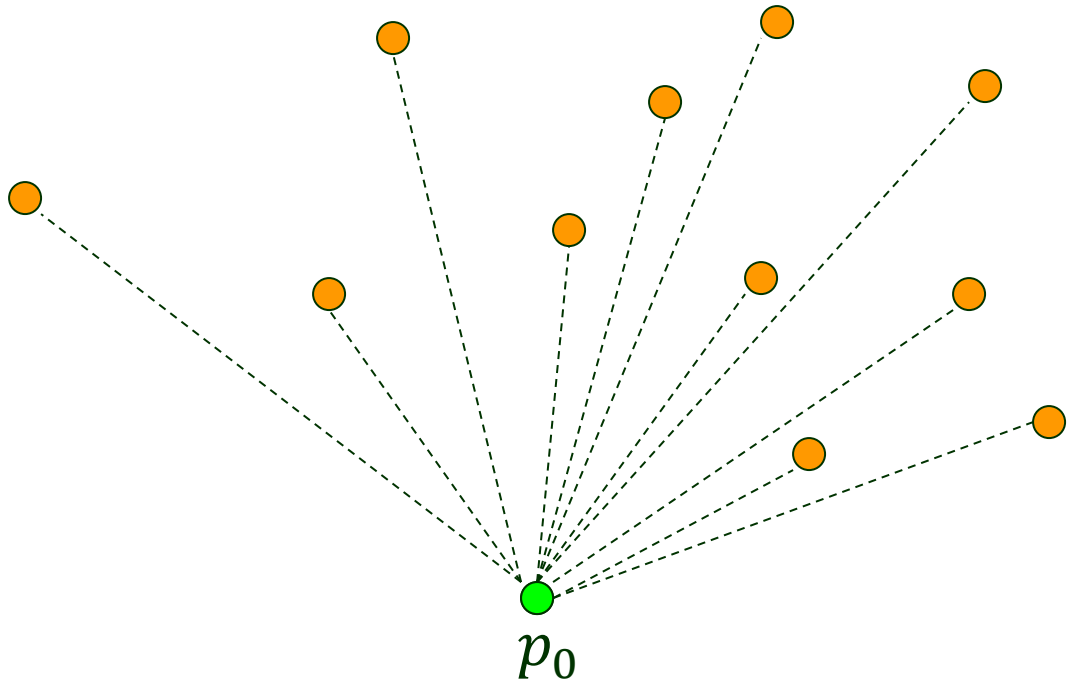

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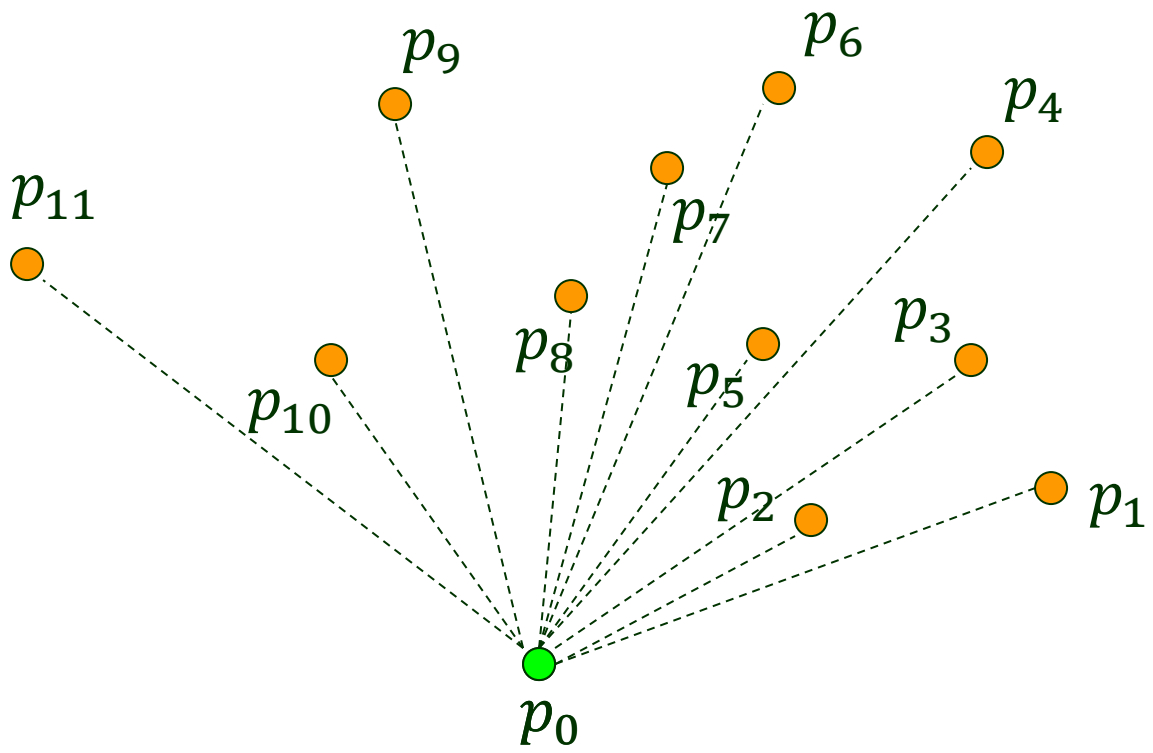

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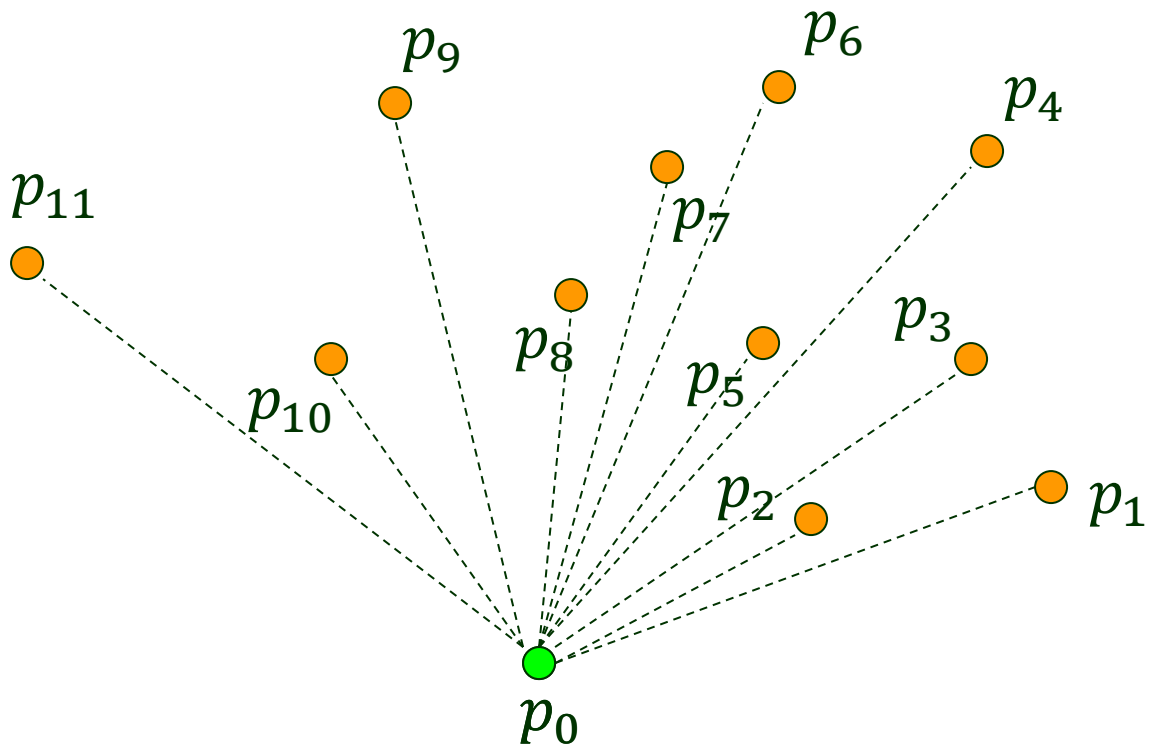
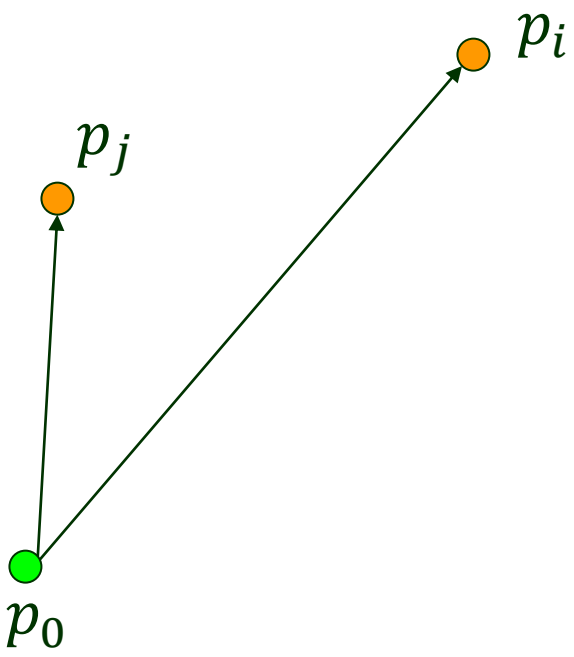


Labels are in the polar angle order.

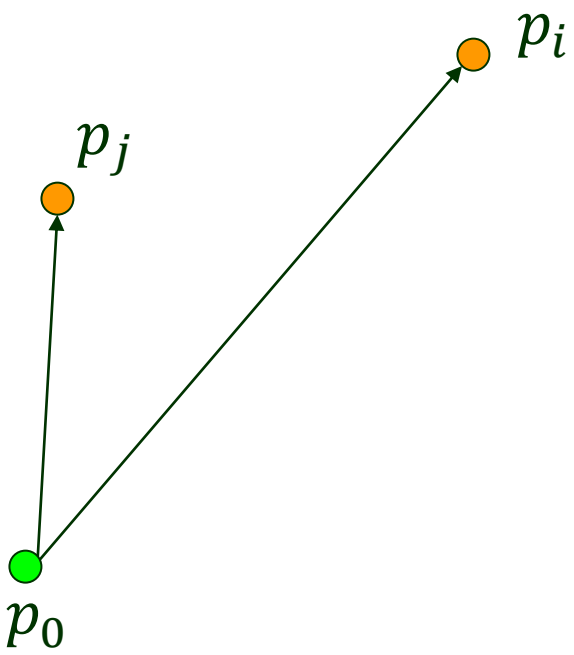
No Polar Angle Evaluation

p_0 is the lowest (and leftmost)



No Polar Angle Evaluation

p_0 is the lowest (and leftmost) ⇒ all polar angles ∈ $[0, \pi)$.



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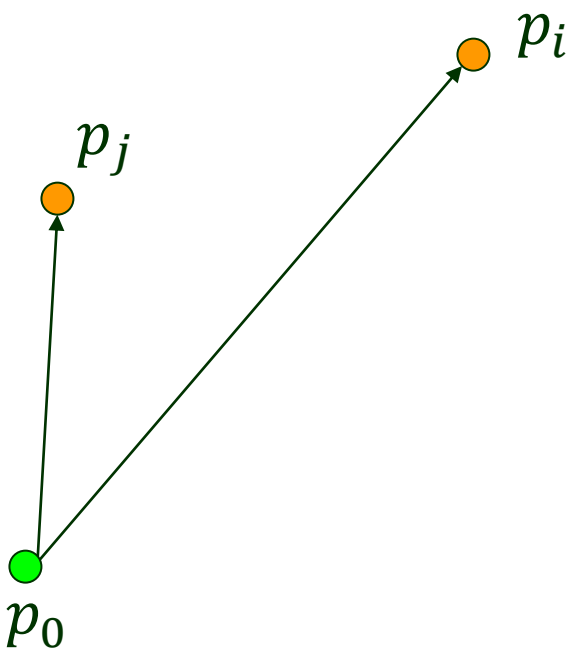
Use cross product!



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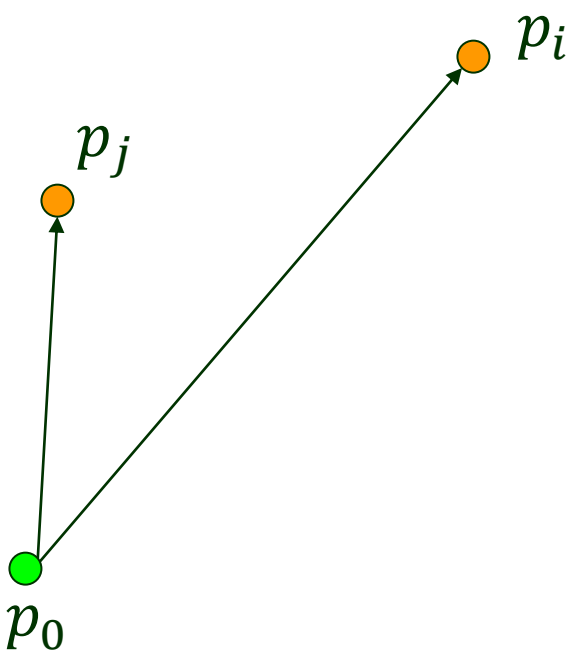
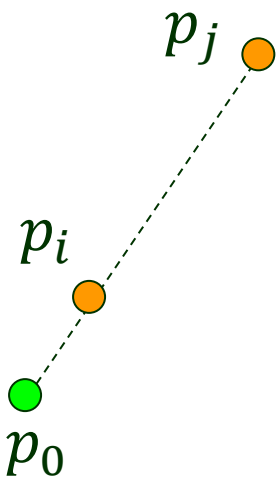
$$p_i < p_j \text{ if } \overrightarrow{p_0 p_i} \times \overrightarrow{p_0 p_j} > 0$$



Tie Breaking (2)

What if p_0, p_i, p_j are on the same line?



Tie Breaking (2)

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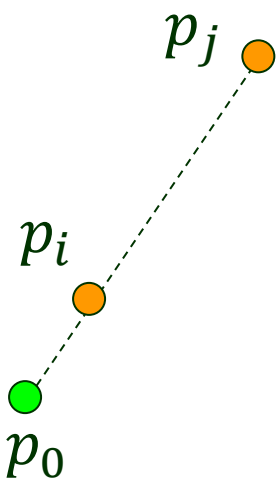
Order them by distance from p_0.



Tie Breaking (2)

What if p_0, p_i, p_j are on the same line?

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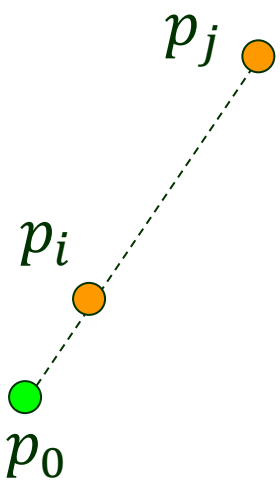
$p_i < p_j$ if $\overrightarrow{p_0p_i} \times \overrightarrow{p_0p_j} = 0$ and

$|\overrightarrow{p_0p_i}| < |\overrightarrow{p_0p_j}|$



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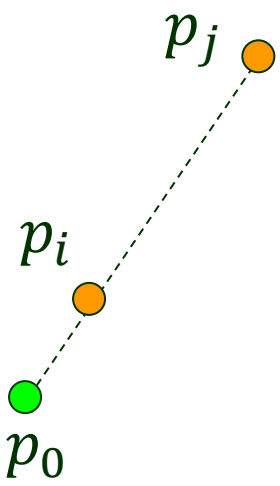

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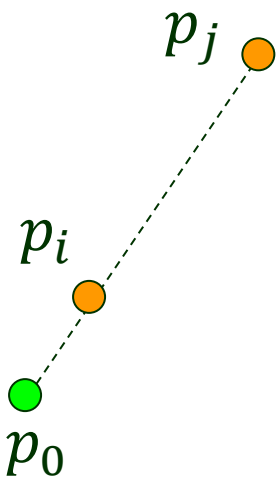
No square roots.
Use dot product!

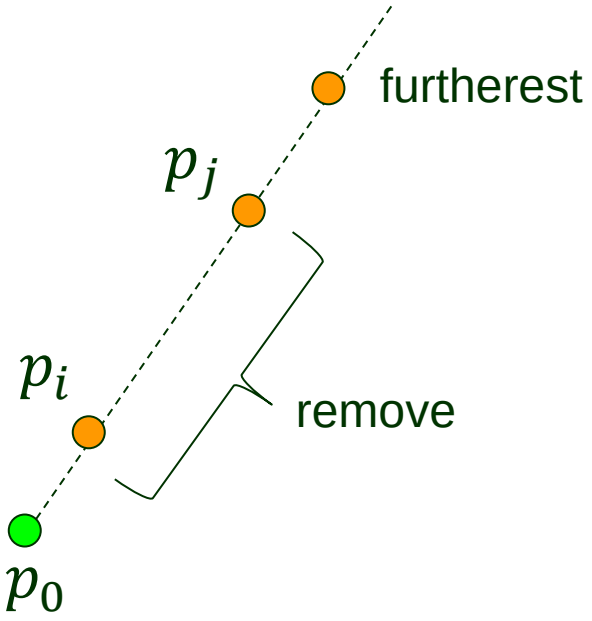
$$\overrightarrow{p_0p_i} \cdot \overrightarrow{p_0p_i} < \overrightarrow{p_0p_j} \cdot \overrightarrow{p_0p_j}$$



Point Elimination

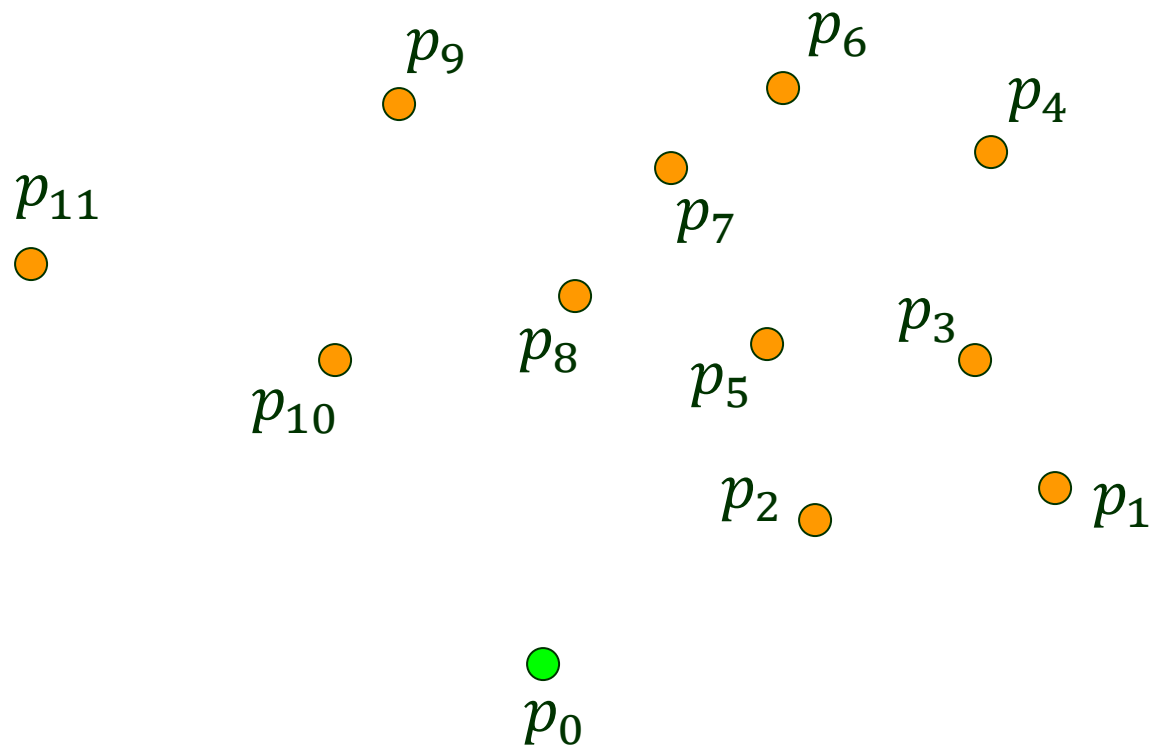
When multiple points have the same polar angle, keep the one *furthest* from p_0.

Remove the rest since they cannot possibly be the hull vertices.



Stack Initialization

3) Scan points in the increasing order of polar angle, maintaining a stack.



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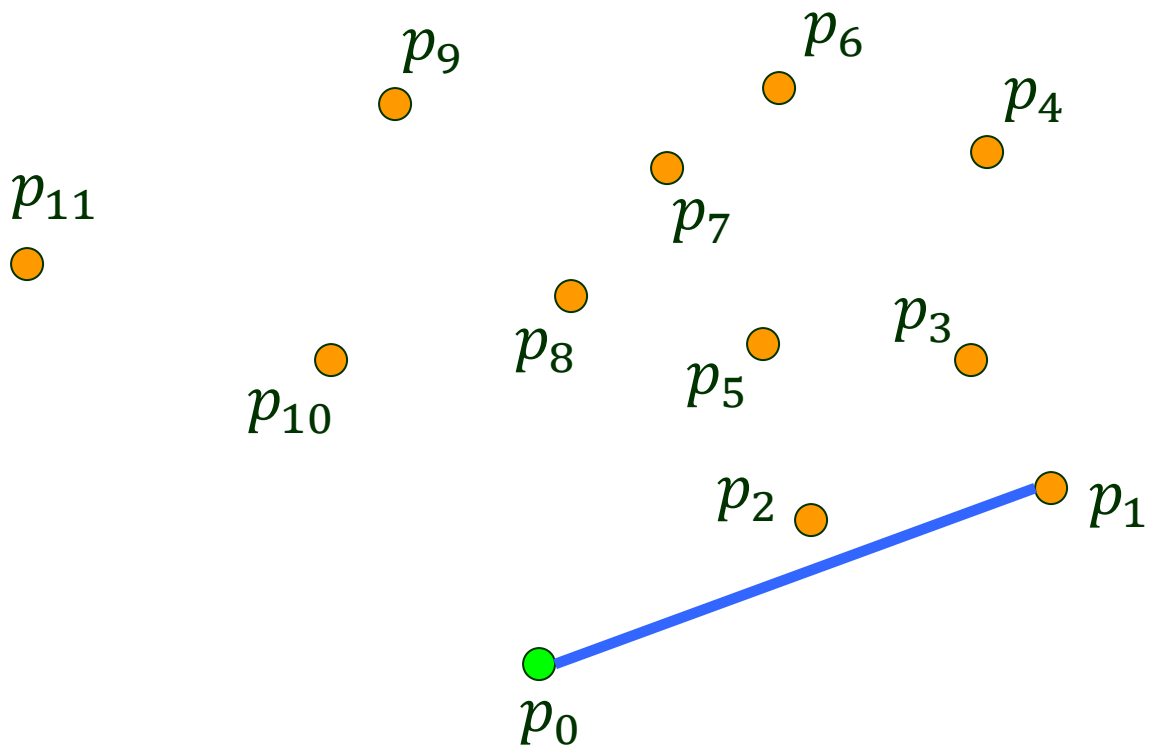

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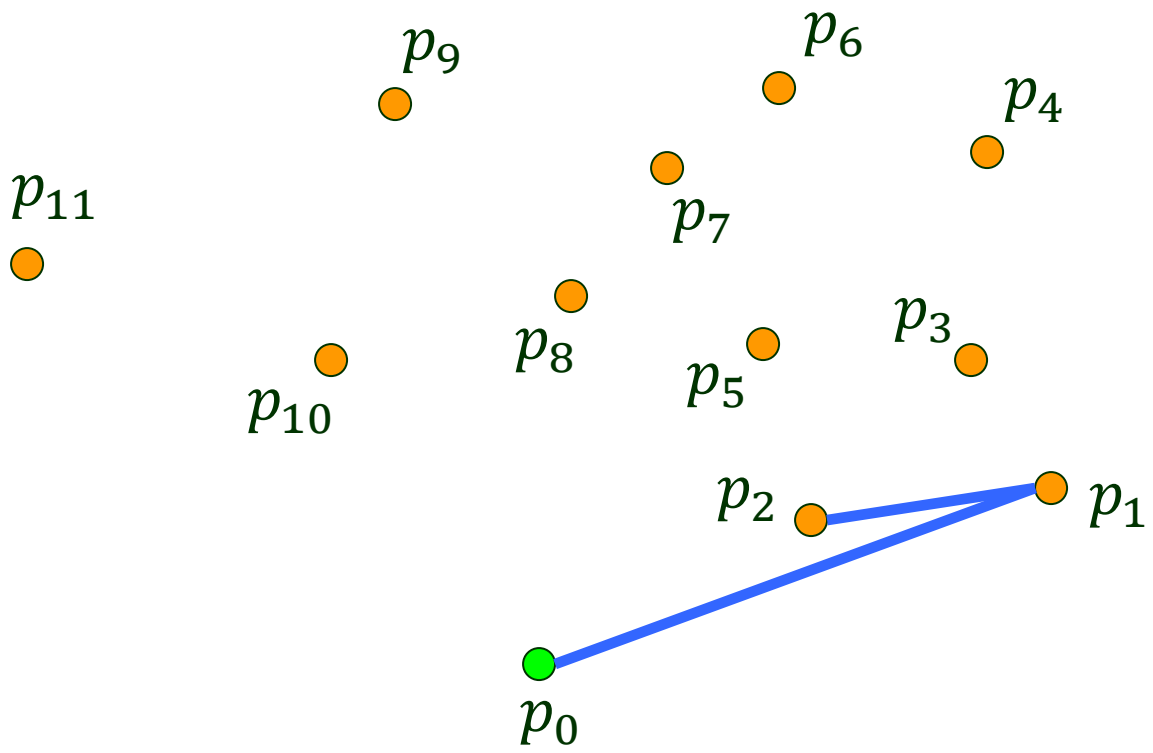

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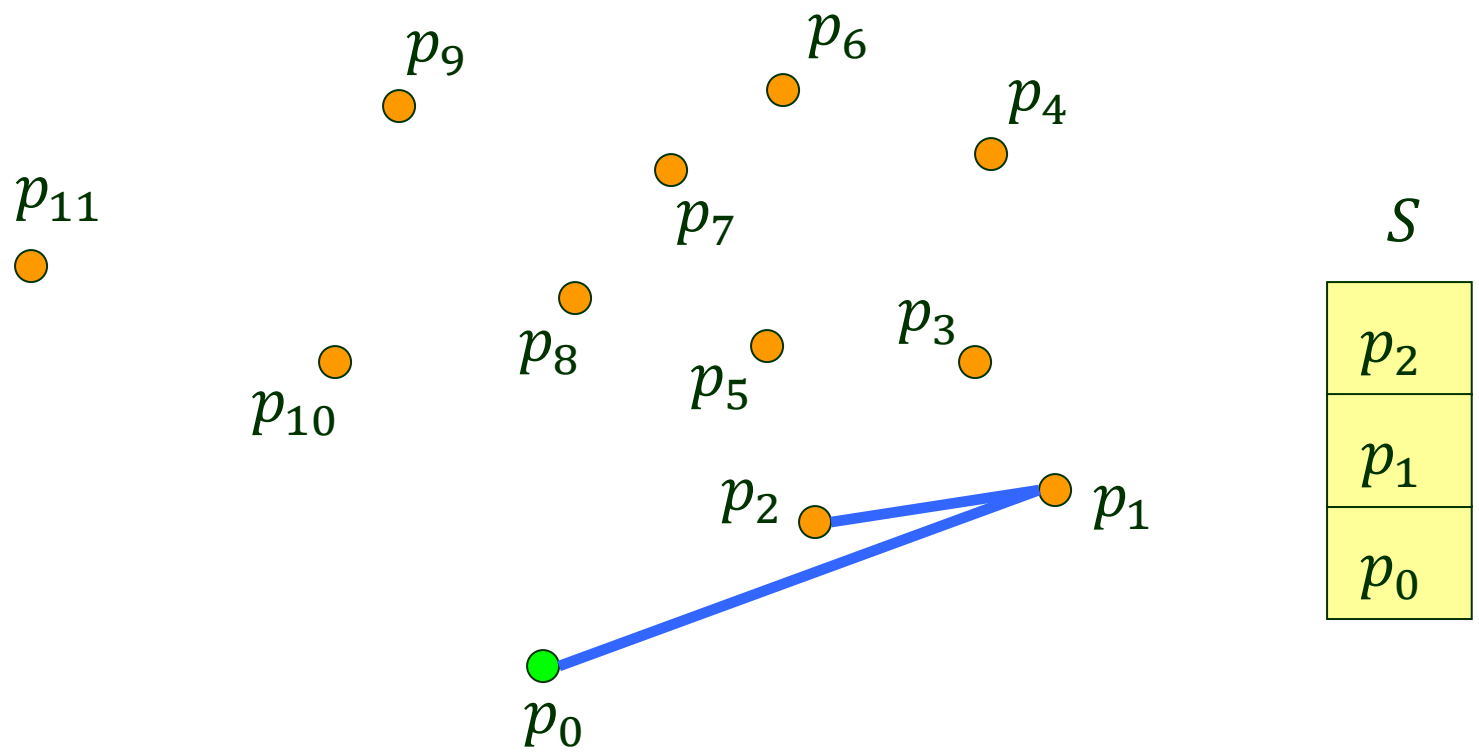

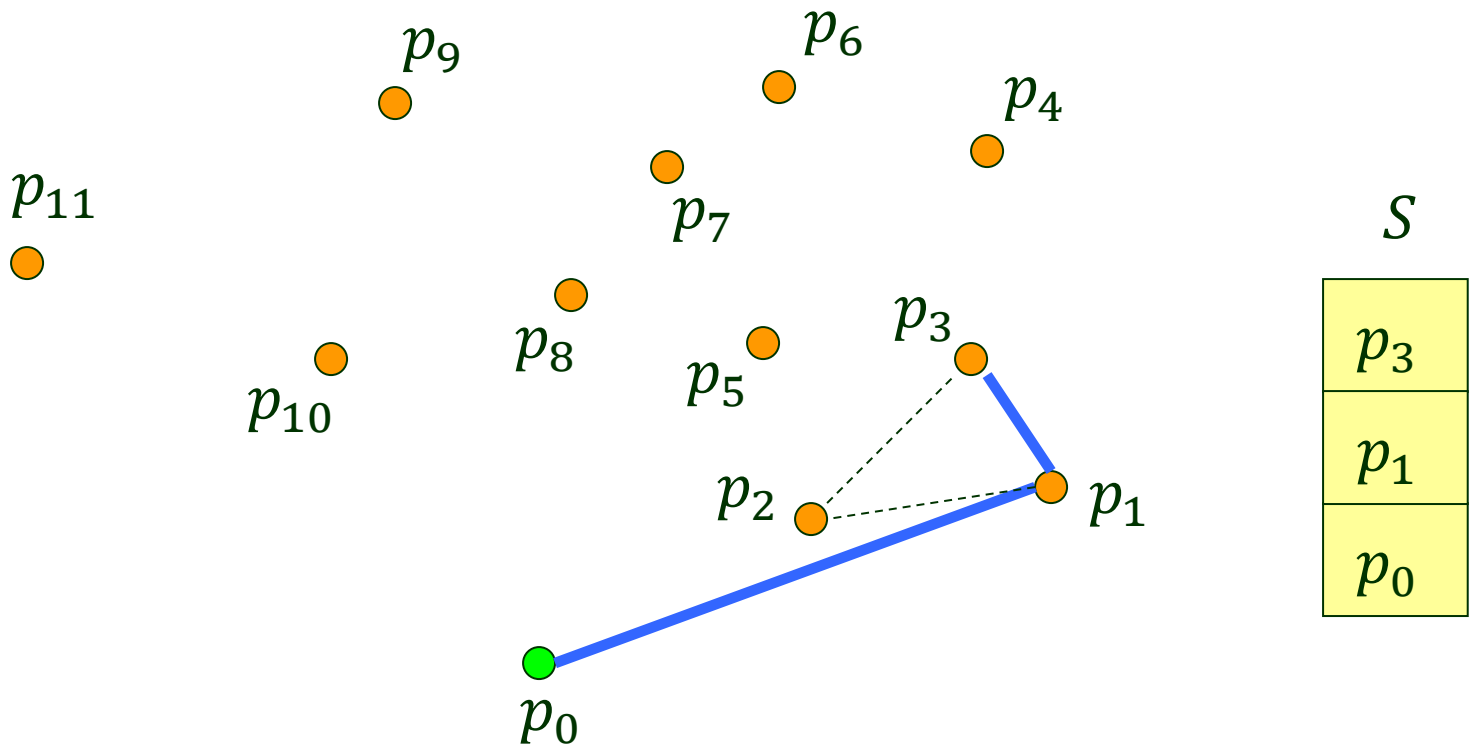
p_9
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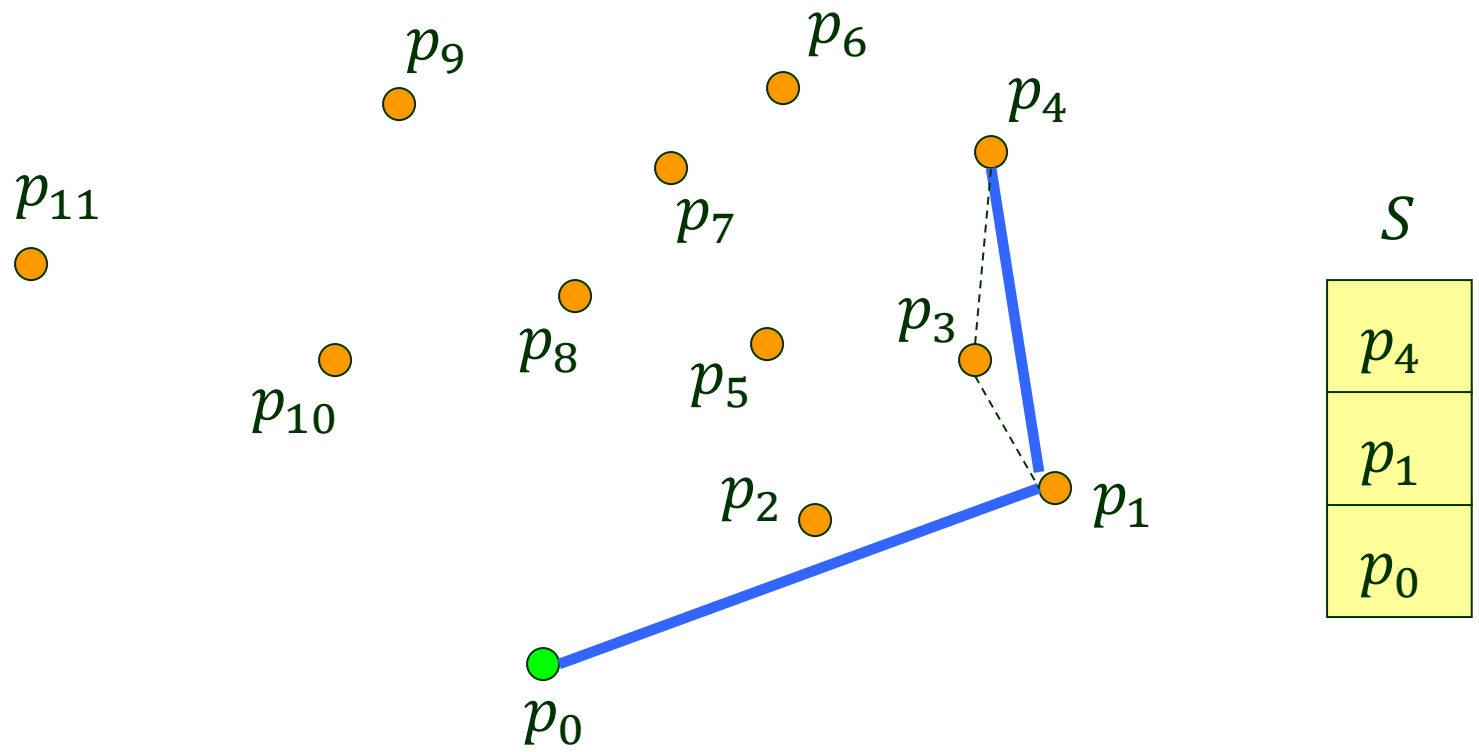
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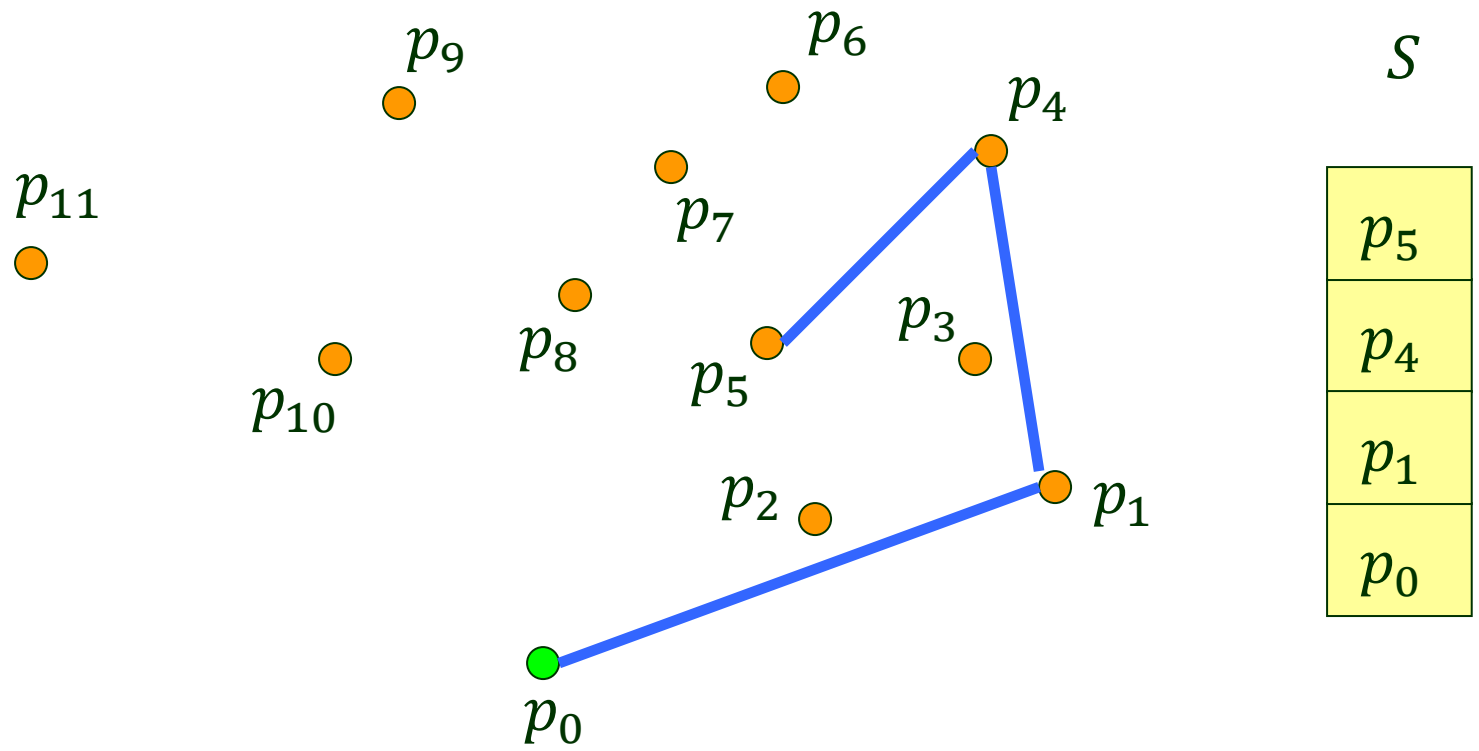
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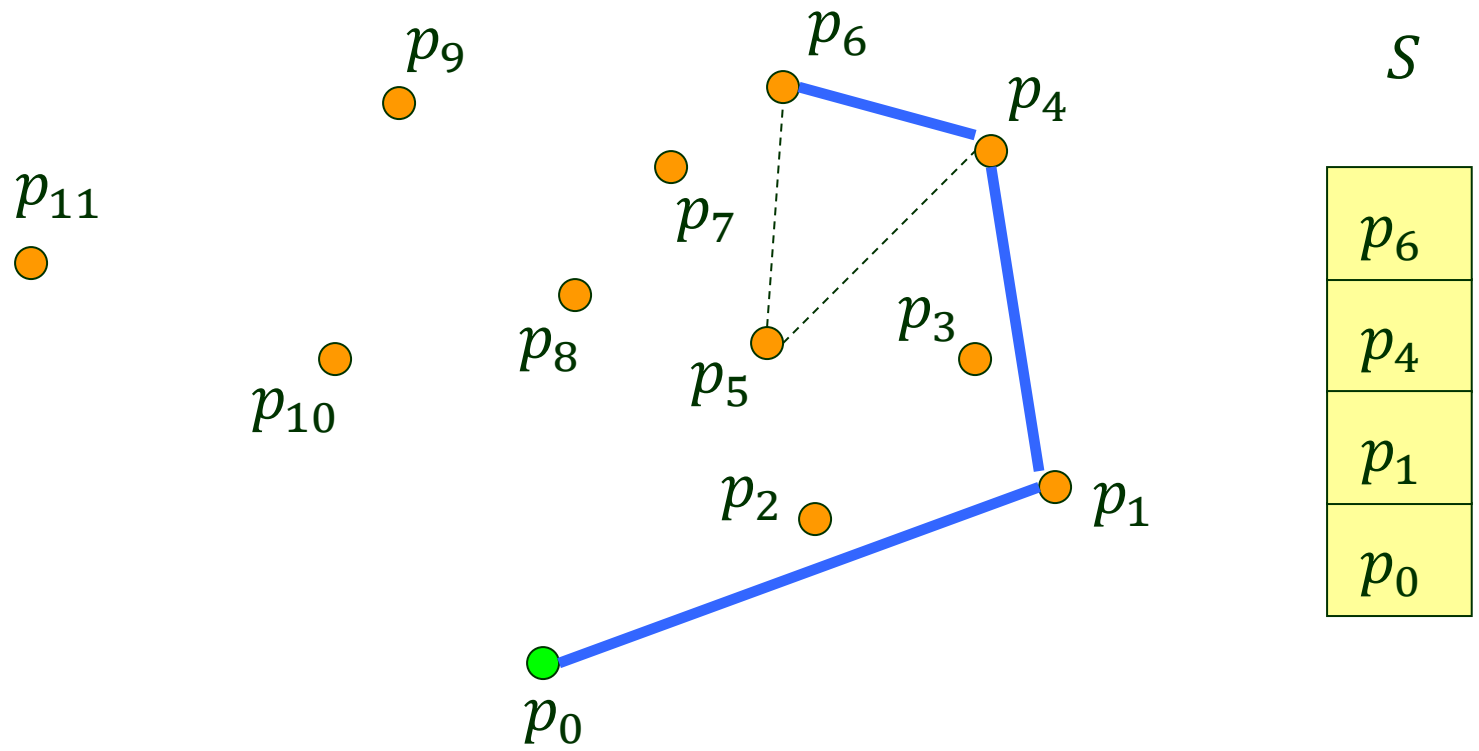
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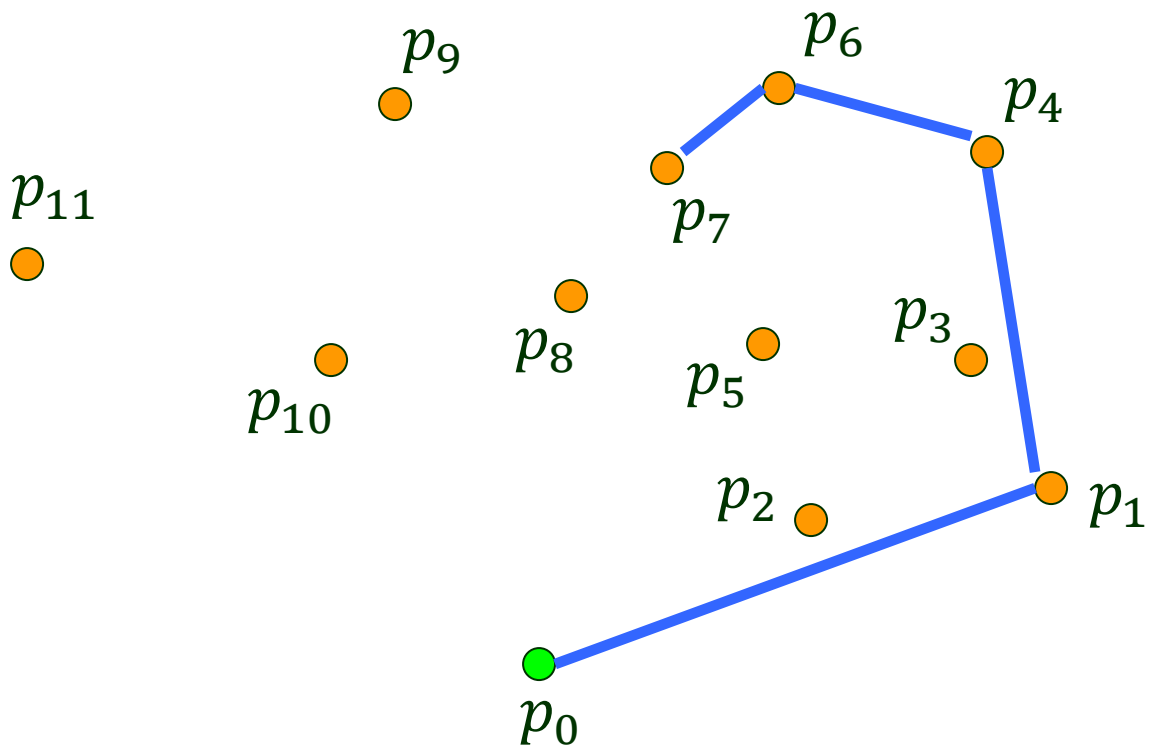
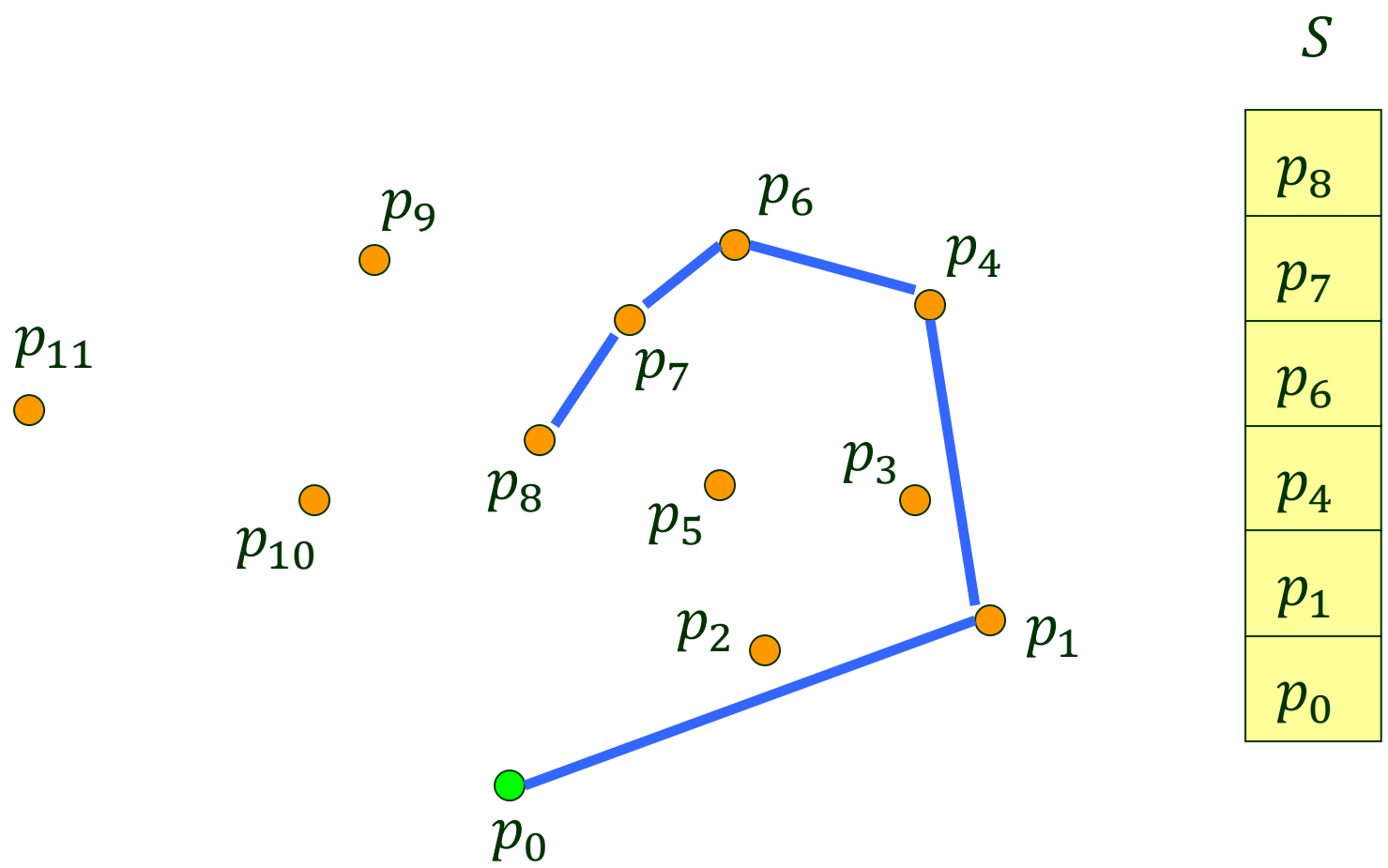


S

p_7
p_6
p_4
p_1
p_0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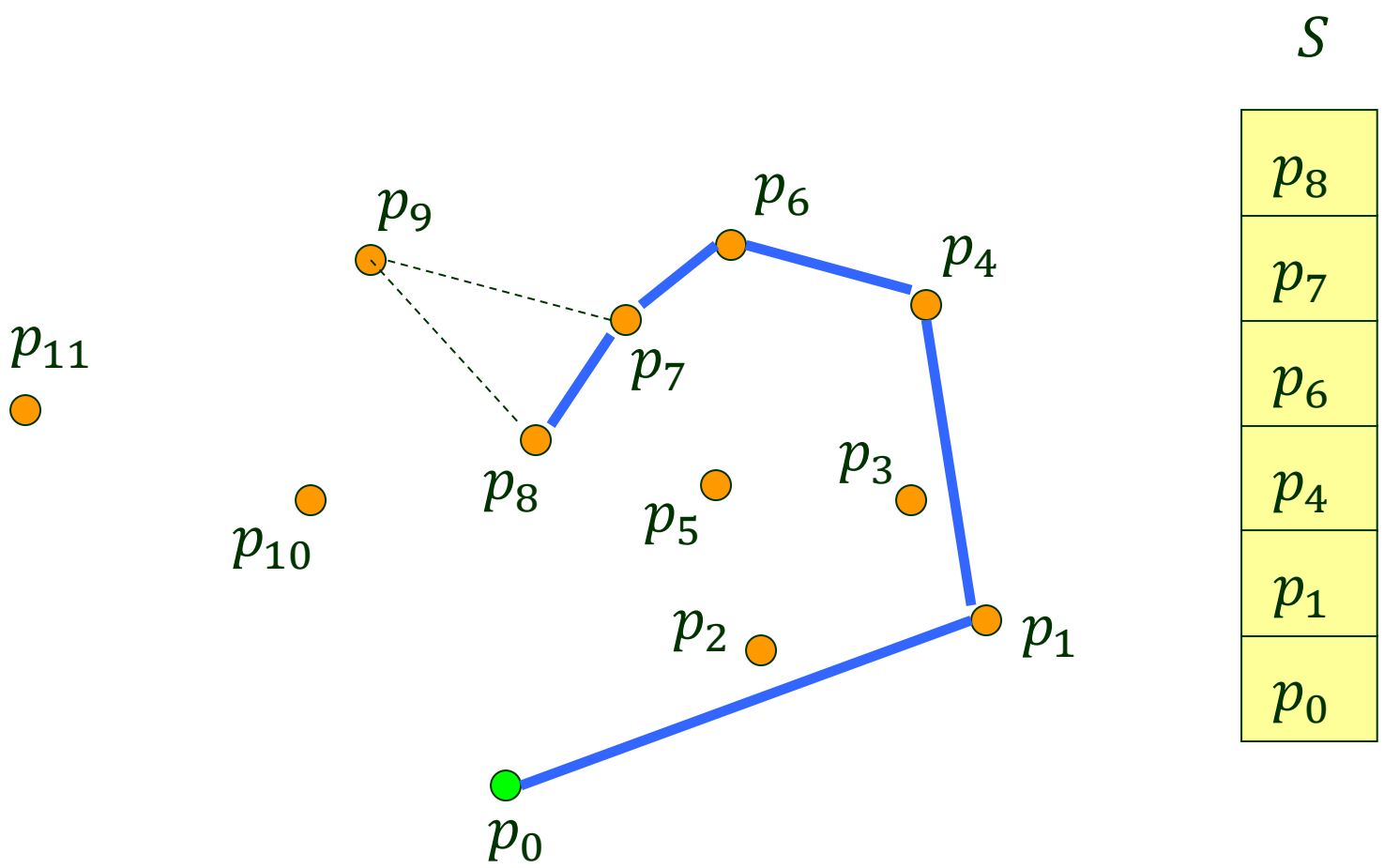
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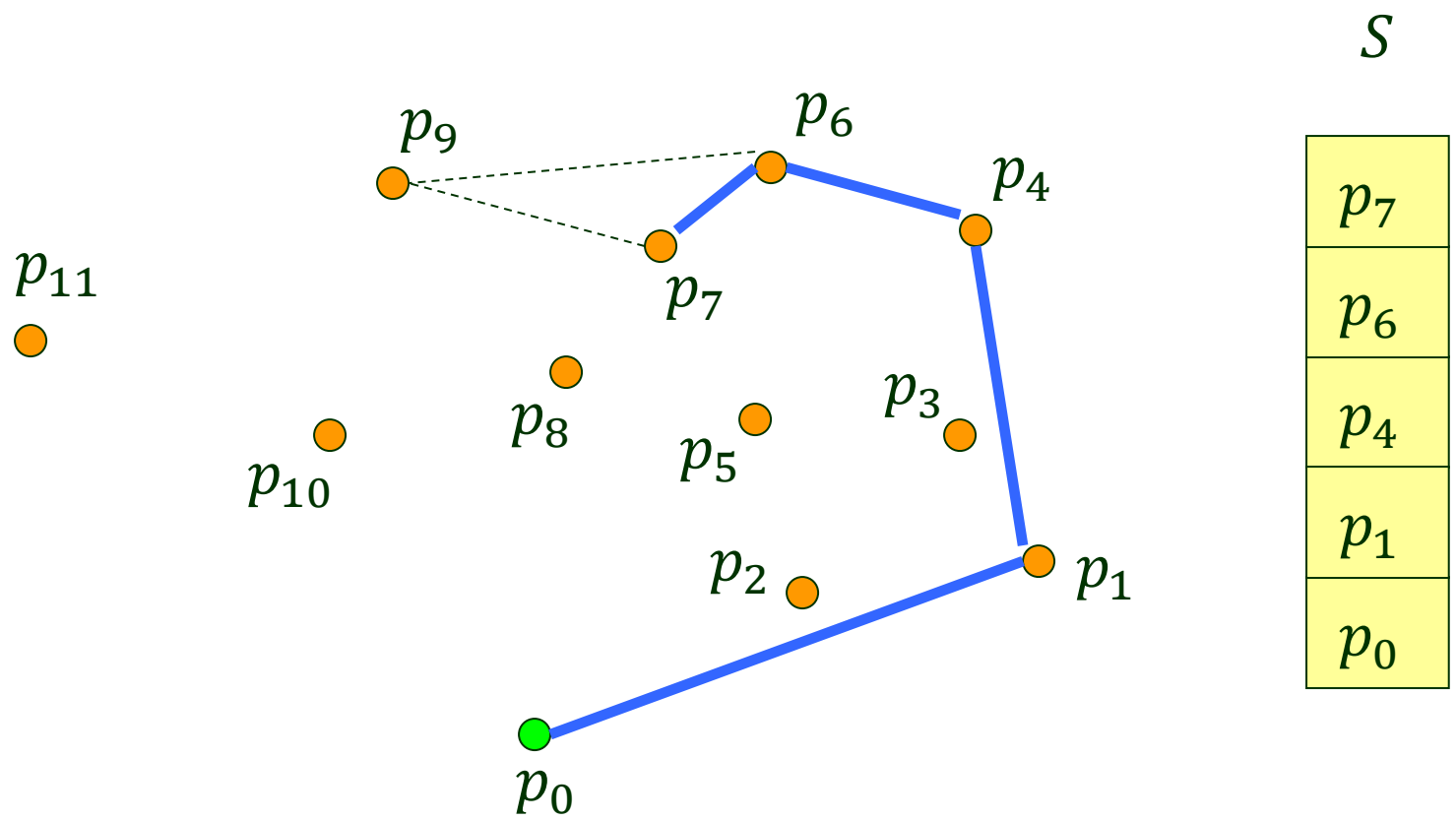
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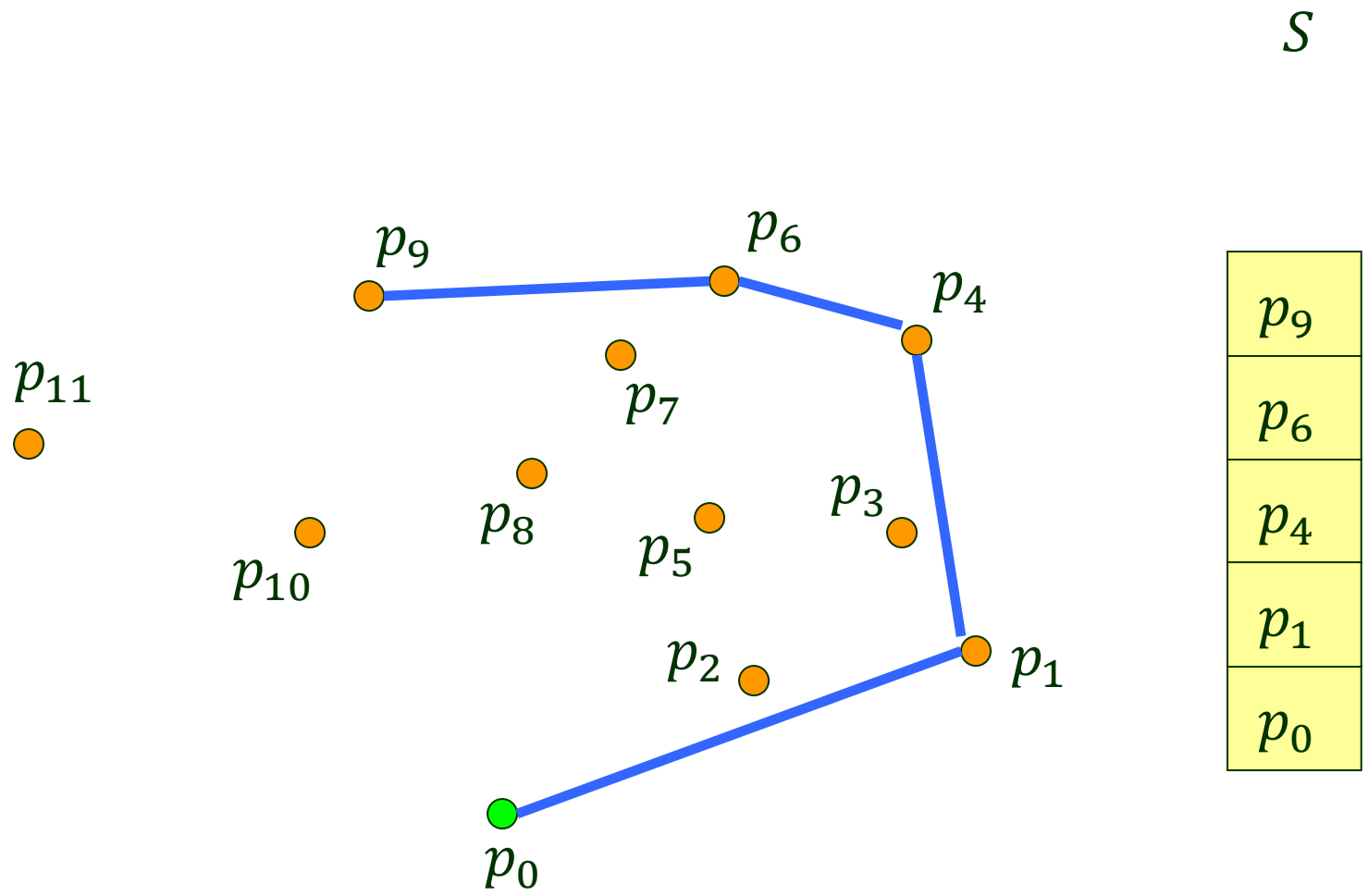
p_1
p_0


S
p_9
p_6
p_4
p_{11}
p_7
p_8
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p_9
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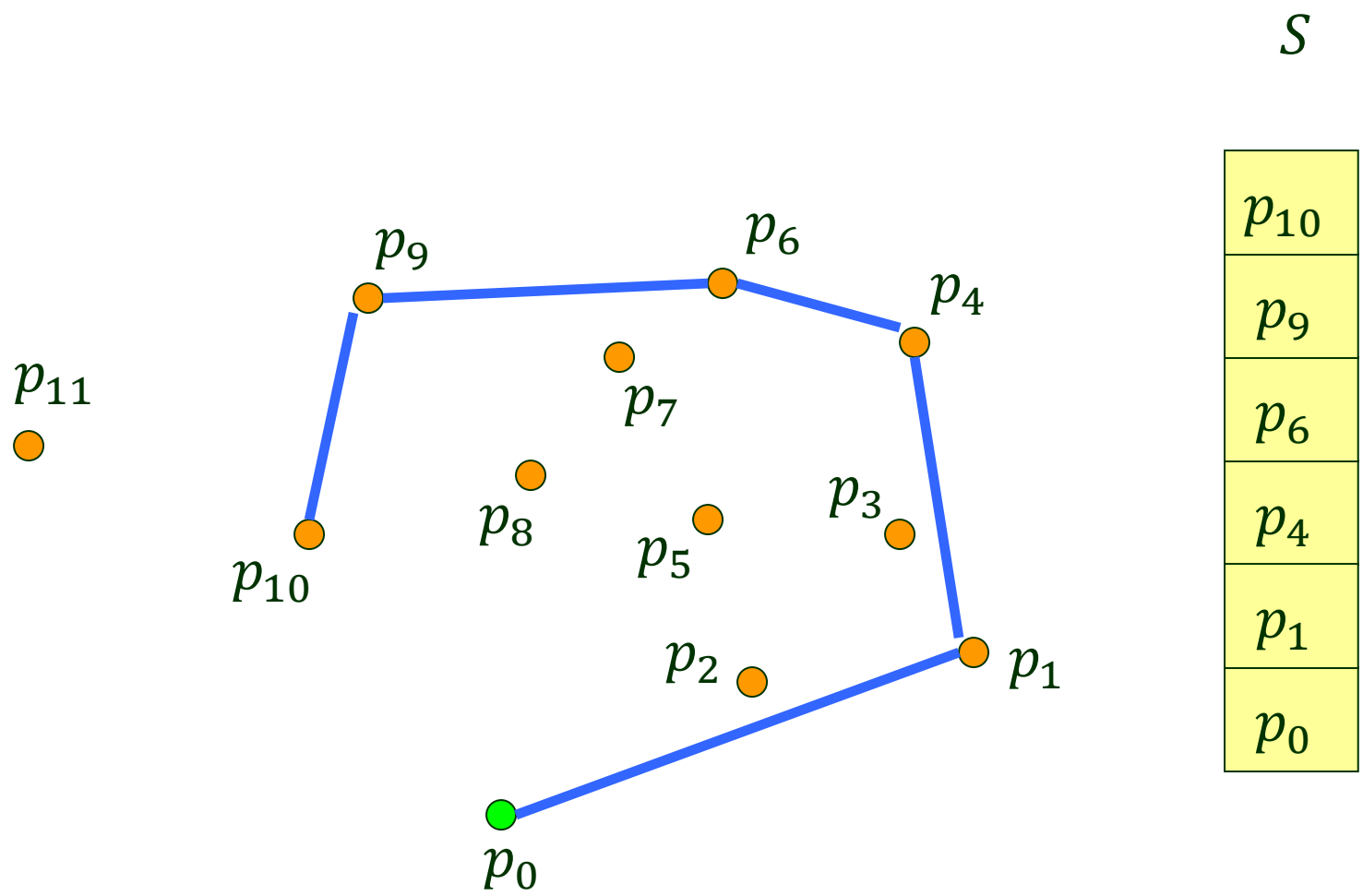
p_0


S
p_{10}
p_9
p_6
p_4
p_1
p_0
p_{11}
p_9
p_6
p_4
p_7
p_8
p_3
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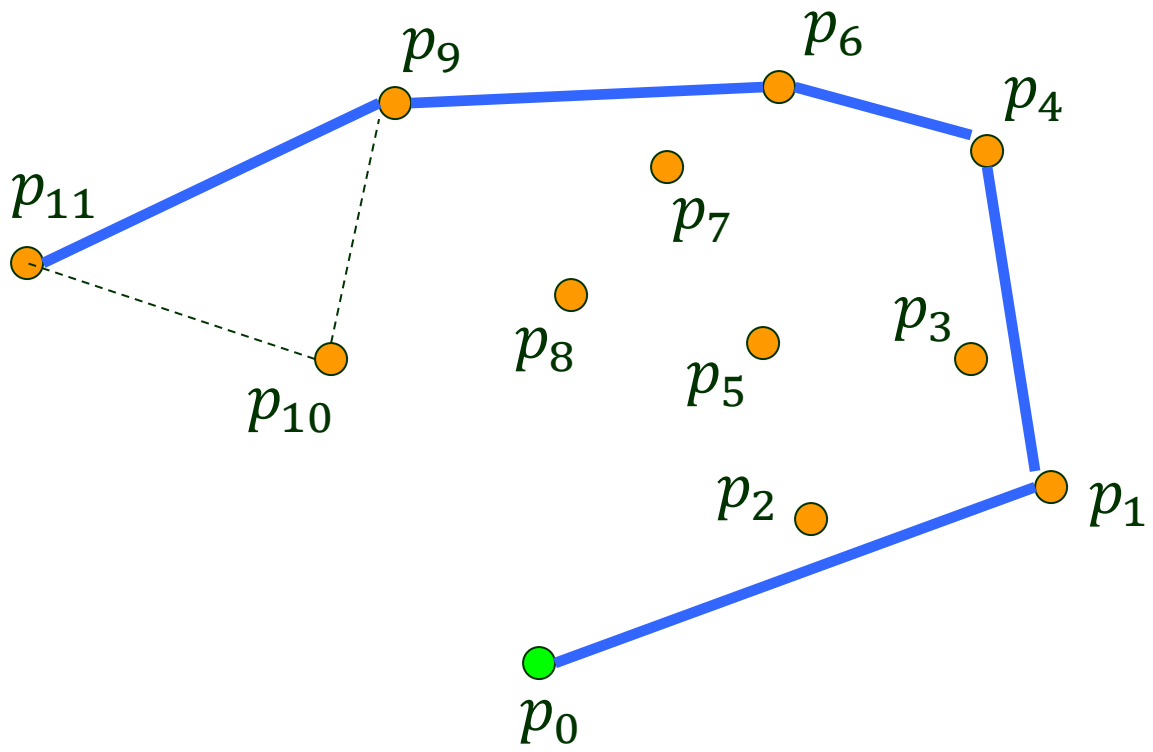
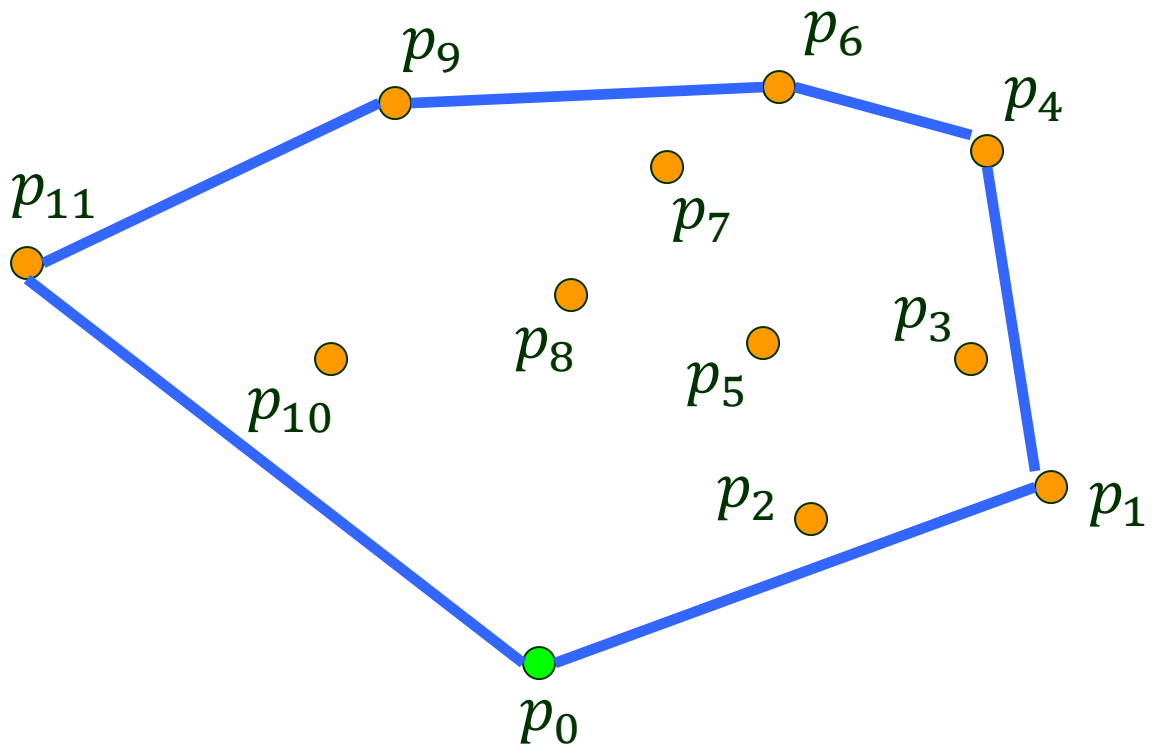
p_{10}
p_2
p_1
p_0



S

p_{11}
p_9
p_6
p_4
p_1
p_0

Finish



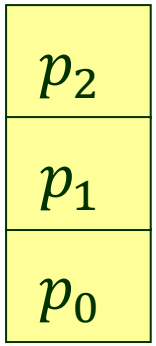
S

p_{11}
p_9
p_6
p_4
p_1
p_0

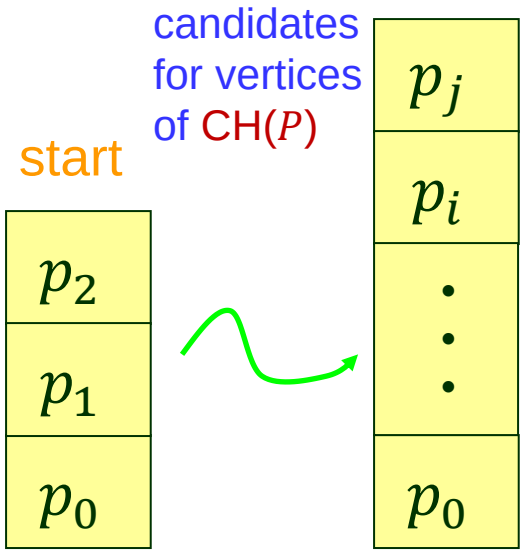
Graham’s Scan

Graham's Scan

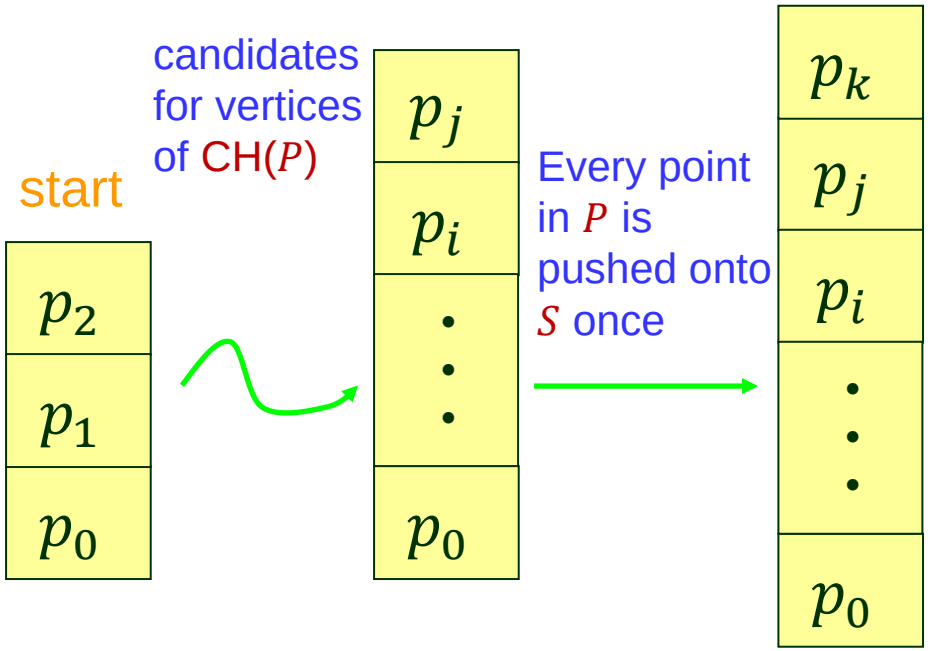
start



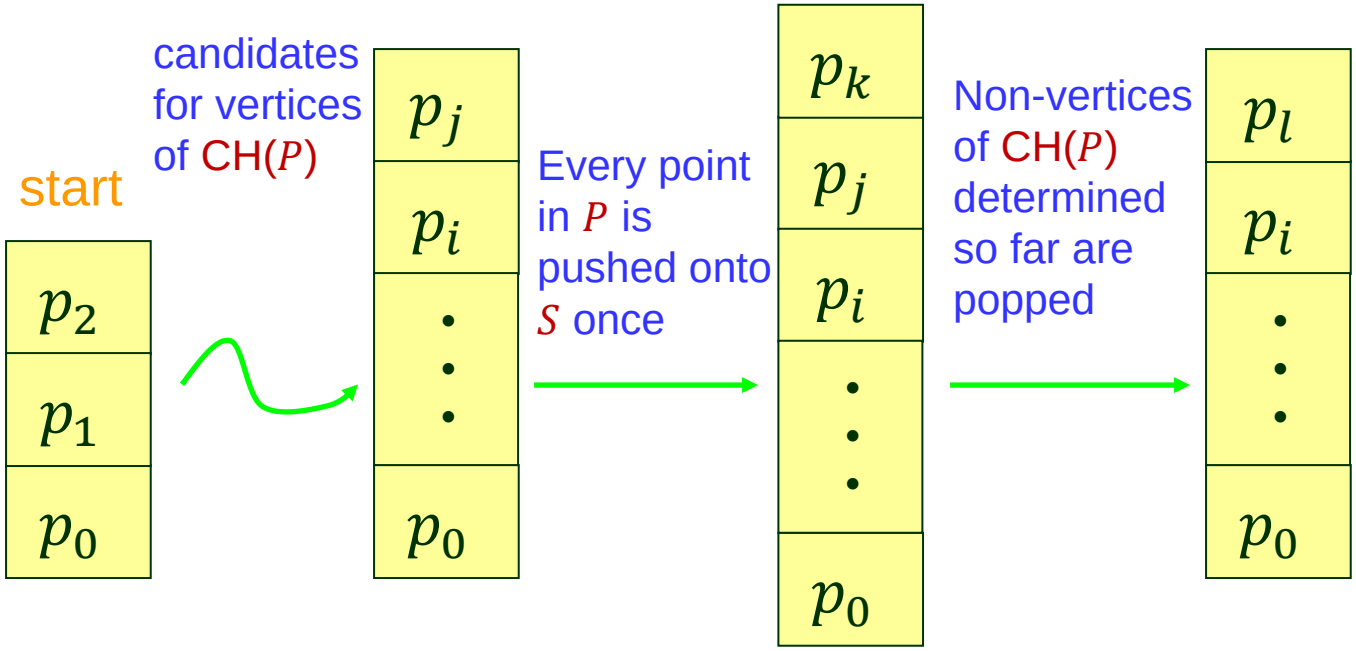
Graham's Scan



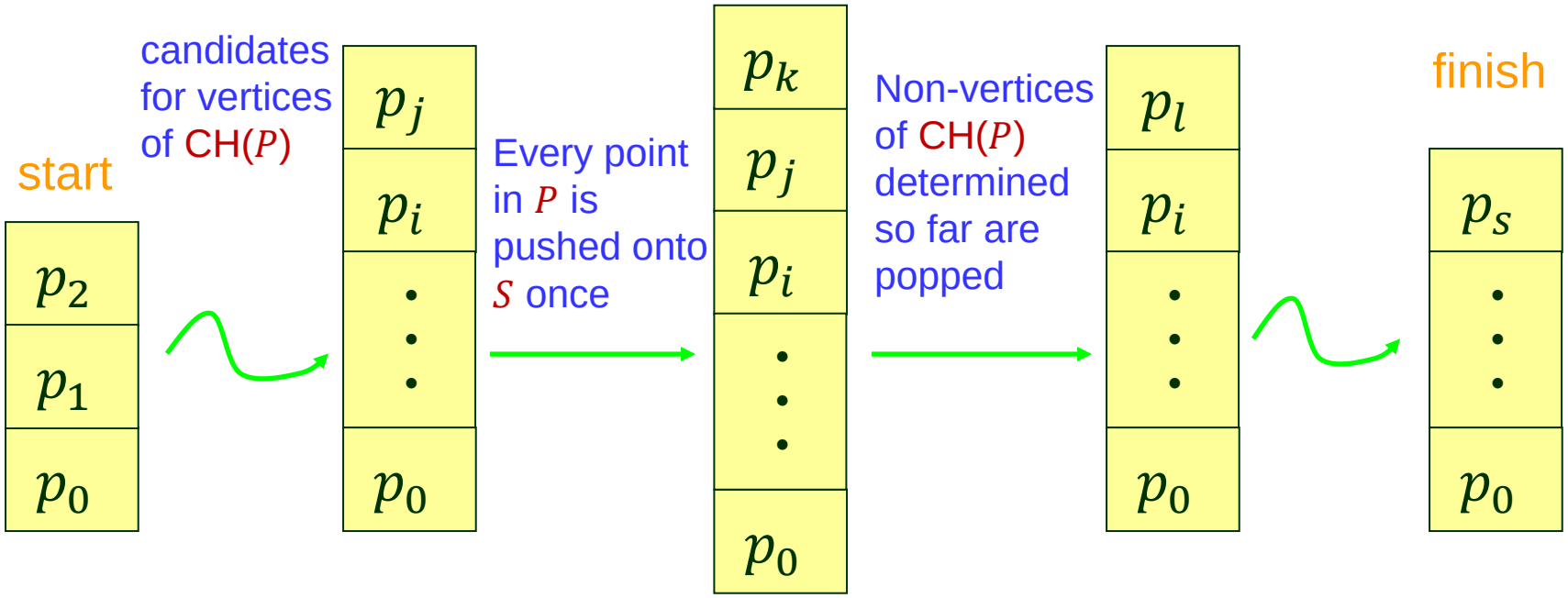
Graham's Scan



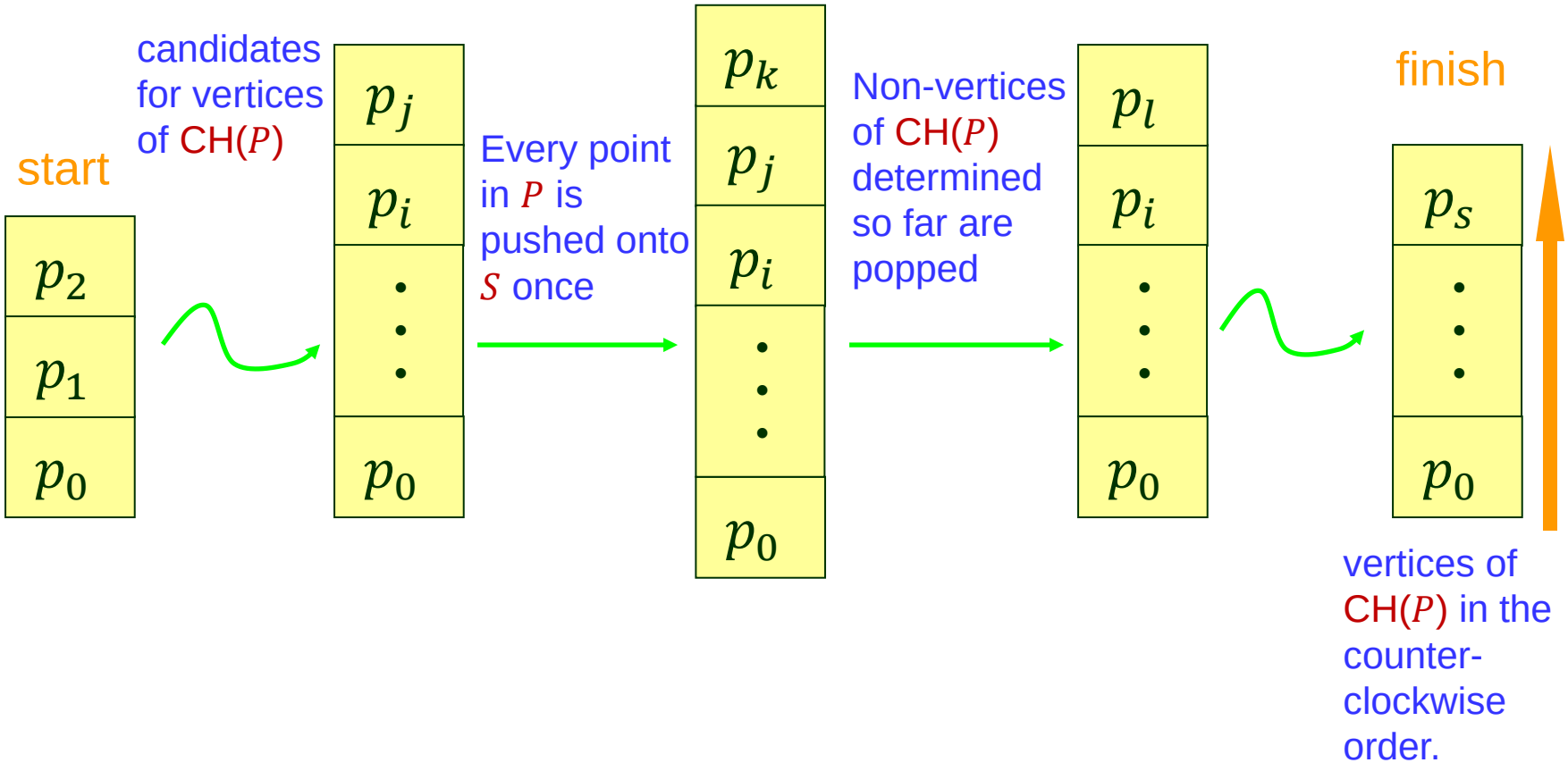
Graham's Scan



Graham's Scan



Graham's Scan



IV. The Graham Scan Algorithm

```
Graham-Scan(P)
    let p_0 be the point in P with minimum y-coordinate
    let ⟨p_1, p_2, …, p_{n−1}⟩ be the remaining points in P
        sorted in counterclockwise order by polar angle around p_0.
    Top[S] ← 0
    Push(p_0, S)
    Push(p_1, S)
    Push(p_2 , S)
    for i = 3 to n − 1
        do while p_i makes a nonleft turn from the line segment
                    determined by Top(S) and Next-to-Top(S)
                do Pop(S)
            Push(S, p_i)
    return S
```

Running time

#operations	time / operation	total

Running time

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Finding p_0	1	$\Theta(n)$	$\Theta(n)$

Running time

	#operations	time / operation	total
Finding p_0	1	$\Theta(n)$	$\Theta(n)$
Sorting	1	$O(n \log n)$	$O(n \log n)$

Running time

	#operations	time / operation	total
Finding p_0	1	$\Theta(n)$	$\Theta(n)$
Sorting	1	$O(n \log n)$	$O(n \log n)$
Push	n	$O(1)$	$\Theta(n)$

Running time

	#operations	time / operation	total
Finding p_0	1	$\Theta(n)$	$\Theta(n)$
Sorting	1	$O(n \log n)$	$O(n \log n)$
Push	n	$O(1)$	$\Theta(n)$
Pop	$\leq n - 2$	$O(1)$	$O(n)$

Running time

	#operations	time / operation	total
Finding p_0	1	$\Theta(n)$	$\Theta(n)$
Sorting	1	$O(n \log n)$	$O(n \log n)$
Push	n	$O(1)$	$\Theta(n)$
Pop	$\leq n - 2$ Why?	$O(1)$	$O(n)$

Running time

	#operations	time / operation	total
Finding p_0	1	$\Theta(n)$	$\Theta(n)$
Sorting	1	$O(n \log n)$	$O(n \log n)$
Push	n	$O(1)$	$\Theta(n)$
Pop	$\leq n - 2$ Why?	$O(1)$	$O(n)$

The running time of Graham's Scan is $\boldsymbol{O(n \log n)}$.

Proof of Correctness

Claim 1 Each point popped from stack S is not a vertex of CH(P).

Proof

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Proof Two cases when p_j is popped:

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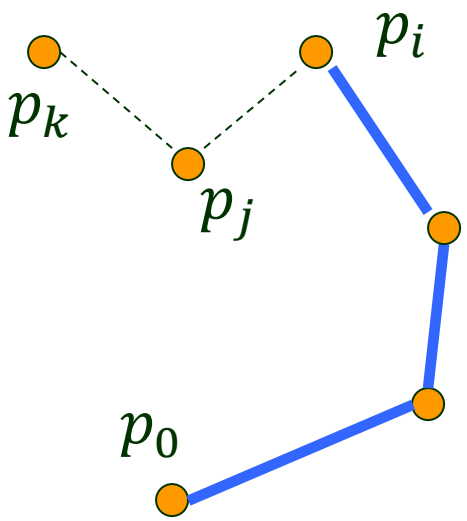

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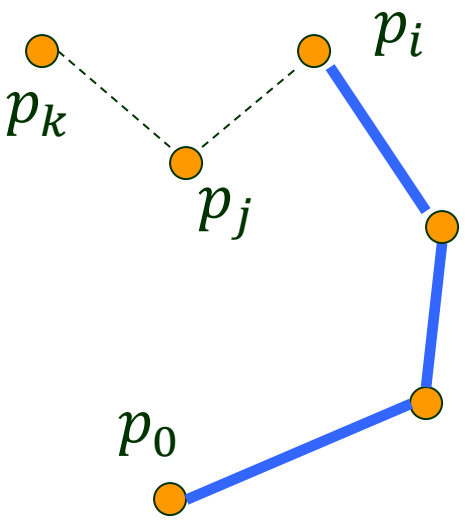

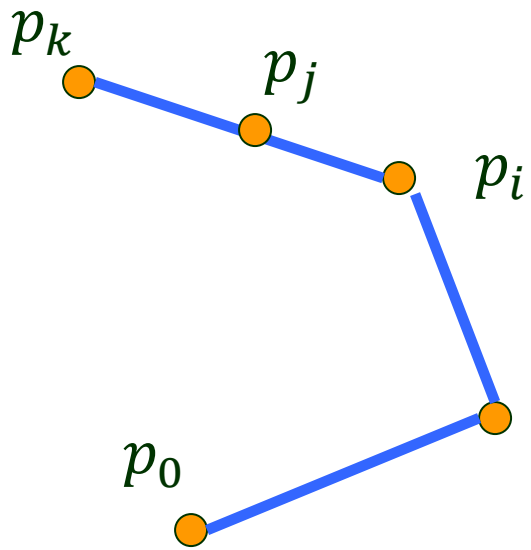

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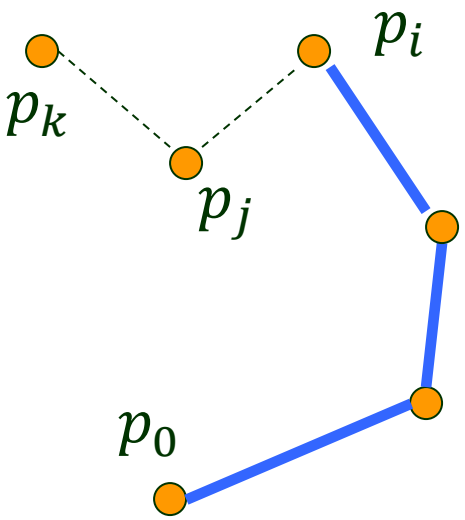

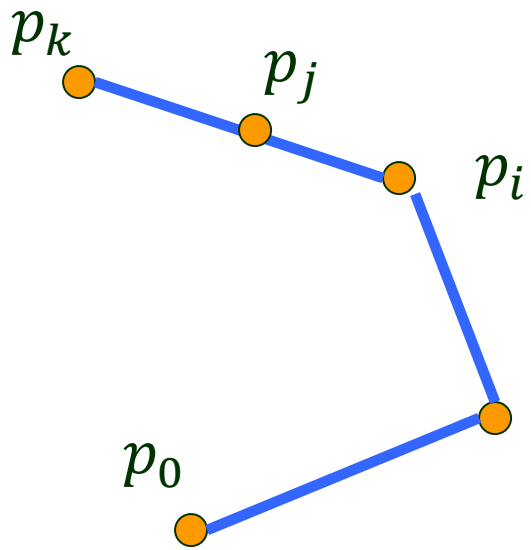

In neither case can p_j become a vertex of CH(*P*).

Claim 2 Graham-Scan maintains the ***invariant*** that the points on stack *S* always form the vertices of a convex polygon in counterclockwise order.

Proof

Claim 2 Graham-Scan maintains the ***invariant*** that the points on stack S always form the vertices of a convex polygon in counterclockwise order.

Proof The claim holds right after initialization of S when p_0, p_1, p_2 form a triangle (which is obviously convex).

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- Popping a point from S preserves the invariant.

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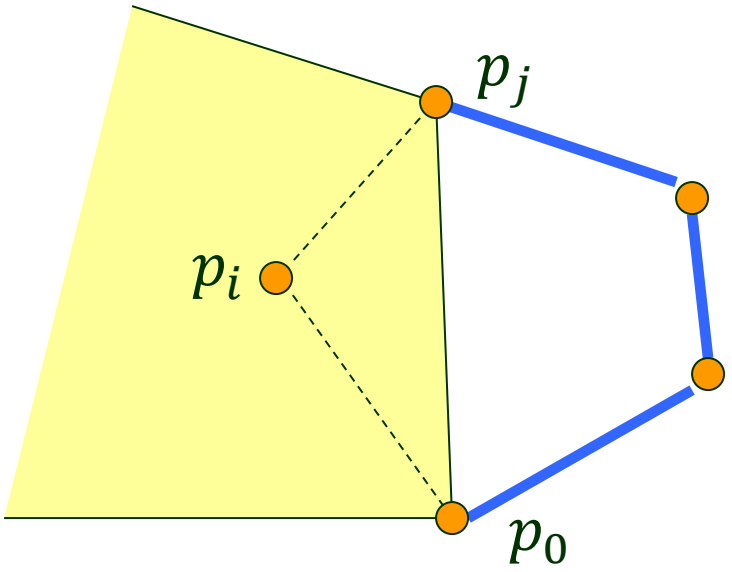

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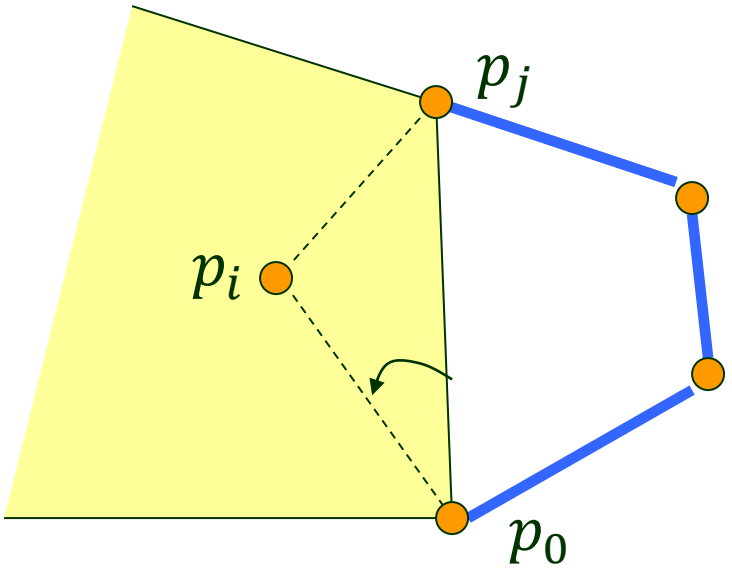

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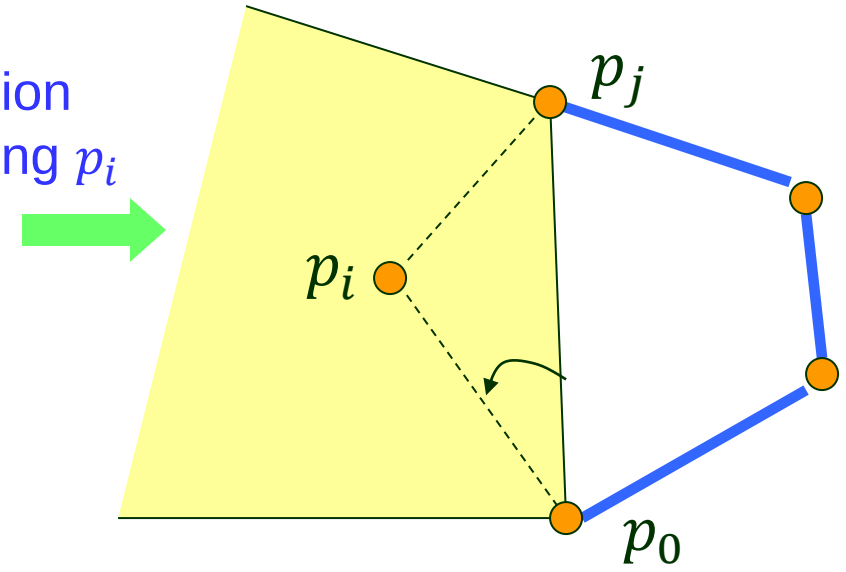
The region containing p_i



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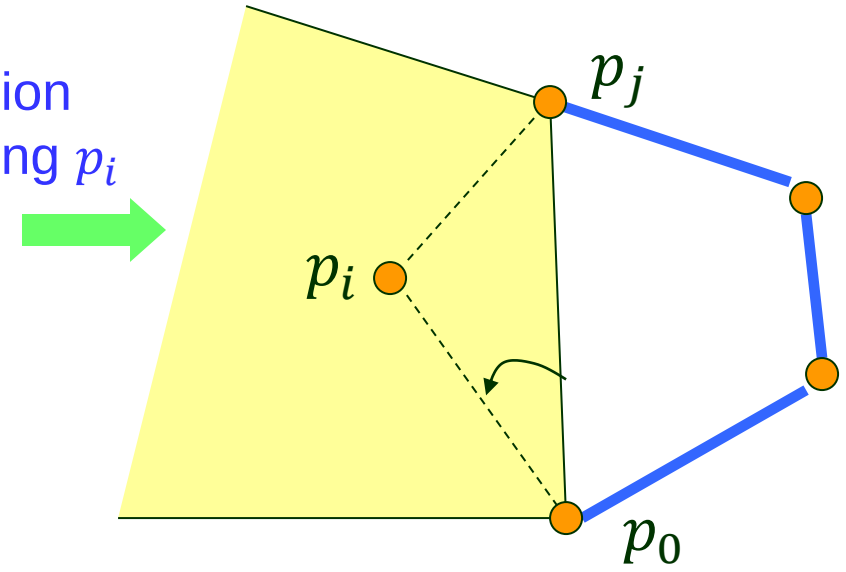


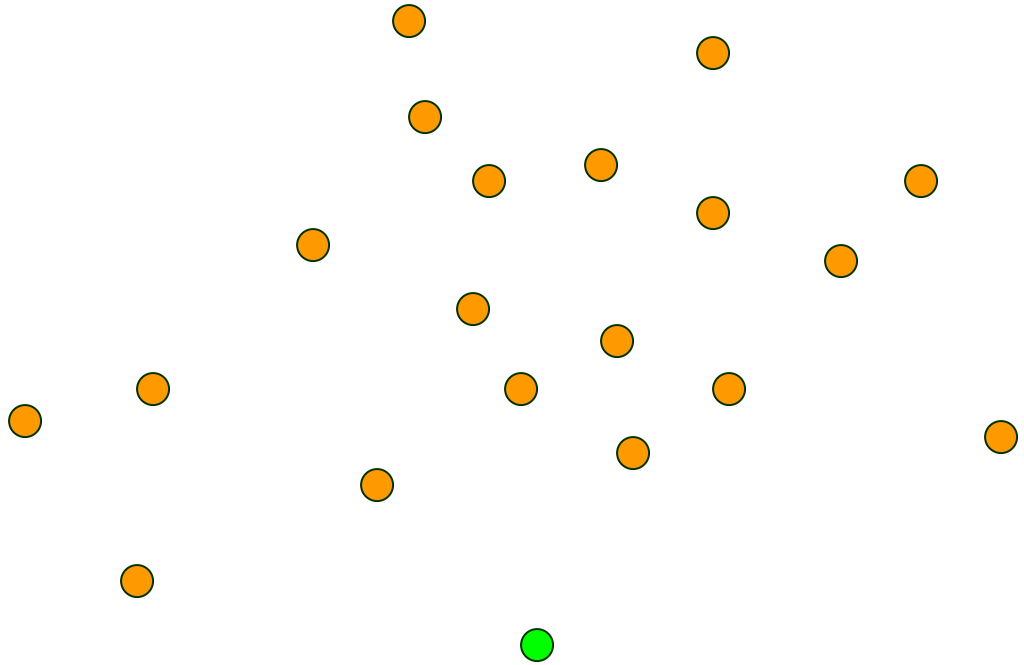
The invariant still holds.

Correctness of Graham's Scan

Theorem If Graham-Scan is run on a set P of at least three points, then a point of P is on the stack S at termination if and only if it is a vertex of CH(P).

V. Jarvis' March

A "package wrapping" technique



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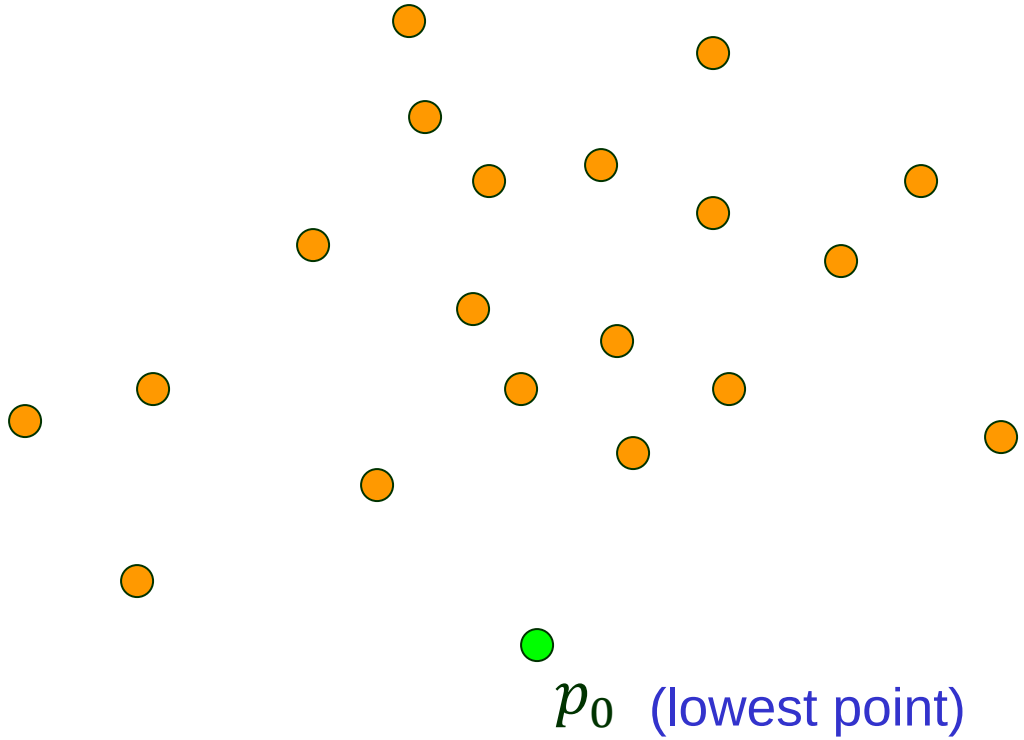

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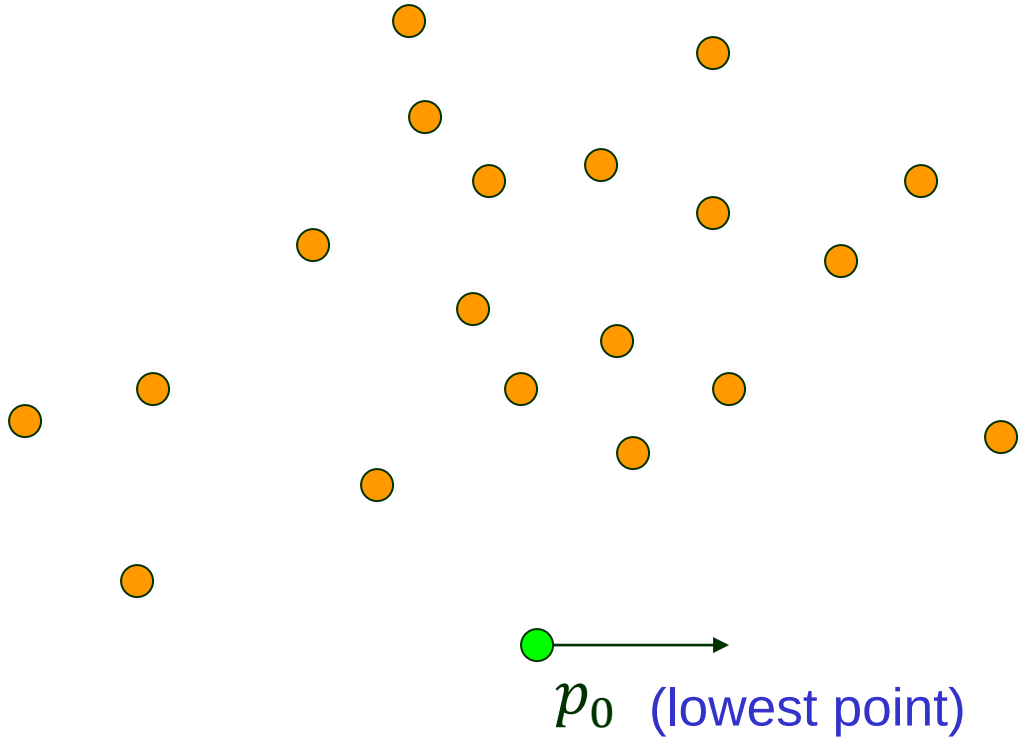

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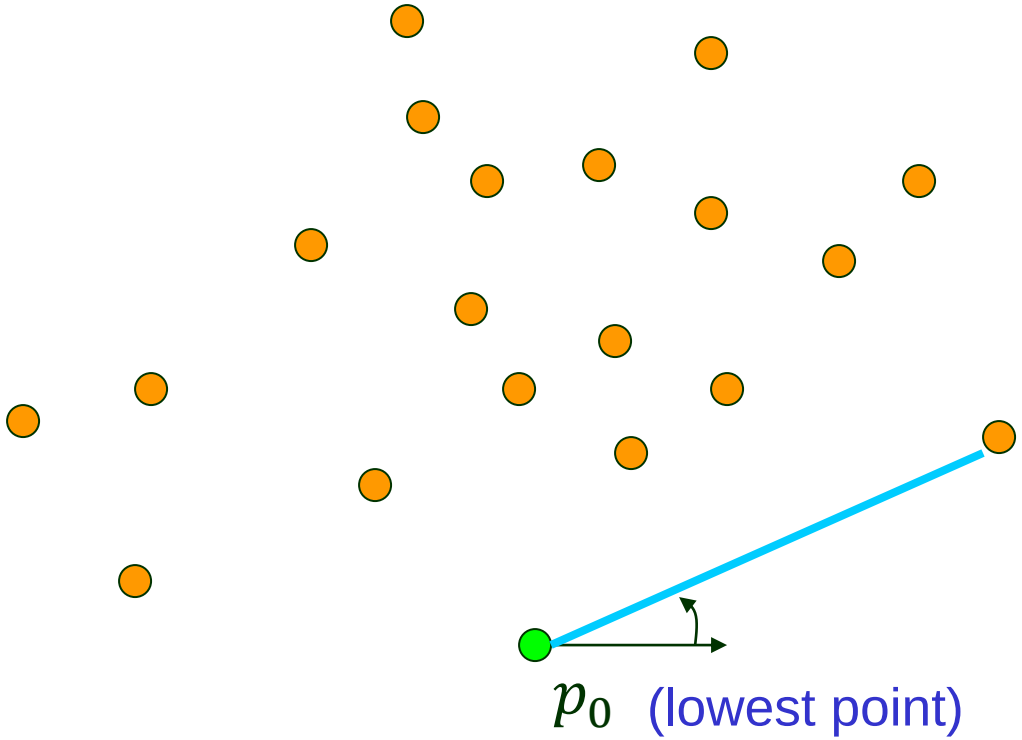

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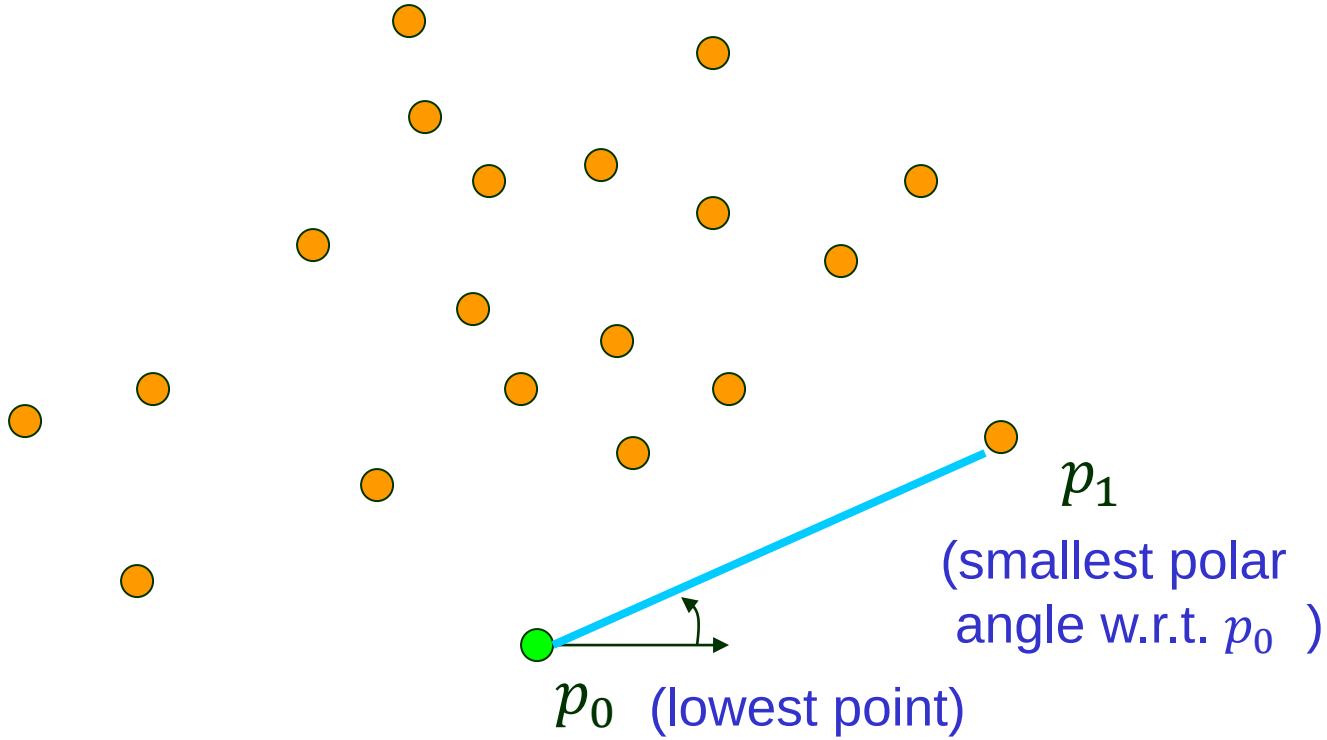

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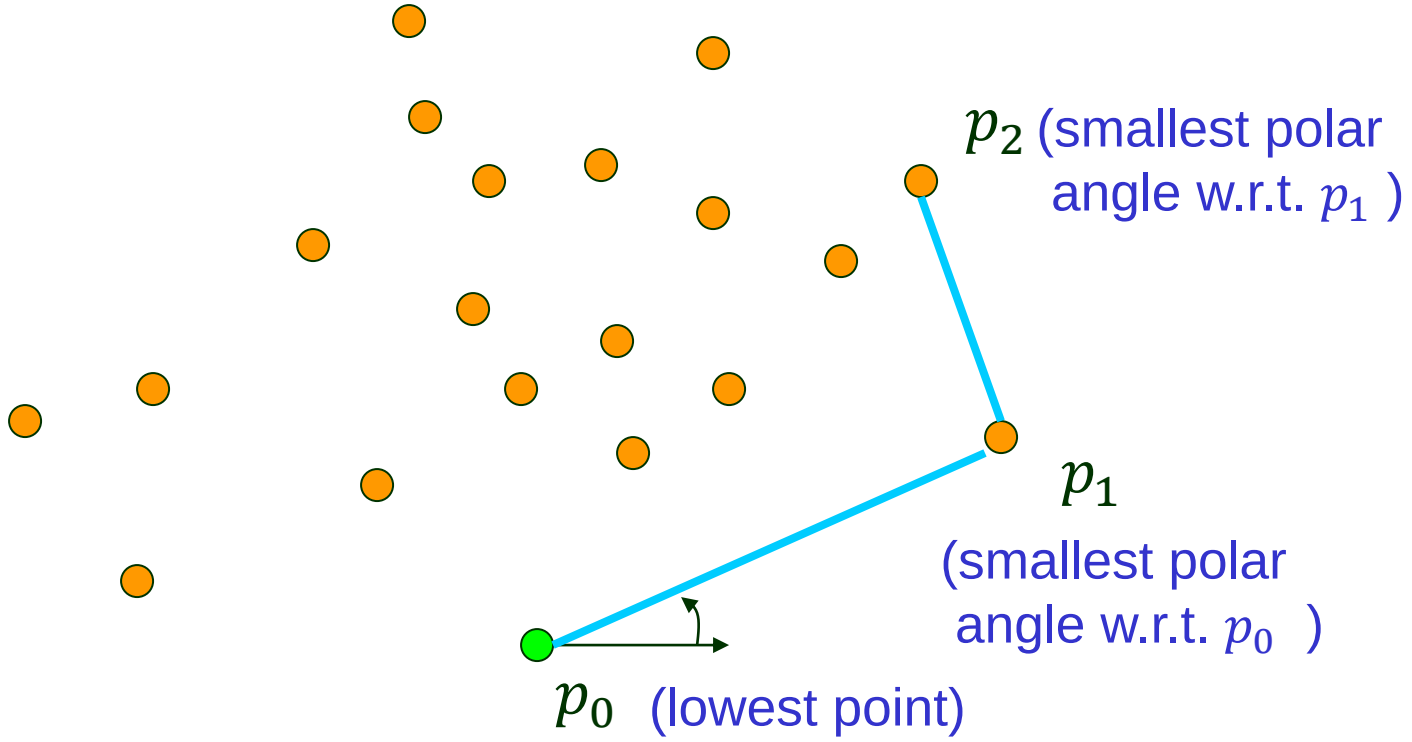

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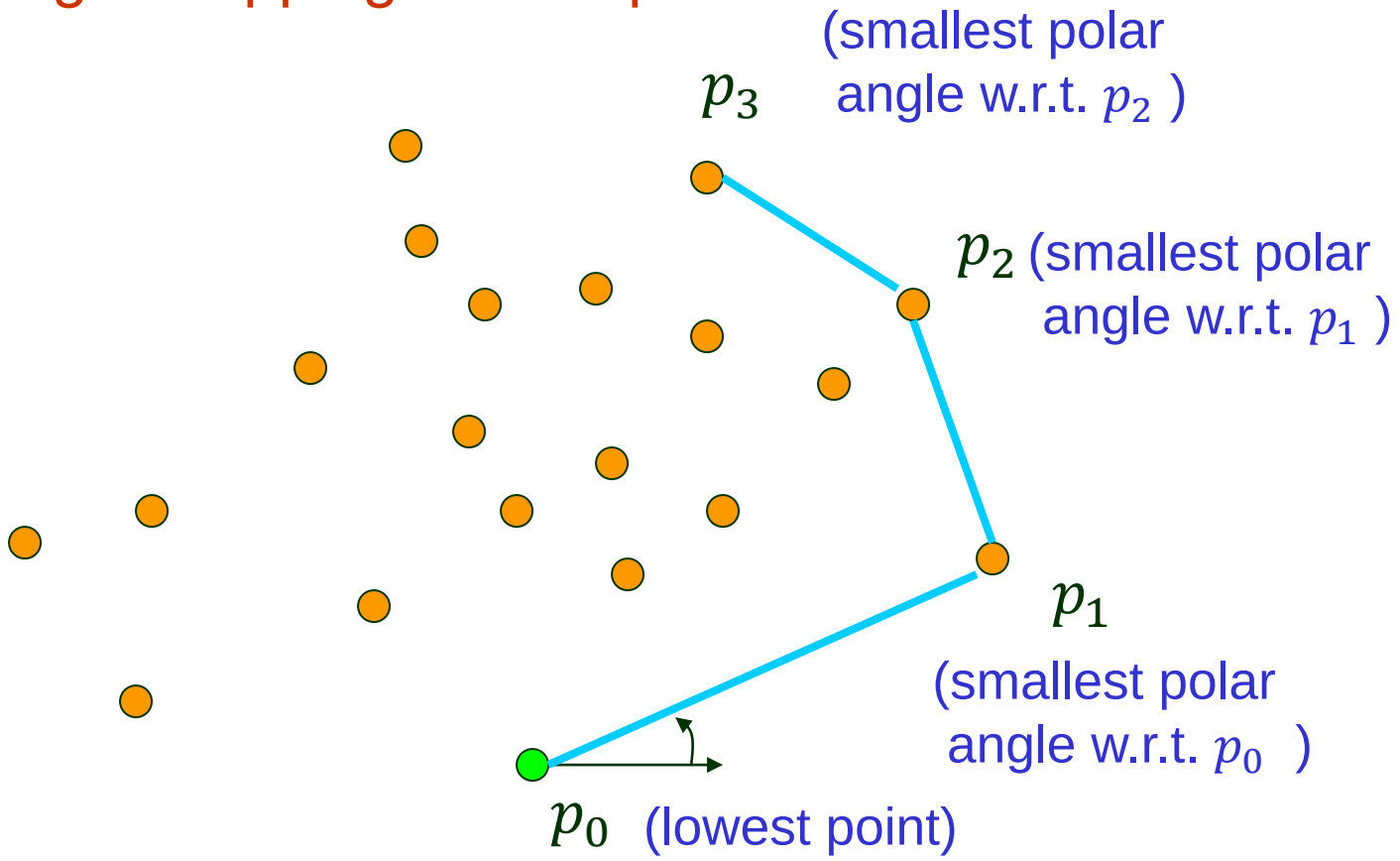

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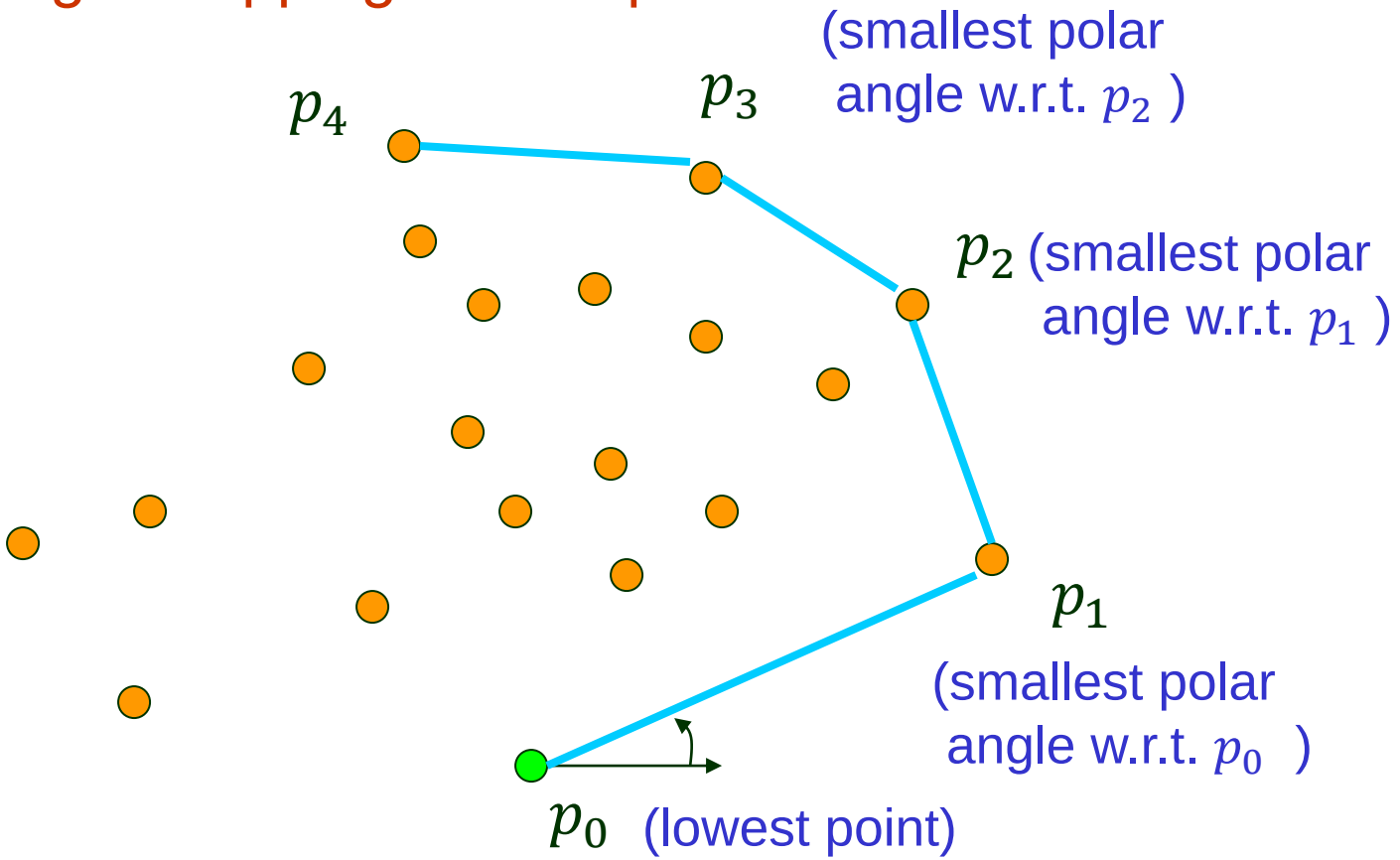

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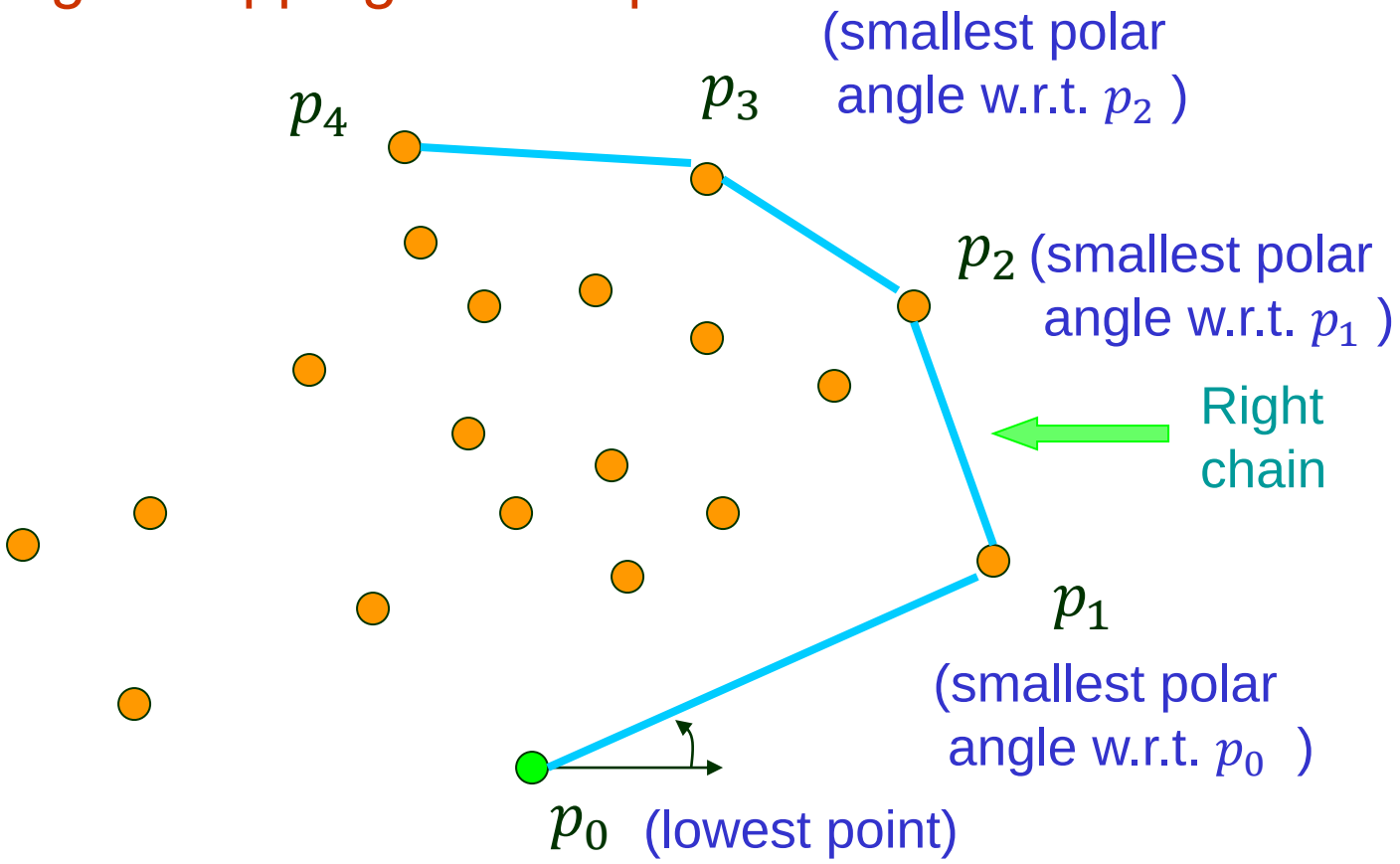

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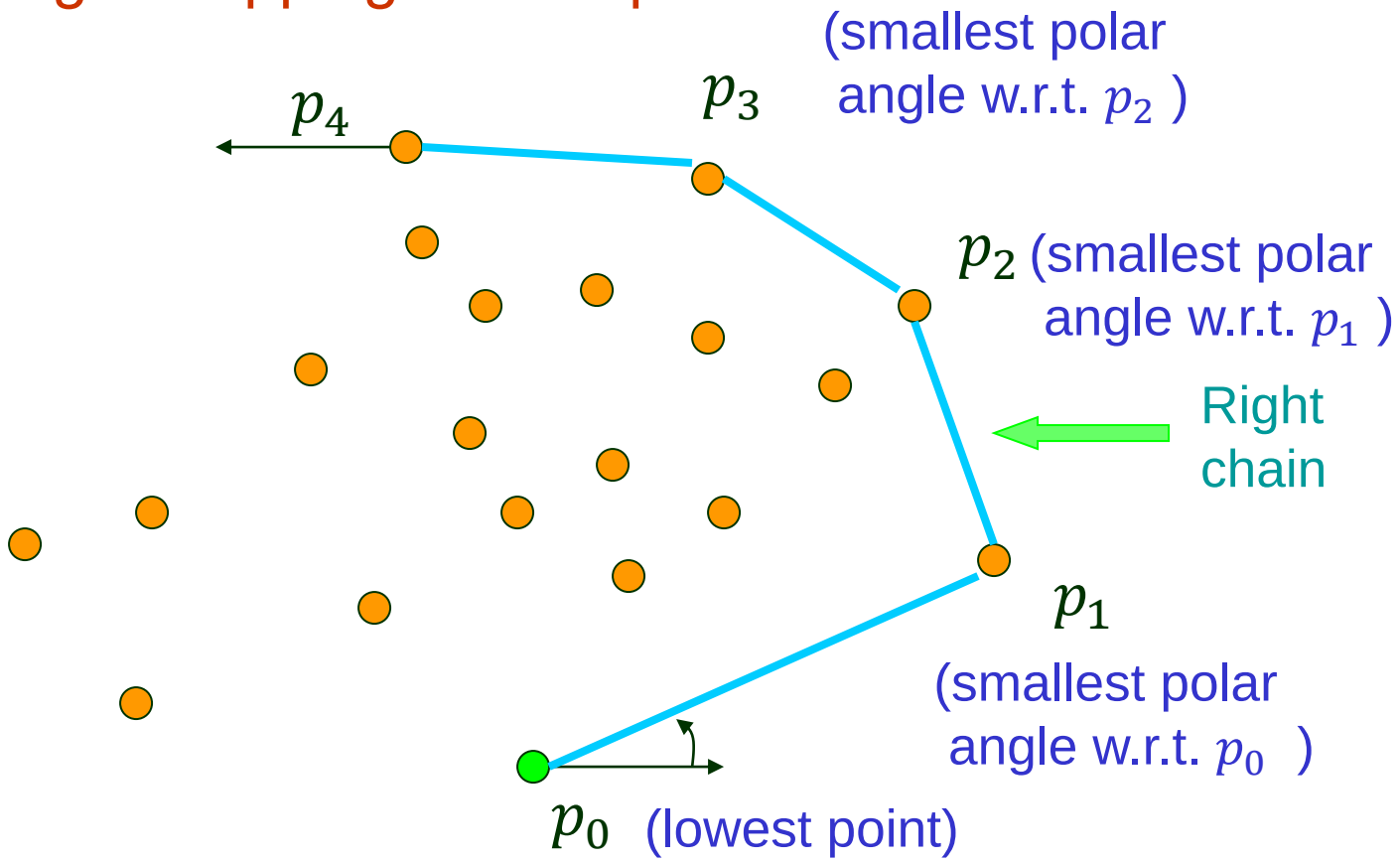

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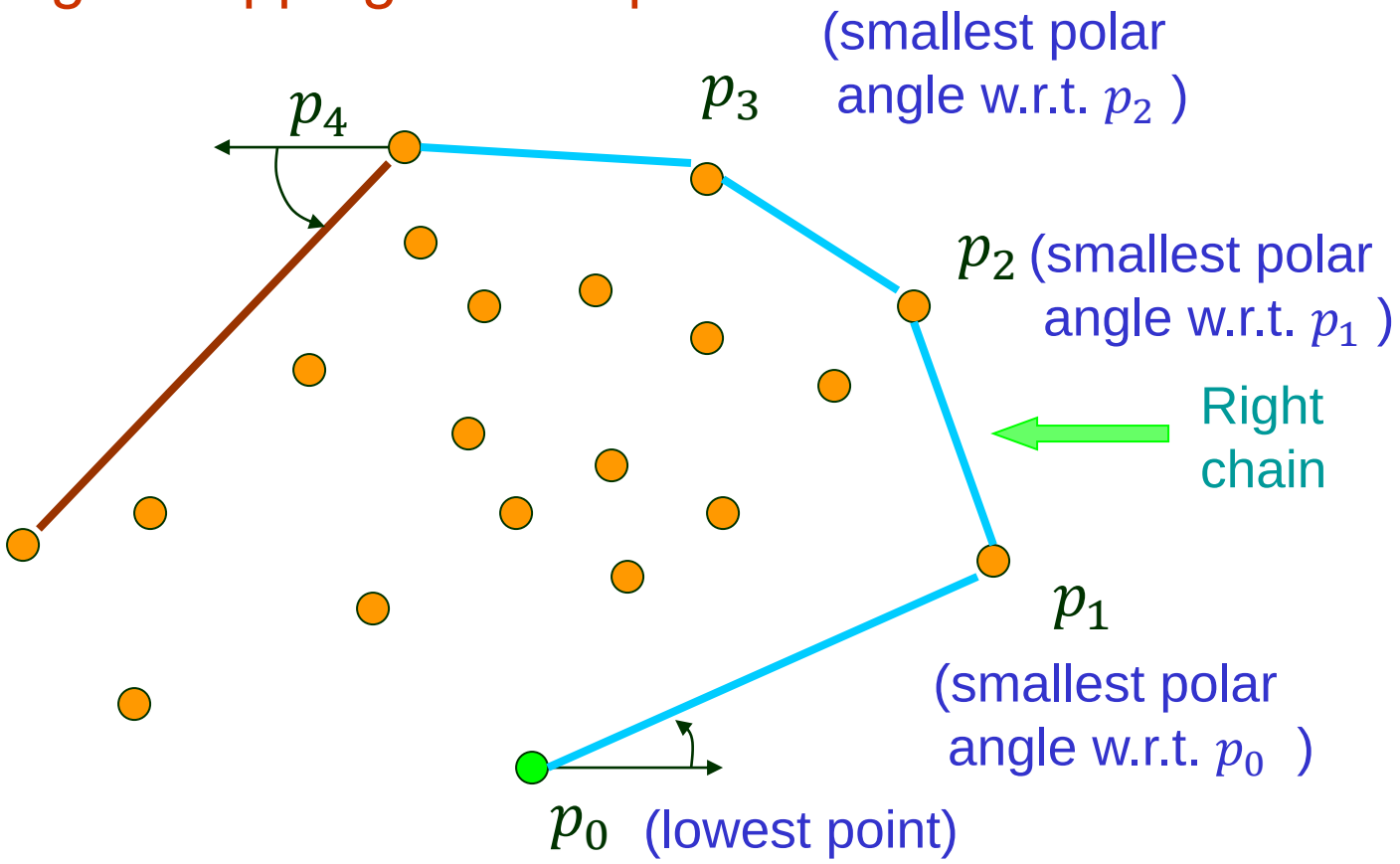

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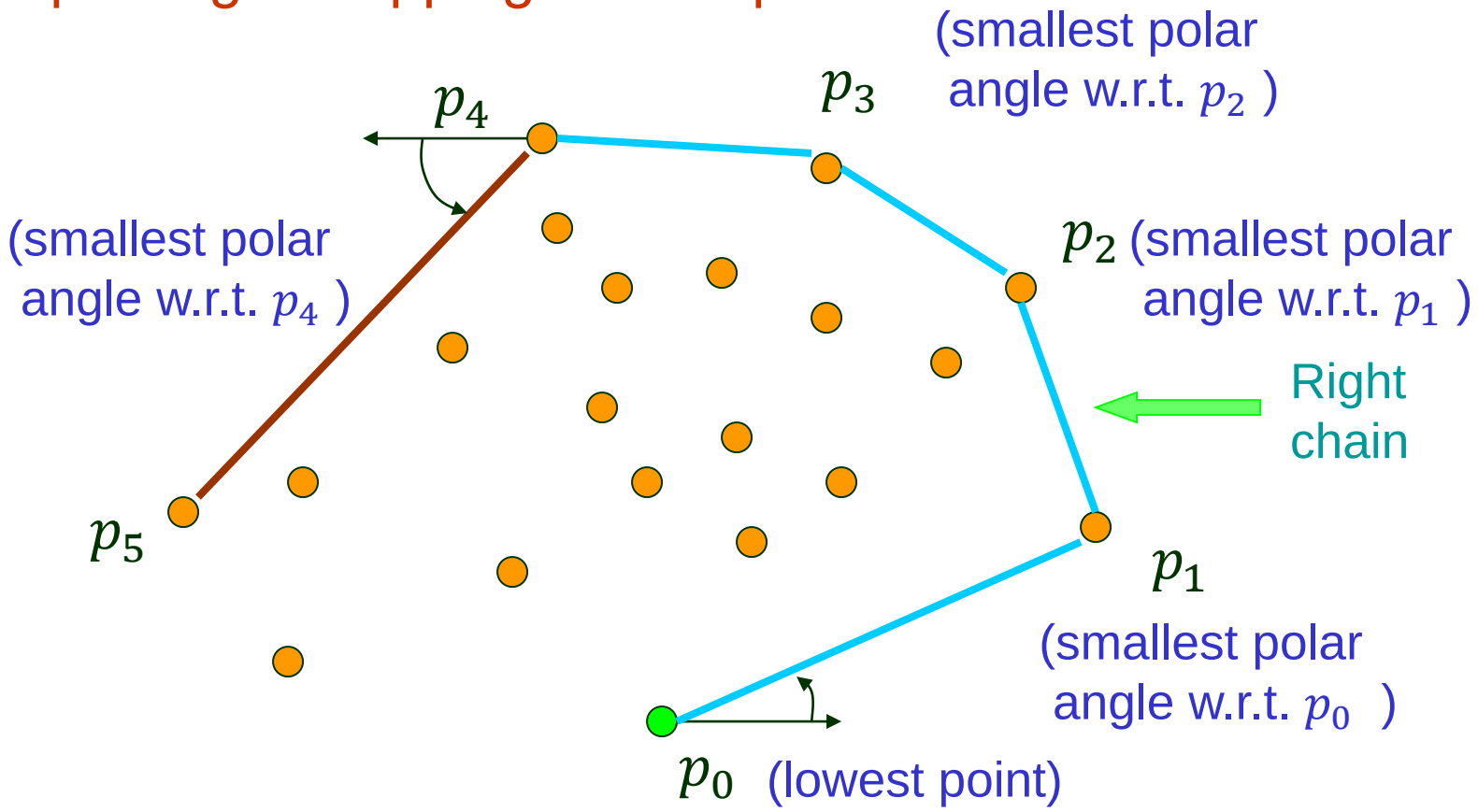

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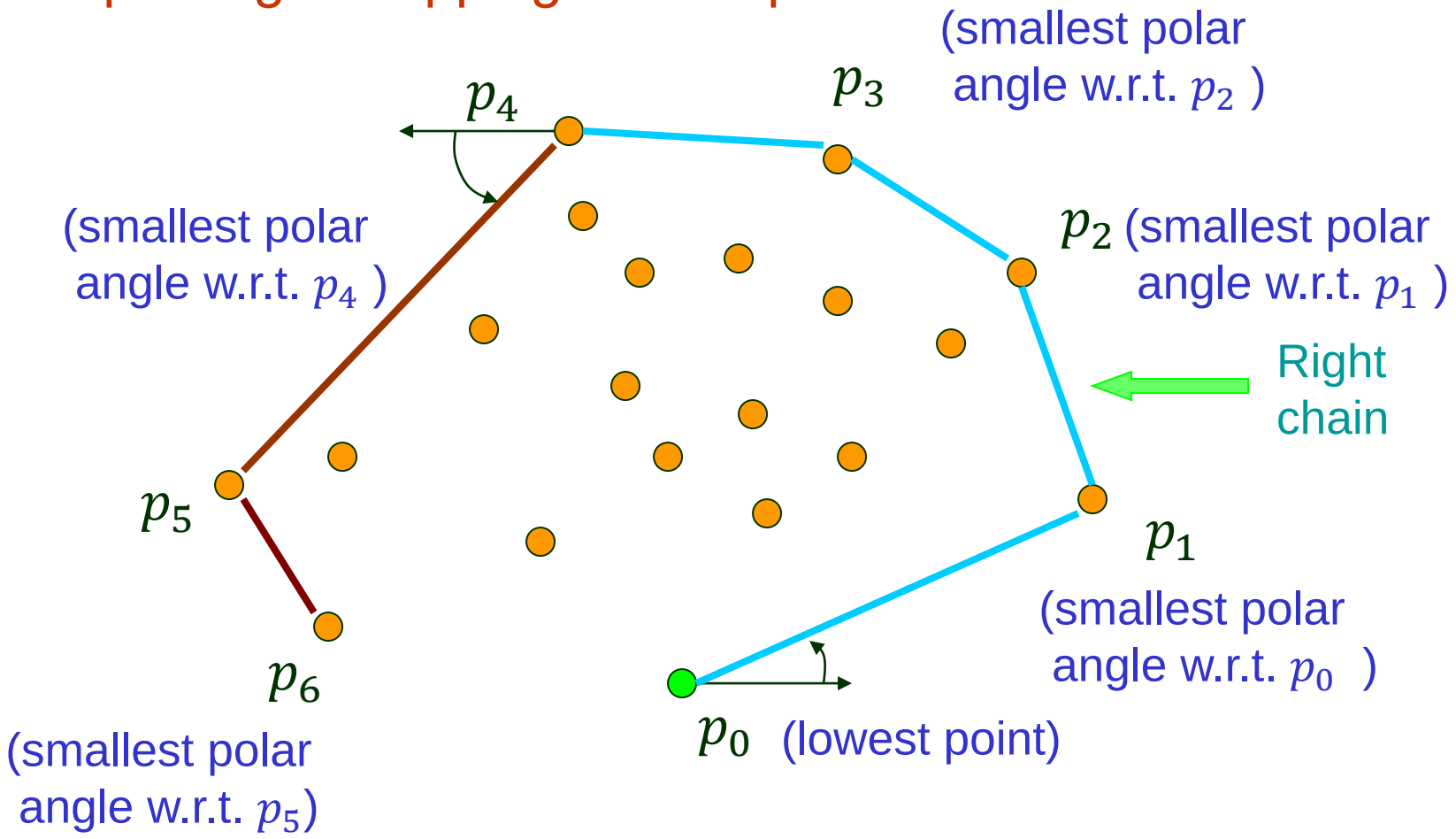

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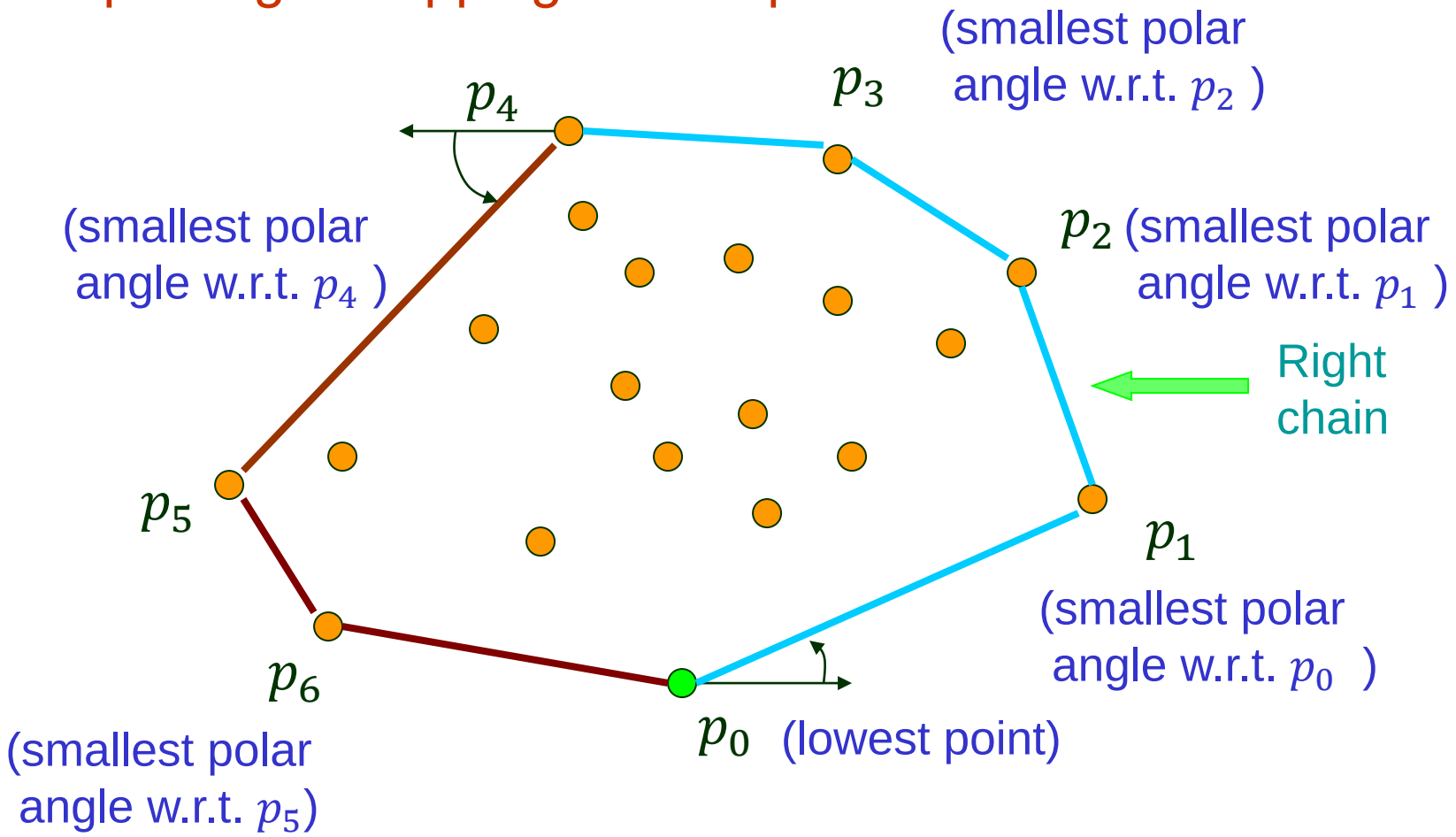

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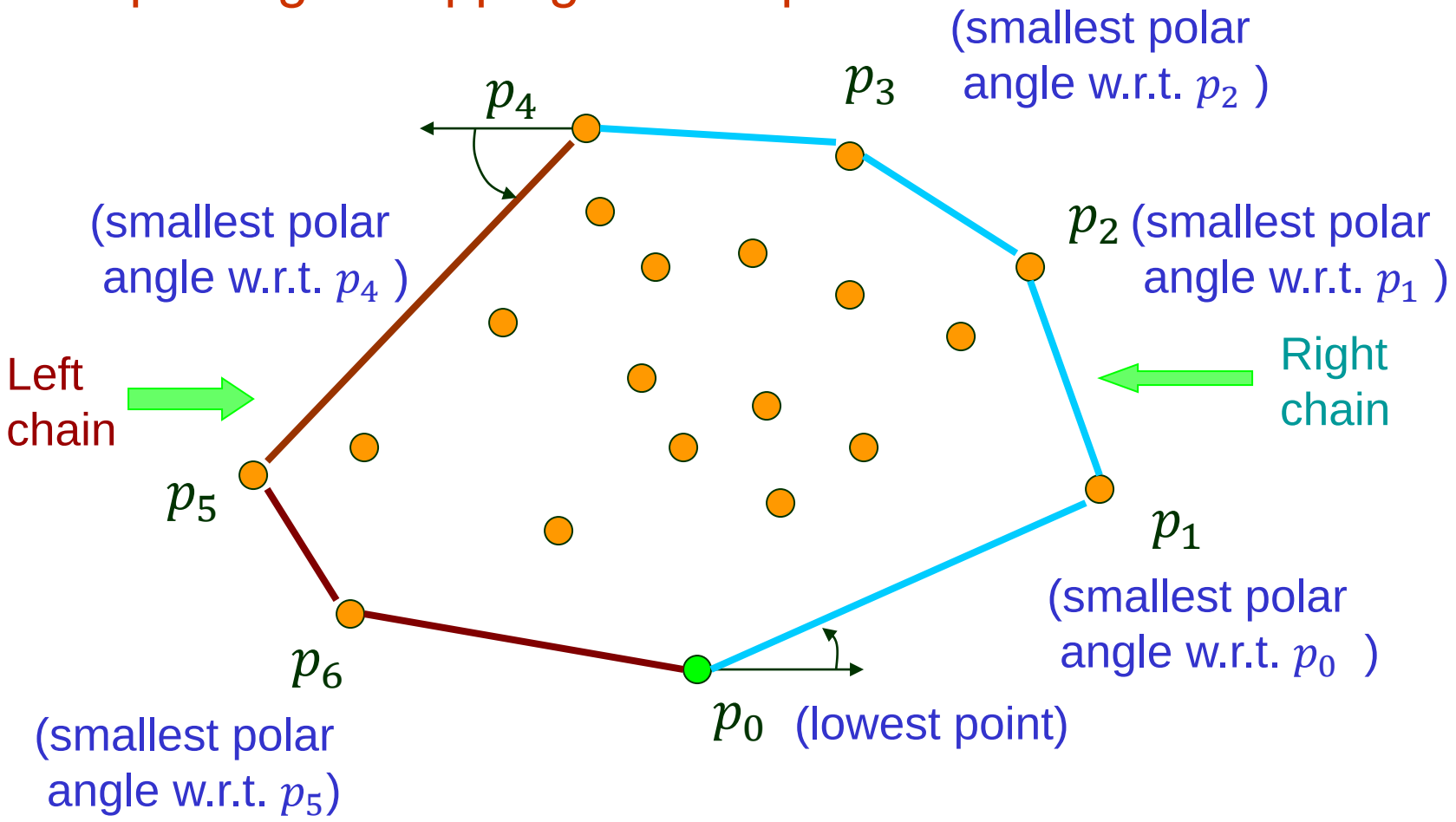

Running Time of Jarvis's March

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Thus $\boldsymbol{O(nh)}$ time in total.