Farthest-Point Voronoi Diagrams

Outline:

I. Farthest site

II. Voronoi cell

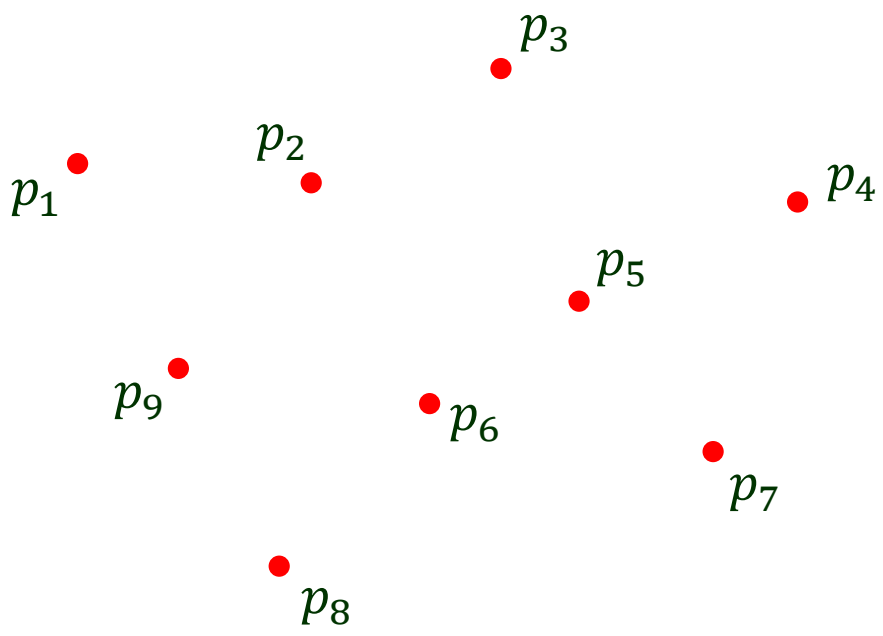
III. Structure of FPVD

IV. Construction

V. Smallest-Width Annulus

I. Farthest Point

$$P = \{\underbrace{p_1, p_2, \dots, p_n}_{\text{Sites}}\}$$



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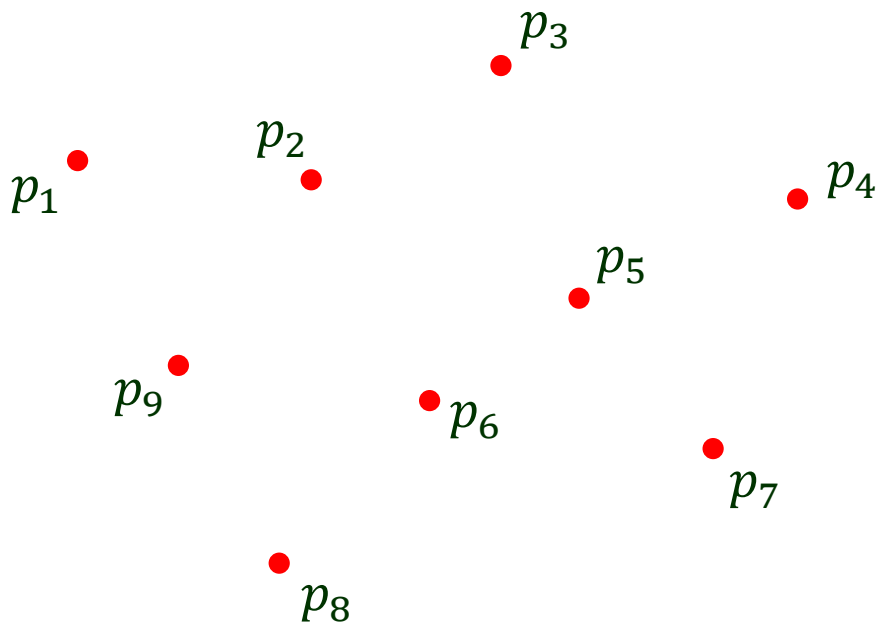

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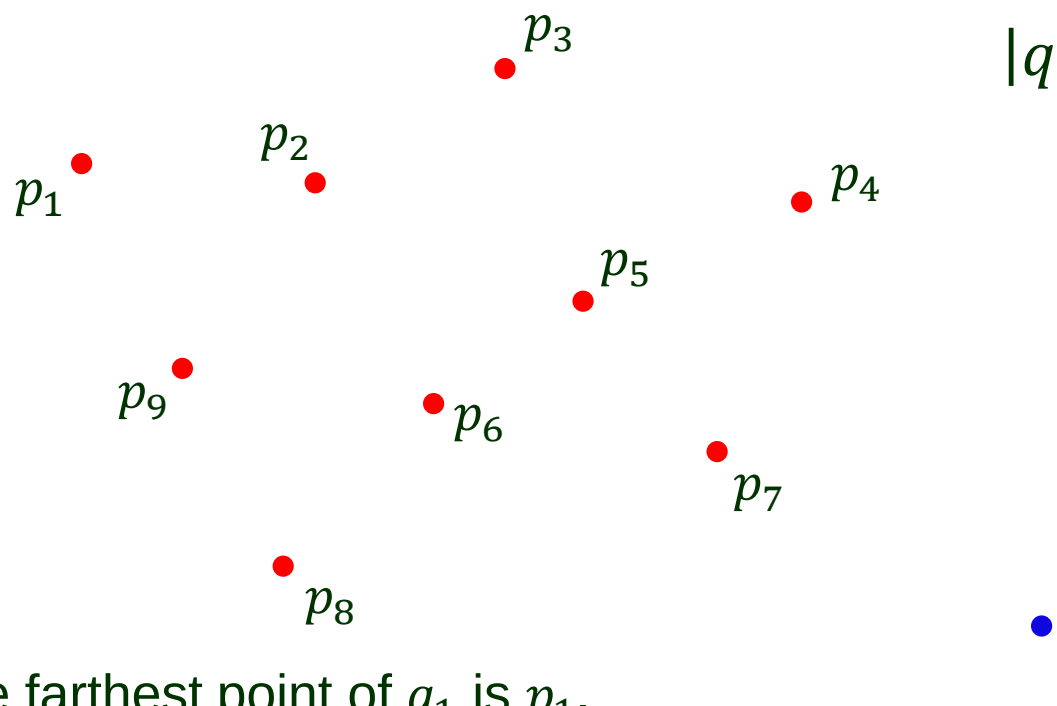

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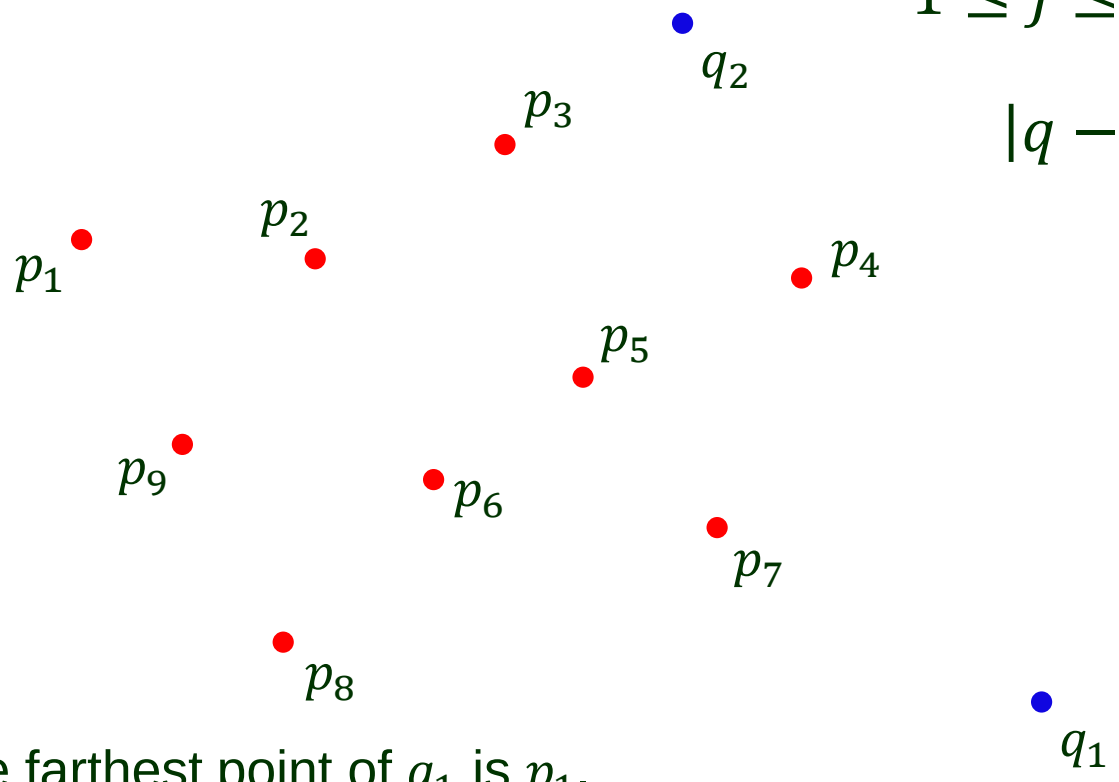
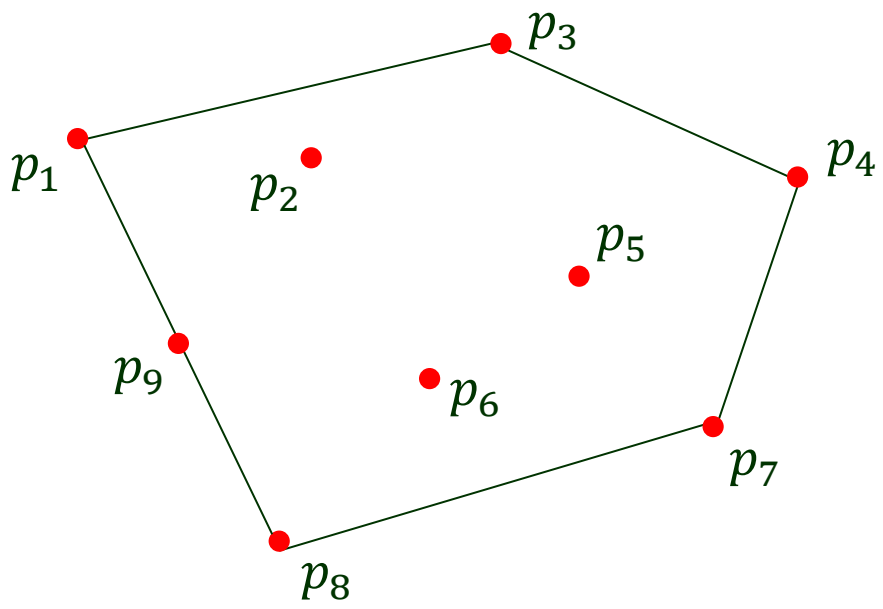


The farthest point of q_1 is p_1.

The farthest point of q_2 is p_8.

Not Every Site Can be the Farthest

Claim A point $p_i \in P$ is the farthest site of some point q in the plane if and only if p_i is a vertex of the convex hull $\mathrm{CH}(P)$ of P.



Proof of Necessity

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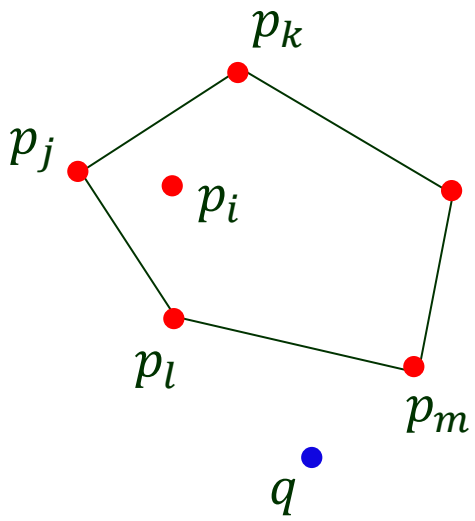

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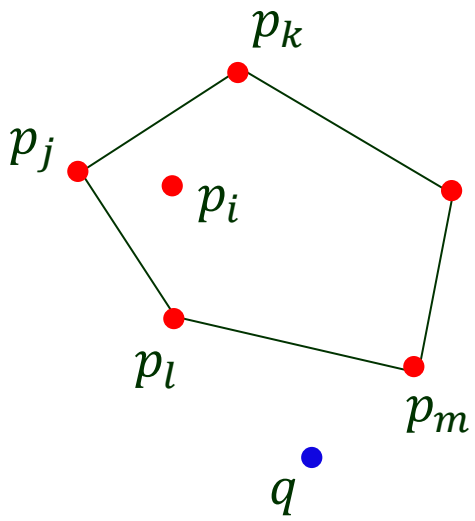

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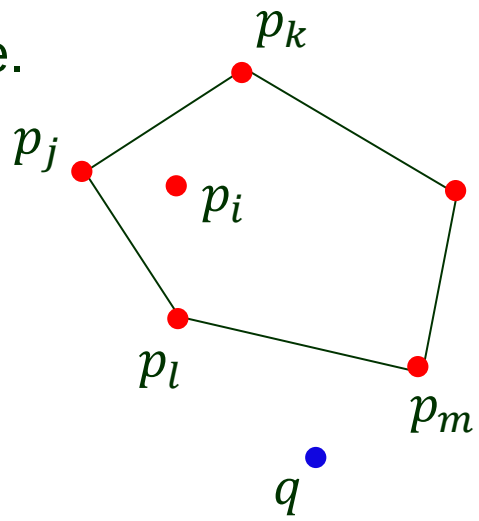

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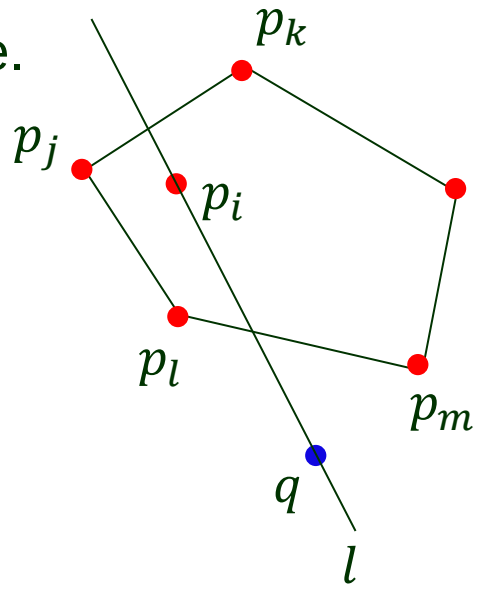

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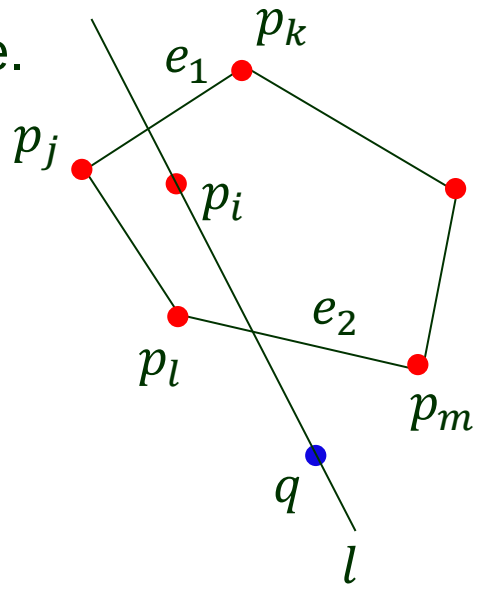

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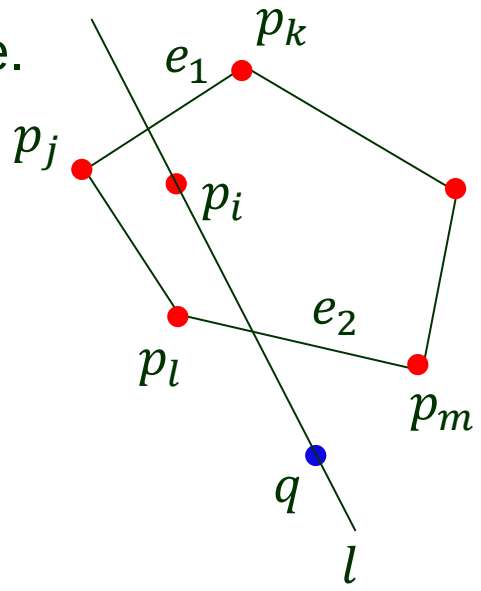

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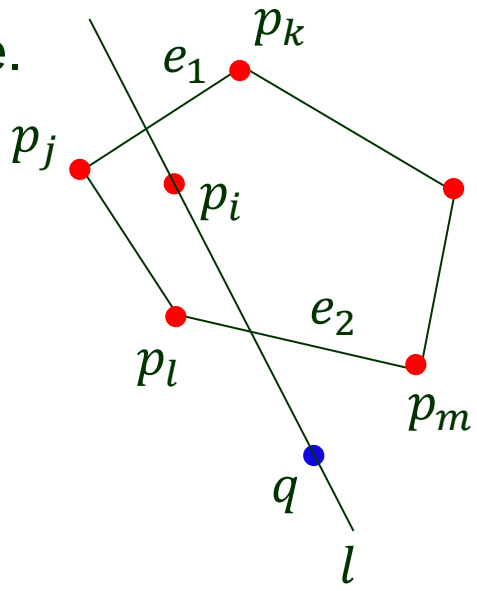
Contradiction.



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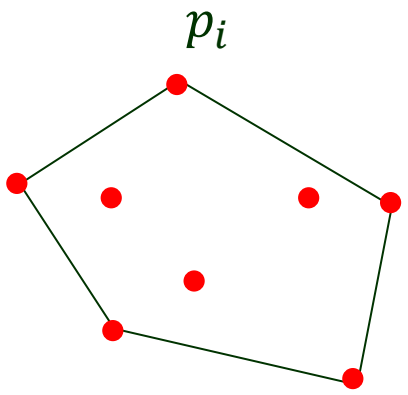

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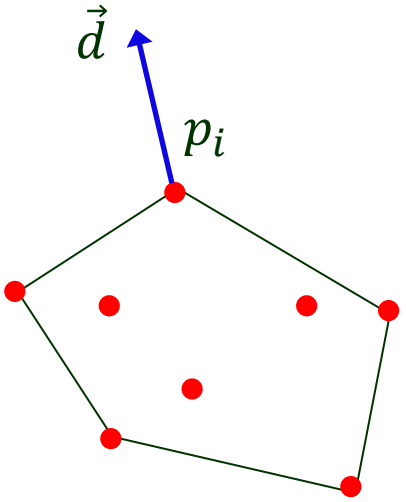
p_i must be extreme in some direction $\vec{d}$.



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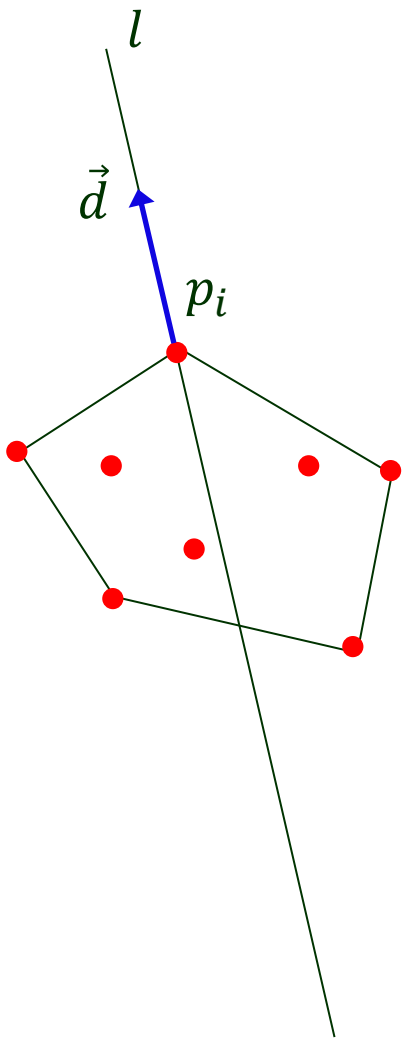

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- Start at p_i.
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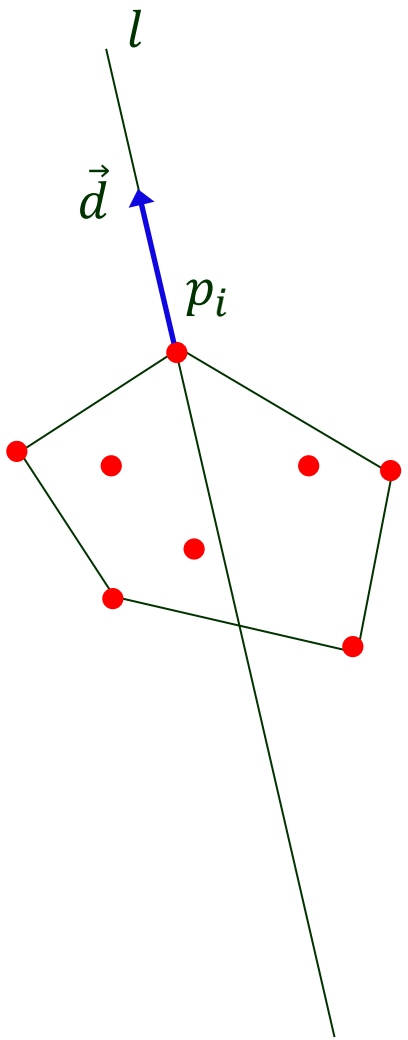

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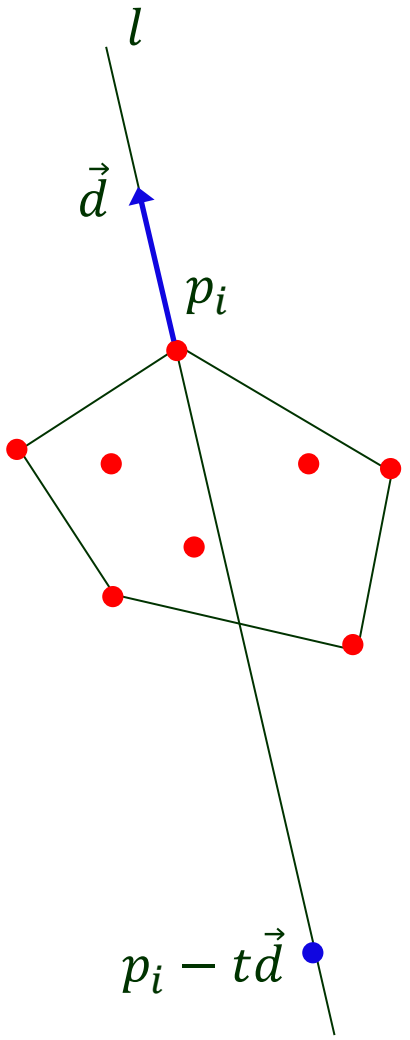

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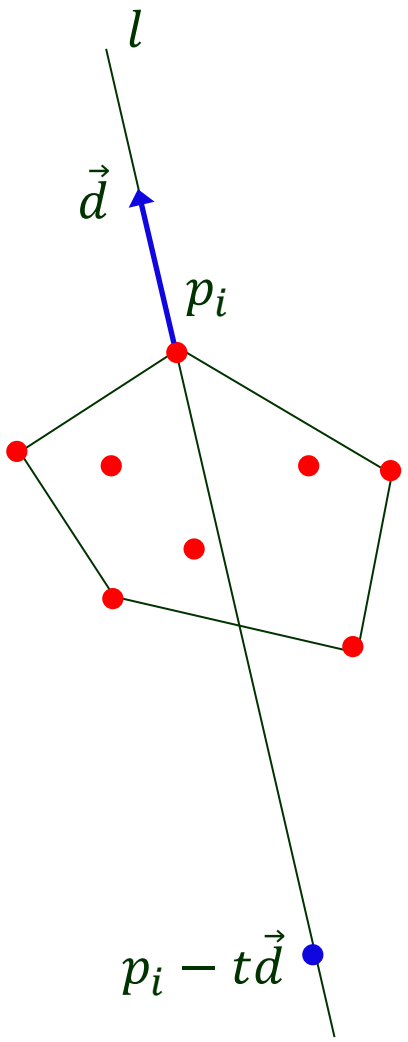
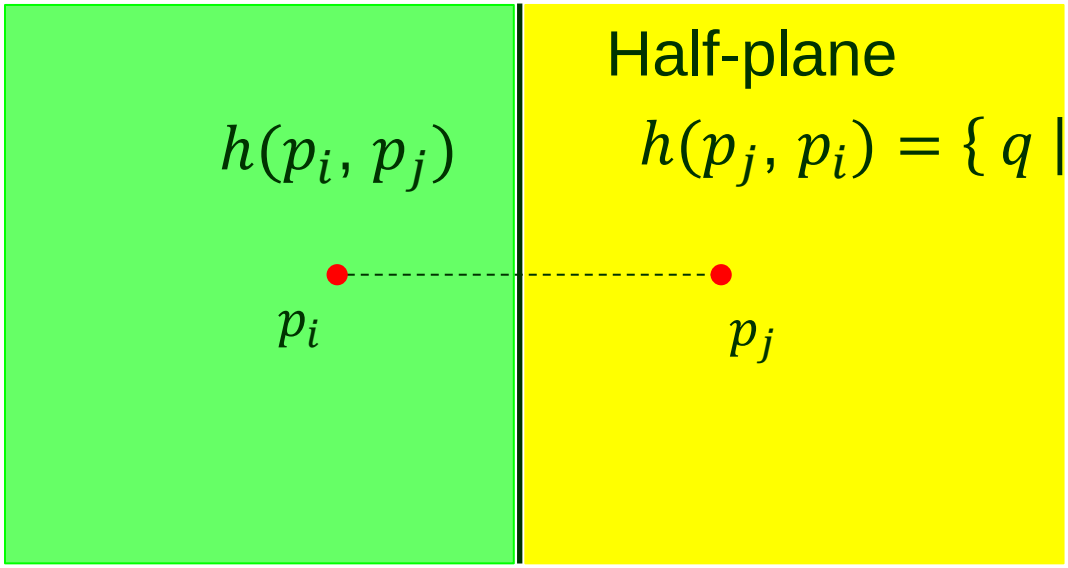
The point $p_i - t\vec{d}$, for large enough t, is farther from p_i than any other site.



II. Two Sites

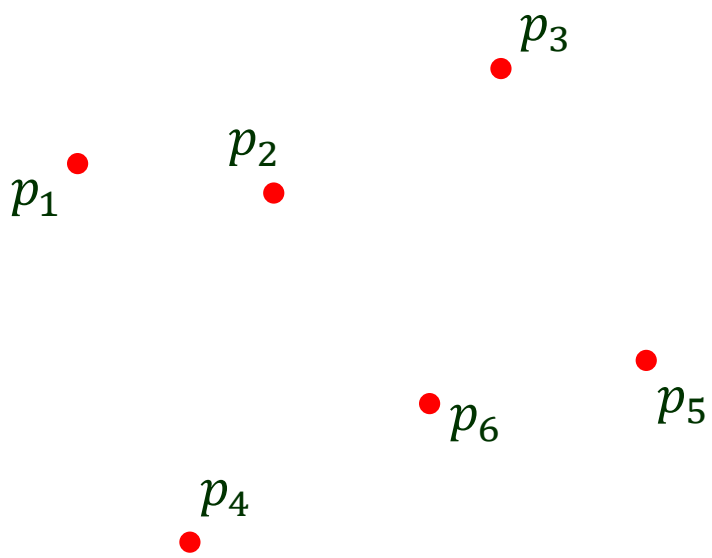


l_{ij}

Perpendicular bisector

Voronoi Cell

$$V'(p_i) = \bigcap_{\substack{1 \le j \le n \\ j \neq i}} h(p_j, p_i)$$



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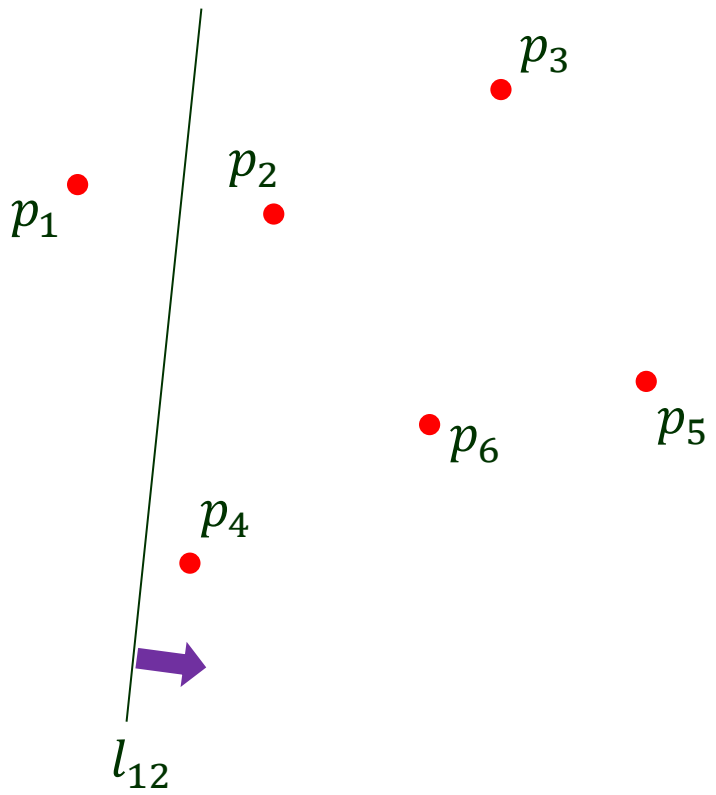

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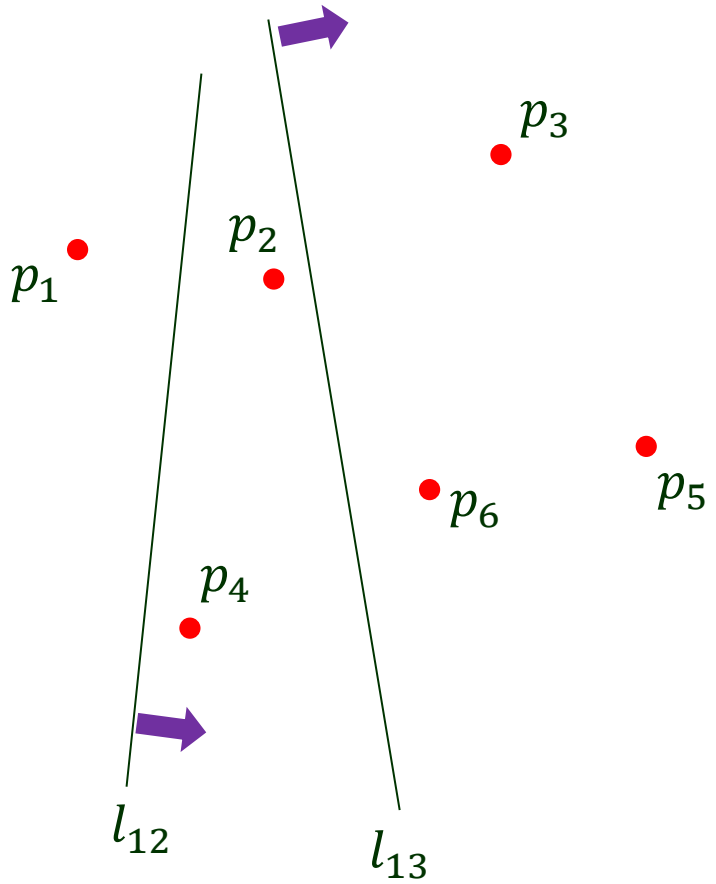

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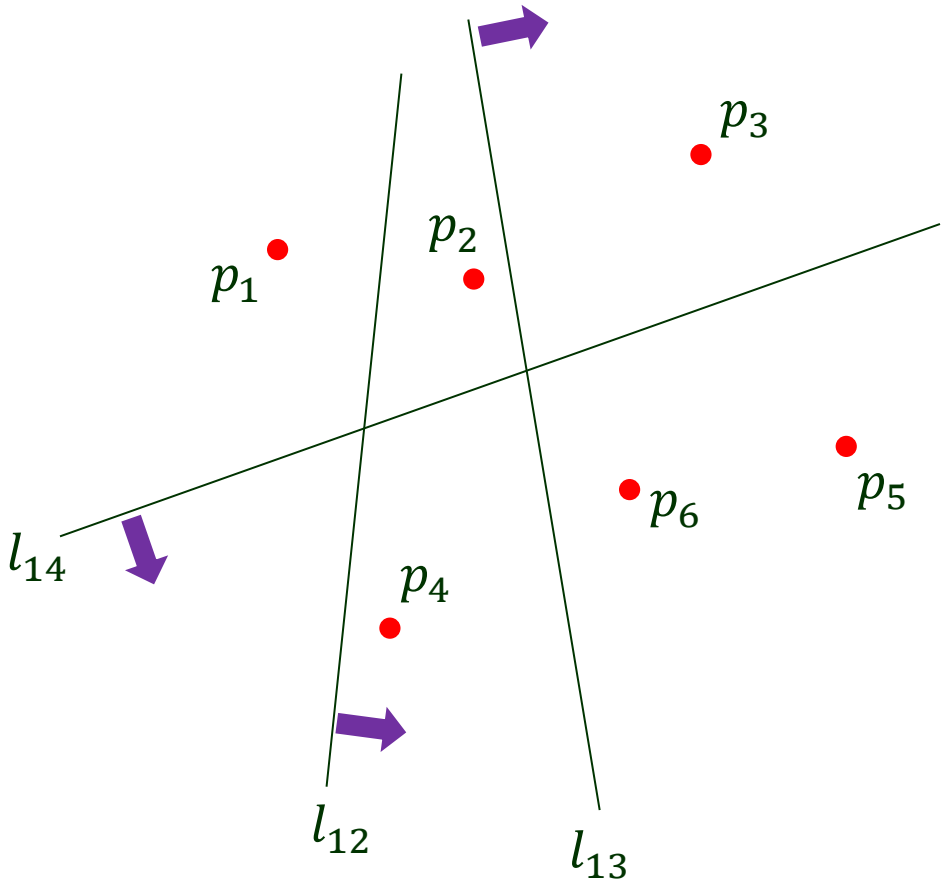

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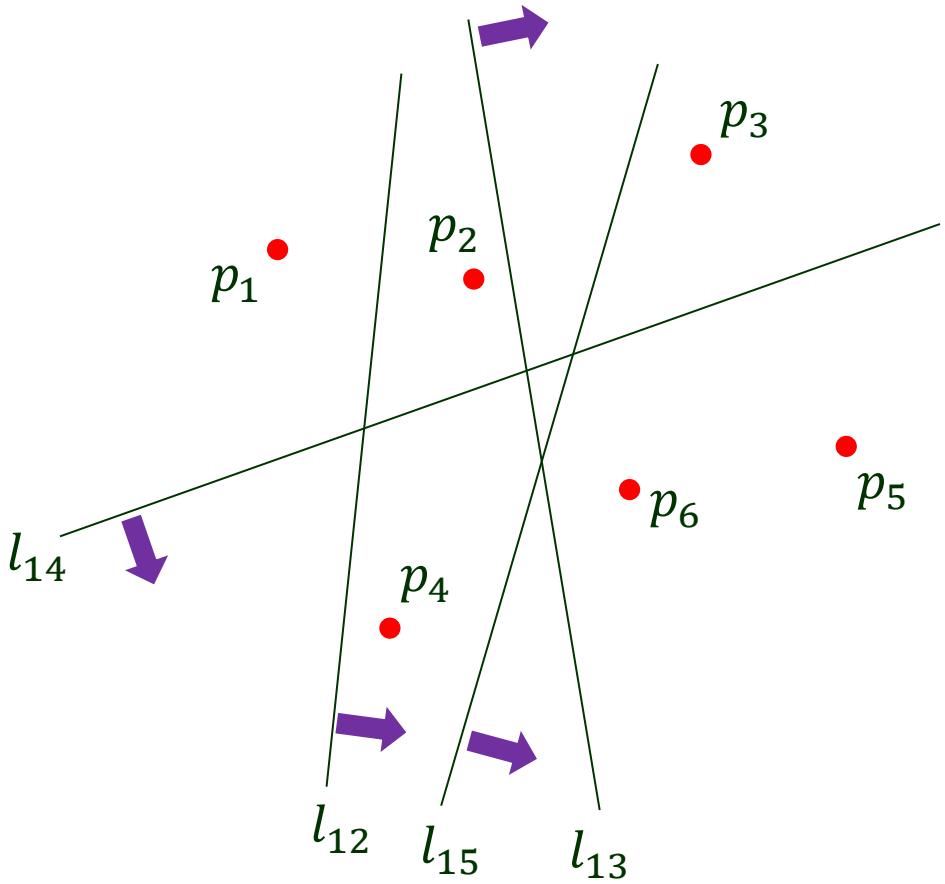

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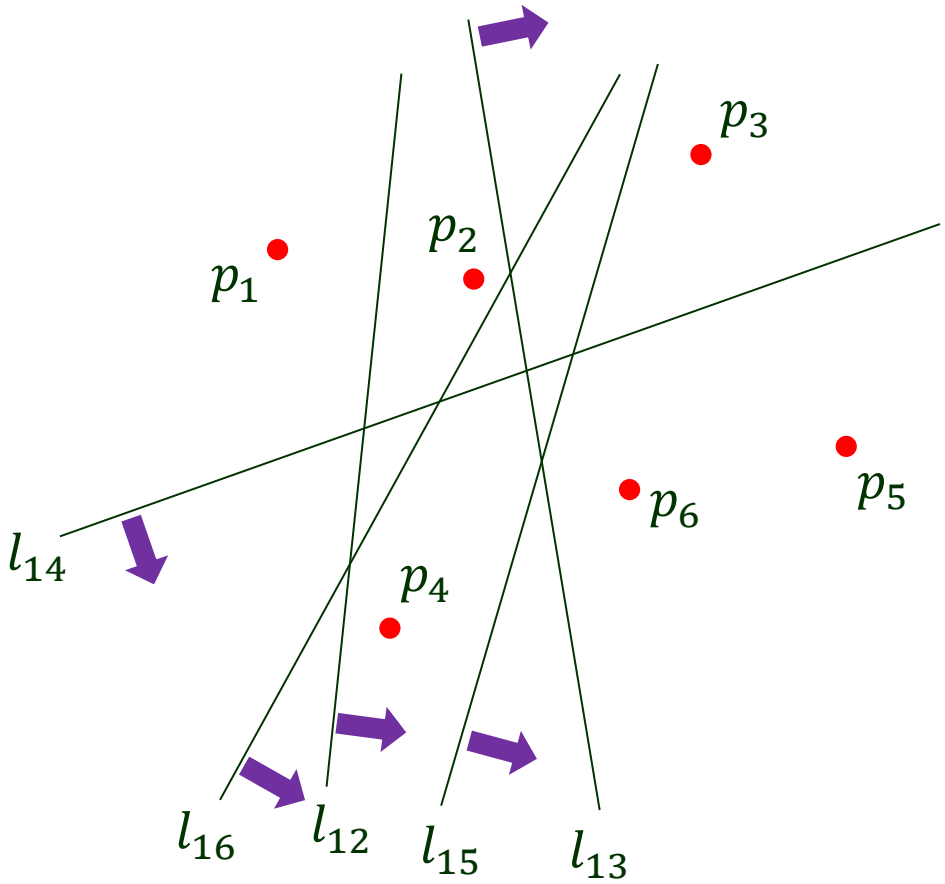

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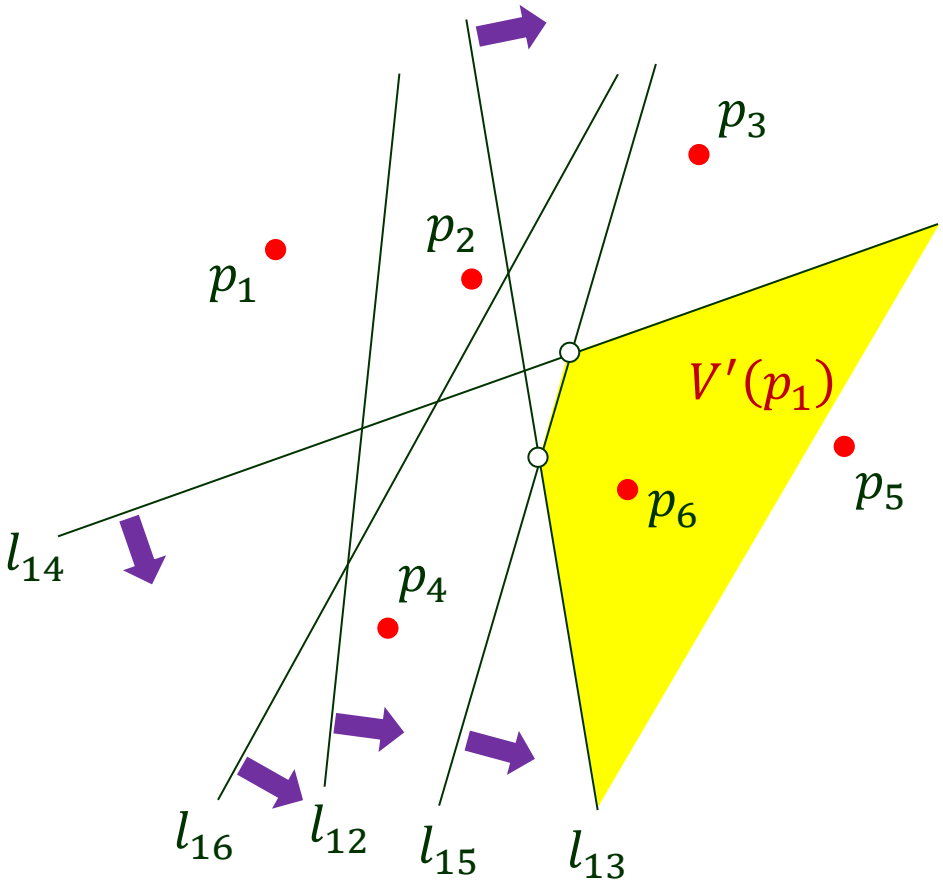

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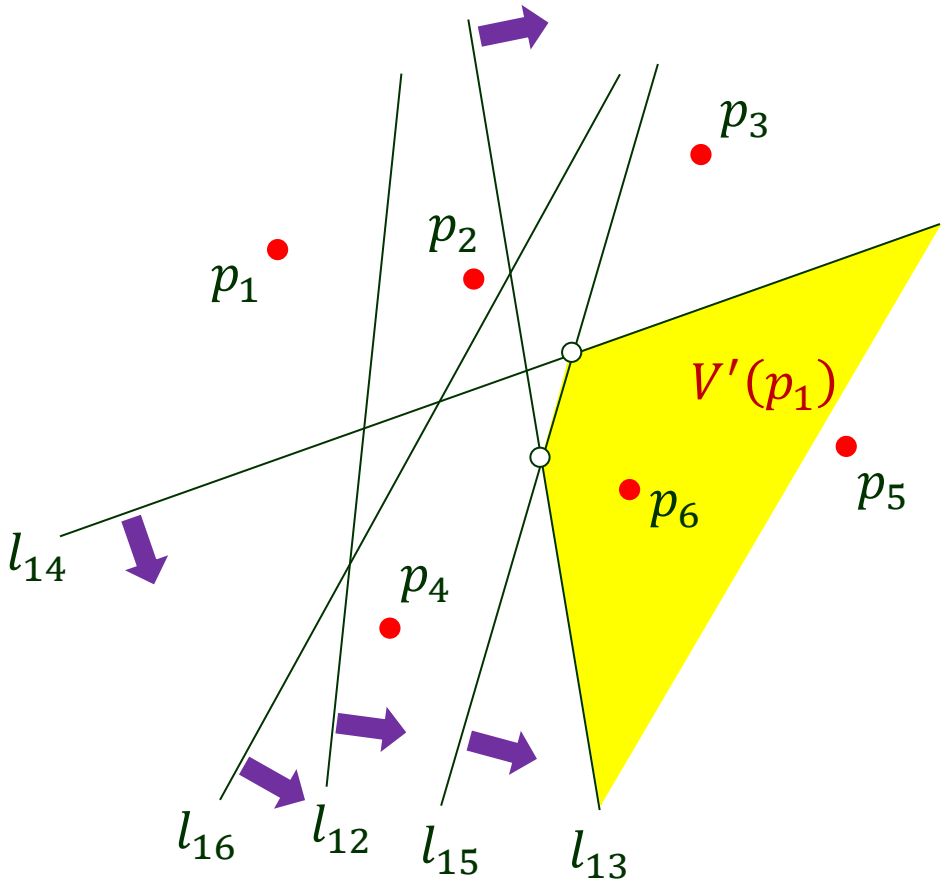
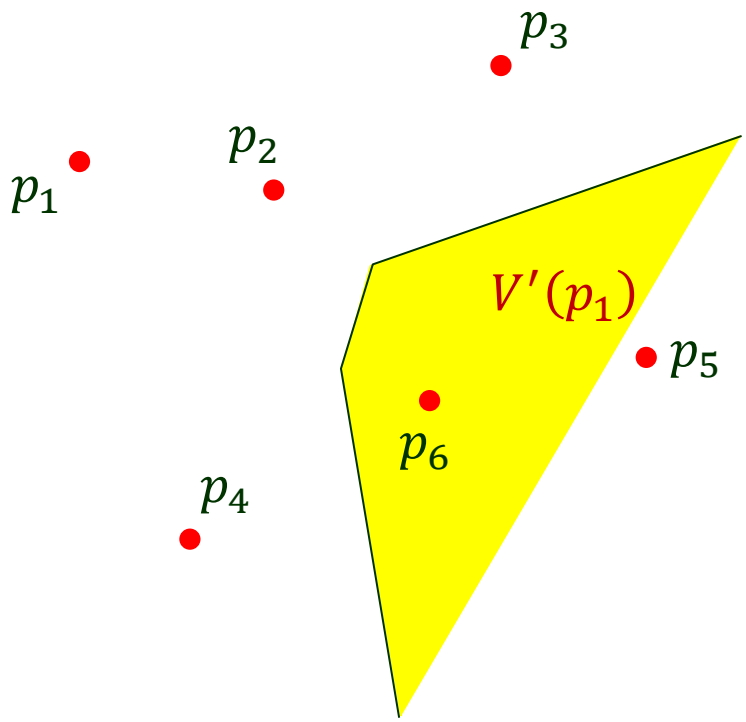


- Open convex region
- $\leq n - 1$ vertices
- $\leq n - 1$ edges

Unboundedness

The cell contains a ray r collinear with p_i.



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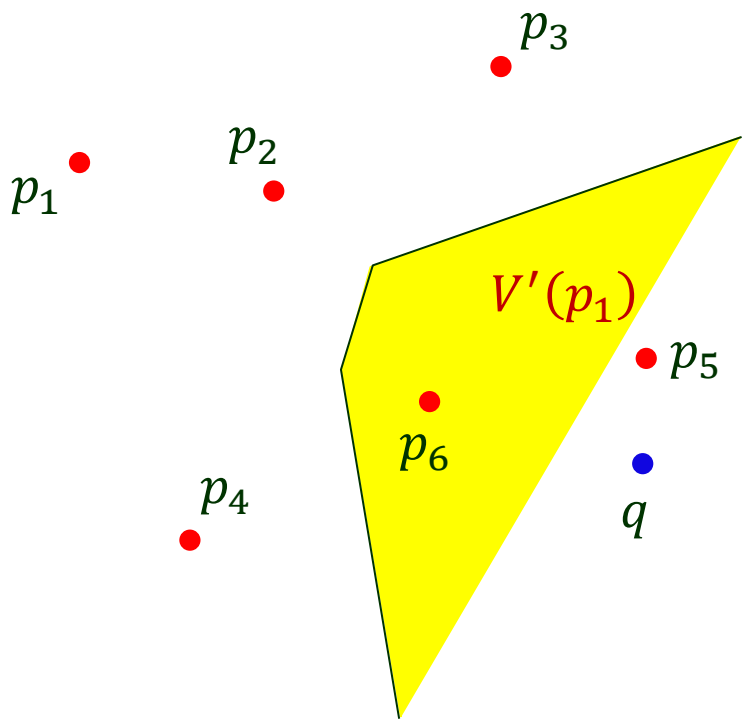

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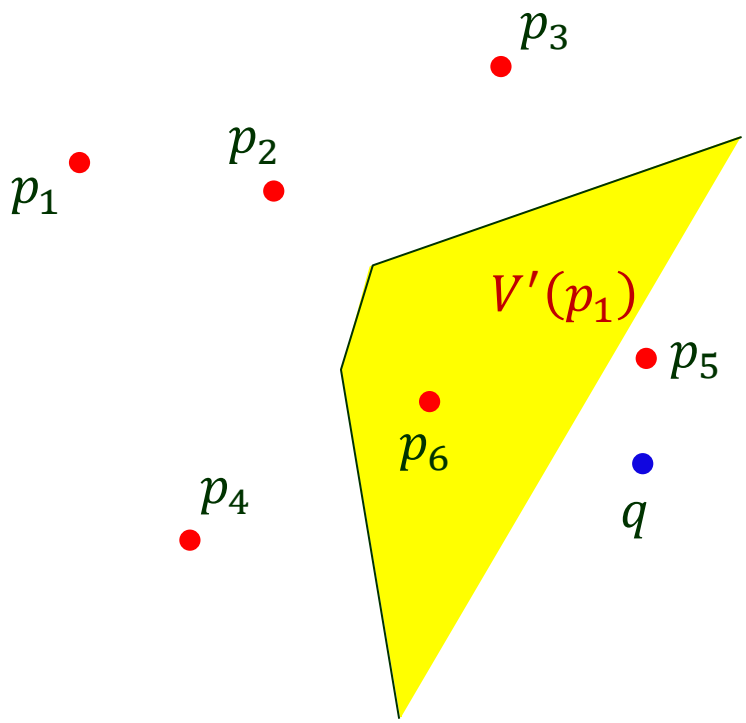
- p_i: farthest point from q.



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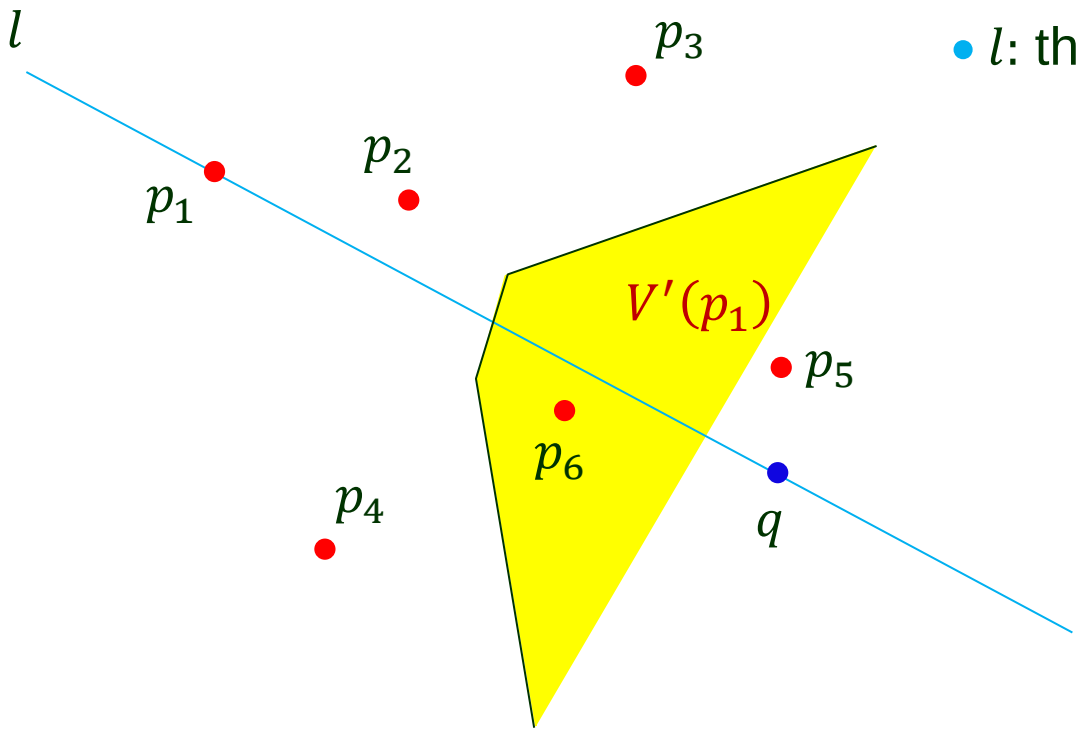

- p_i: farthest point from q.
- l: the line through p_i and q.

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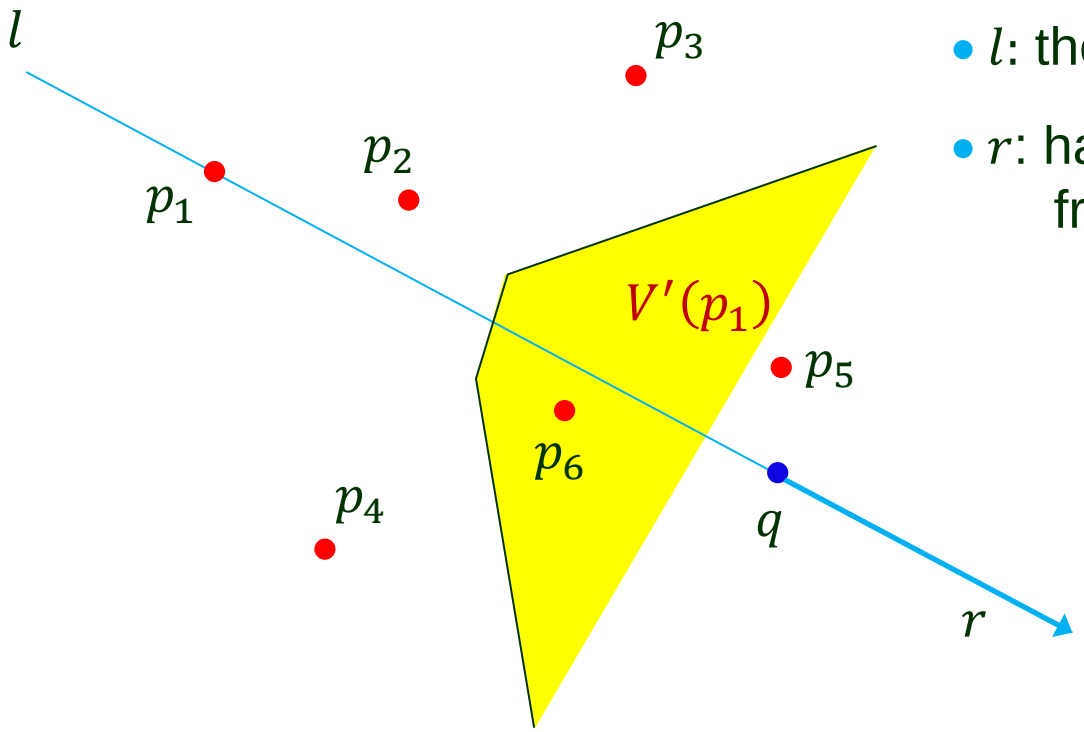

- p_i: farthest point from q.
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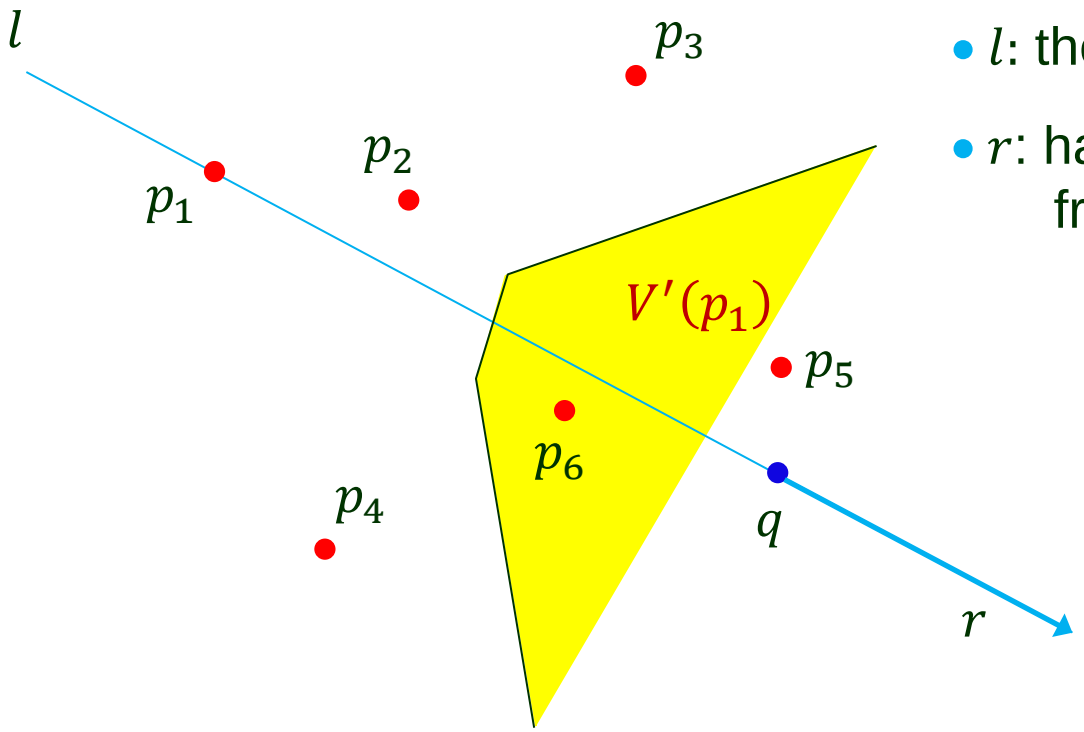
- r: half-line starting at q and away from p_i.

Unboundedness

The cell contains a ray r collinear with p_i.



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All the points on r have p_i as the farthest point!

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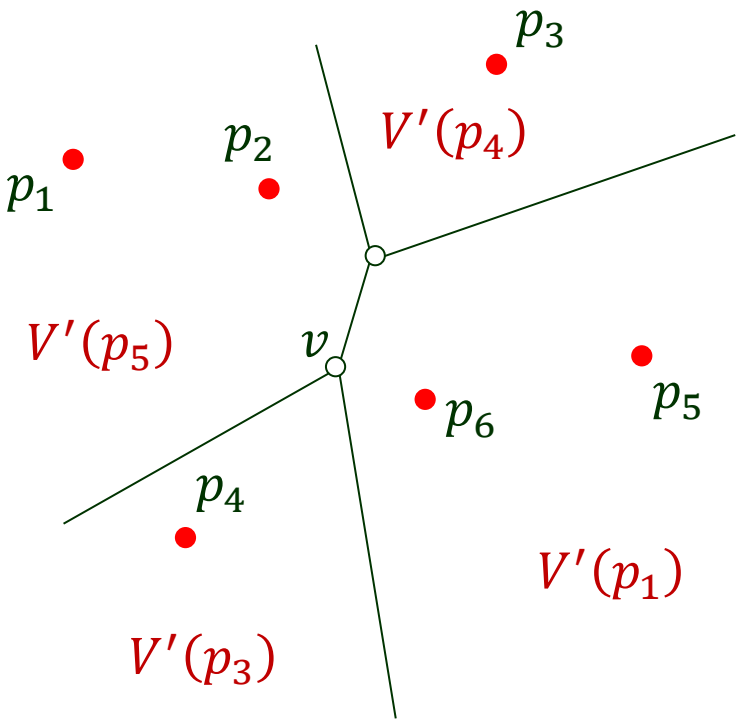

Tree-like structure

- Edges include segments and half-infinite lines.

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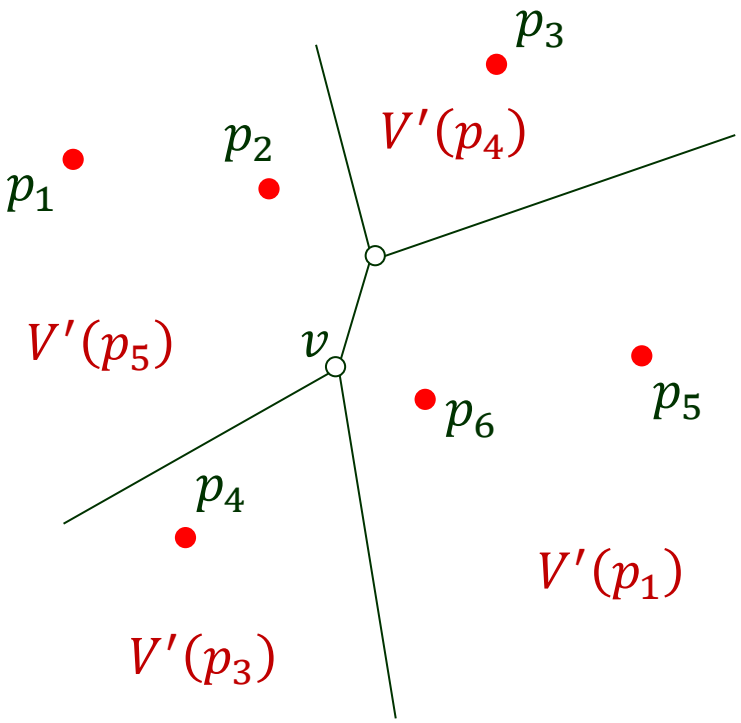

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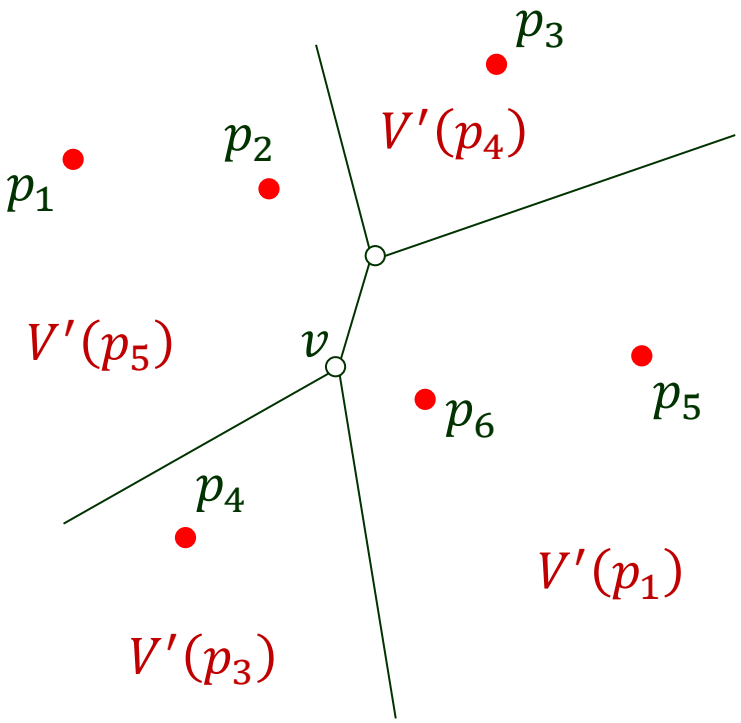

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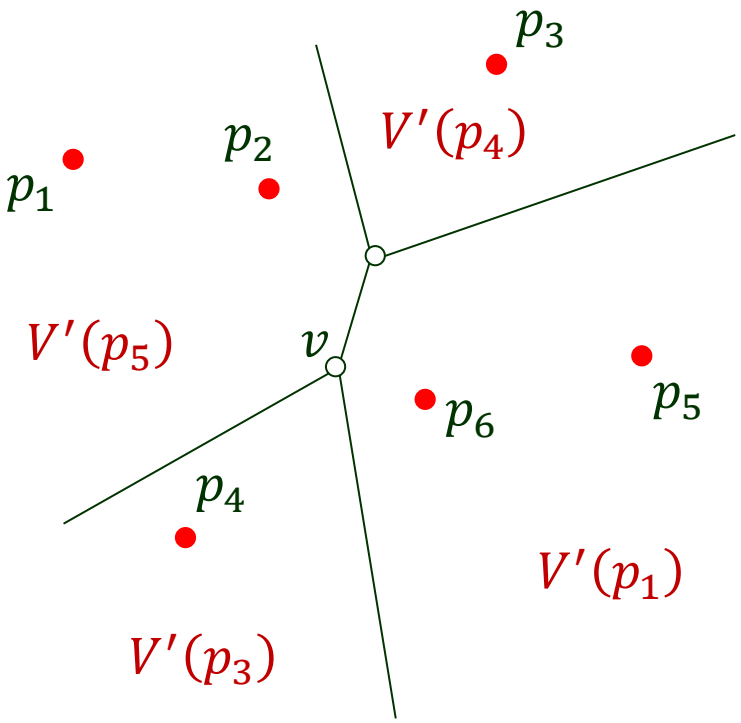

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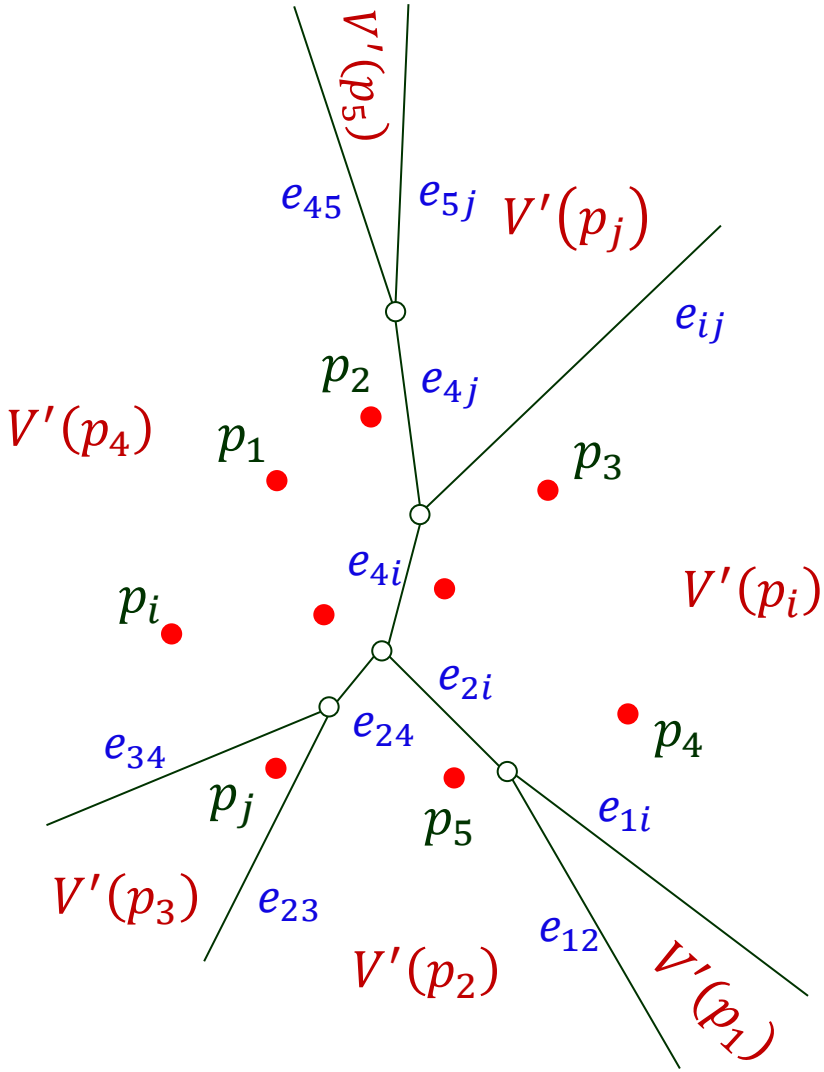
  A cycle would imply a bounded cell.
- A vertex has ≥ 3 farthest sites.

More Properties



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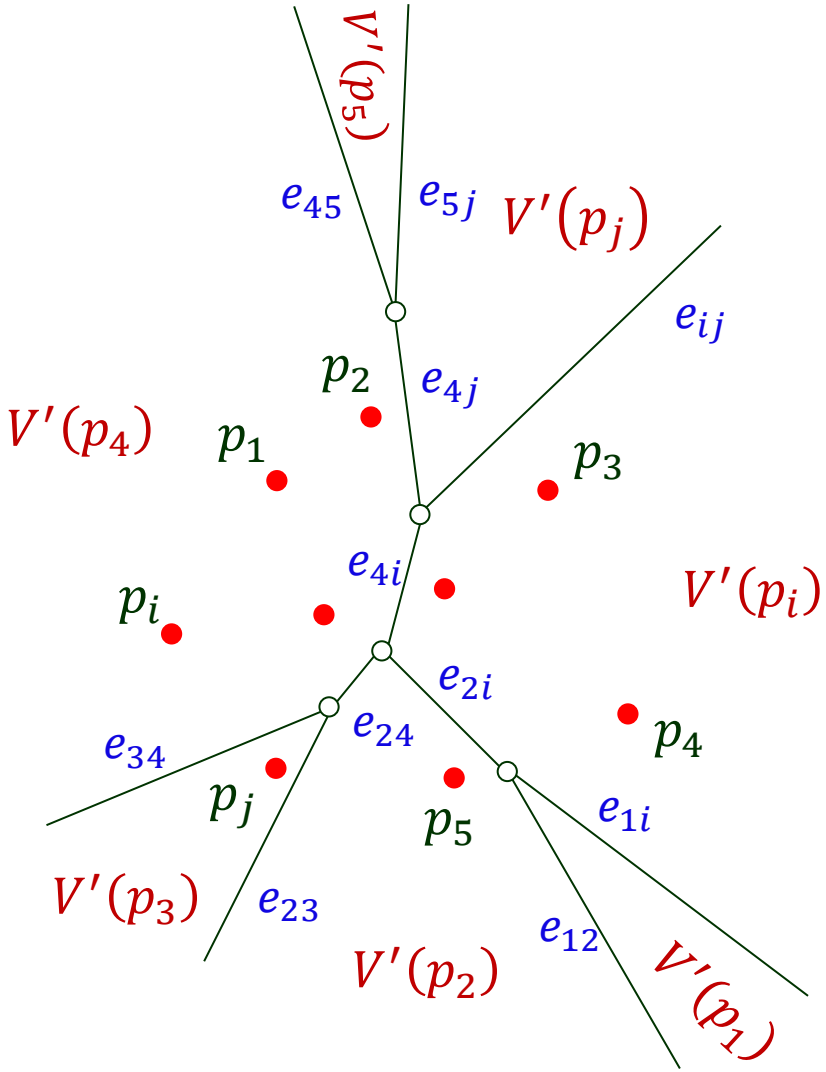

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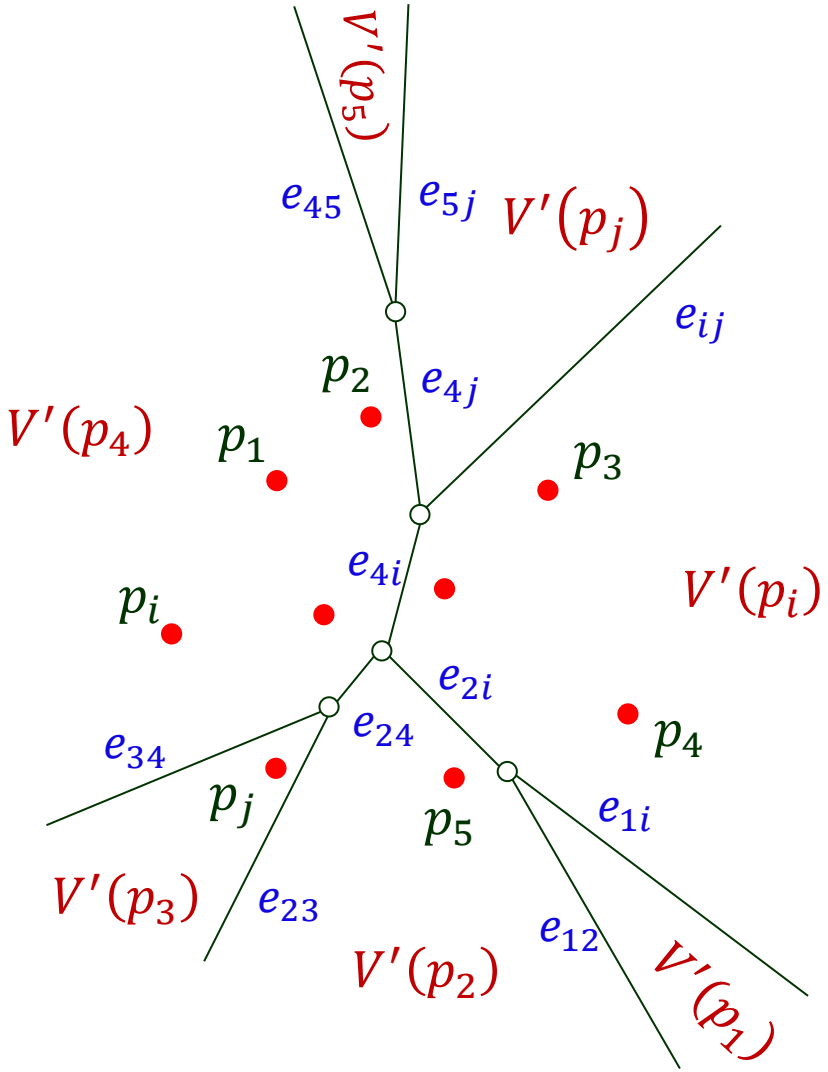
⇩

It contributes no Voronoi edge.

More Properties



- Any site that is not a vertex of the convex hull has *no Voronoi cell*.

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- Every Voronoi edge is part of a bisector of two convex hull vertices.

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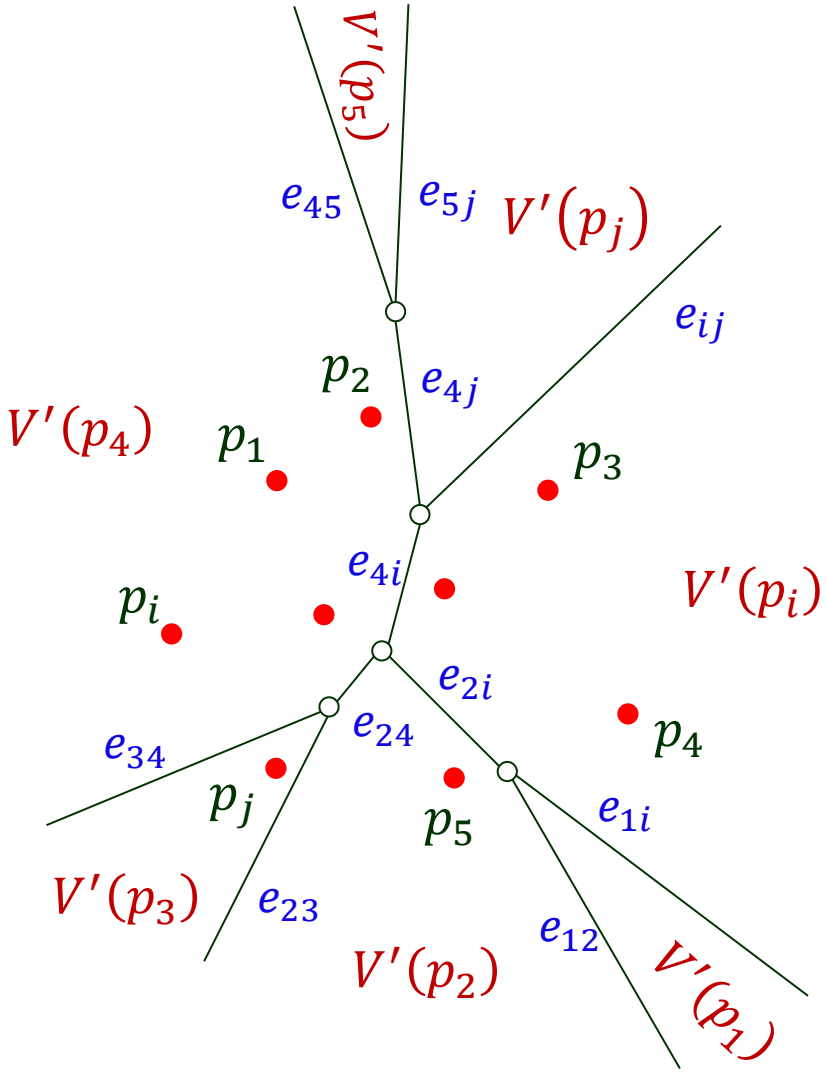

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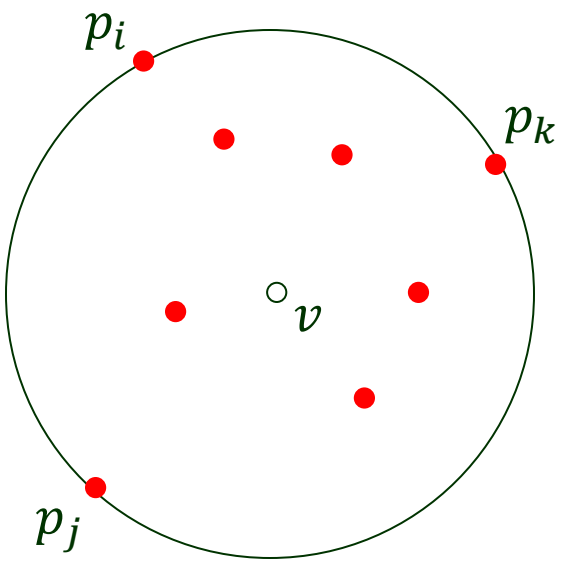
- $O(n)$ vertices, edges and cells

Center of Smallest Enclosing Disk



Two possibilities:

- Vertex ⇒ ≥ 3 equidistant farthest sites

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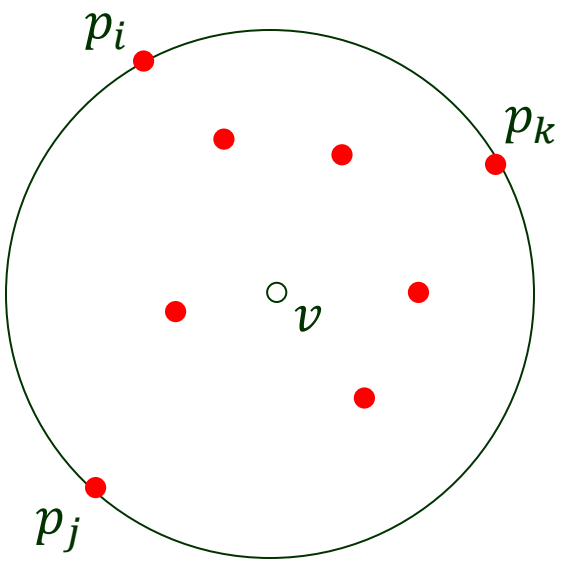
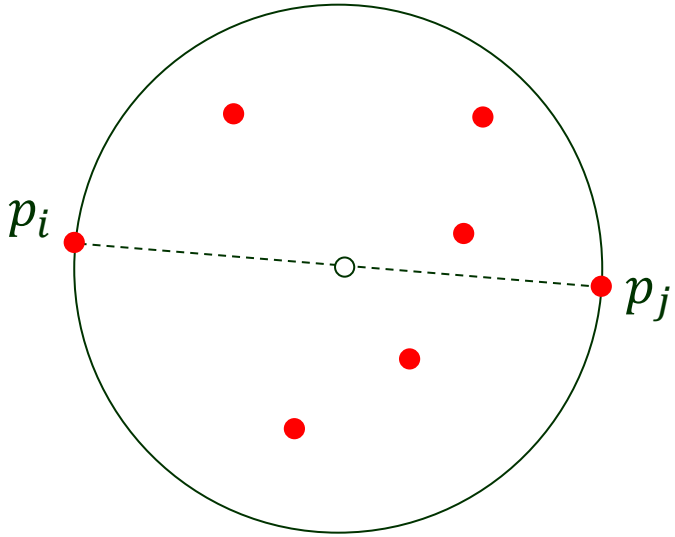


Two possibilities:

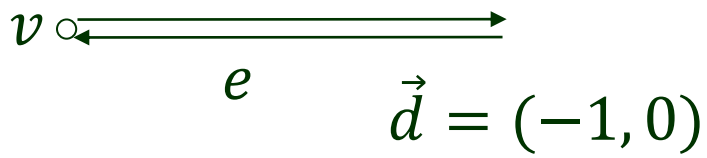
- Vertex ⇒ ≥ 3 equidistant farthest sites
- Midpoint of two sites defining an edge ⇒ two equidistant farthest sites



Storage

Doubly-connected edge list (DCEL) with modifications

Half-infinite edge



- If no origin, stores the direction of the edge ($\vec{d}$) instead of coordinates.

- Either next(e) or prev(e) is undefined.

IV. Preprocessing for Construction

1. Compute the convex hull CH(P) with h vertices.

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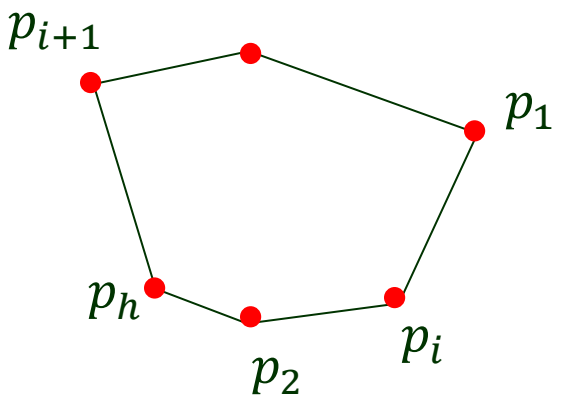

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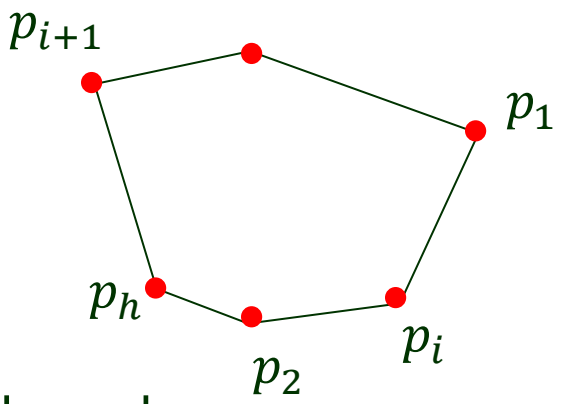

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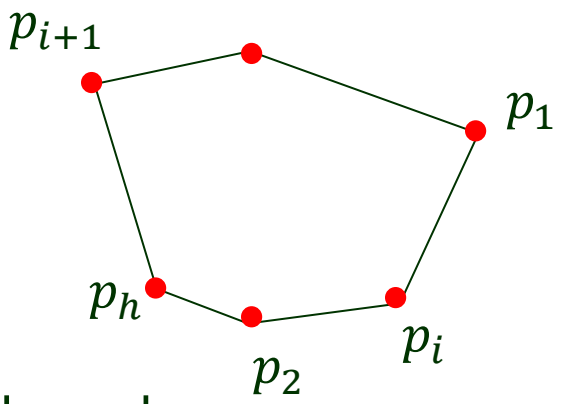

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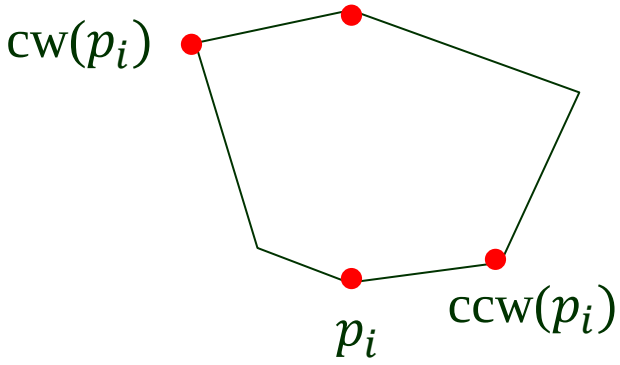
- For each p_i, store its clockwise neighbor cw(p_i) and counterclockwise neighbor ccw(p_i) at the time of removal.



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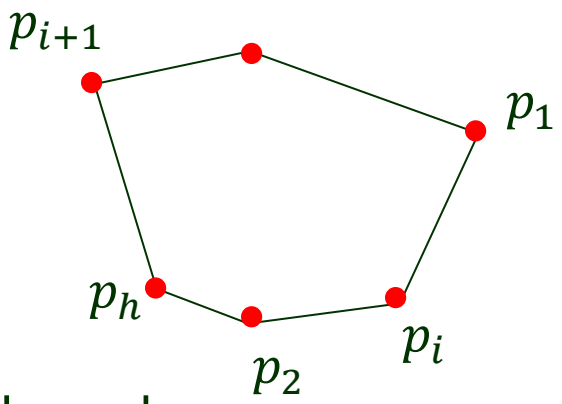

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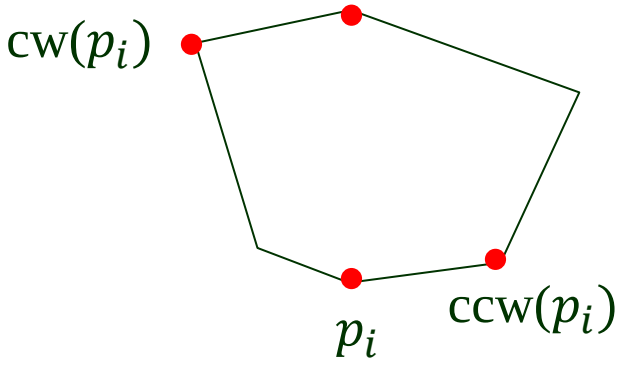
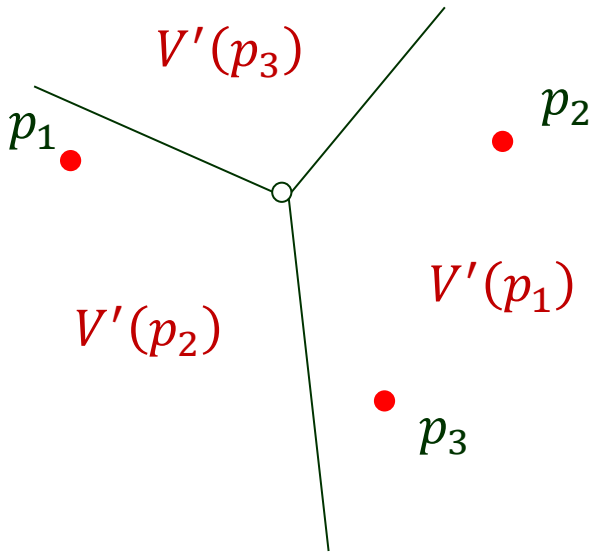
⇩

p_i cannot be a neighbor of any point removed later.



Construction

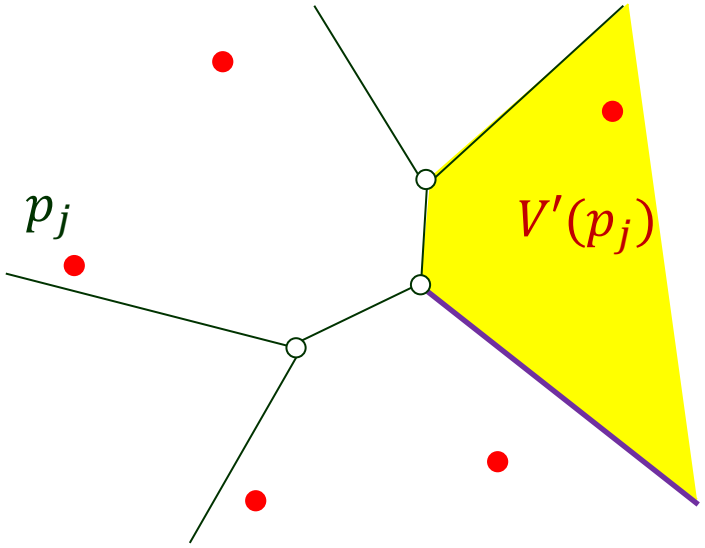
1. Initialize with the FPVD of p_1, p_2, p_3.



Construction (cont'd)

2. Insert $p_4, p_5, \ldots, p_h$ one by one in the order.

$FPVD_{i-1}$ for $\{p_1, \ldots, p_{i-1}\}$:



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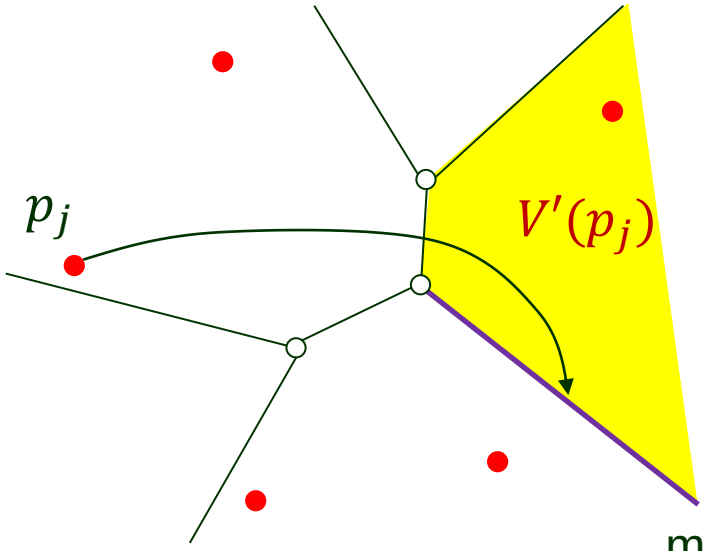
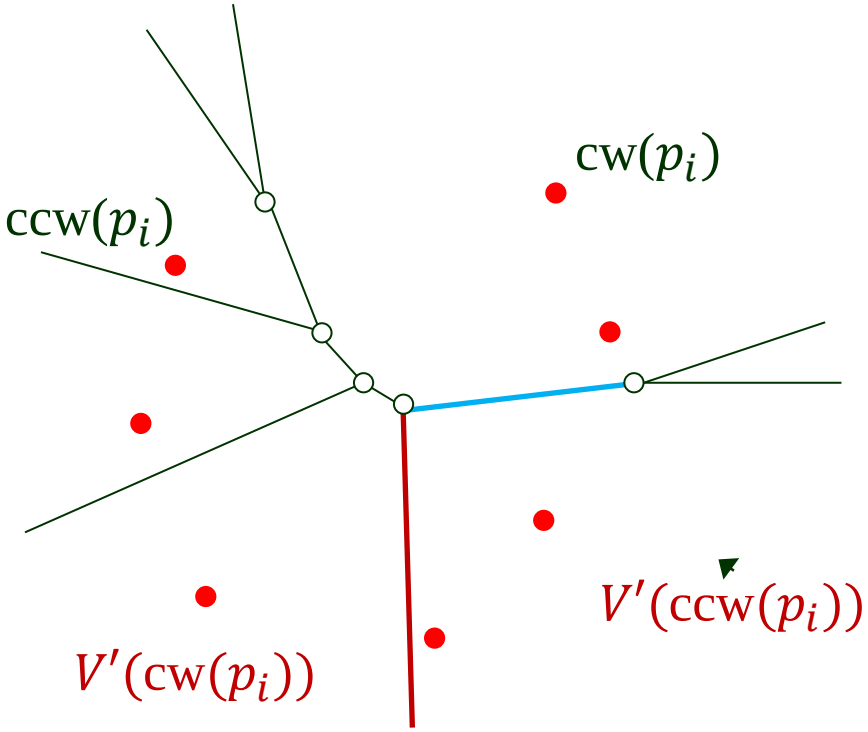


most counterclockwise half-edge in a traversal of the boundary of $V'(p_j)$

How to Add p_i?



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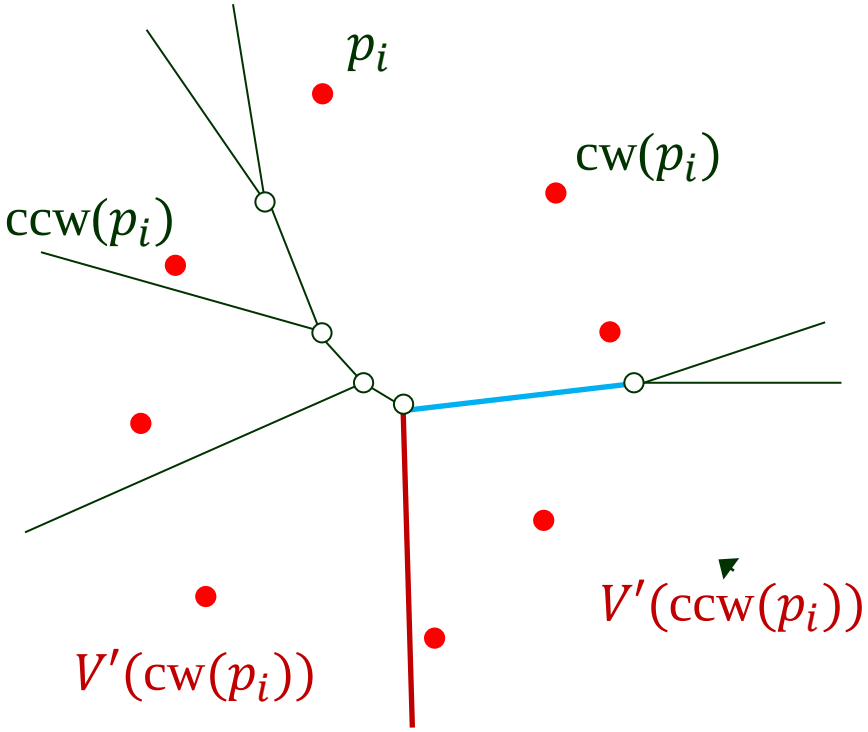

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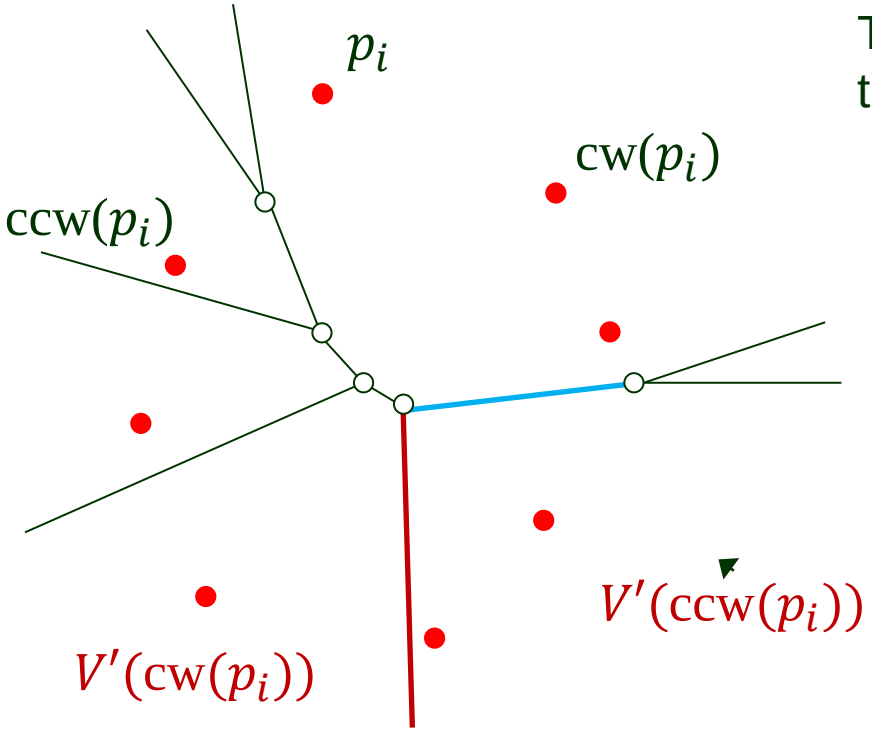

The cell $V'(p_i)$ of p_i will come in between the adjacent cells $V'(\mathrm{cw}(p_i))$ and $V'(\mathrm{ccw}(p_i))$.

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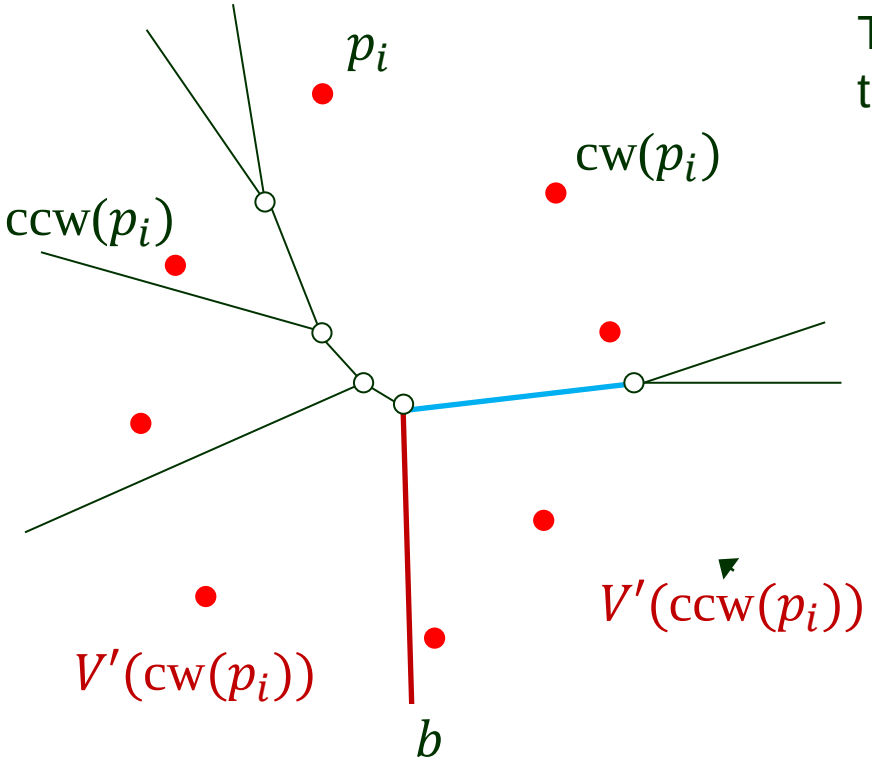

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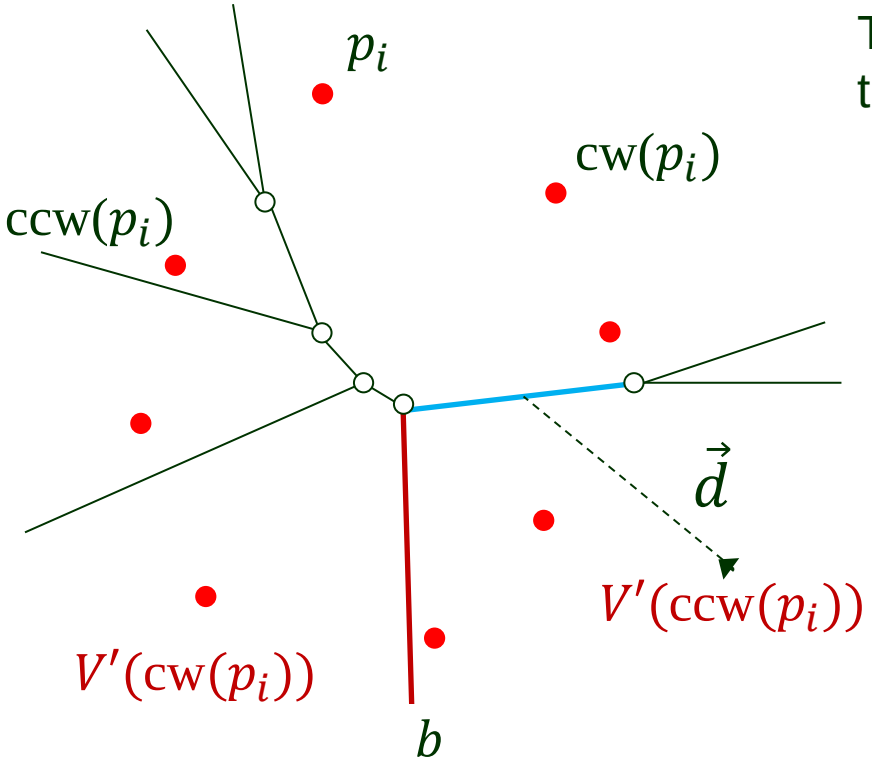

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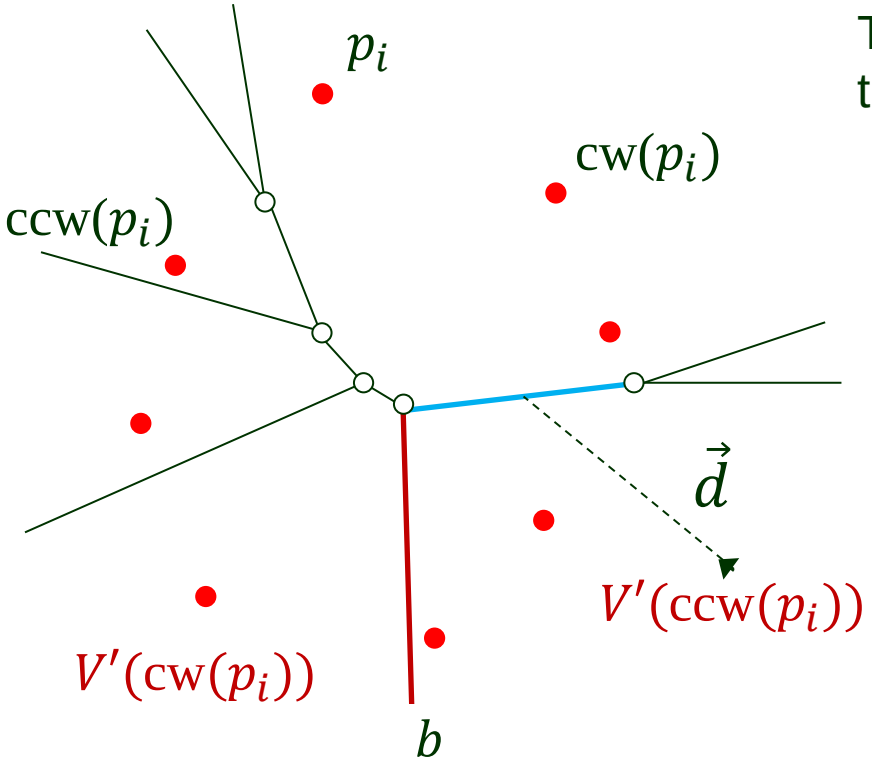

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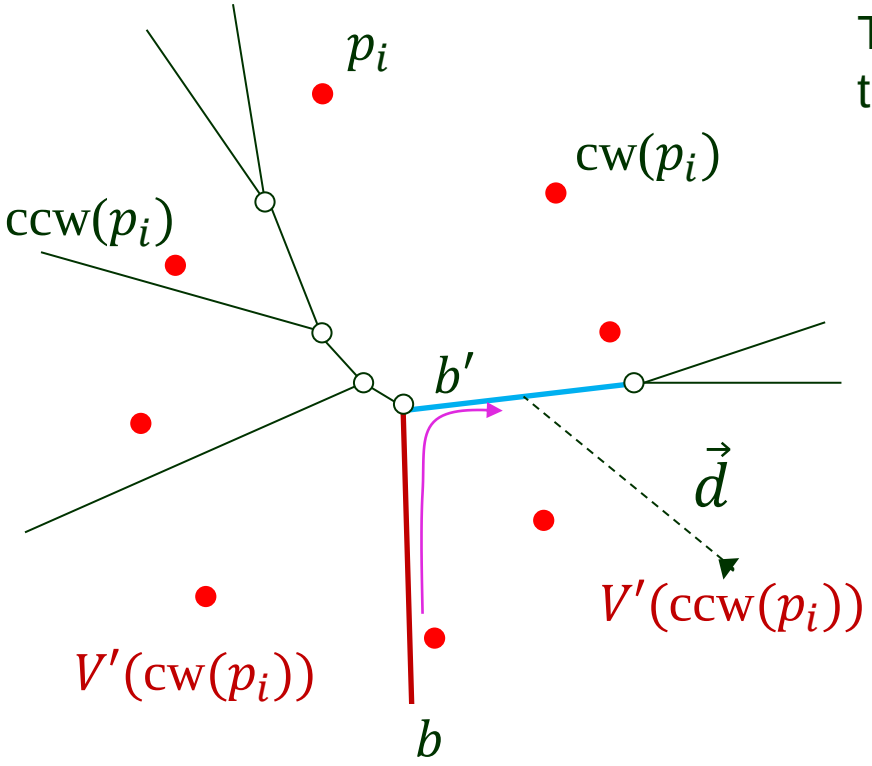

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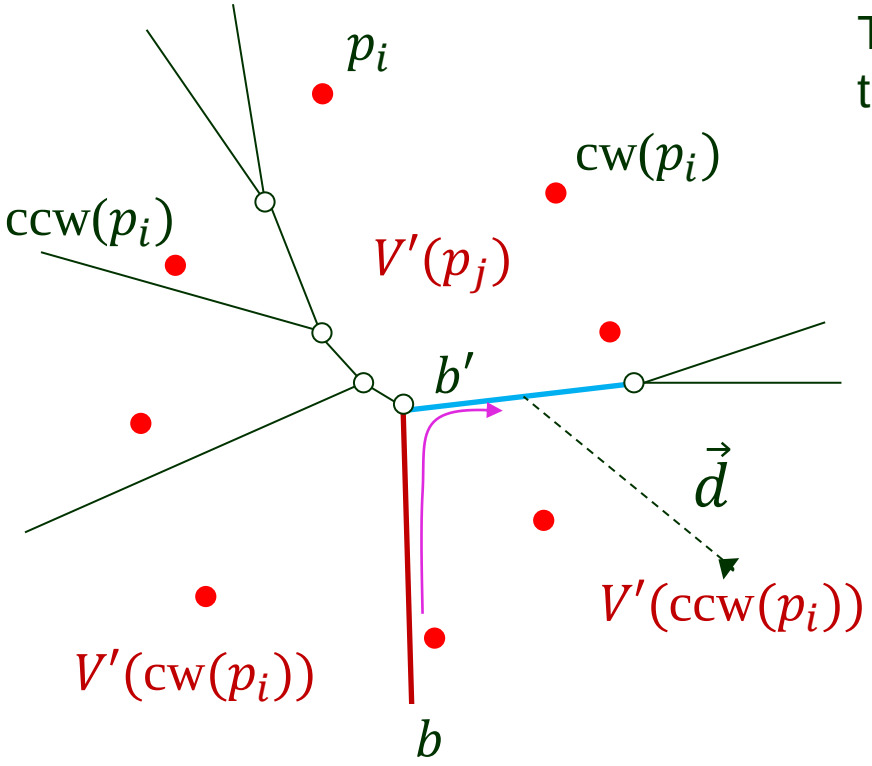

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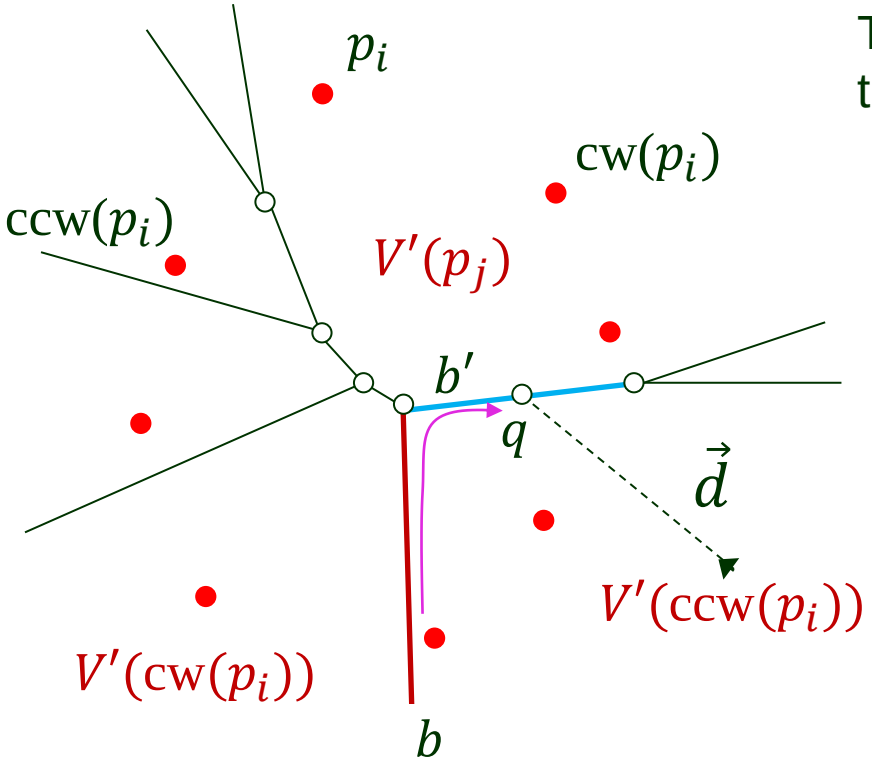

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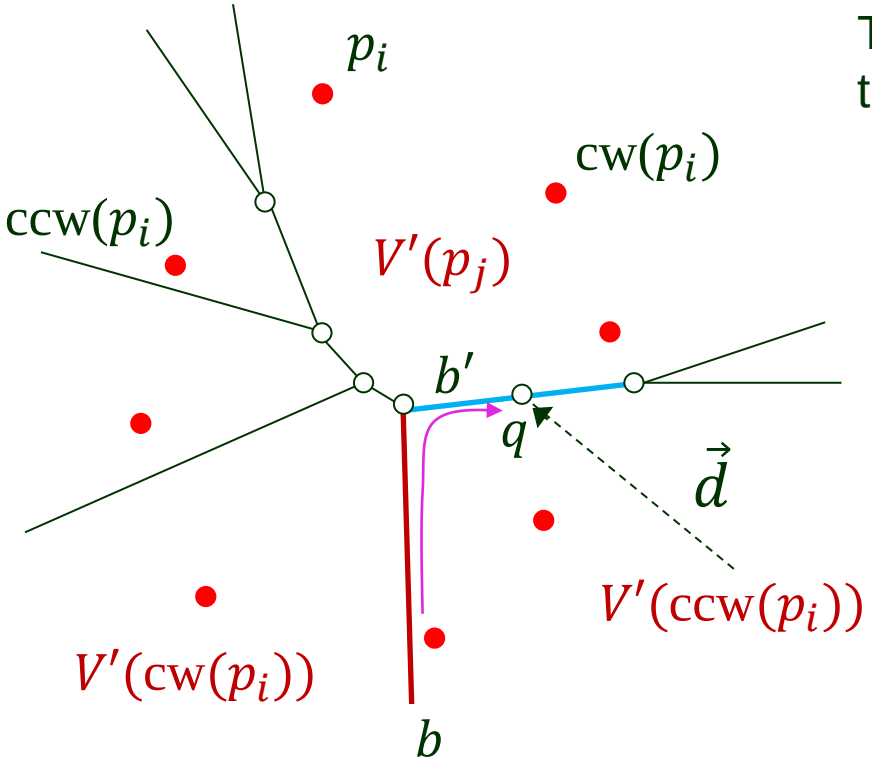

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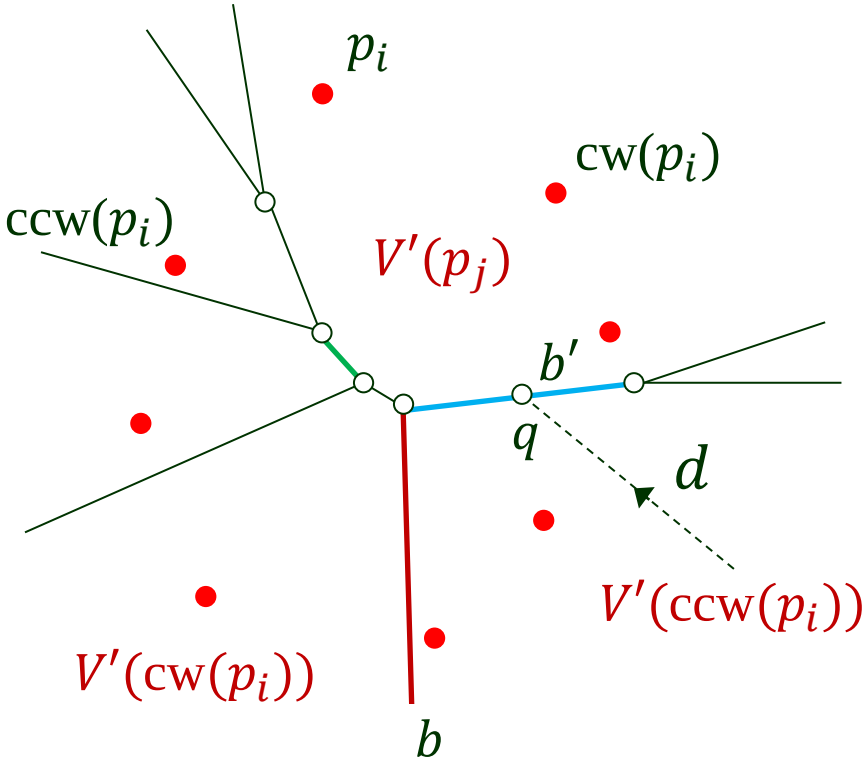

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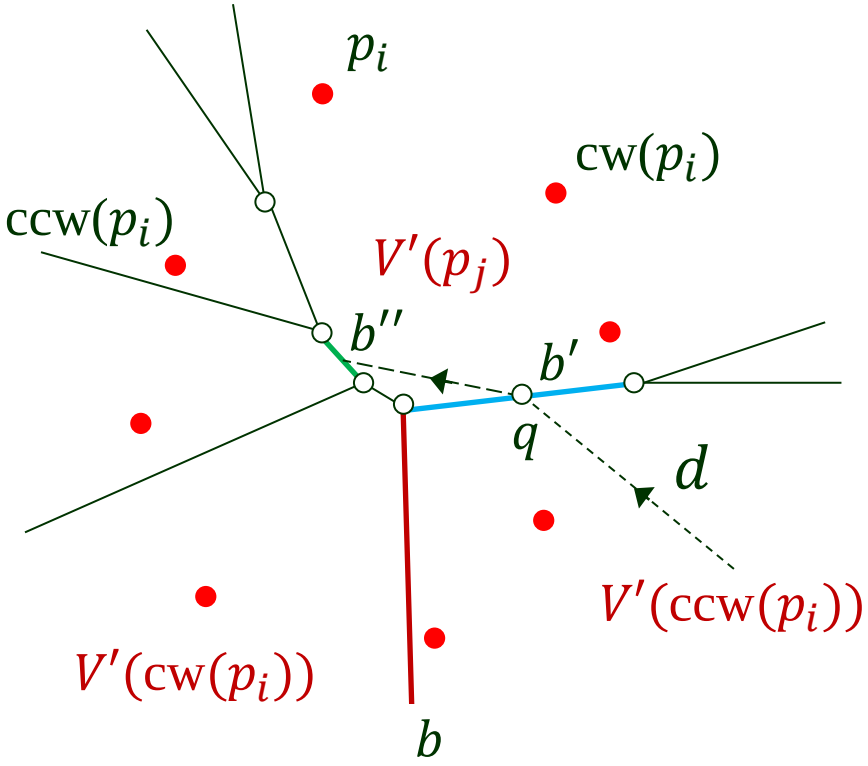

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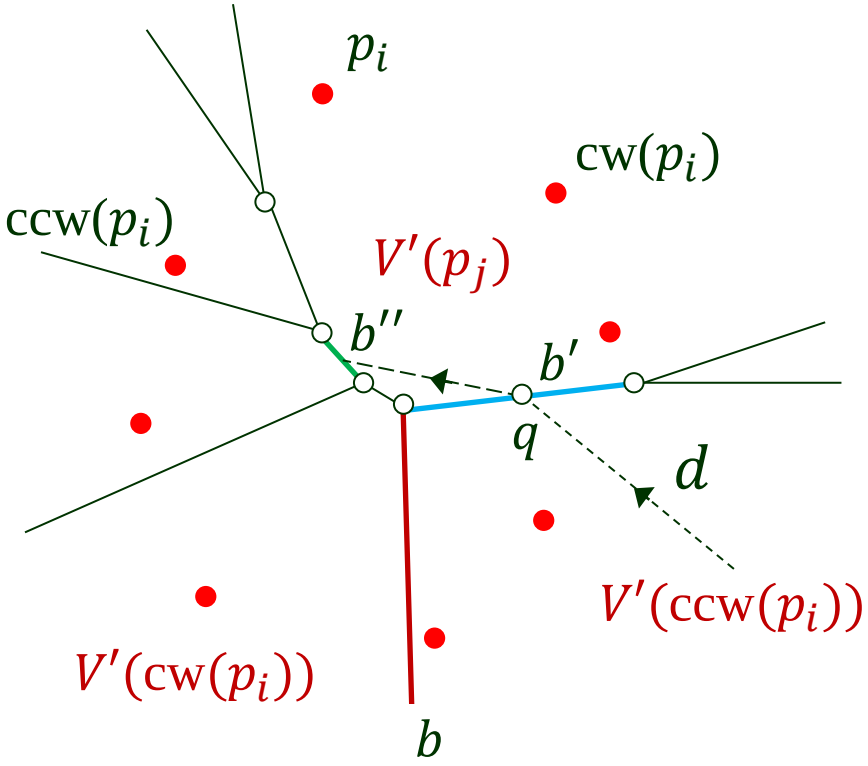

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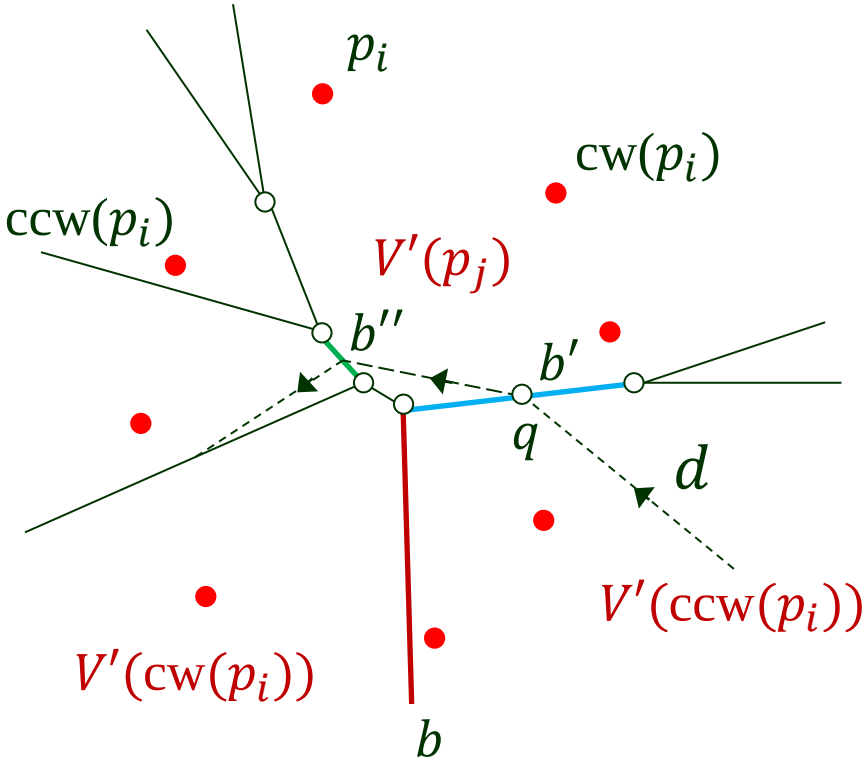

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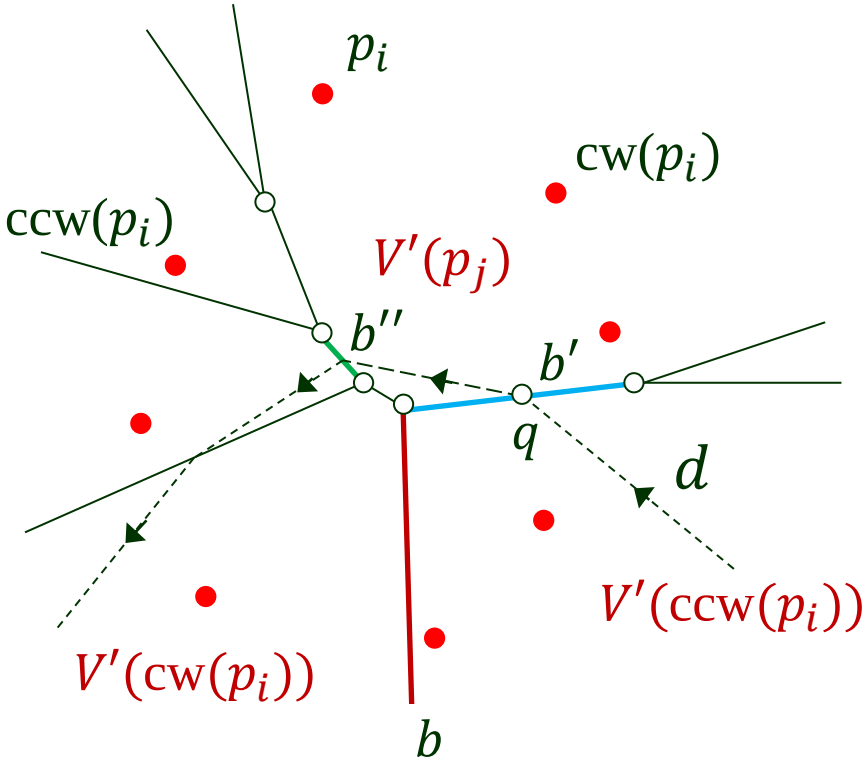

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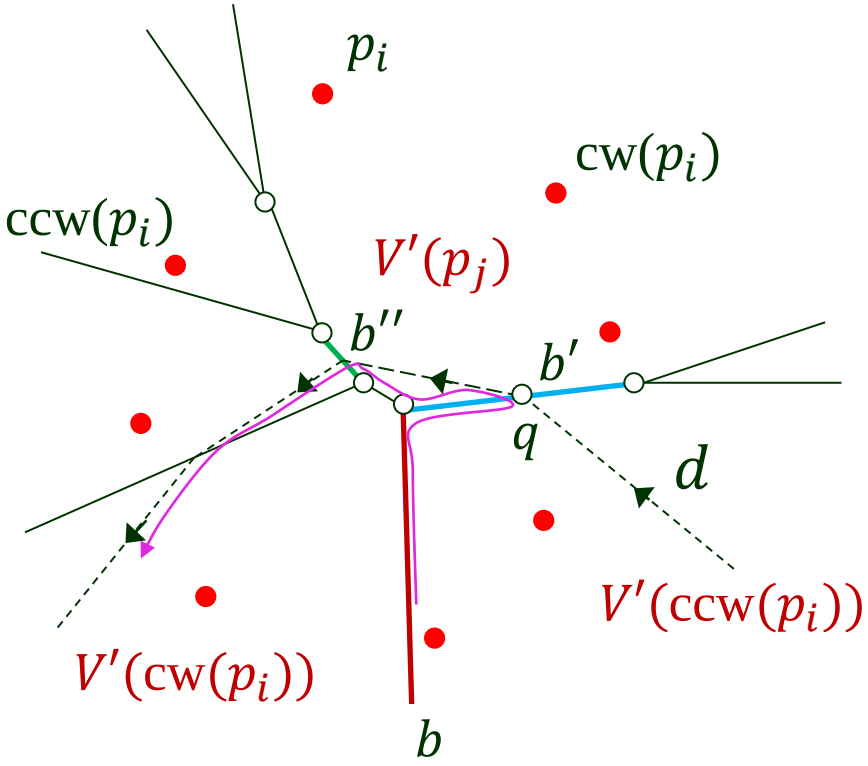

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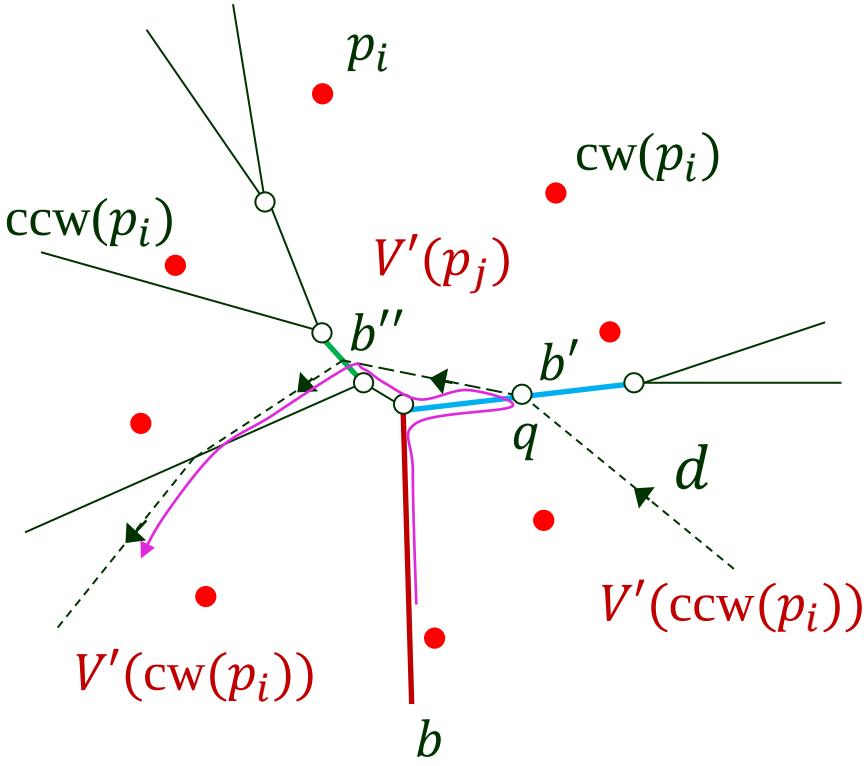

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- Last bisector will be between p_i and $\mathrm{cw}(p_i)$.

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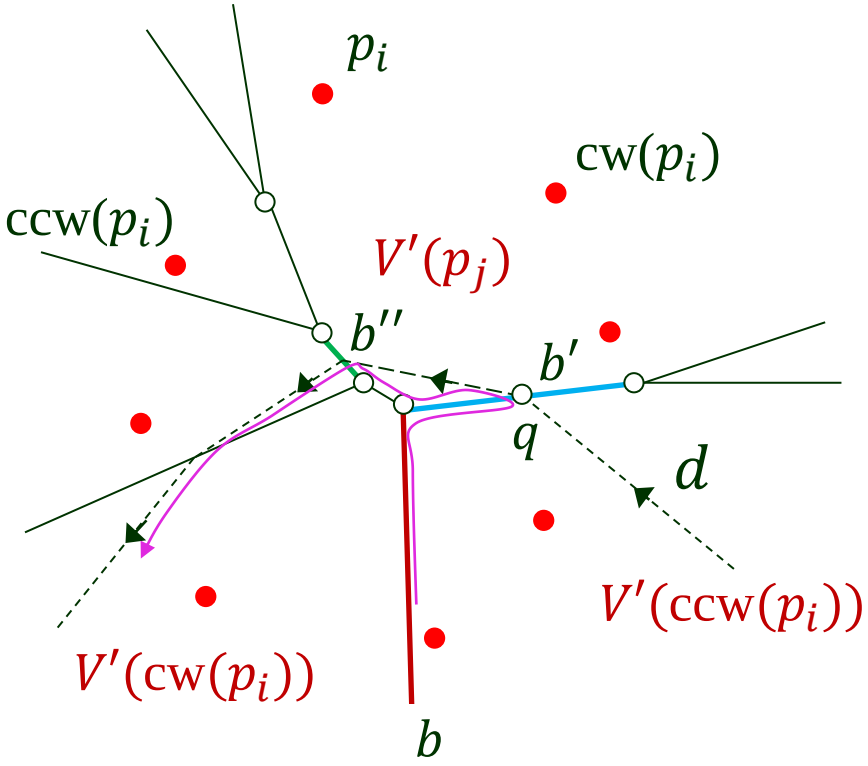

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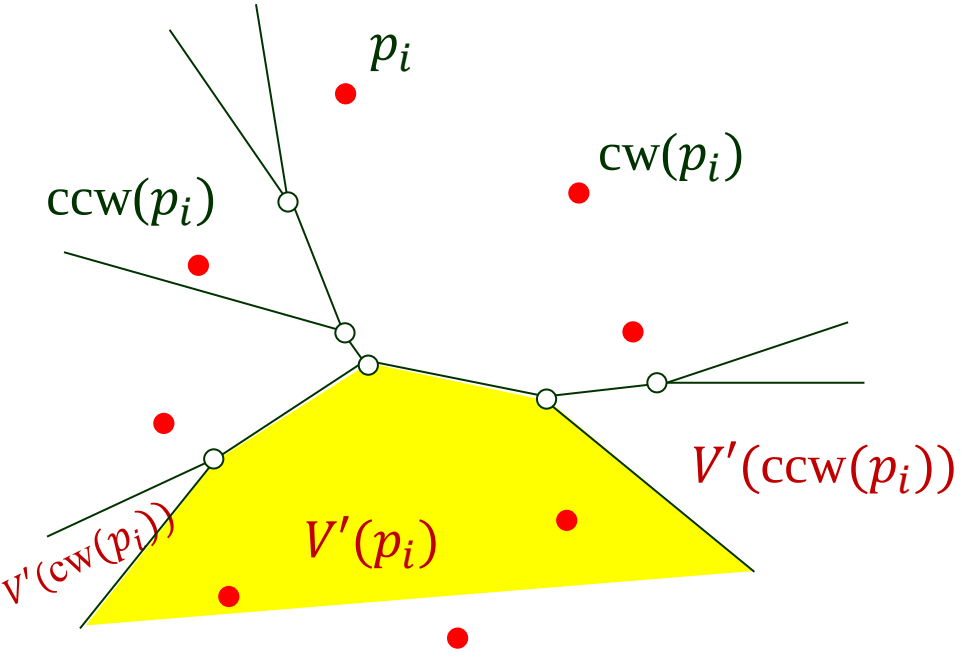
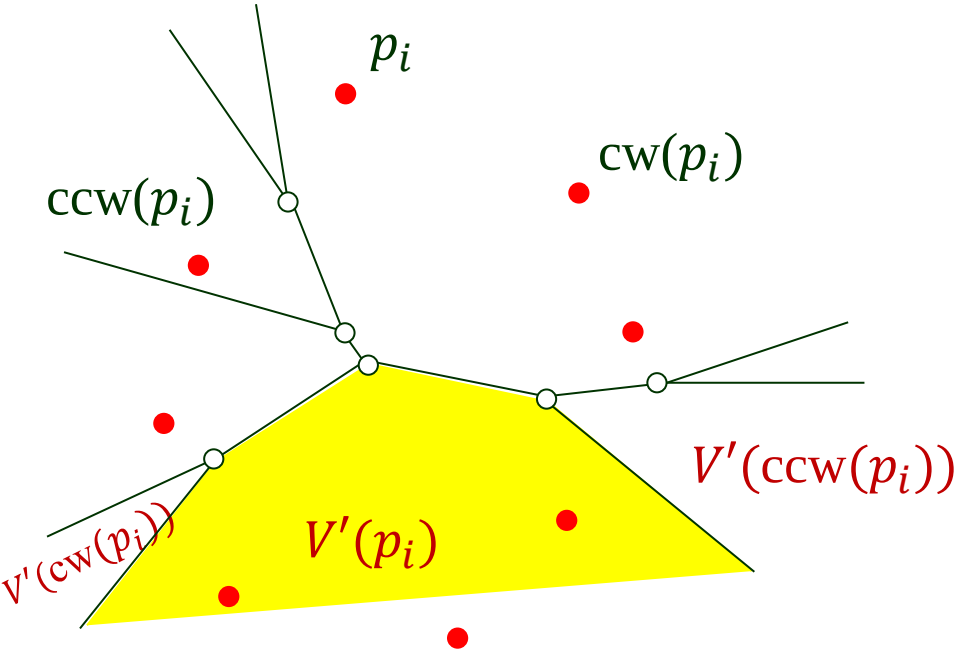
Trace out the boundary of $V'(p_i)$ by traversing a sequence of cells, each in a clockwise way.

Summary



- All new edges are added to DCEL.
- Afterward, all the edges lying inside the cell of p_i are removed.

Summary



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Theorem FPVD can be constructed in $O(n \log n)$ expected time using $O(n)$ storage.

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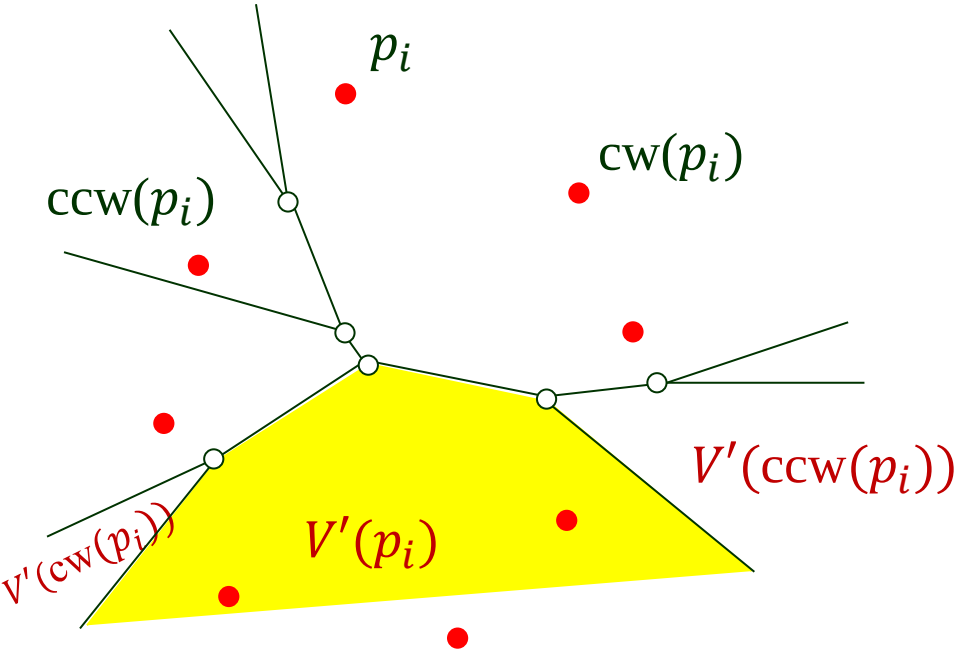

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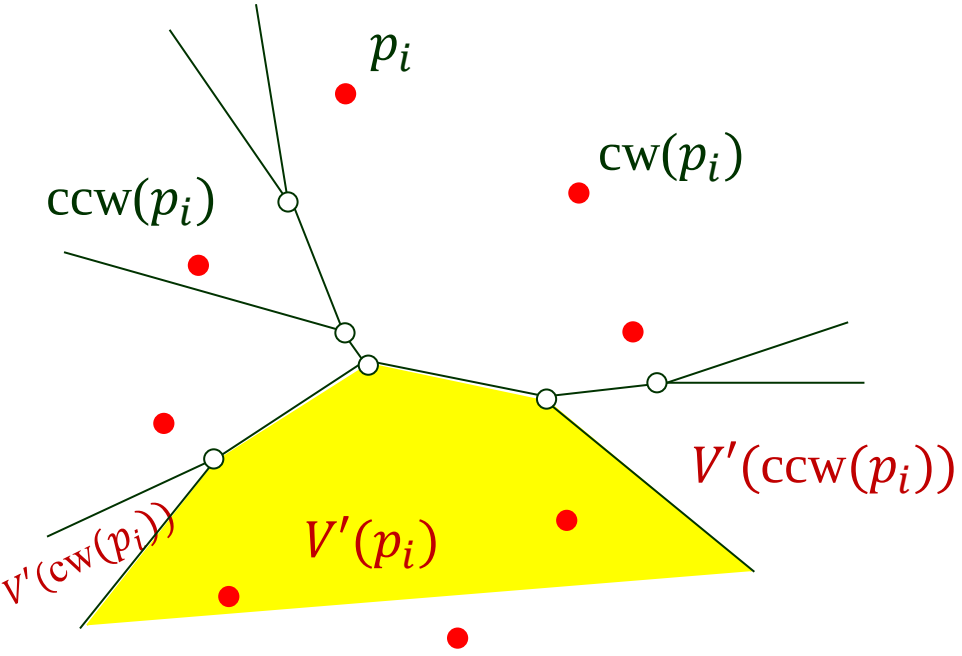

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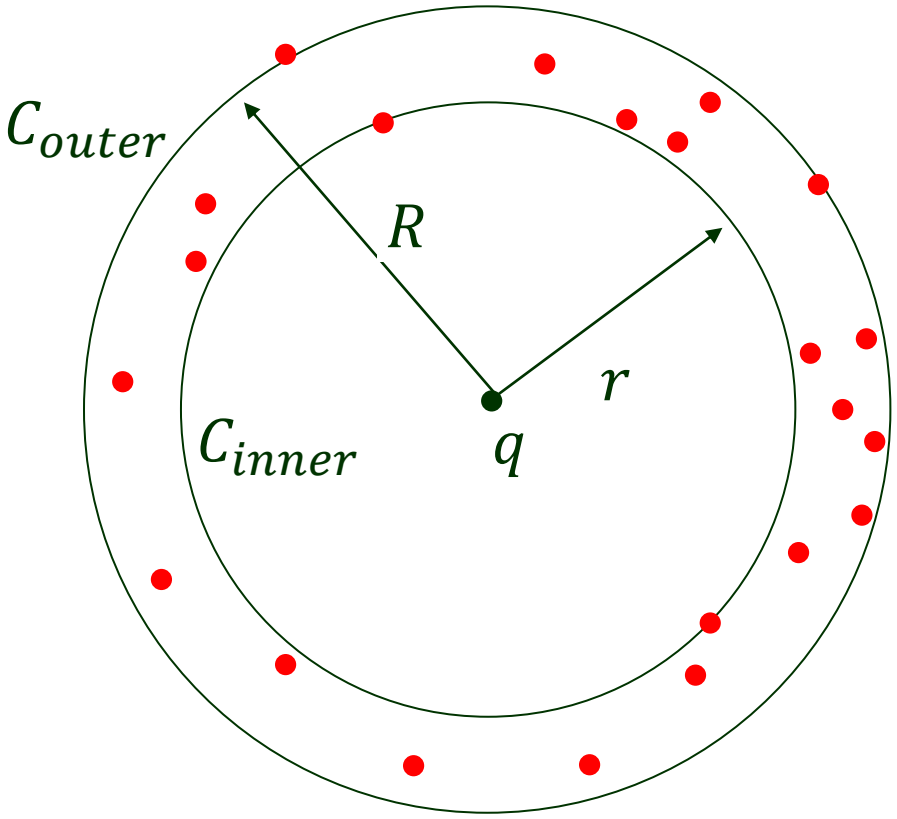
Theorem FPVD can be constructed in $O(n \log n)$ expected time using $O(n)$ storage.

- $O(n \log n)$ time to compute the convex hull.
- $O(n)$ time to construct FPVD (backward analysis).

V. Roundness of a Point Set

The ***roundness*** of a set of points is measured by the *minimum width* of any annulus that contains the points.



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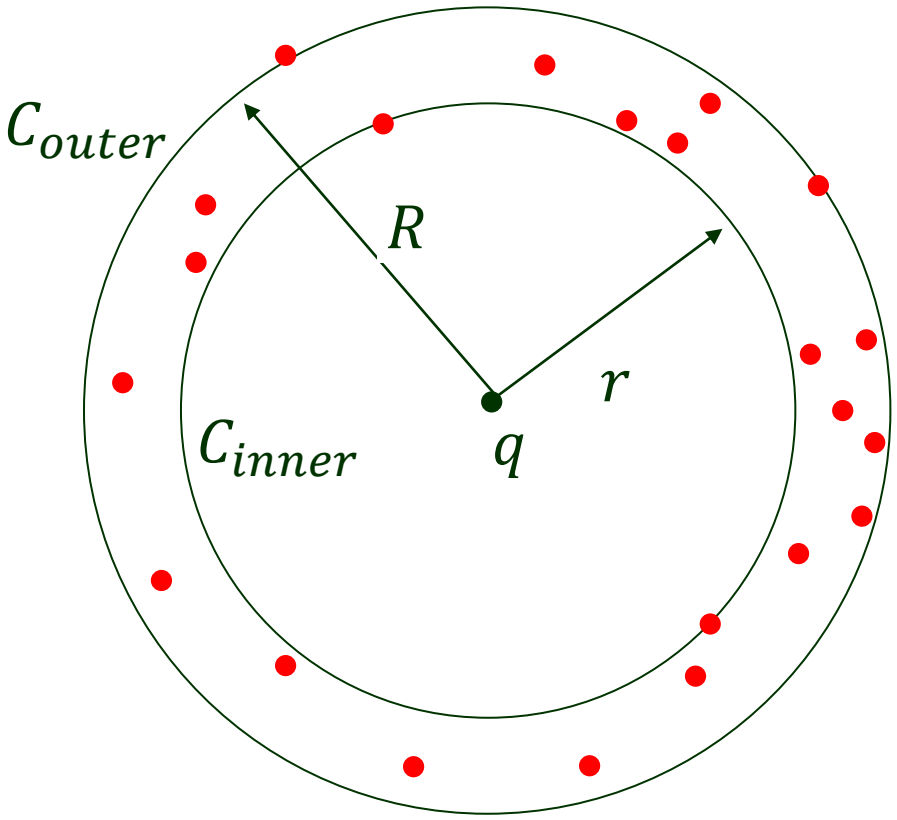

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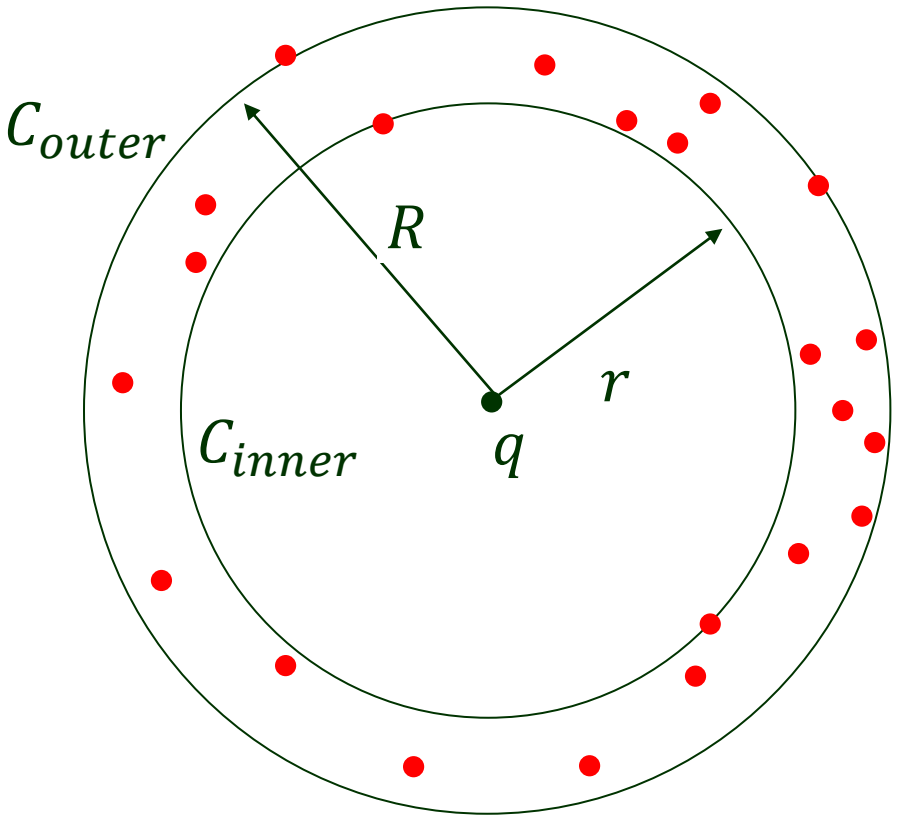

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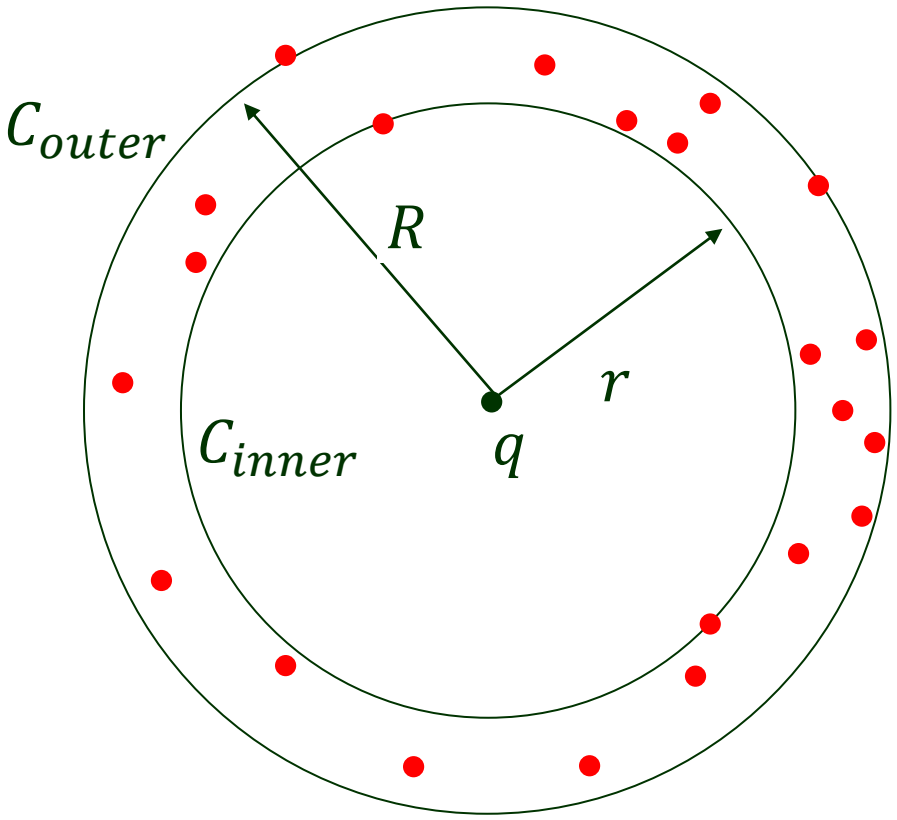

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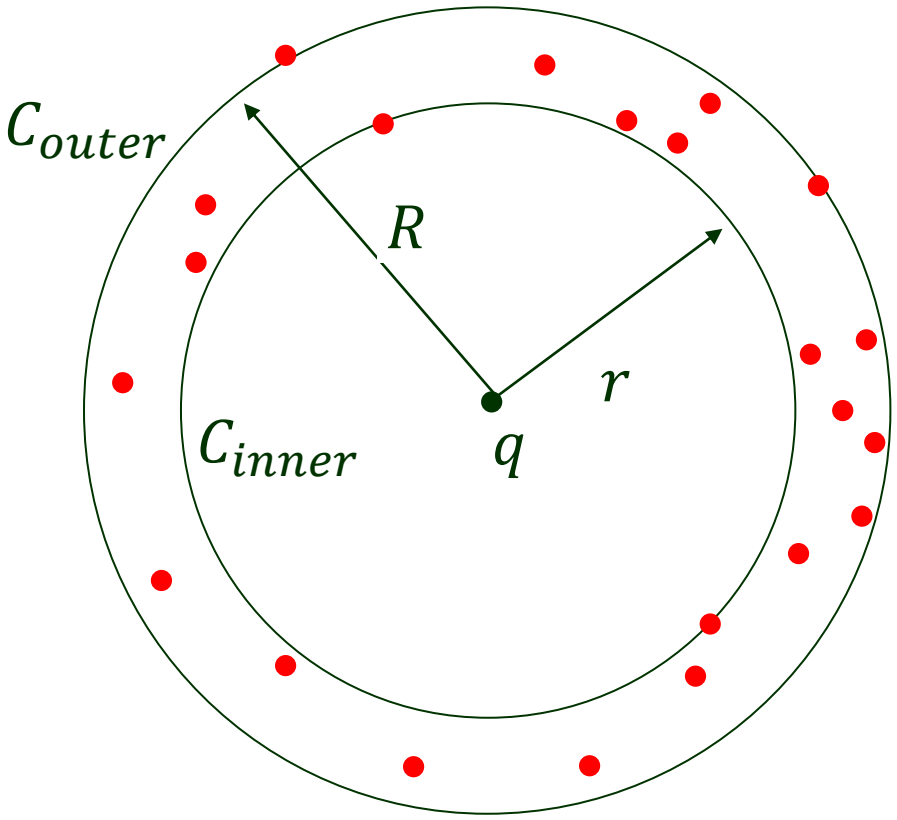

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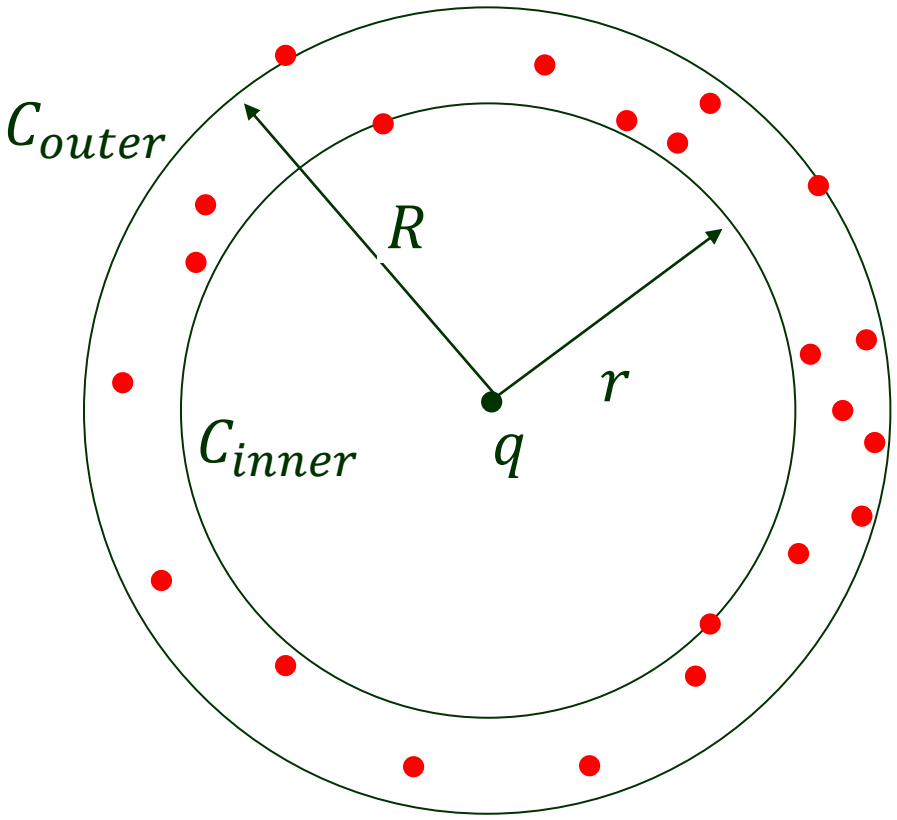

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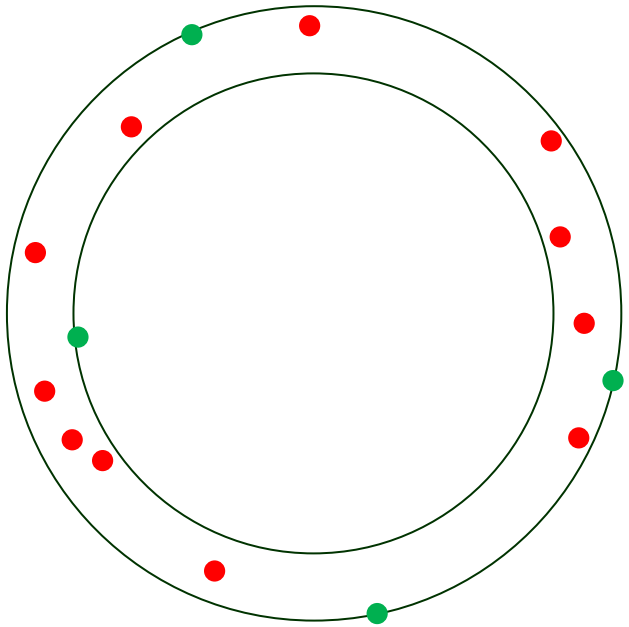
Need 4 constraints!

Only Three Different Cases

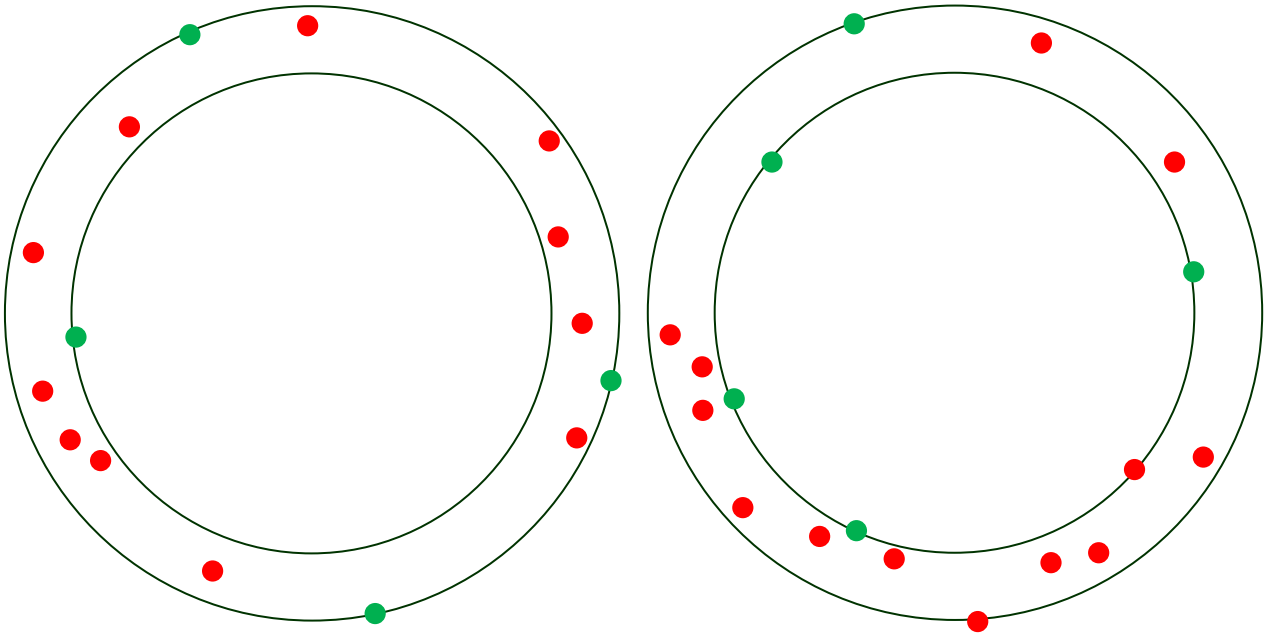


(a)

≥ 3 points on C_{outer}

≥ 1 point on C_{inner}

Only Three Different Cases



(a)

≥ 3 points on C_{outer}

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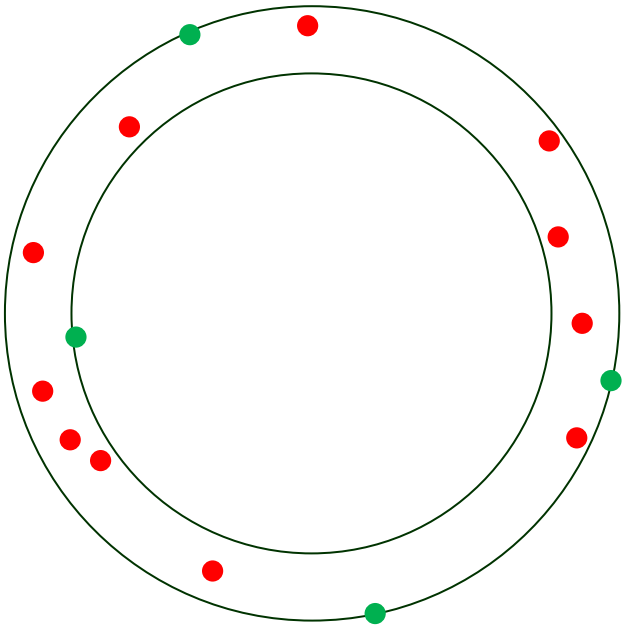
(b)

≥ 1 point on C_{outer}

≥ 3 points on C_{inner}

Only Three Different Cases



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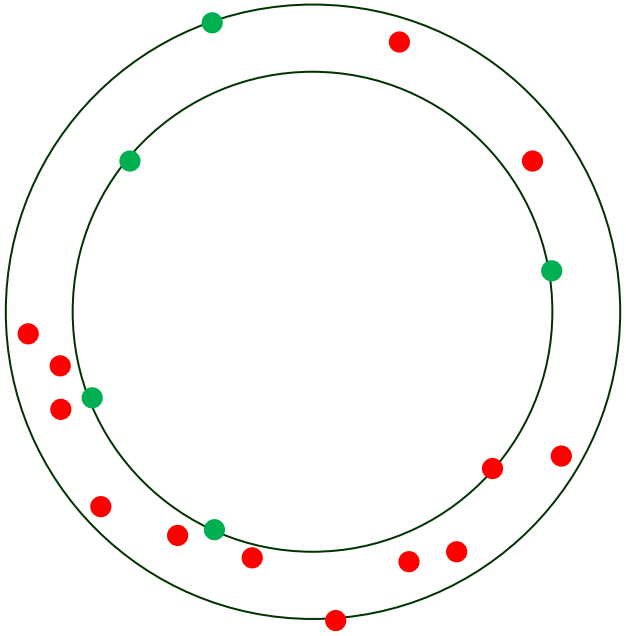
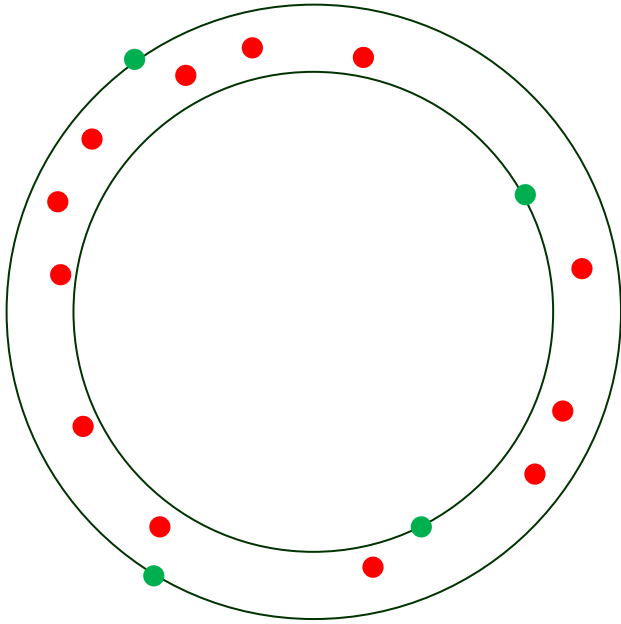


(b)

≥ 1 point on C_{outer}

≥ 3 points on C_{inner}



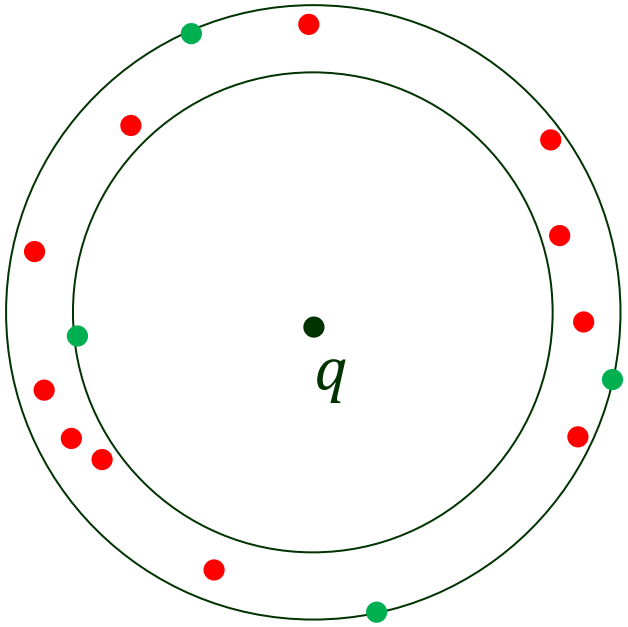
(c)

2 points on C_{outer}

2 points on C_{inner}

Smallest-Width Annulus – Case (a)

The problem is equivalent to finding the center point q of the annulus.



(a)

In case (a)

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$\geq$ 1 point on C_{inner}

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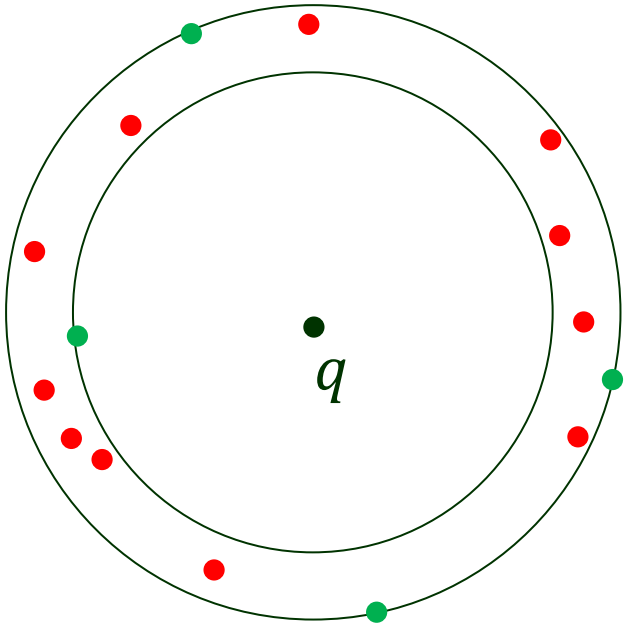

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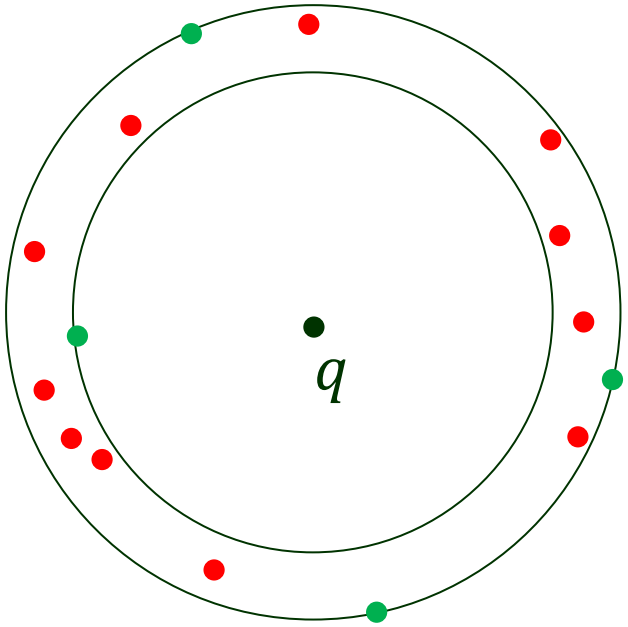

(a)

$\geq$ 3 points on C_{outer}

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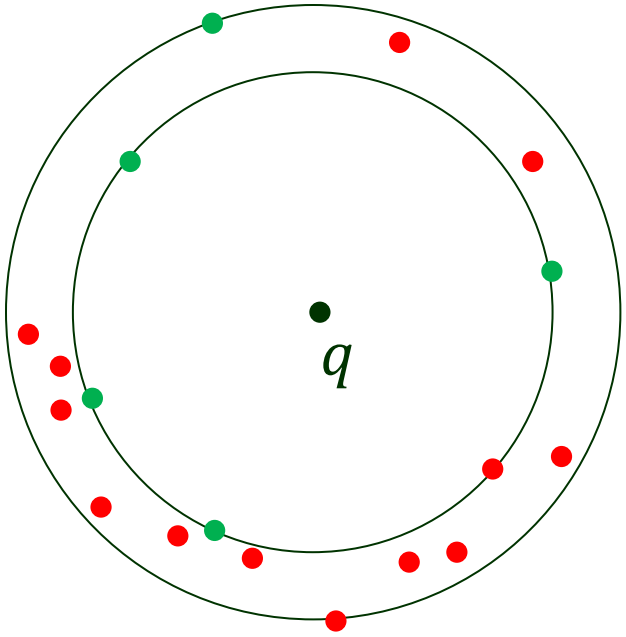
In case (a)

$\geq$ 3 points on C_{outer}

$\Downarrow$

q must be a vertex of the farthest-point Voronoi diagram.

Case (b)



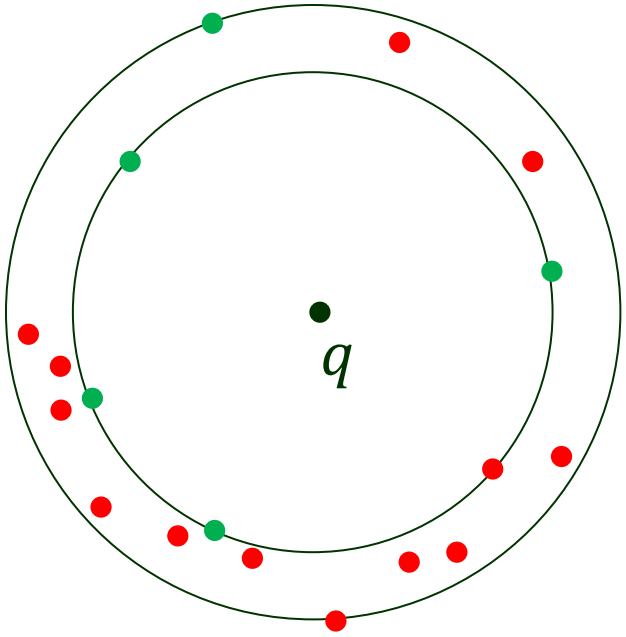
(b)

≥ 1 point on C_{outer}

≥ 3 points on C_{inner}

≥ 3 points on C_{inner}

Case (b)



(b)

$\geq$ 1 point on C_{outer}

$\geq$ 3 points on C_{inner}

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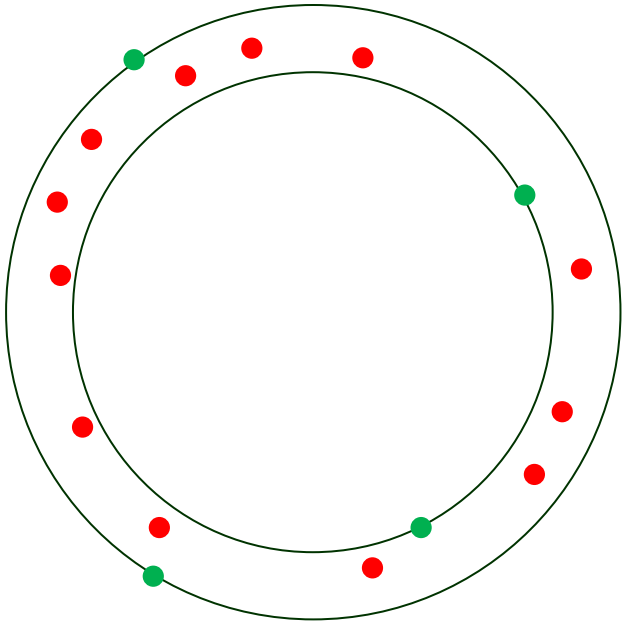
q must be a vertex of the (nearest-point) Voronoi diagram.

Case (c)

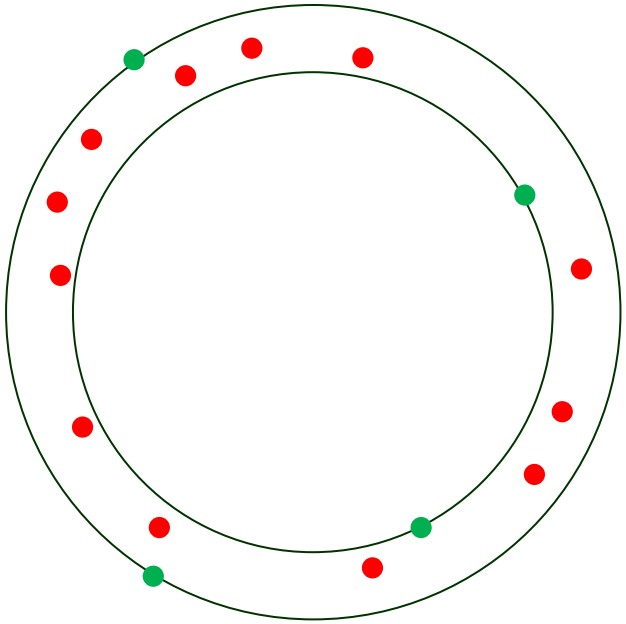


(c)

2 points on C_{outer}

2 points on C_{inner}

Case (c)

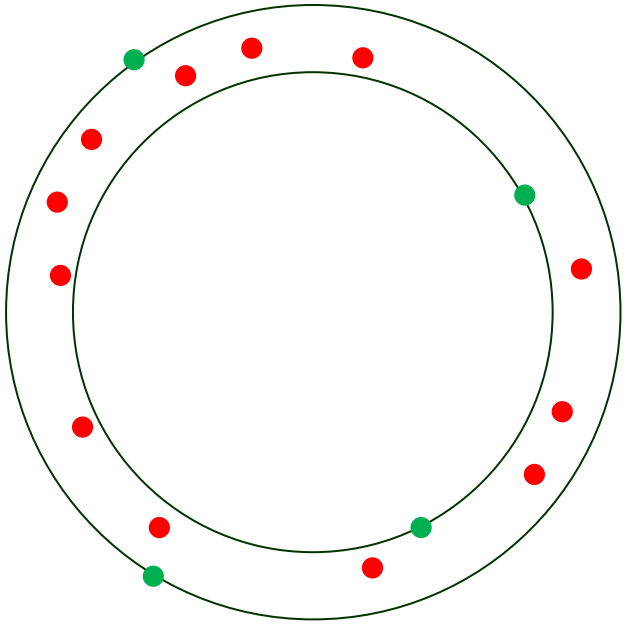


(c)

2 points on C_{outer} $\Longrightarrow$ q must be on an edge of the FPVD.

2 points on C_{inner}

Case (c)



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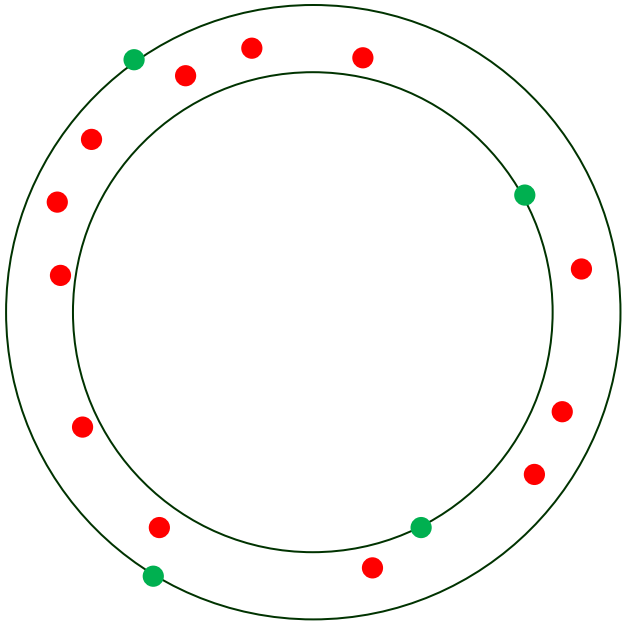
2 points on C_{outer} $\Longrightarrow$ q must be on an edge of the FPVD.

2 points on C_{inner} $\Longrightarrow$ q must be on an edge of the VD.

Case (c)



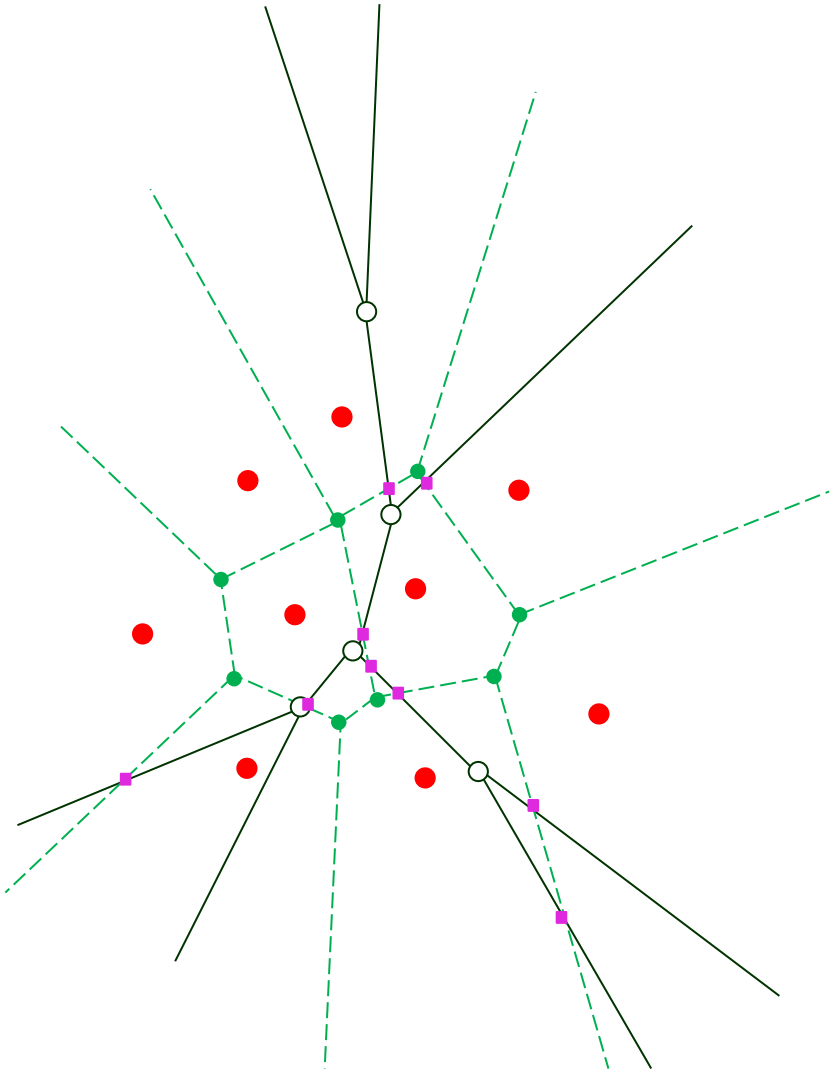
(c)

q must be at the intersection of an VD edge with an FPVD edge.

2 points on C_{outer} $\Longrightarrow$ q must be on an edge of the FPVD.

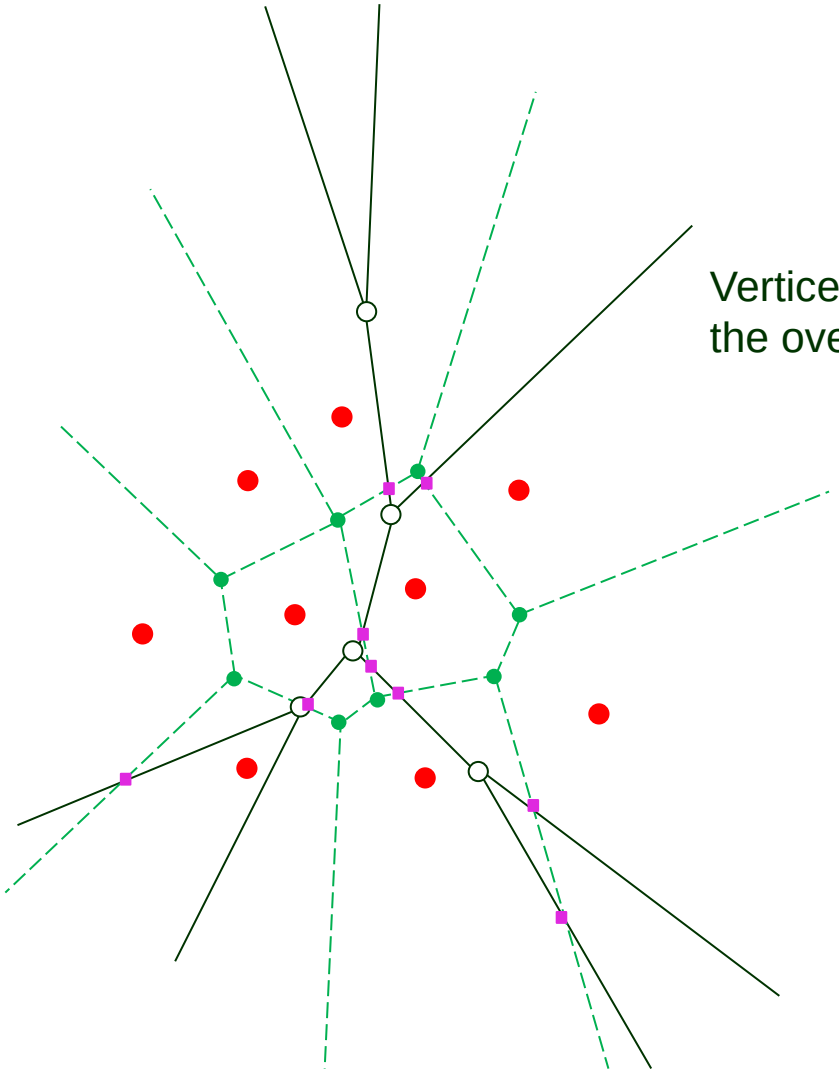
2 points on C_{inner} $\Longrightarrow$ q must be on an edge of the VD.

Overlay of VD and FPVD



- Site
- Vertex of VD
- Vertex of FPVD
- Intersection of VD and FPVD

Overlay of VD and FPVD

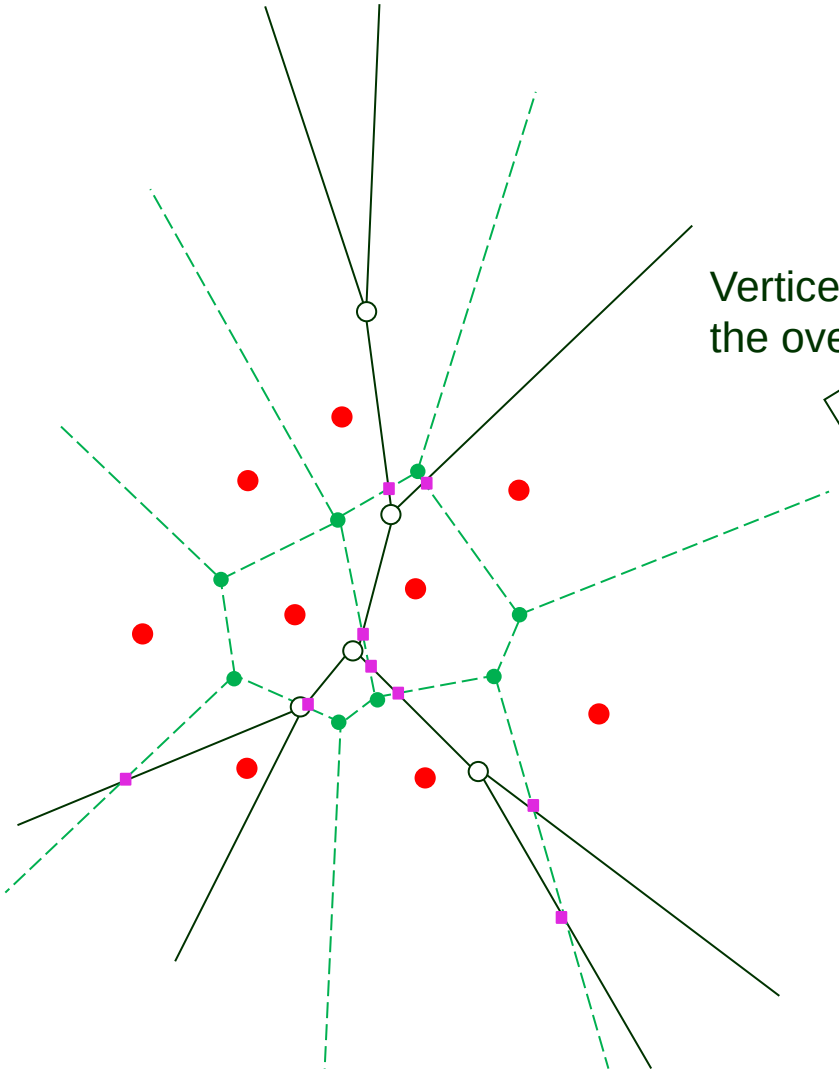


Site

Vertices of the overlay

- Vertex of VD
- Vertex of FPVD
- Intersection of VD and FPVD

Overlay of VD and FPVD



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Vertices of the overlay

Intersection of VD and FPVD

Exactly the candidate centers of the smallest-width annulus.

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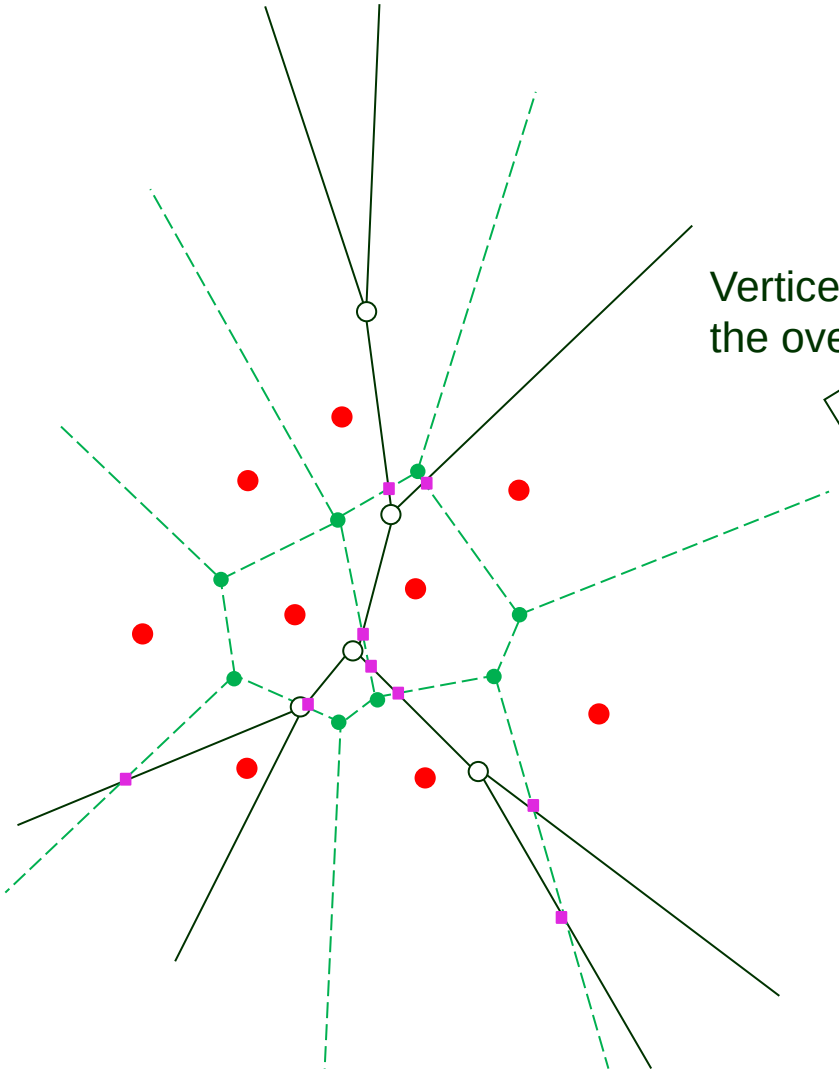

Site

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Vertex of FPVD

Intersection of VD and FPVD

Vertices of the overlay

Exactly the candidate centers of the smallest-width annulus.

No need to compute the overlay!

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Algorithm (cont'd)

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Theorem Given a set of n points in the plane, the smallest-width annulus can be determined in $O(n^2)$ time using $O(n)$ storage.