Yan-Bin Jia, Iowa State University

Computational Geometry

The systematic study of algorithms and data structures for geometric objects, with a focus on exact algorithms that are asymptotically fast.

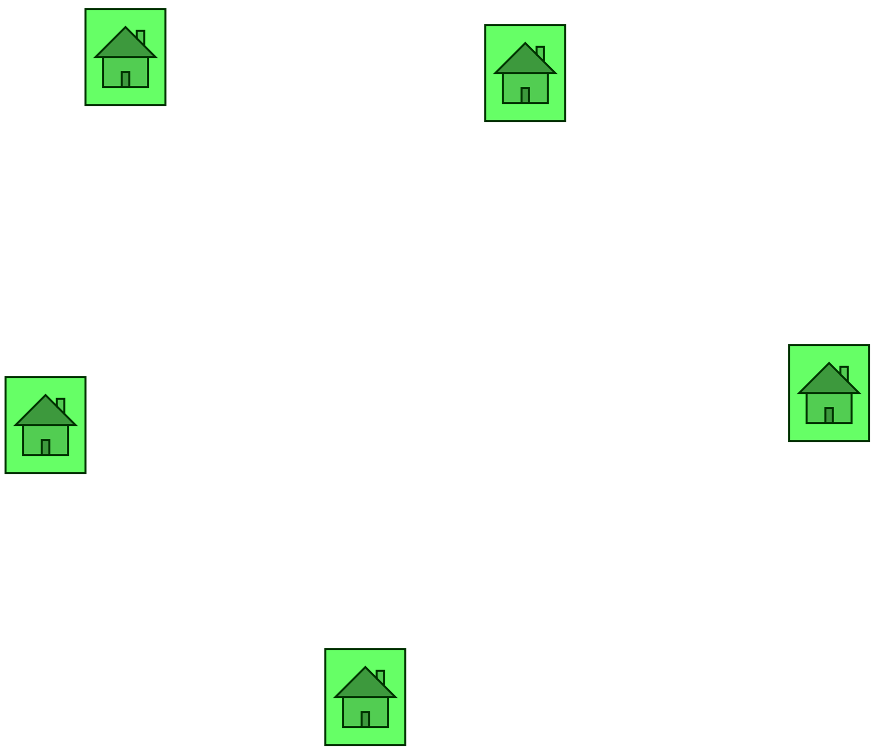
Two key ingredients of a good algorithmic solution:

- ♦ Thorough understanding of the problem geometry
- ♦ Proper application of algorithmic techniques and data structures

Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

Example 1 – Proximity

Closest café on campus?



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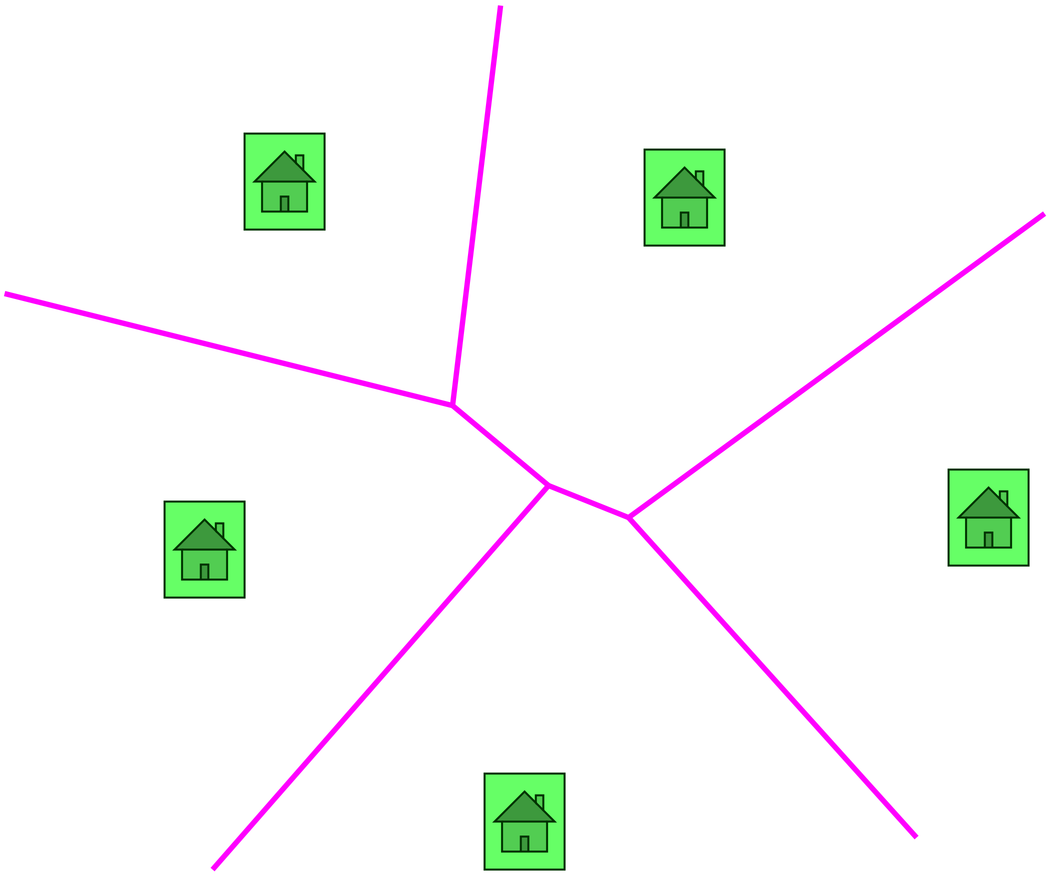

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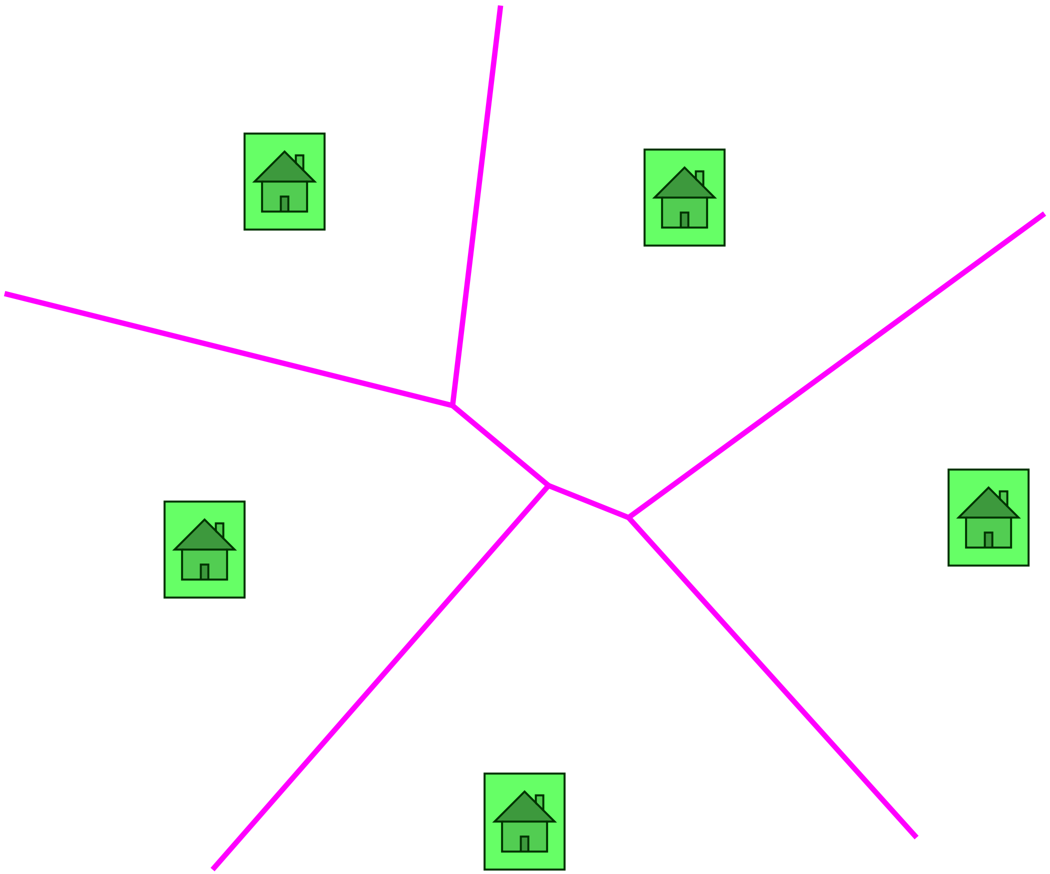

Voronoi diagram

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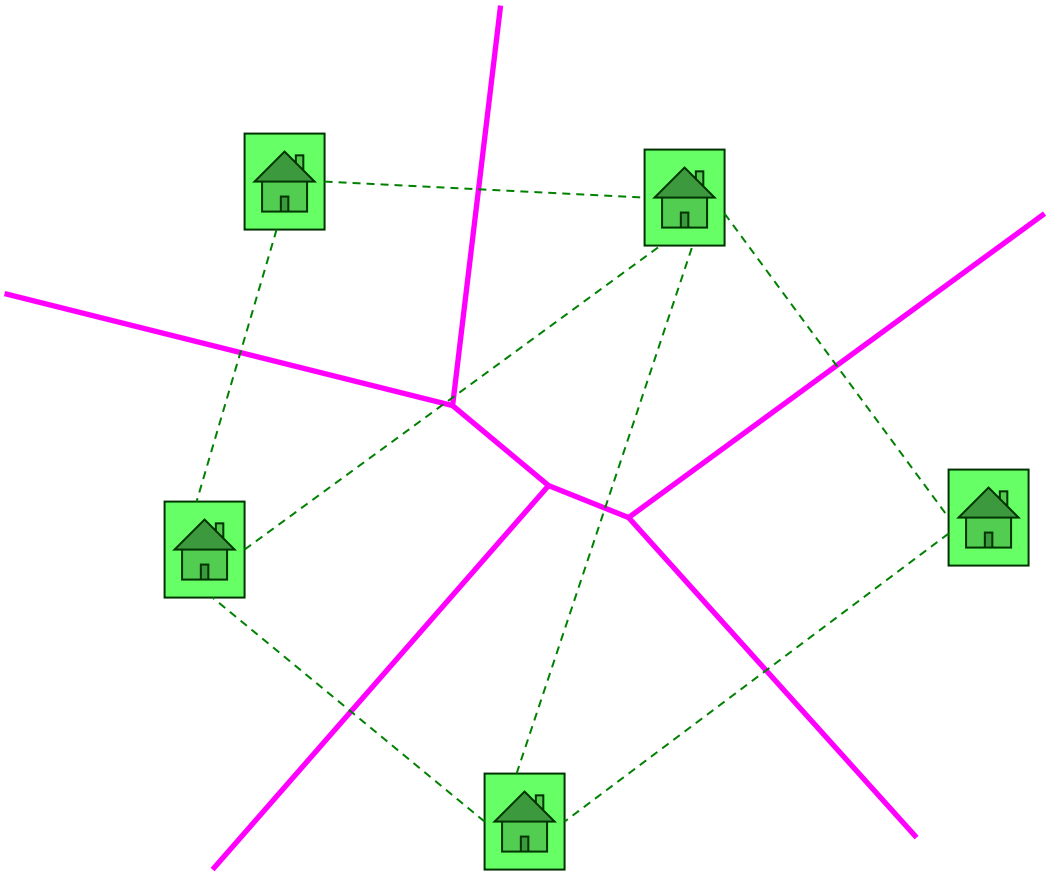

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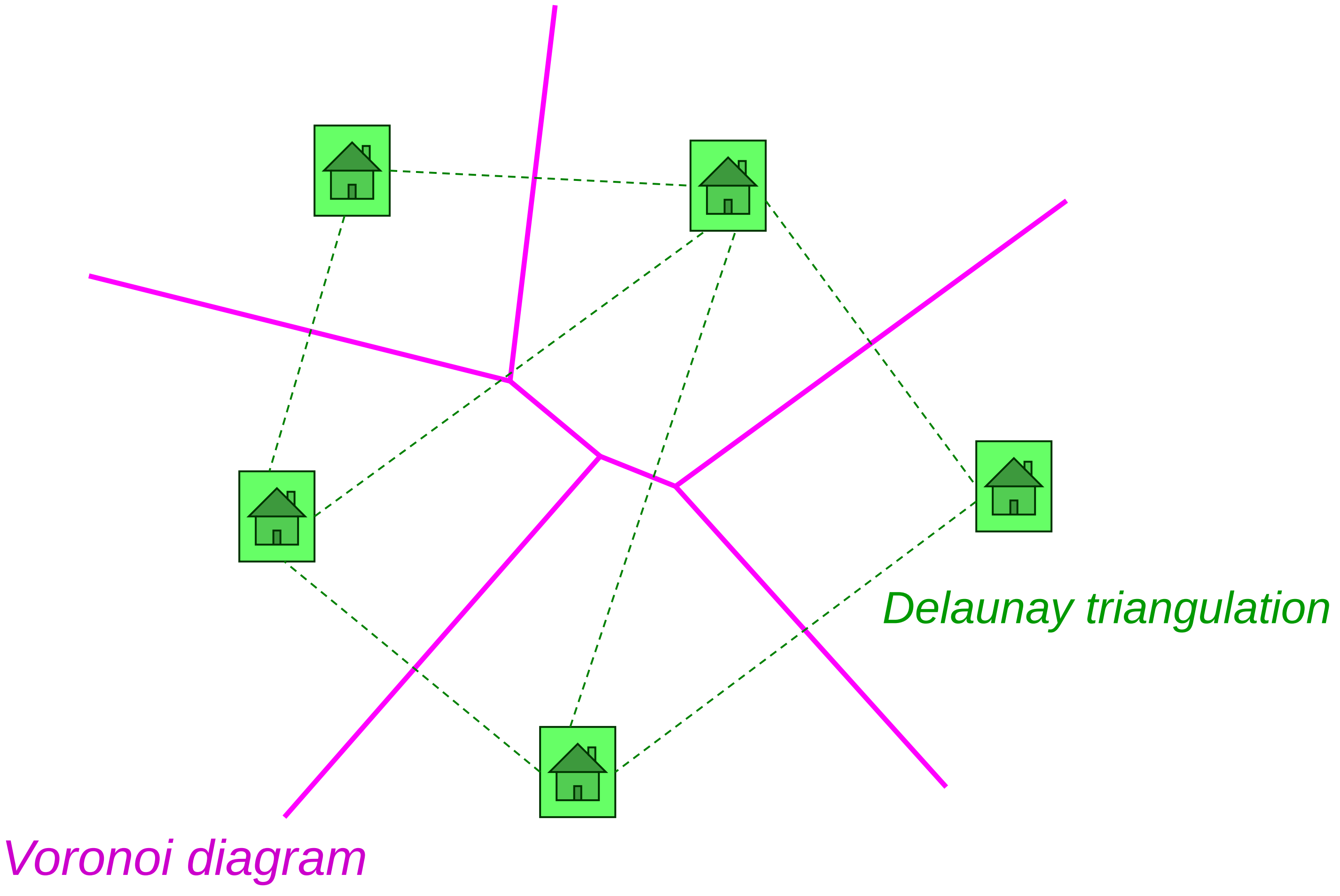
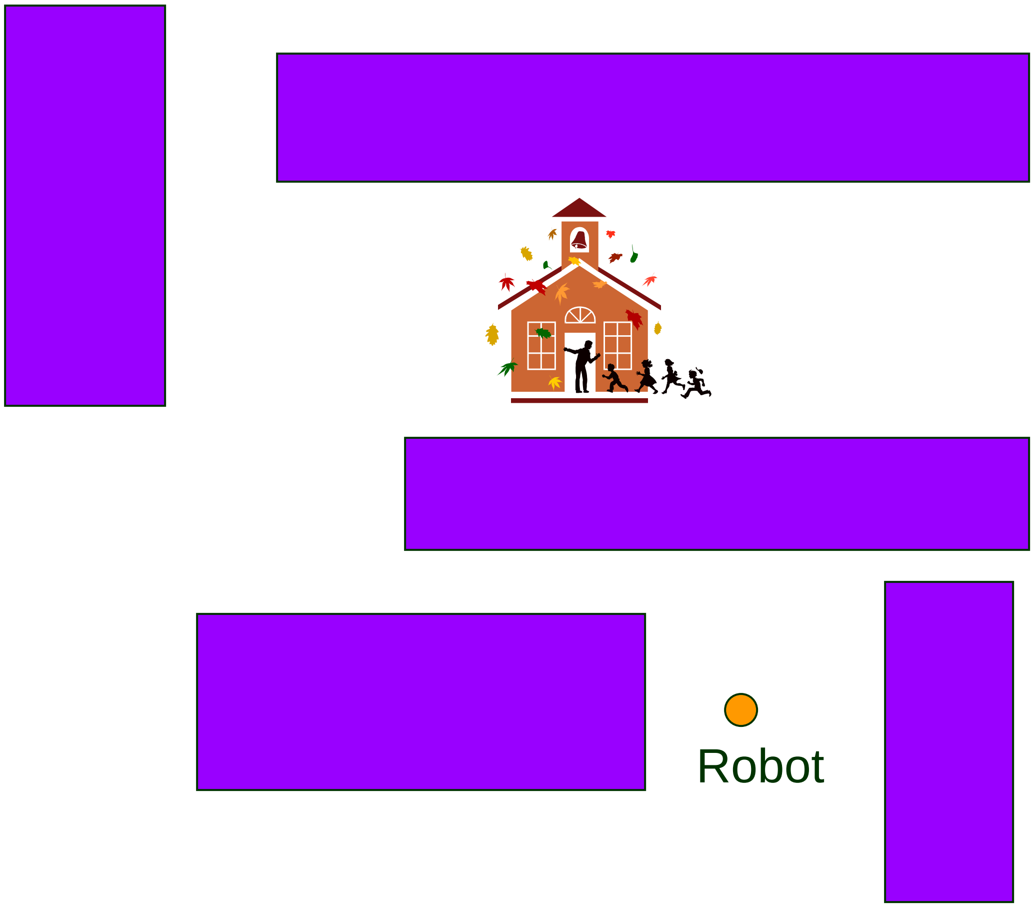


Example 2 – Path Planning

How can a robot find a *short* route to the destination that avoids all obstacles?



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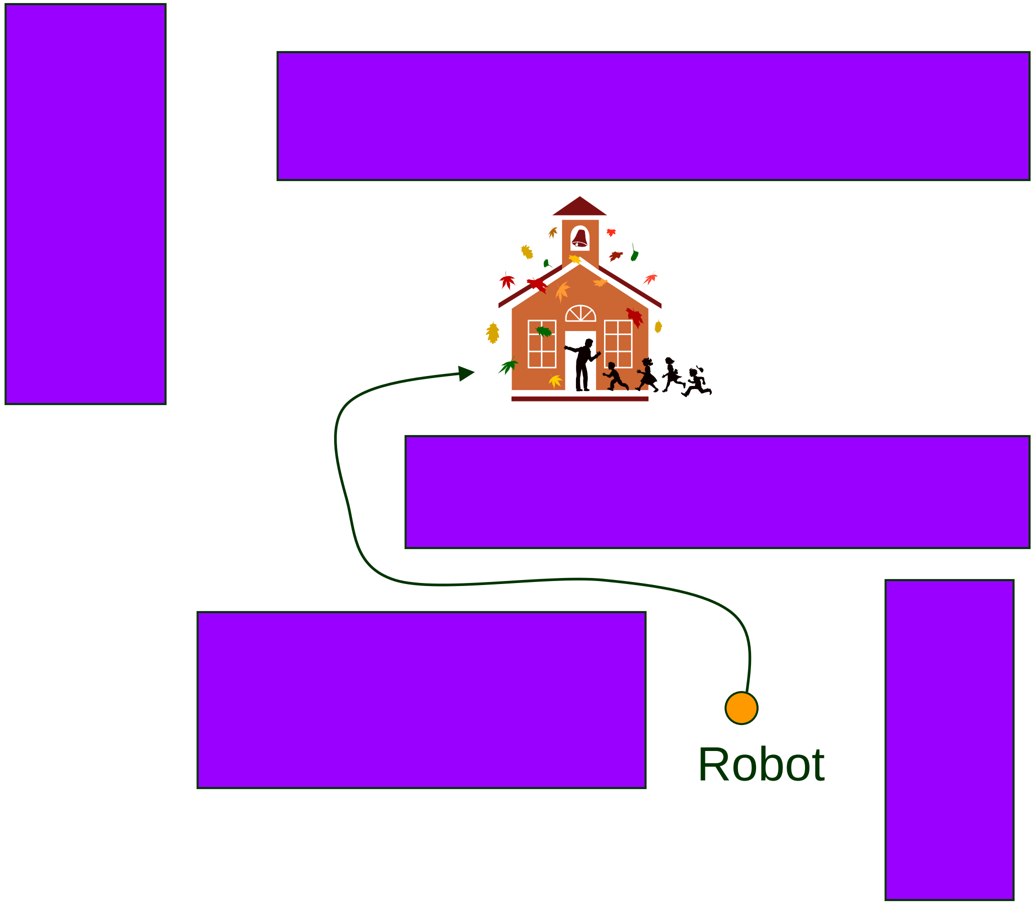


Applications in Computer Graphics

Creating scene images on a display device.

- Intersect geometric primitives (lines, polygons, polyhedra, etc.).
- Determine primitives lying in a region.
- *Hidden surface removal* – determine the visible part of a 3D scene while discard the occluded part from a viewpoint.
- Create realistic-looking scenes – taking into account lighting and computing shadows.
- Deal with moving objects and detect collisions.

Applications in Robotics

How the robot perceives, understands, and acts upon its environment:

- Motion/path planning
- Grasping
- Parts orienting
- Optimal placement

Applications in GIS

Storage of geographical data (contours of countries, height of mountains, course of rivers, population, roads, electricity lines, etc.)

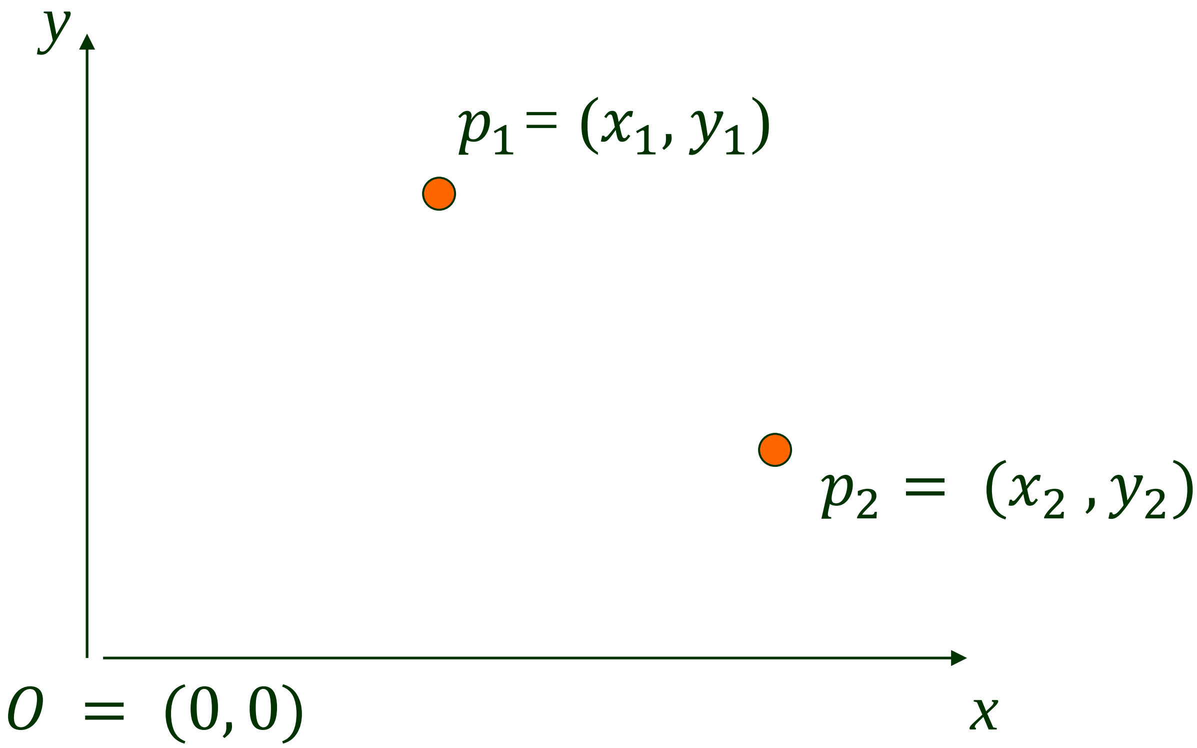
- Large amount of data – requiring efficient algorithms
- Geographic data storage (e.g., map of roads for car positioning or computer display)
- Interpolation between nearby sample data points
- Overlay of multiple maps

Applications in CAD/CAM

Design of products with a computer

- Intersection, union, and decomposition of objects
- Testing on product specifications
- Testing design for feasibility
- *Design for assembly* – modeling and simulation of assembly

Line Segments & Vectors



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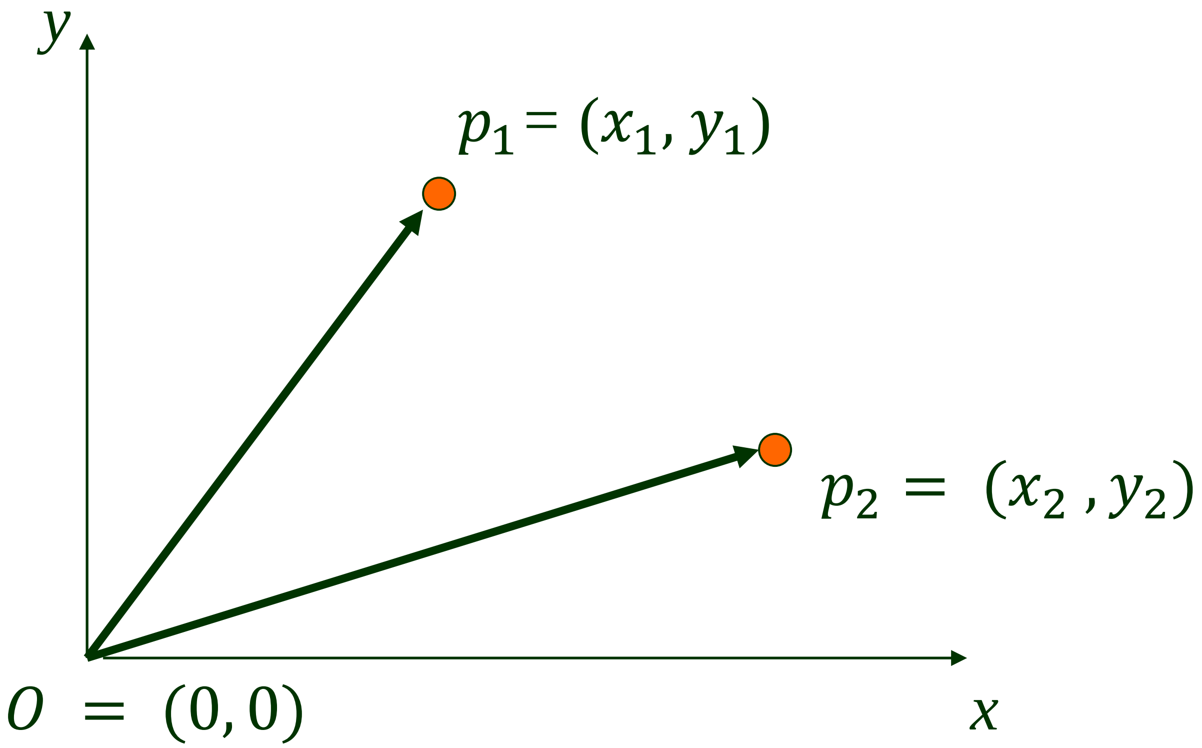

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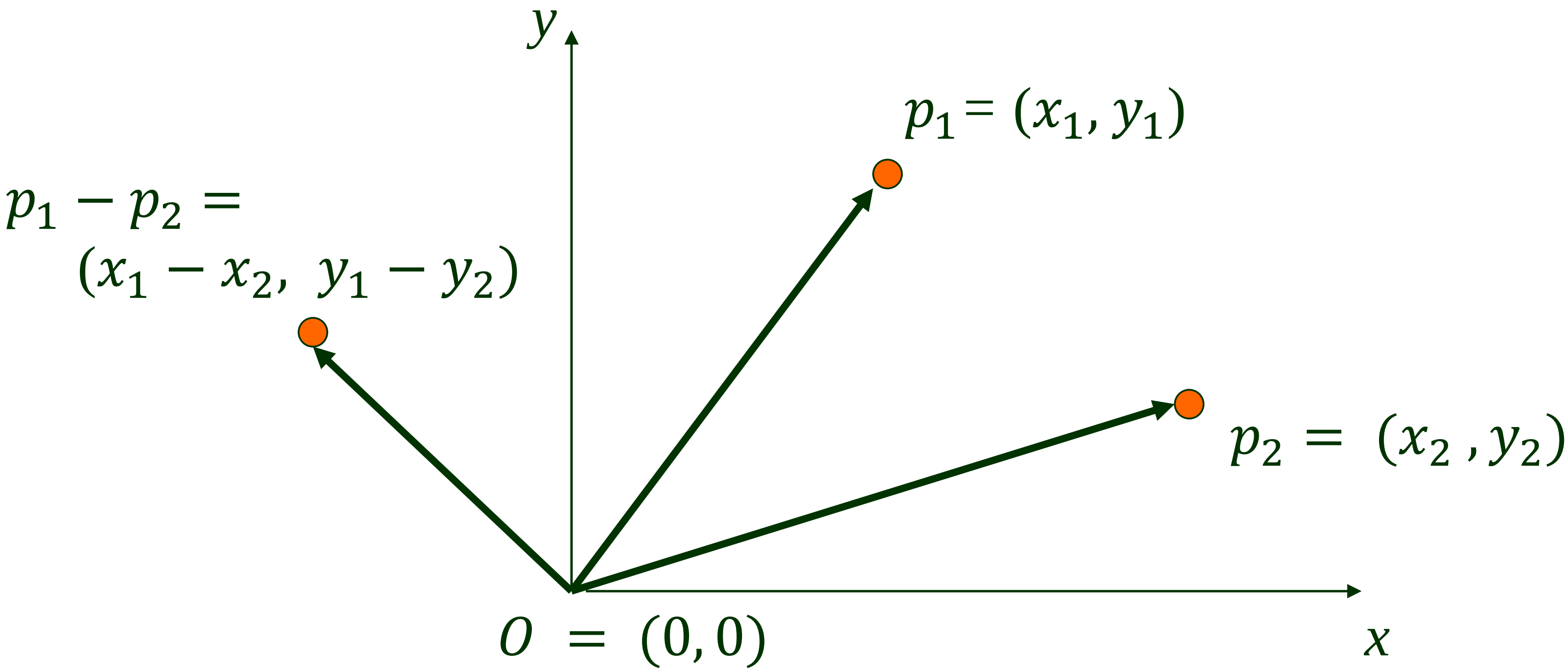

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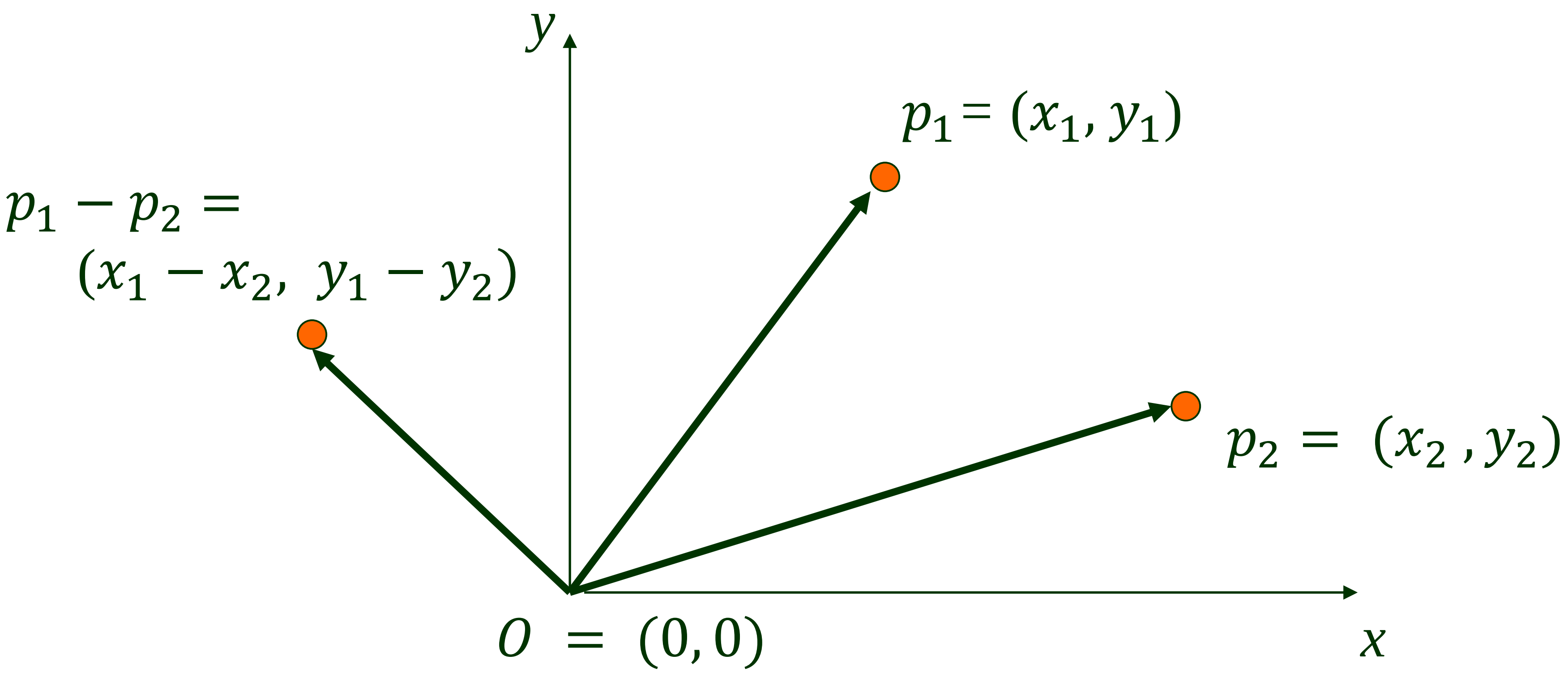

Points (vectors): $p_1, p_2, p_1 - p_2 = \overrightarrow{p_2 p_1}$

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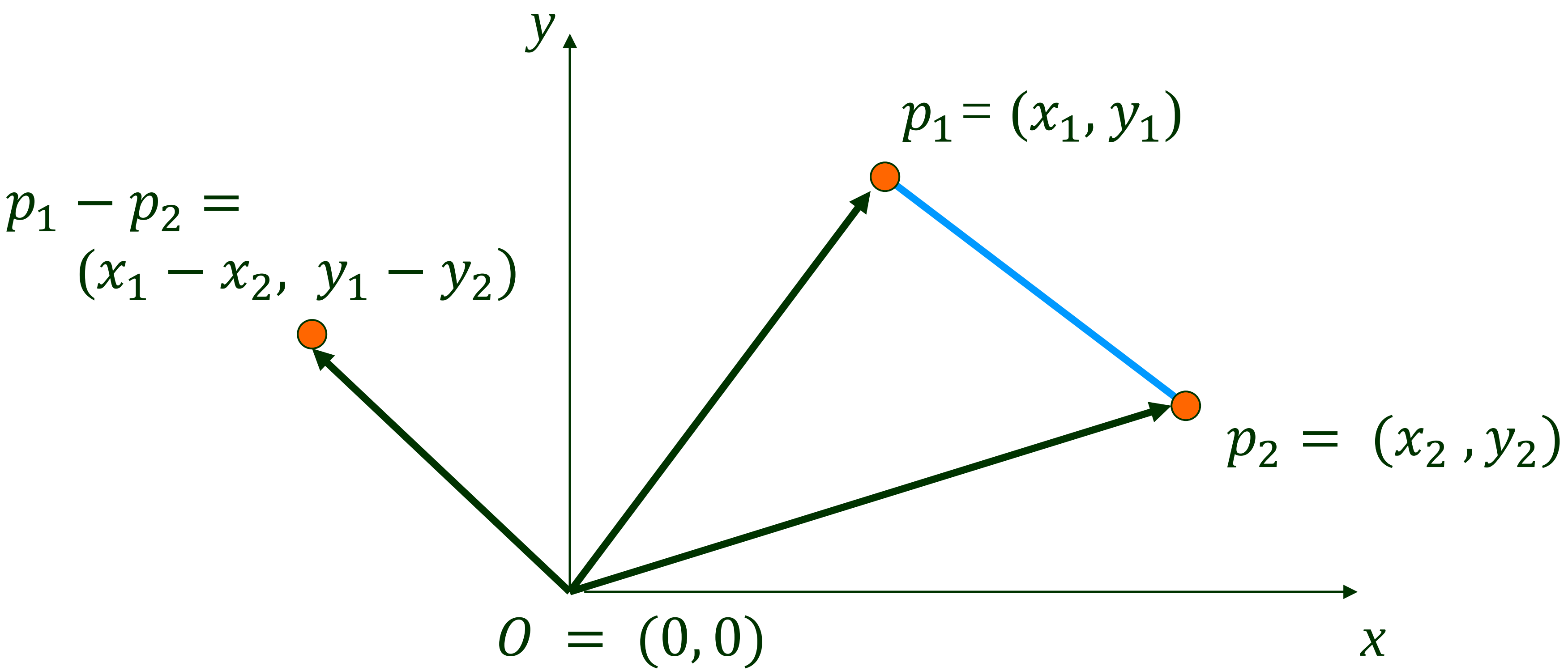

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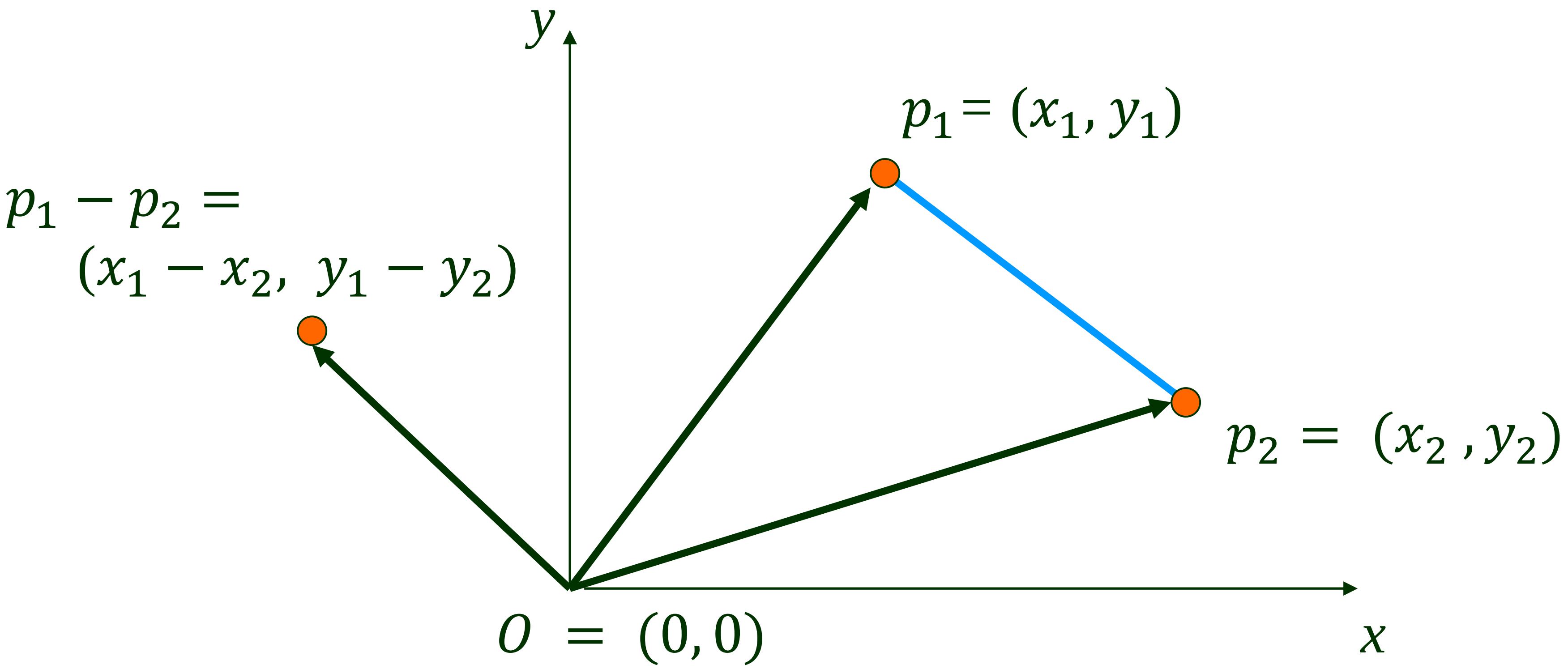
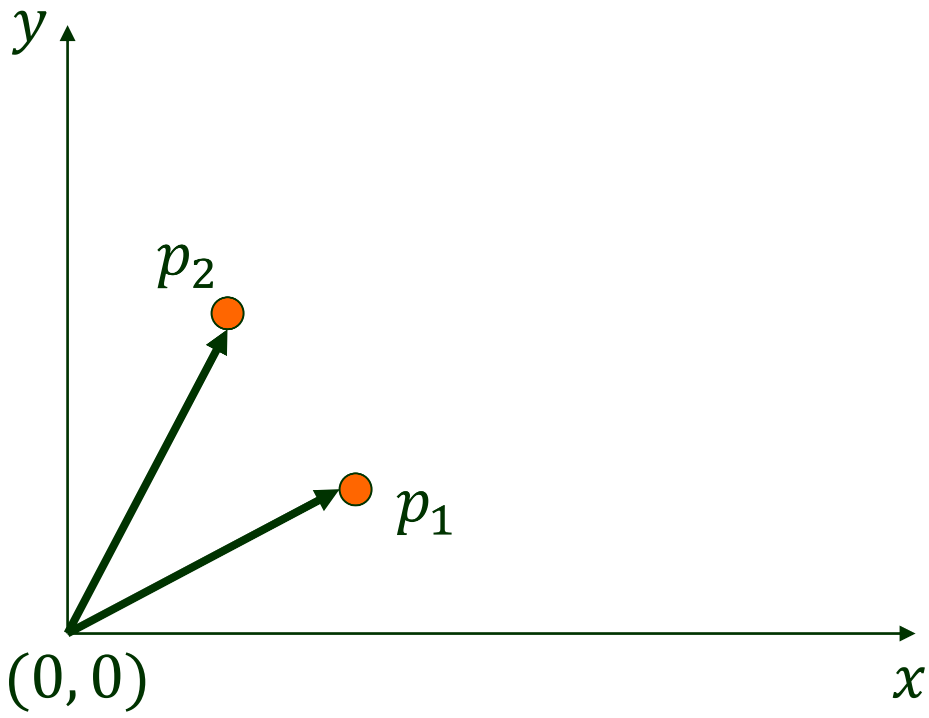


Points (vectors): $p_1, p_2, p_1 - p_2 = \overrightarrow{p_2 p_1}$

Line segment: $\overline{p_2 p_1} = \overline{p_1 p_2}$

Dot (Inner) Product



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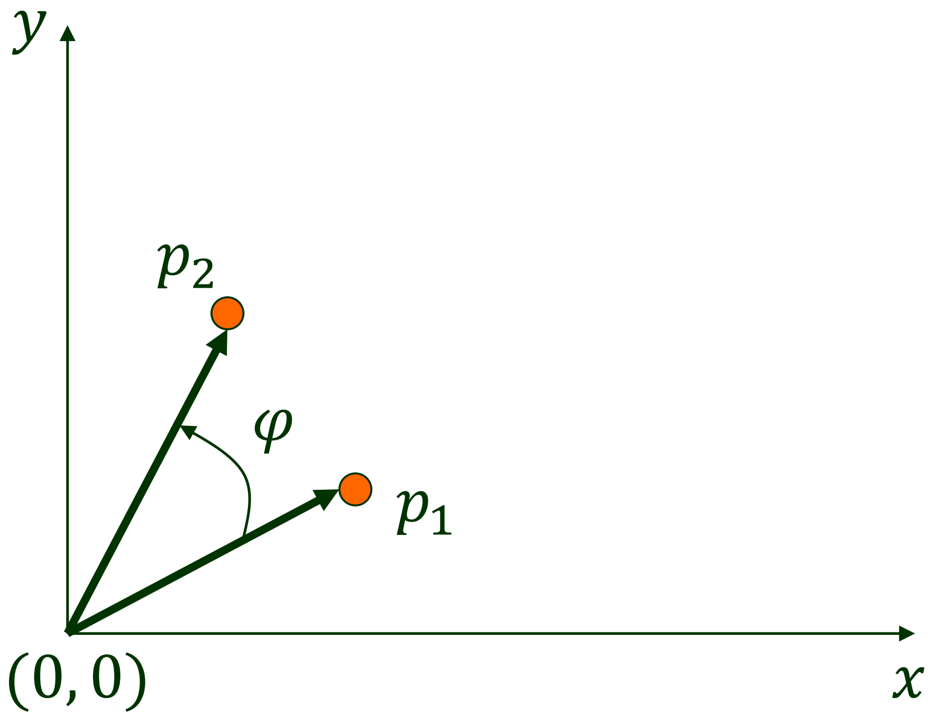

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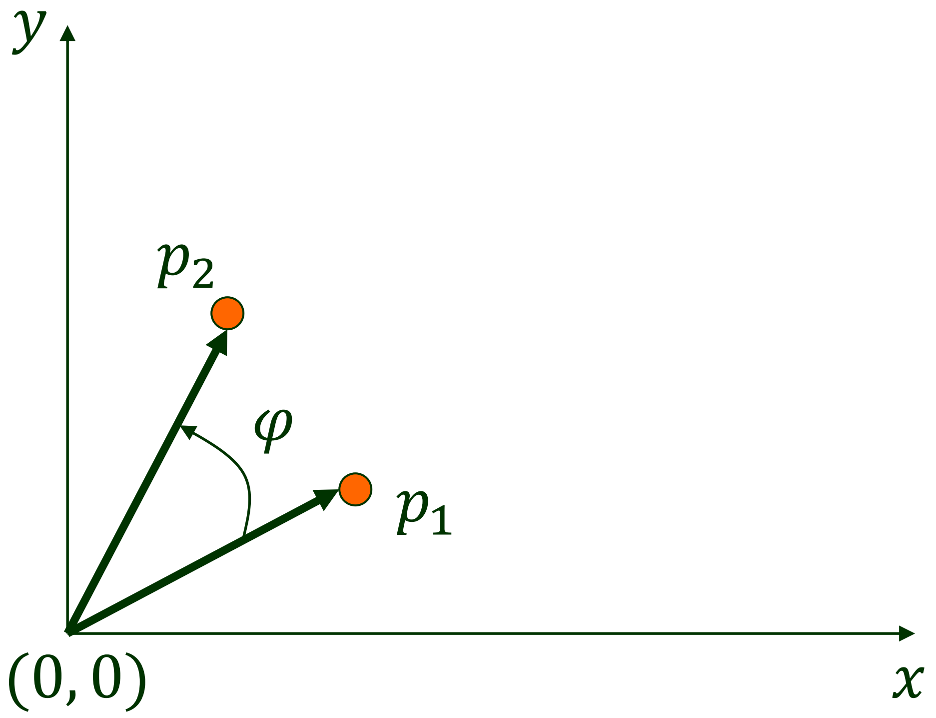
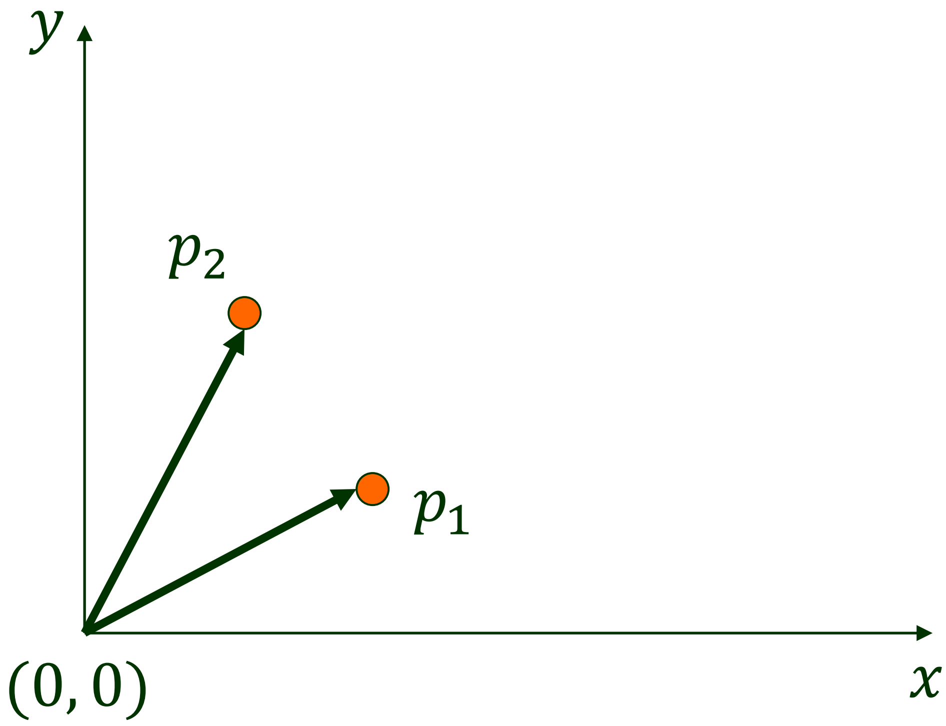


$$p_1 \cdot p_2 = x_1 x_2 + y_1 y_2 = p_2 \cdot p_1 = |p_1|\ |p_2|\ \cos\varphi$$

Cross (Vector) Product



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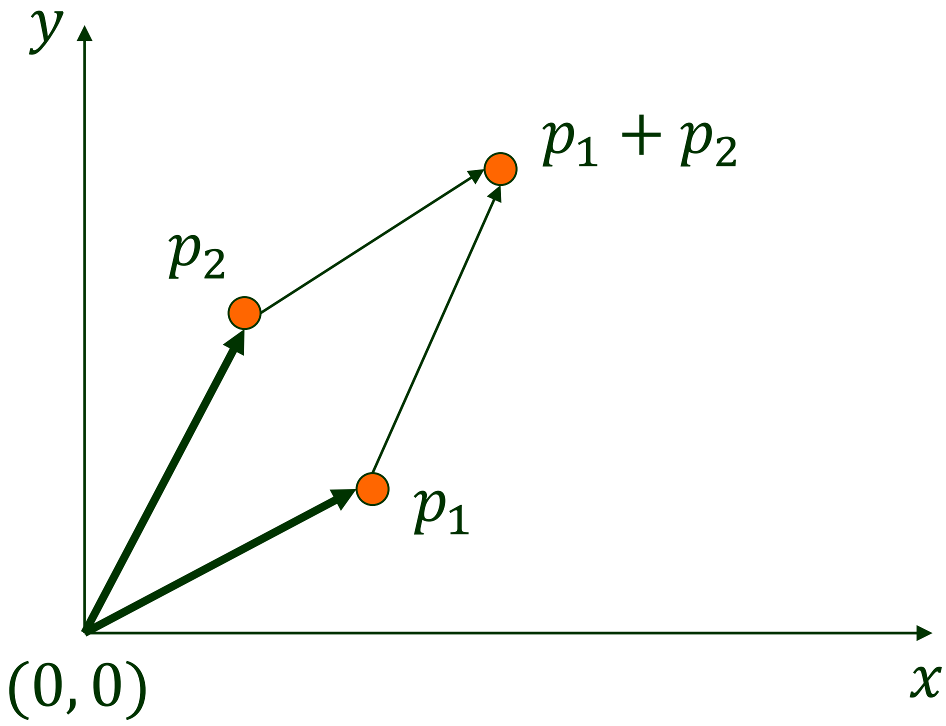

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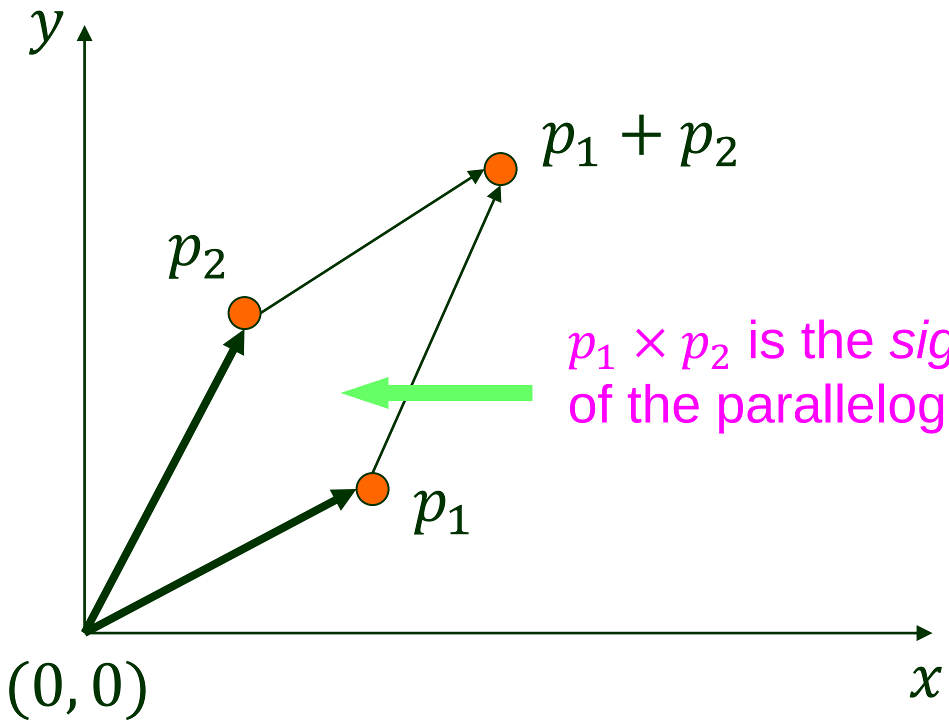

$p_1 \times p_2$ is the *signed* area of the parallelogram.

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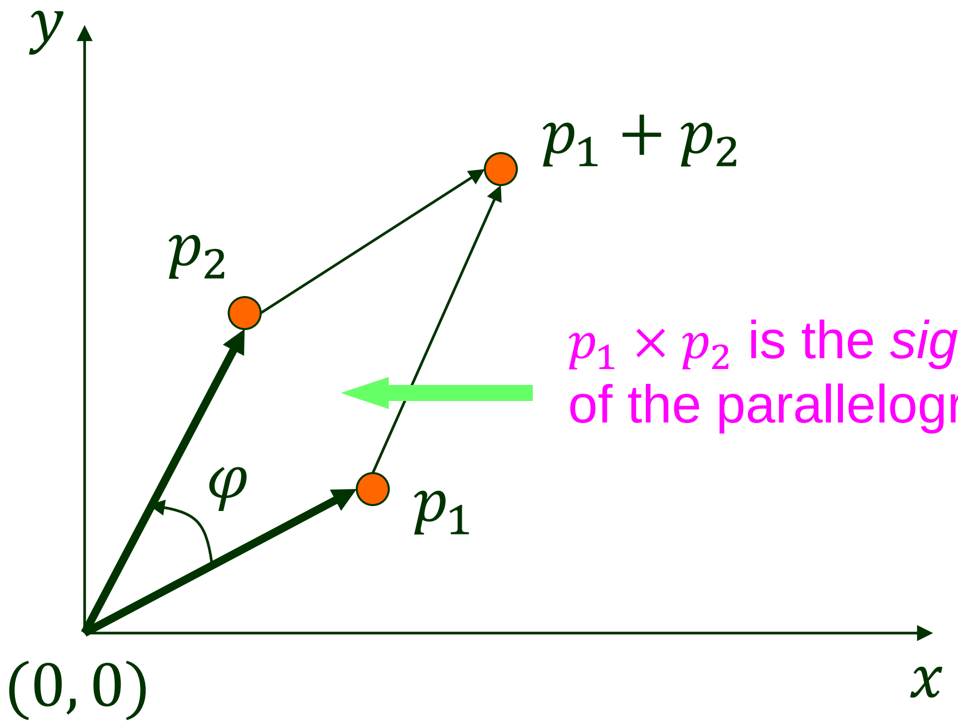

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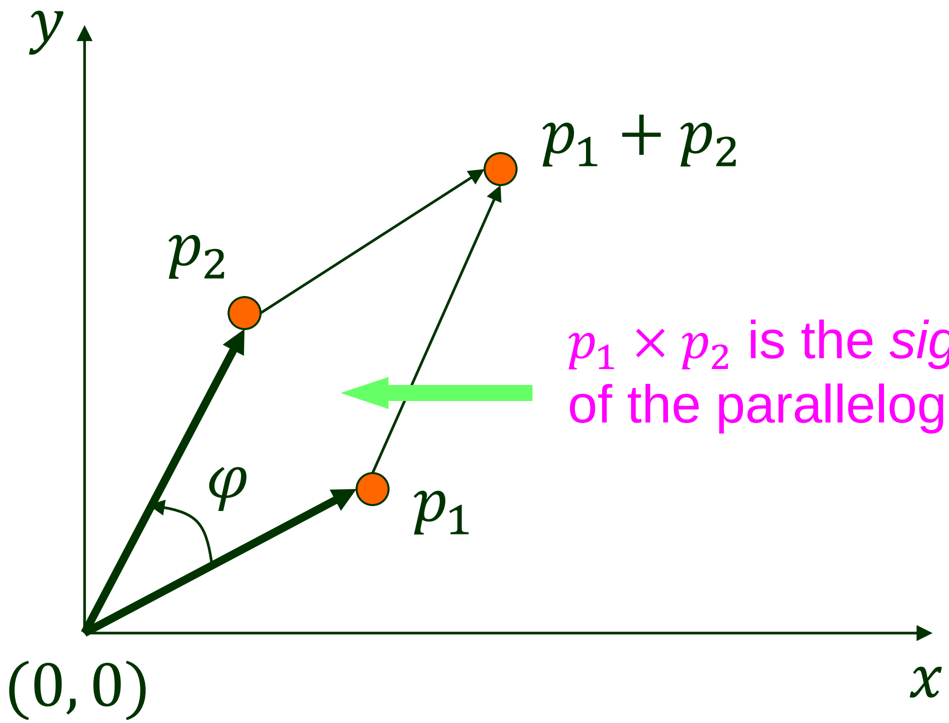

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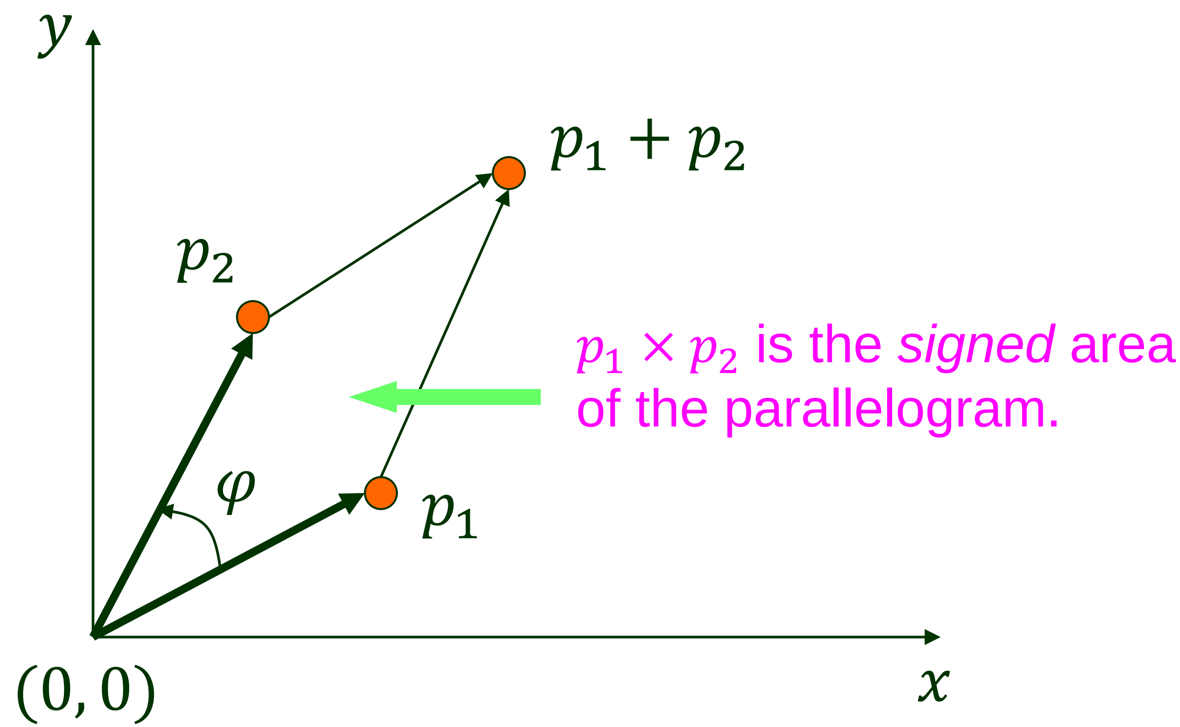

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p_1 and p_2 are *collinear* with the origin iff $p_1 \times p_2 = 0$.

Turning of Consecutive Segments

Segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$. Move from p_0 to p_1 then to p_2.

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Counterclockwise Clockwise Turn of 0 or π

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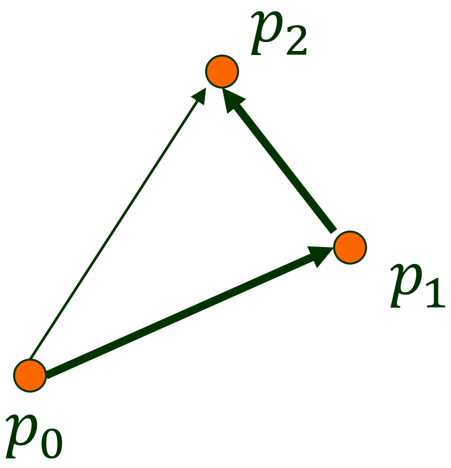

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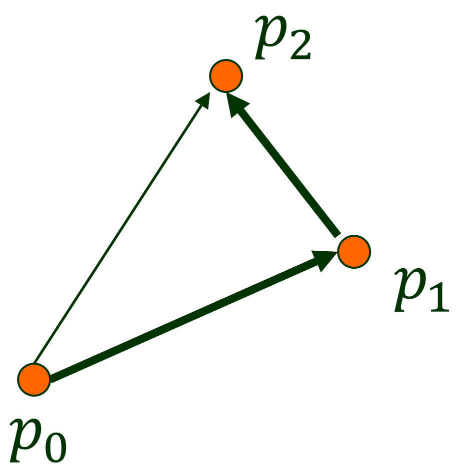

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$\overrightarrow{p_0 p_1} \times \overrightarrow{p_1 p_2} > 0$

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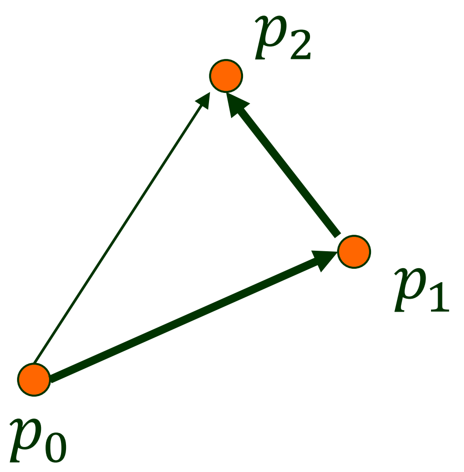

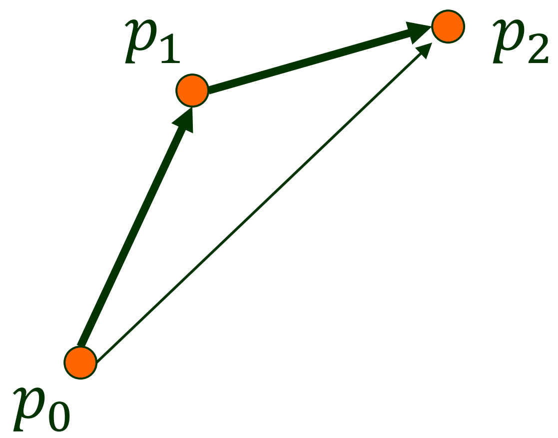

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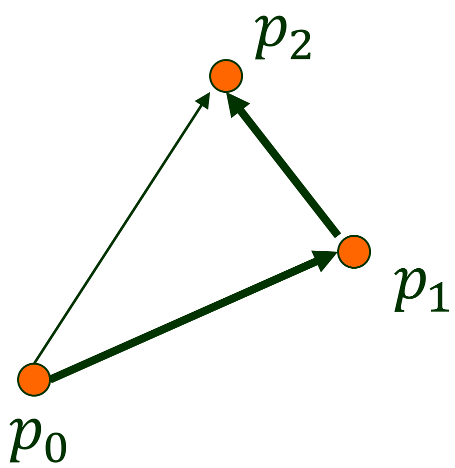

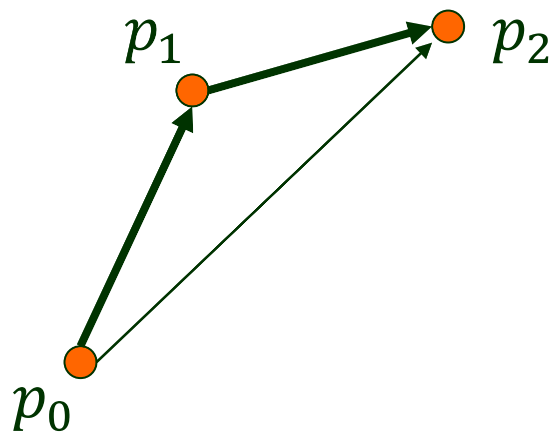

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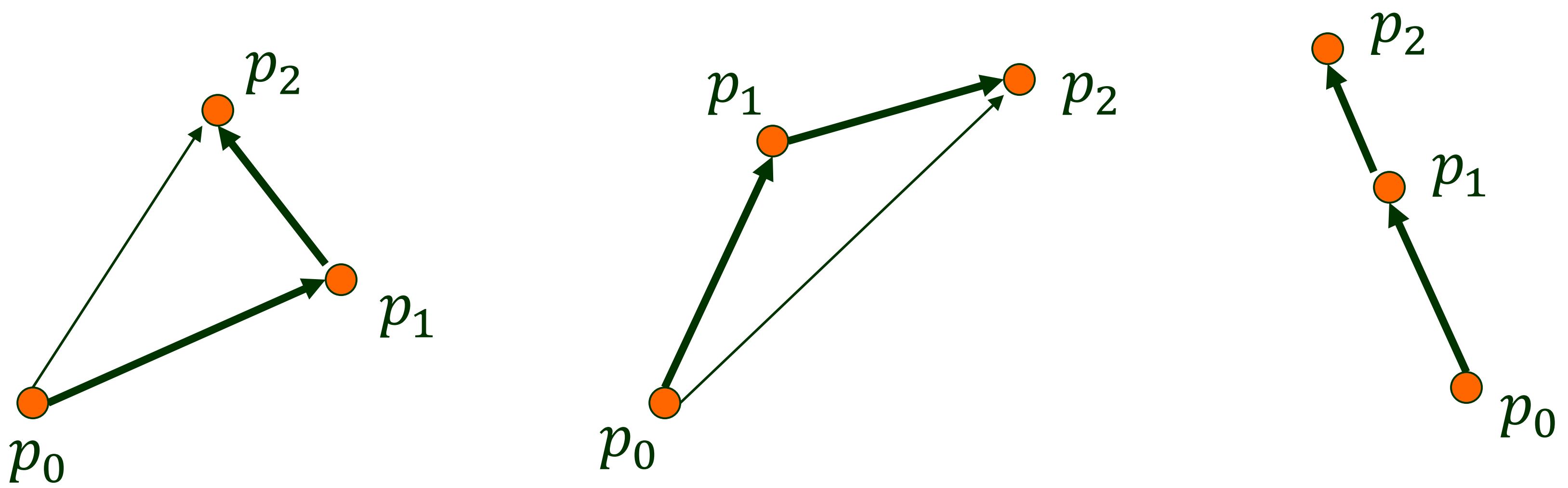

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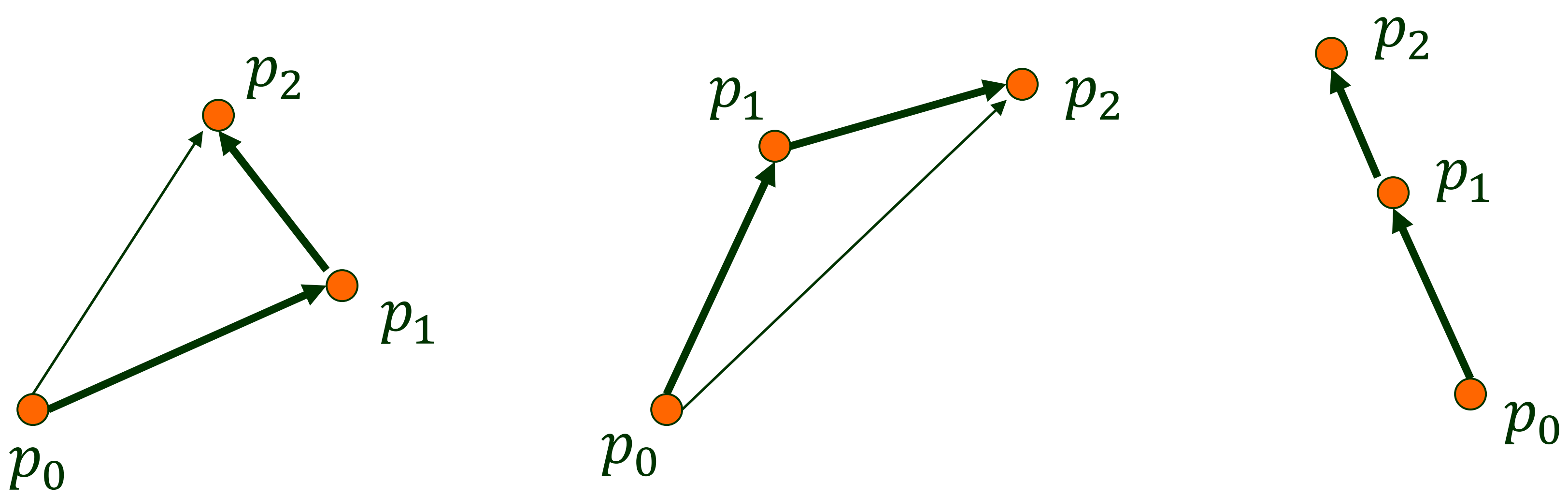

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Case of Collinearity

$\overrightarrow{p_0 p_1} \times \overrightarrow{p_1 p_2} = 0 \Longrightarrow p_1, p_2, p_3$ are collinear.

No change of direction

Direction reversal

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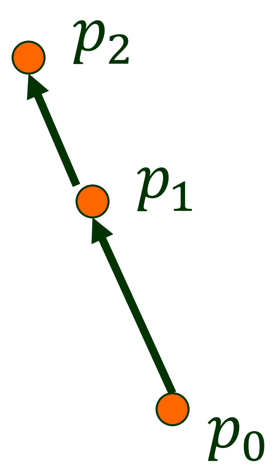

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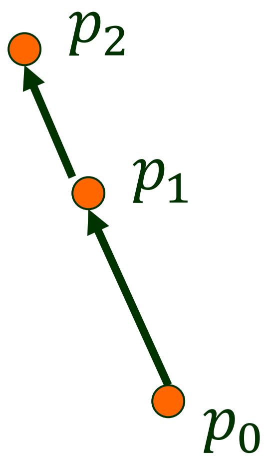

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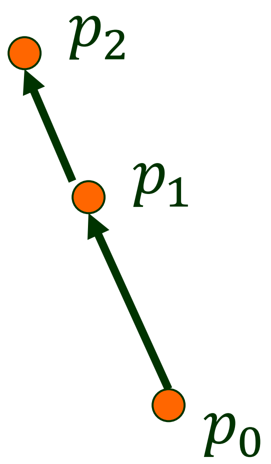

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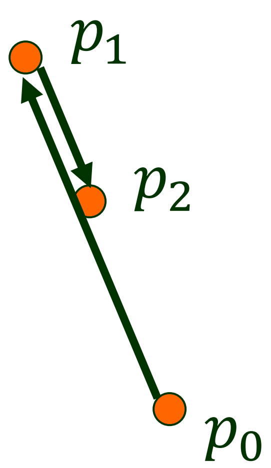

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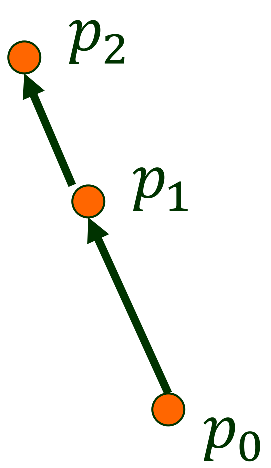

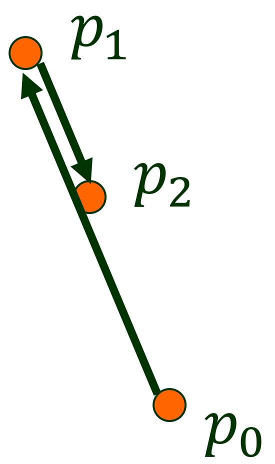

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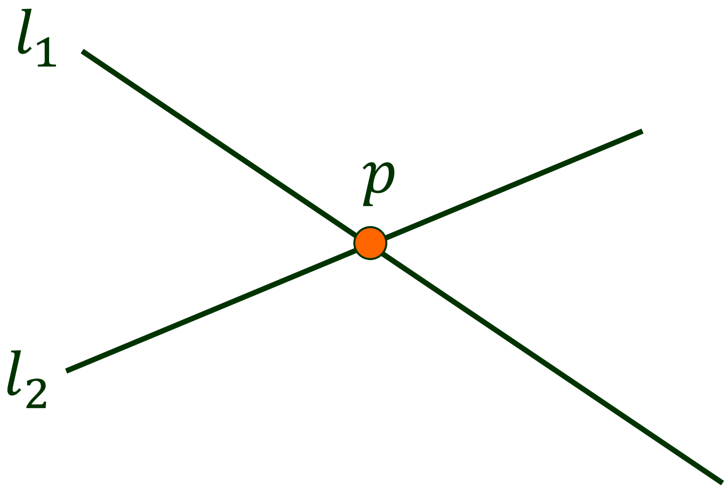
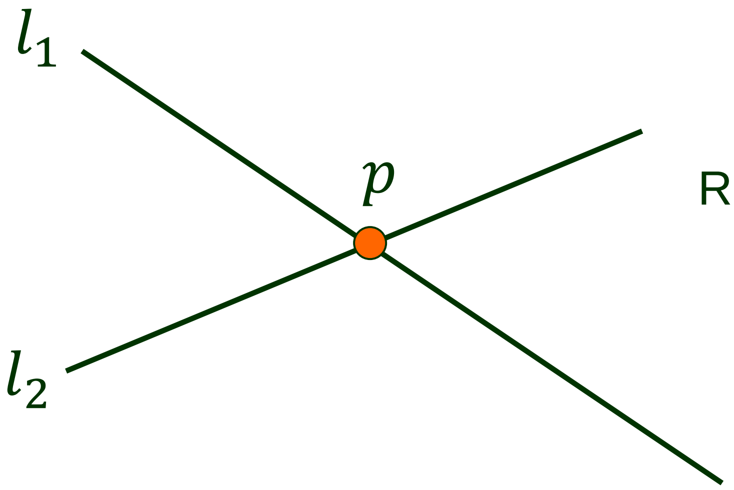
Intersection of Two Lines*



$$l_1: a_1 x + b_1 y + c_1 = 0$$
$$l_2: a_2 x + b_2 y + c_2 = 0$$

* Optional material

Intersection of Two Lines*



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Rewrite the above equations as dot products:

$$(a_1, b_1, c_1) \cdot (x, y, 1) = 0$$

$$(a_2, b_2, c_2) \cdot (x, y, 1) = 0$$

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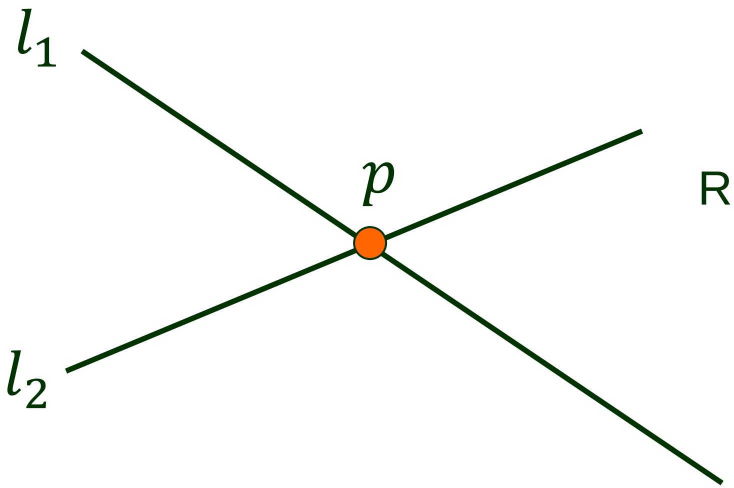

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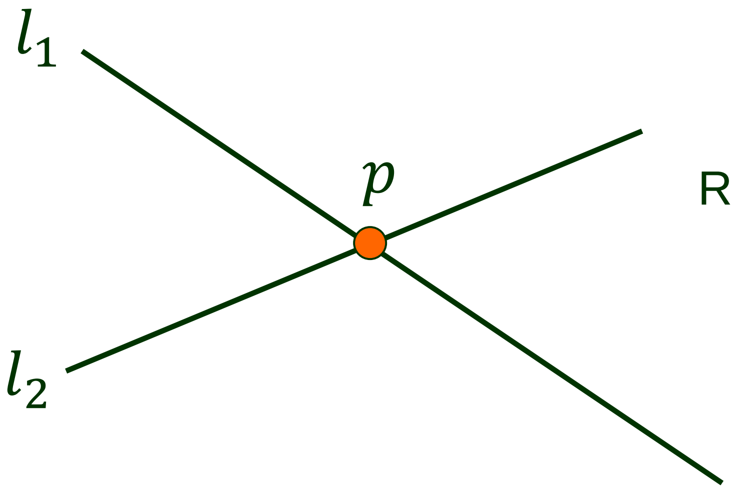

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Hence $(x, y, 1)$ is perpendicular to both (a_1, b_1, c_1) and (a_2, b_2, c_2).

$(x, y, 1)$ must be parallel to their cross product!

* Optional material

Homogeneous Coordinates*

Represent every point (x, y) as $\underline{(x, y, 1)}$.

homogeneous coordinates of (x, y)

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$$(a_1, b_1, c_1) \times (a_2, b_2, c_2) = (b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, a_1 b_2 - a_2 b_1)$$

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homogeneous coordinates of intersection

Normalize the z-coordinate of the cross product to 1.

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}, 1\right)$$

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Hence the intersection point:

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

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Find the intersect point of the lines:

$$x - 7y + 8 = 0 \qquad 3x - 4y + 1 = 0$$

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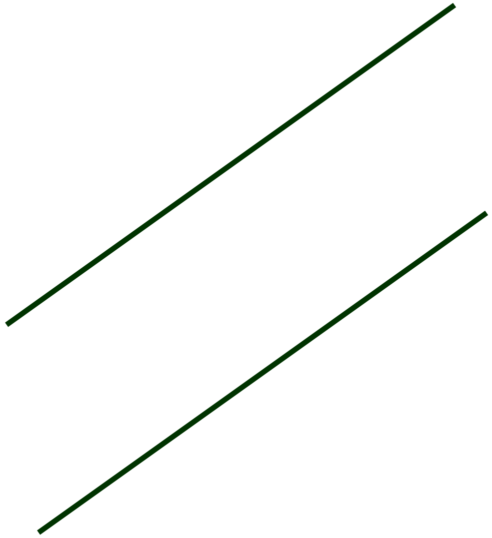
Take the cross product of the two coefficient vectors:

$$(1,-7,8) \times (3,-4,1) = (25,23,17)$$

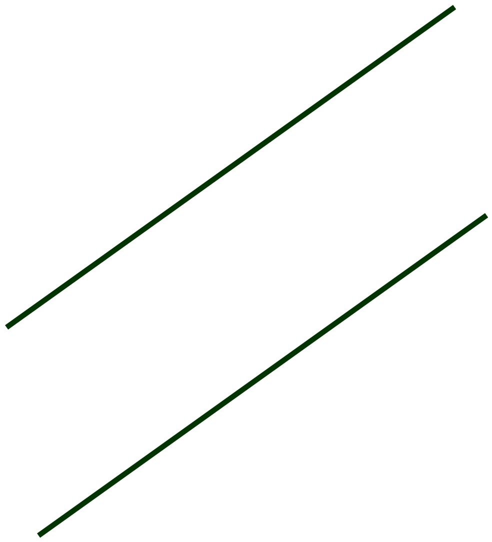
Intersection point: $\left(\frac{25}{17}, \frac{23}{17}\right)$

Two Parallel Lines*



$$l_1: a_1 x + b_1 y + c_1 = 0$$
$$l_2: a_2 x + b_2 y + c_2 = 0$$

Two Parallel Lines*



$$l_1: a_1 x + b_1 y + c_1 = 0$$

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$$\boxed{a_1 b_2 - b_1 a_2 = 0}$$

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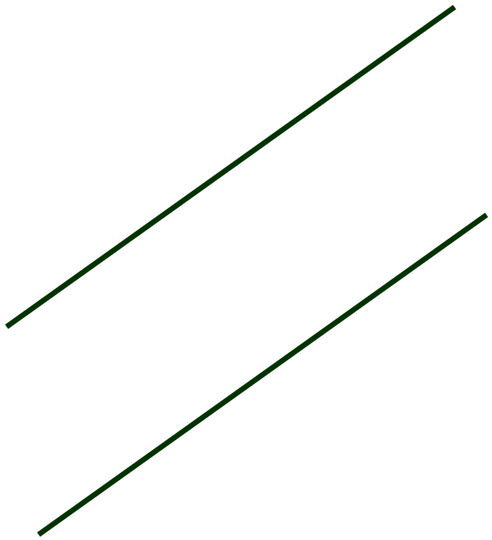


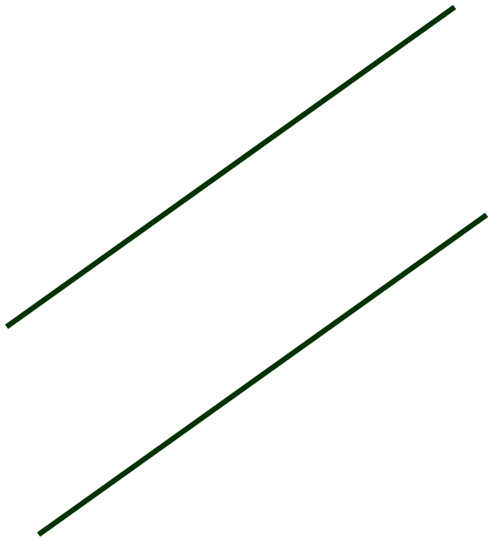
$$l_1: a_1 x + b_1 y + c_1 = 0$$

$$l_2: a_2 x + b_2 y + c_2 = 0$$

$$\boxed{a_1 b_2 - b_1 a_2 = 0}$$

$$(a_1, b_1, c_1) \times (a_2, b_2, c_2) = (b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, a_1 b_2 - a_2 b_1)$$

Two Parallel Lines*



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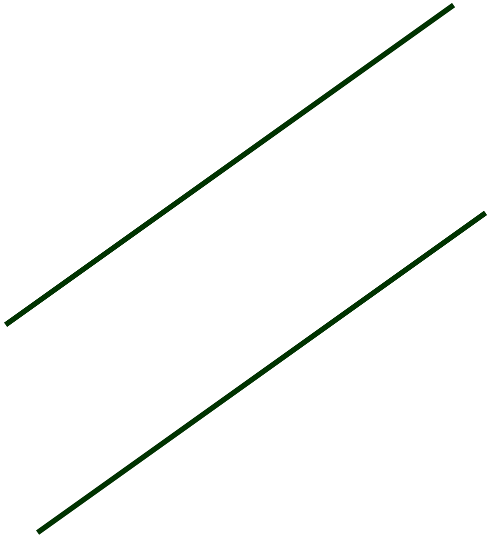
$$= (b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, \mathbf{0})$$

Two Parallel Lines*



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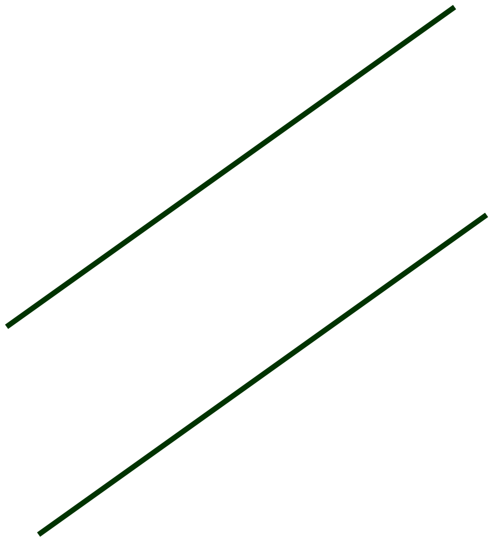
point at infinity

Two Parallel Lines*



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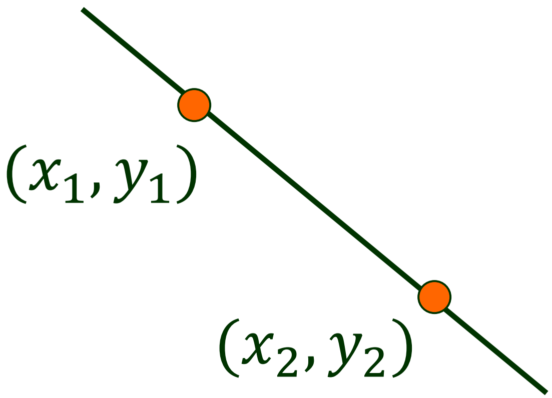
↑ point at infinity

Two parallel lines “intersect” at infinity.

Line through Two Points*

$ax + by + c = 0?$



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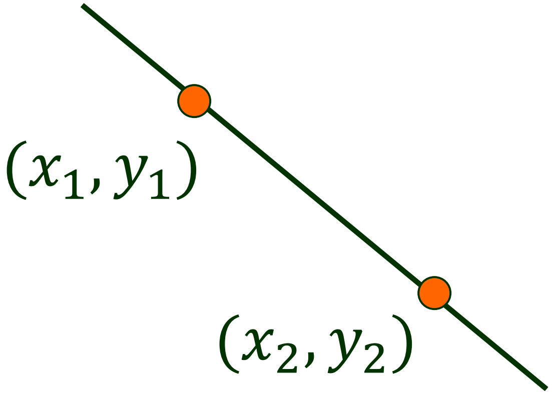

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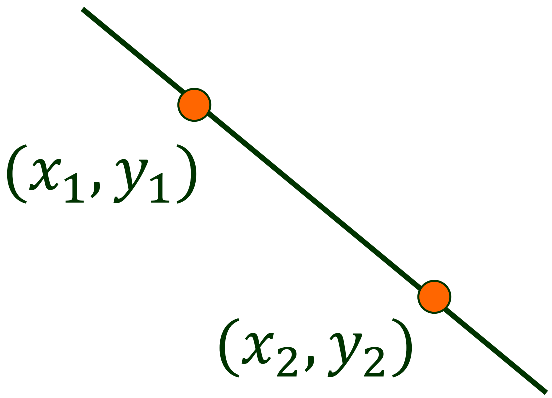

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$

$$\Downarrow$$

$$(a, b, c) \cdot (x_1, y_1, 1) = 0$$
$$(a, b, c) \cdot (x_2, y_2, 1) = 0$$

Line through Two Points*

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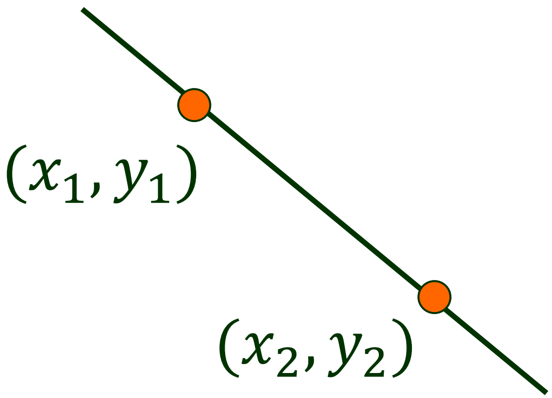

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$$\boxed{(a, b, c) = (x_1, y_1, 1) \times (x_2, y_2, 1)}$$

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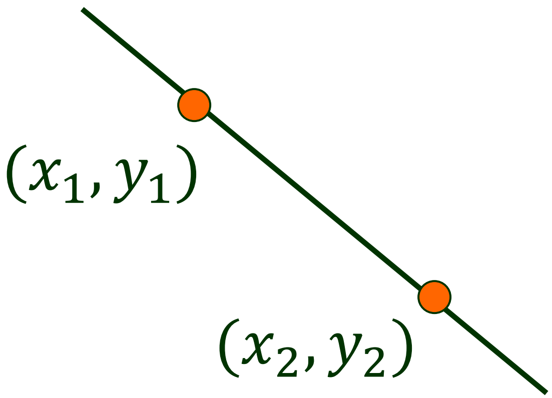

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Scaling yields the same line!

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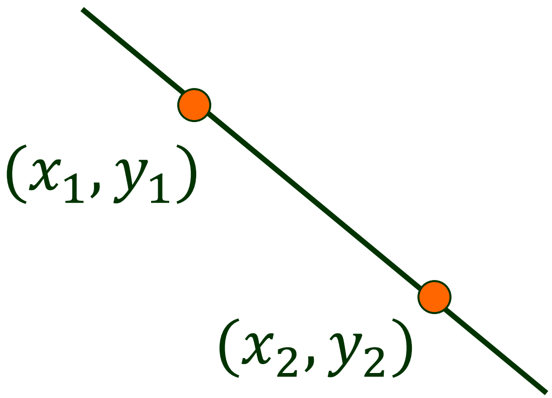

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Ex. The line through (3, 1) and $(-4, 5)$ has coefficients (i.e., homogeneous coordinates)

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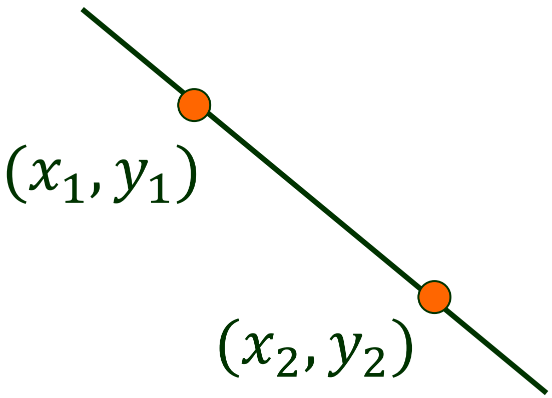

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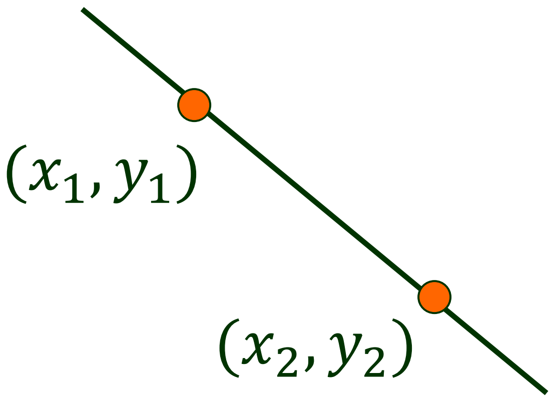
Scaling yields the same line!

Ex. The line through (3, 1) and $(-4, 5)$ has coefficients (i.e., homogeneous coordinates)

$$(3,1,1) \times (-4,5,1) = (-4,-7,19)$$

Line through Two Points*

$ax + by + c = 0?$



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Scaling yields the same line!

Ex. The line through (3, 1) and $(-4, 5)$ has coefficients (i.e., homogeneous coordinates)

$(3,1,1) \times (-4,5,1) = (-4,-7,19)$ and equation $-4x - 7y + 19 = 0$

Advantages of Homogeneous Coordinates*

- Scaling independent:

$$(25,23,17),\quad (50,46,34),\quad \left(\frac{25}{17},\frac{23}{17},1\right)$$

all represent the same point $\left(\frac{25}{17},\frac{23}{17}\right)$

For more, see https://faculty.sites.iastate.edu/jia/files/inline-files/homogeneous-coords.pdf.

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- No need to solve equations – just take a cross product

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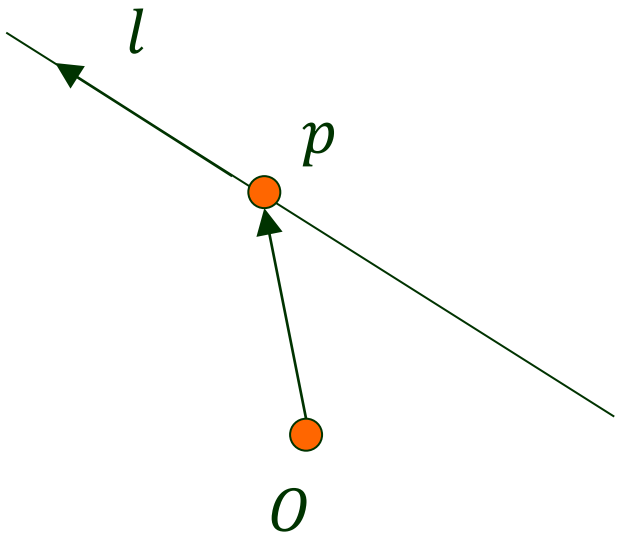
Lines in the Space*

Use Plucker coordinates: $(l, p \times l)$



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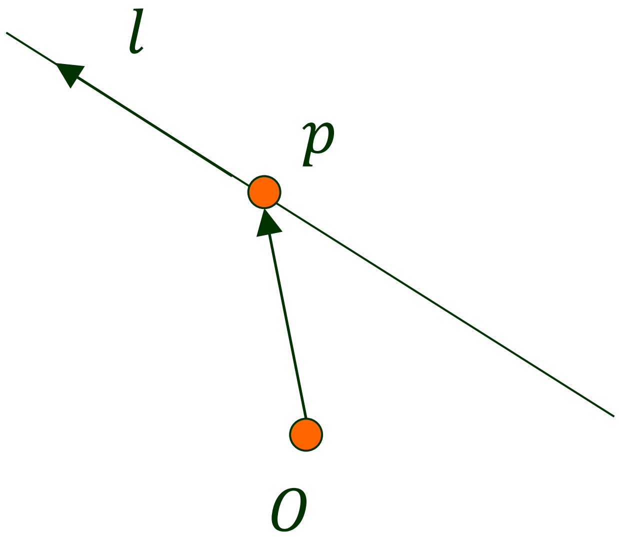

- Compute the distance between two lines.

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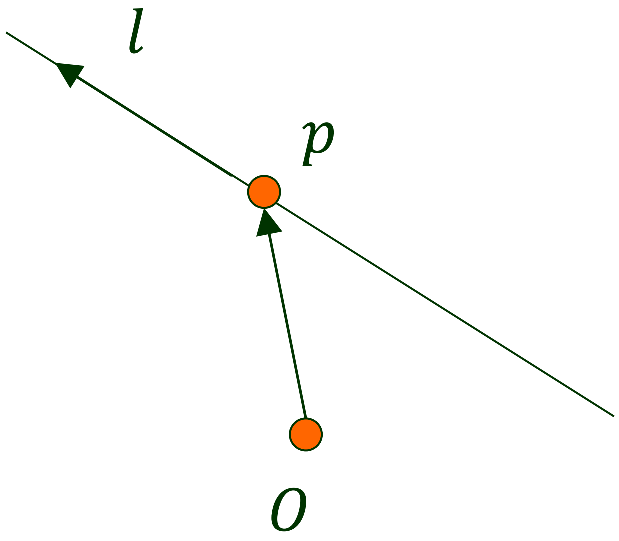

- Compute the distance between two lines.
- Find their common perpendicular.

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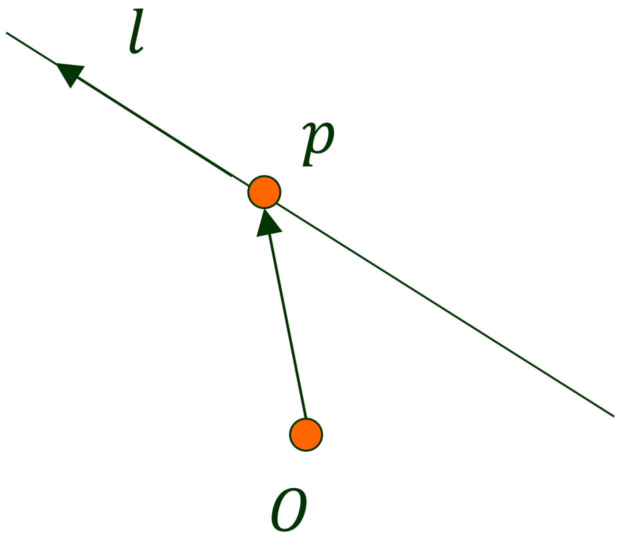

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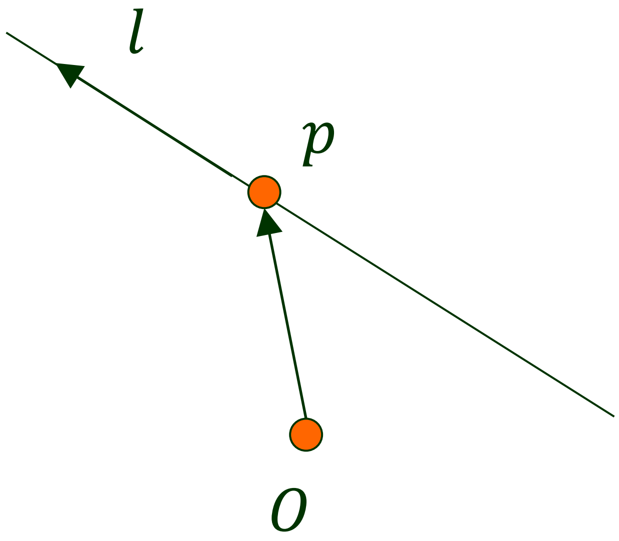

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(When the lines intersect, the two intersections coincide.)

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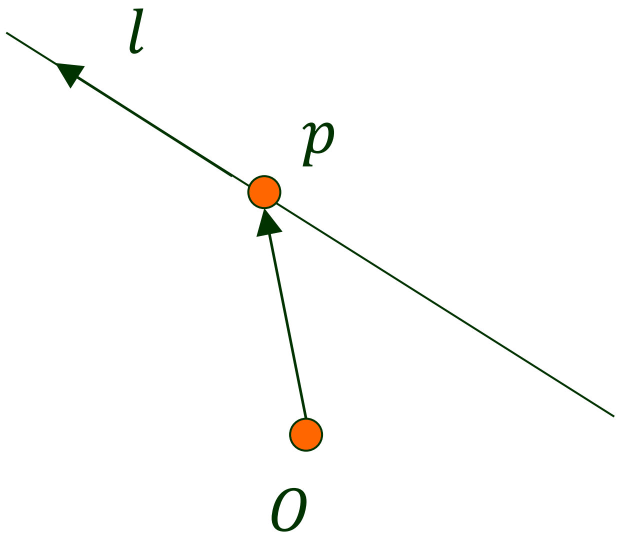

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