Yan-Bin Jia, Iowa State University

Voronoi Diagrams

Outline:

I. Problem definition

II. Voronoi cells

III. Delaunay triangulations

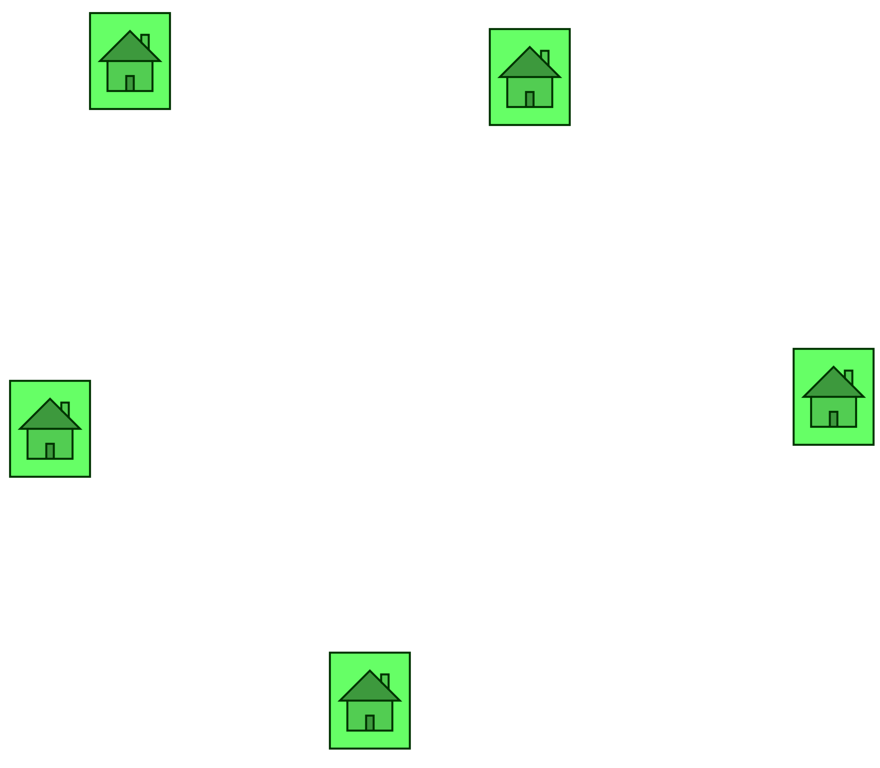
IV. Geometric complexity

V. Beach line in construction

VI. Site event

Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

I. Closest Café on Campus



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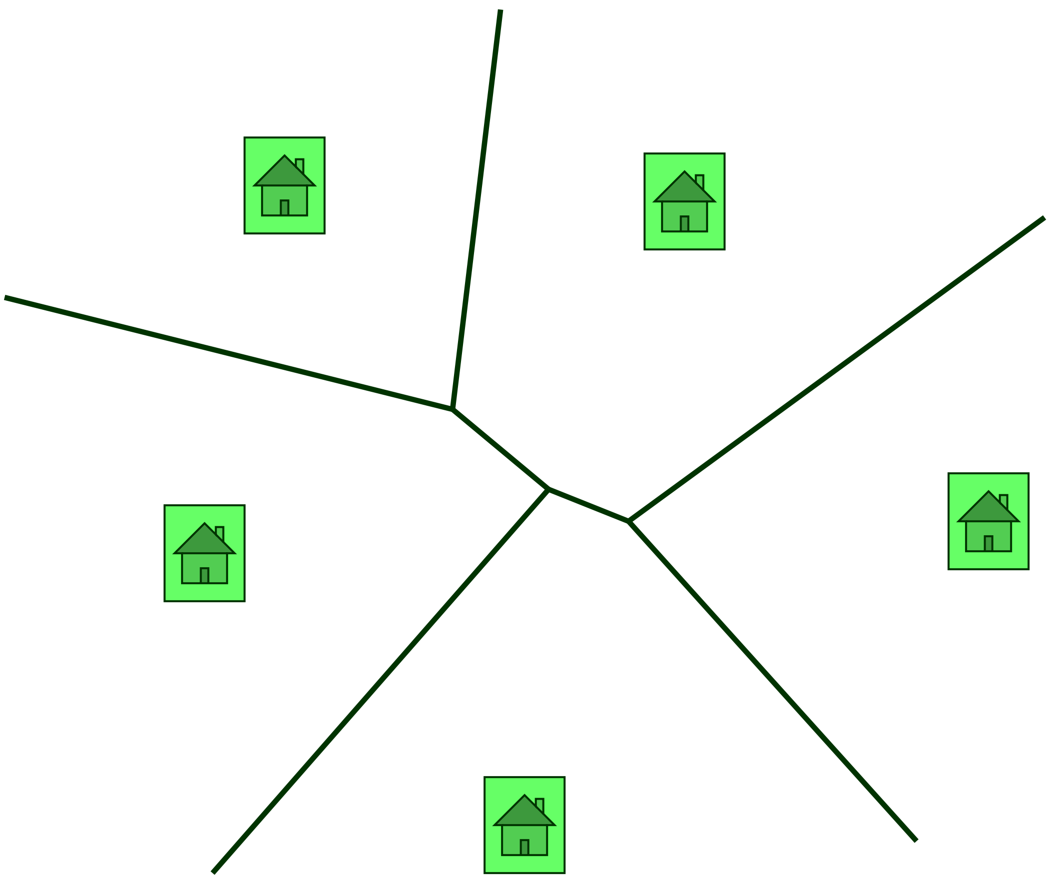

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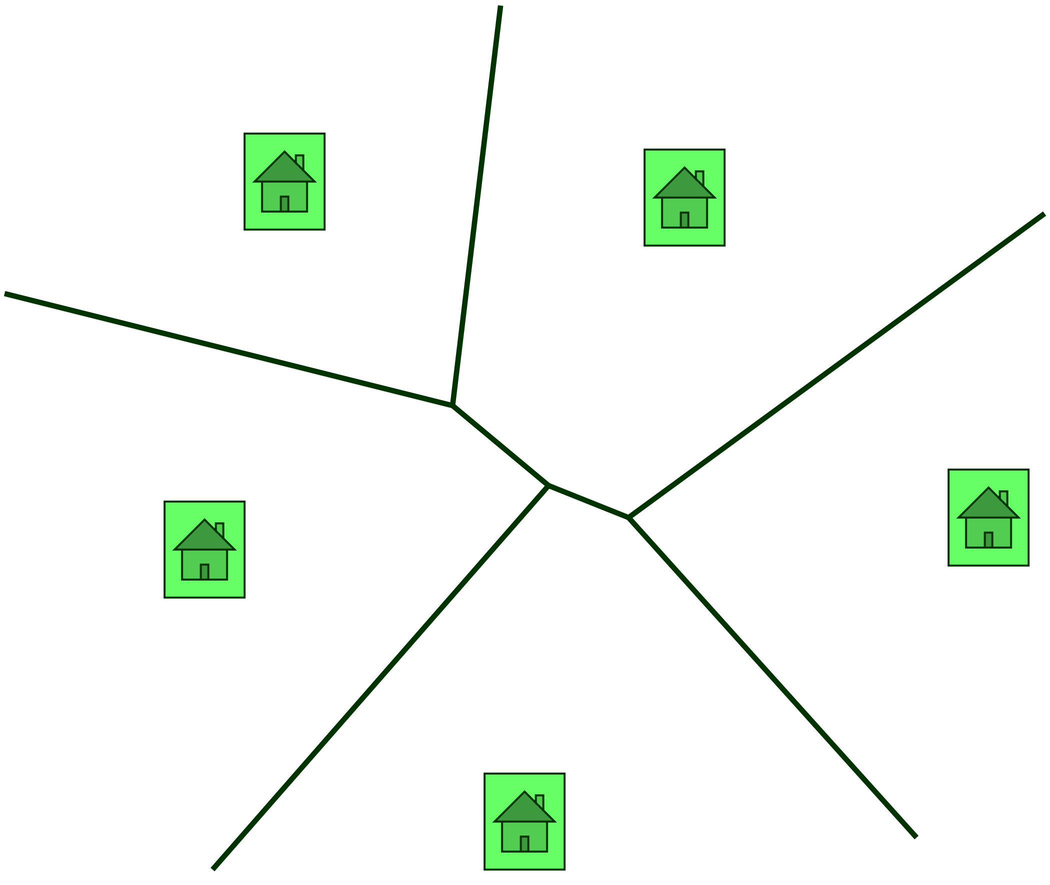

Voronoi diagram

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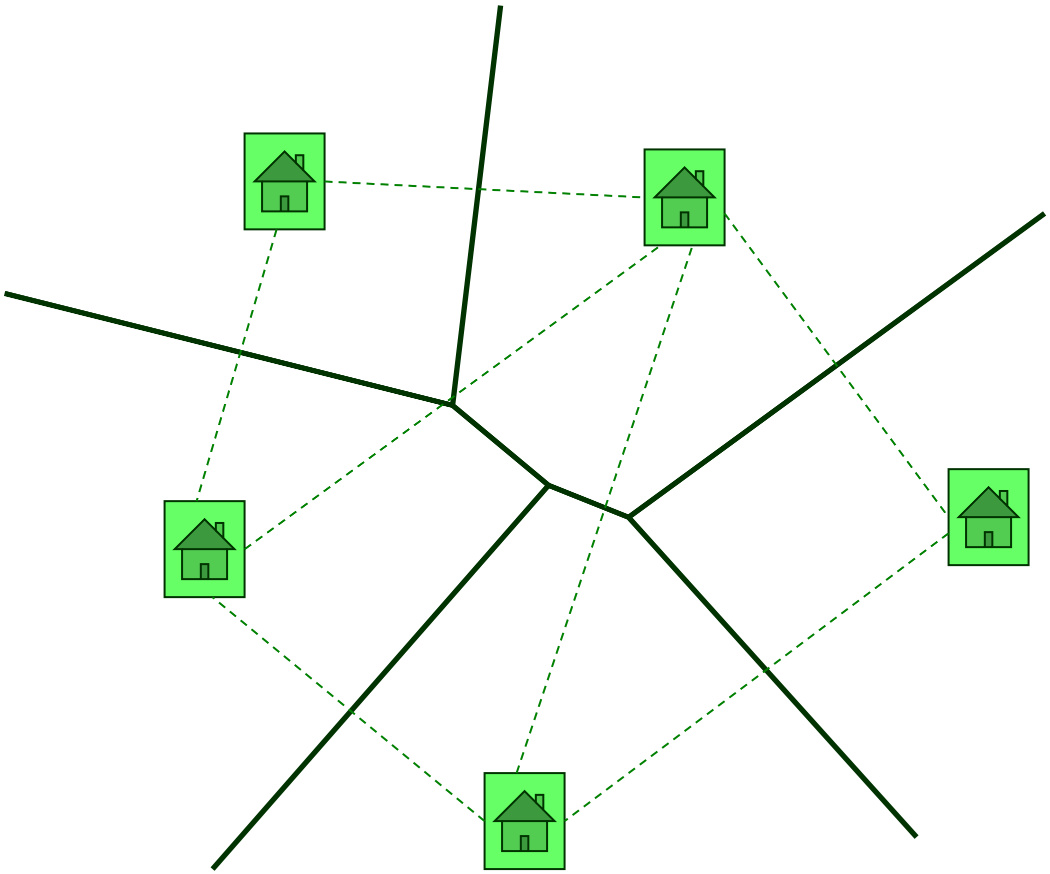

Voronoi diagram

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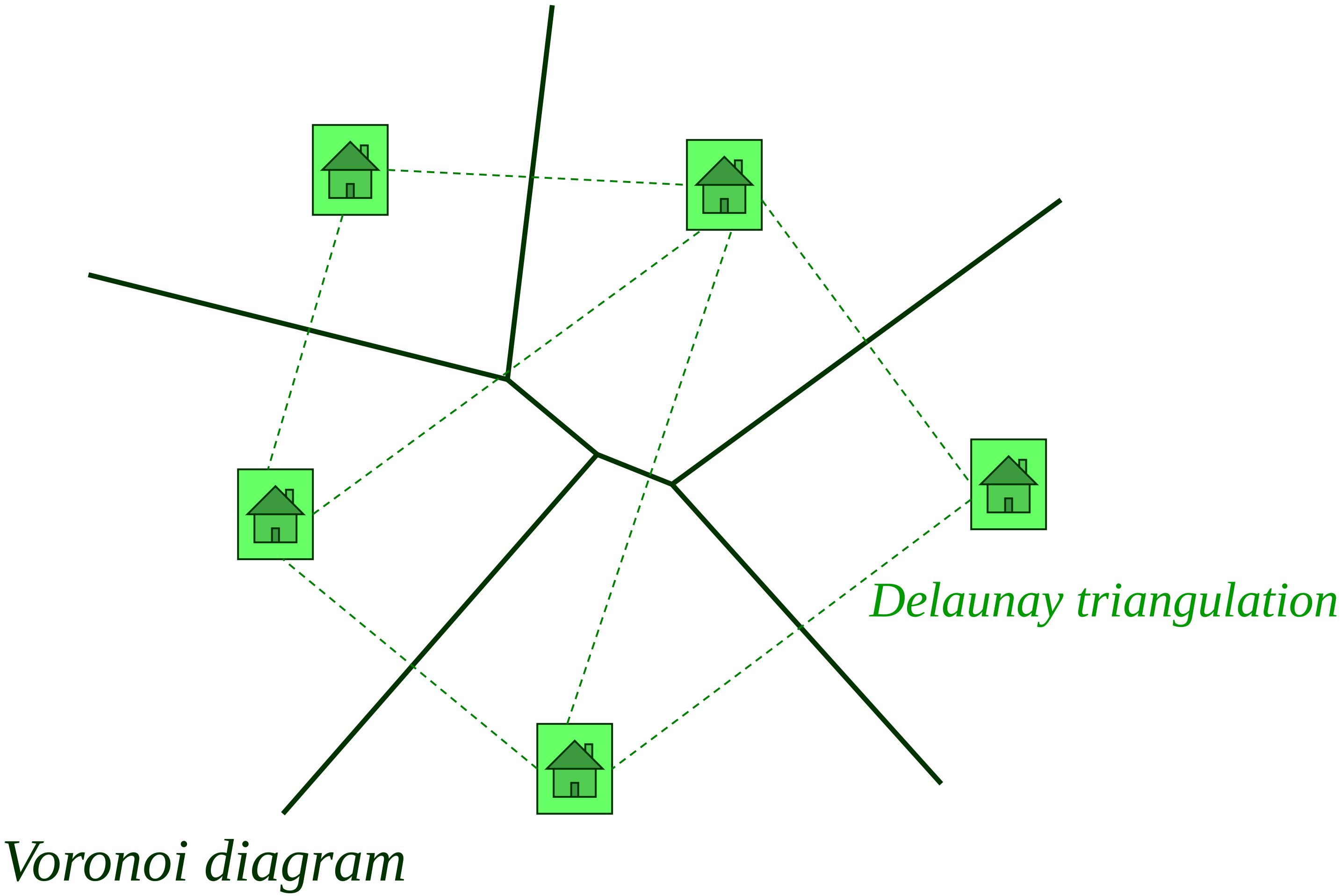
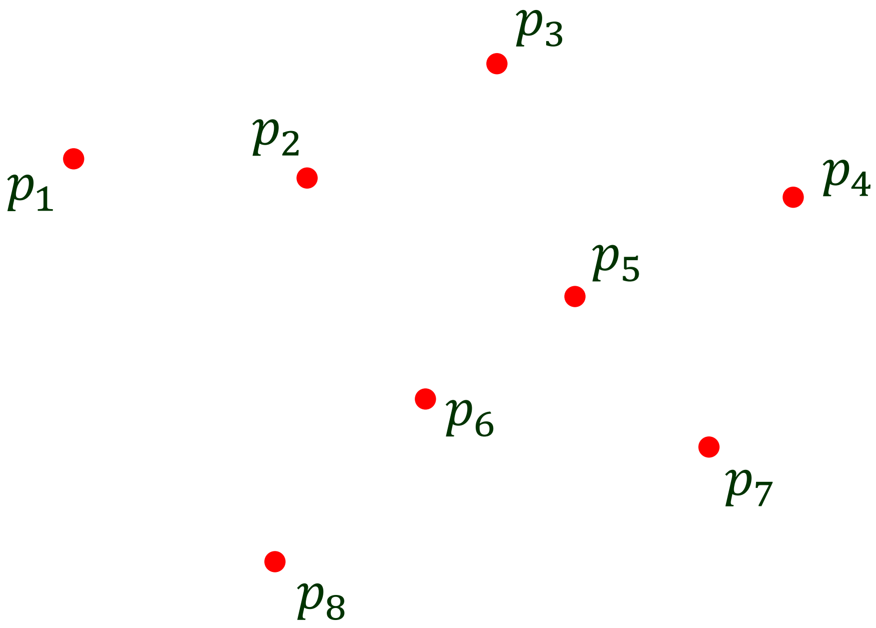


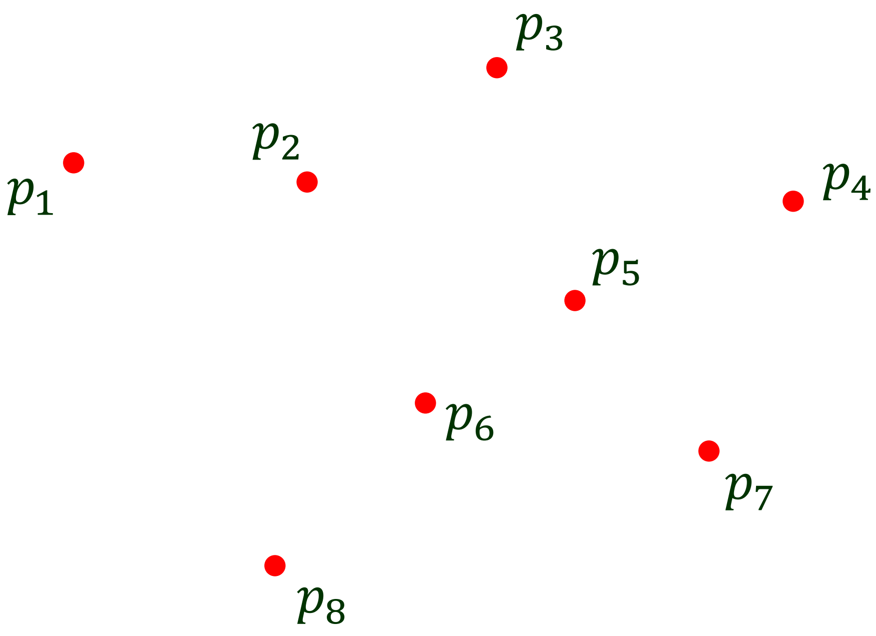
Input: Point Set

$P = \{p_1, p_2, \ldots, p_n\}$

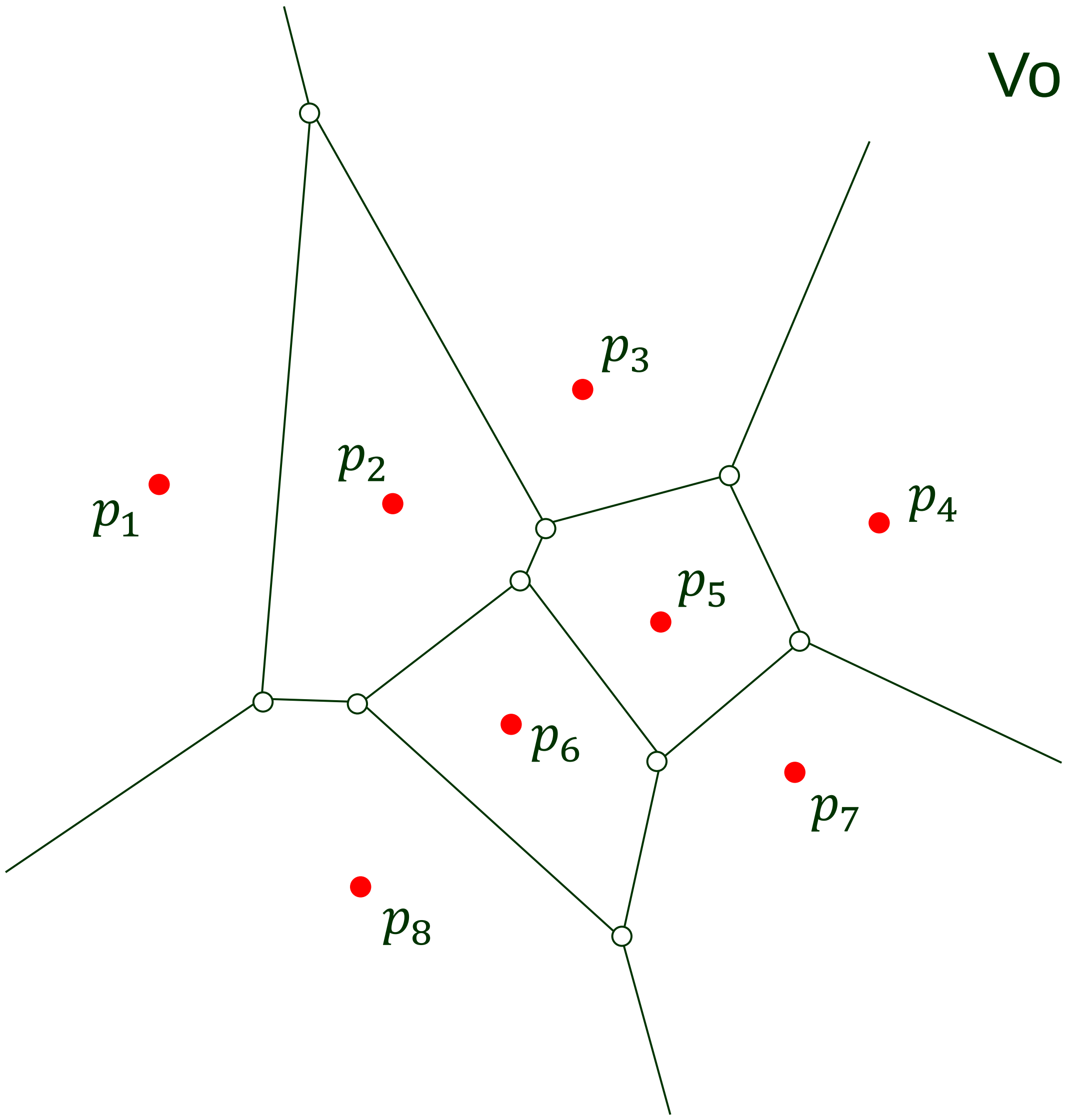


Input: Point Set

$$P = \{\underbrace{p_1, p_2, \ldots, p_n}_{\text{Sites}}\}$$

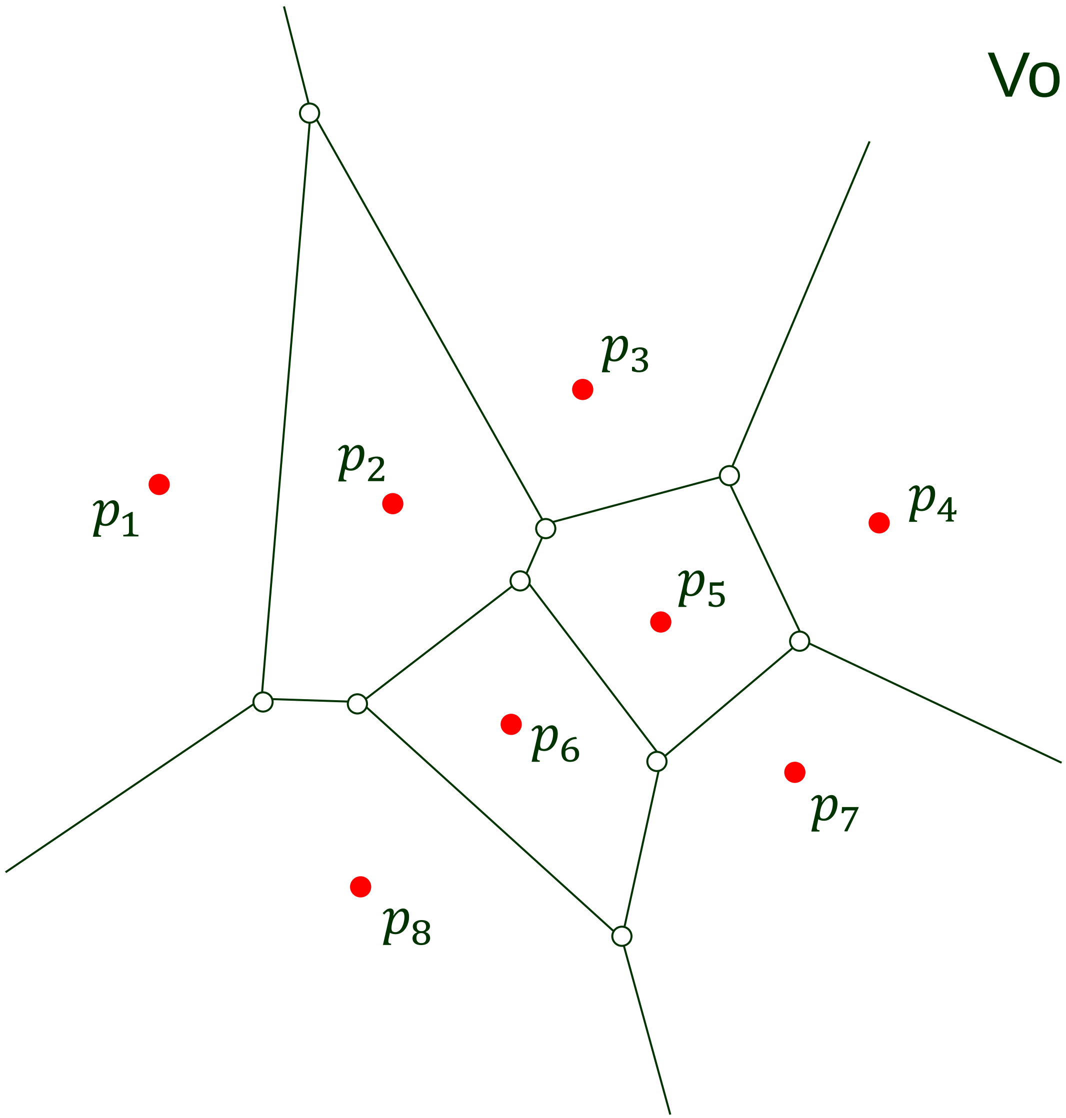


Output: Voronoi Diagram



Vor(P): subdivision

Output: Voronoi Diagram

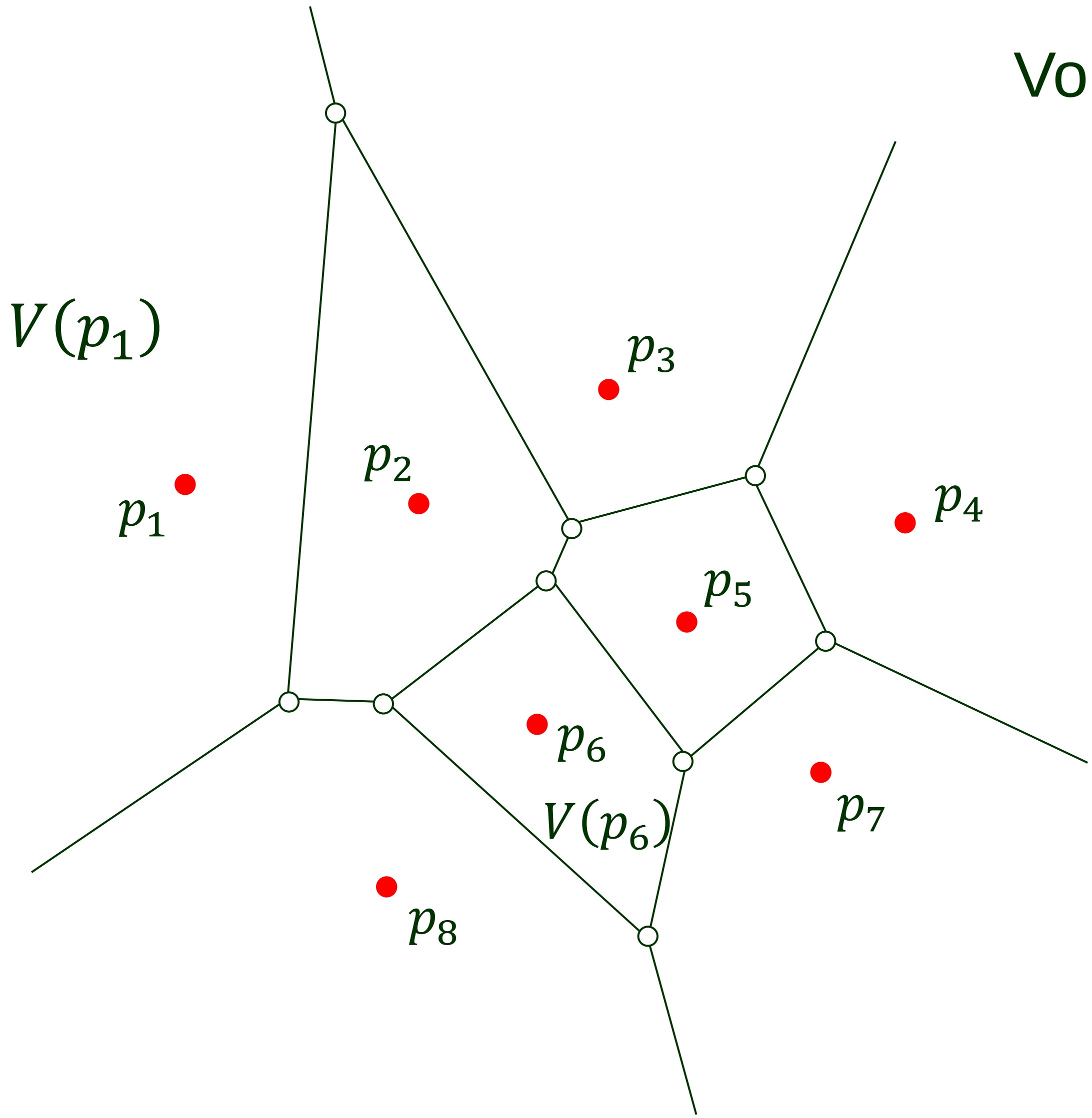


Vor(P): subdivision

n cells $(1 \leq i \leq n)$

$V(p_i)$

Output: Voronoi Diagram



Vor(P): subdivision

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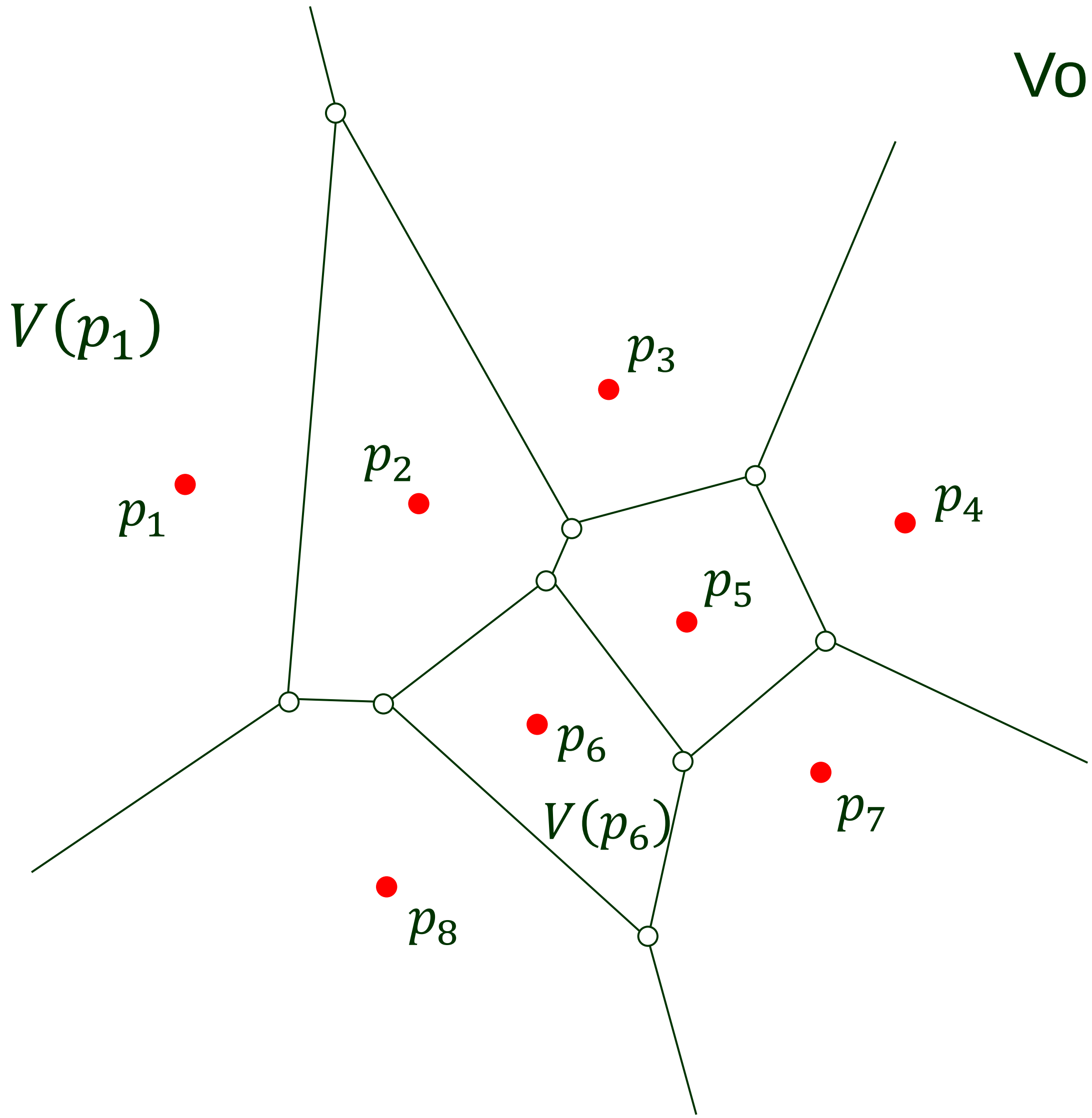
Voronoi cell

Output: Voronoi Diagram



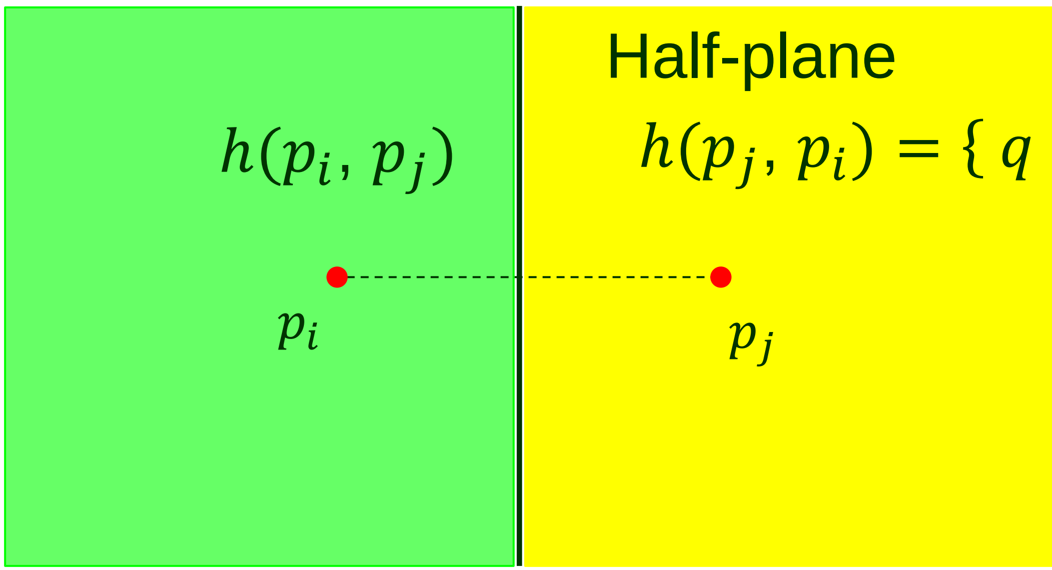
Vor(P): subdivision

n cells $(1 \leq i \leq n)$

$V(p_i) = \{\, q \mid q$ ***closer*** to p_i
than to any p_j
with $j \neq i\}$

Voronoi cell

Two Sites



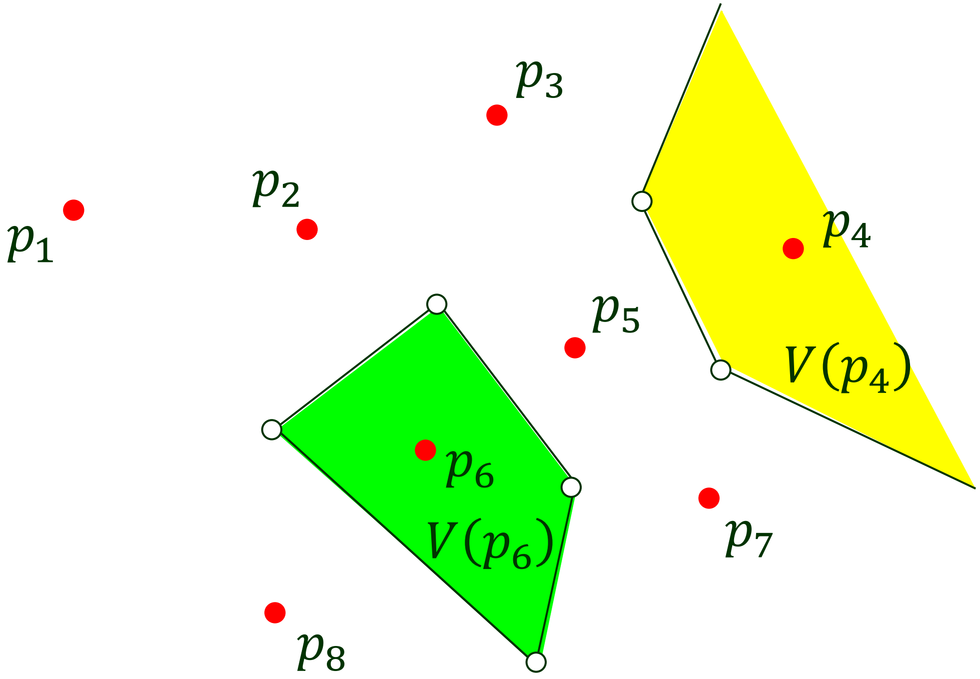
Half-plane

$h(p_j, p_i) = \{\, q \mid q$ closer to p_j than to $p_i \}$

Perpendicular bisector

II. Voronoi Cells

$$V(p_i) = \bigcap_{\substack{1 \le j \le n \\ j \ne i}} h(p_i, p_j)$$



II. Voronoi Cells

$V(p_6)$ is determined by p_2, p_5, p_7, p_8 only.

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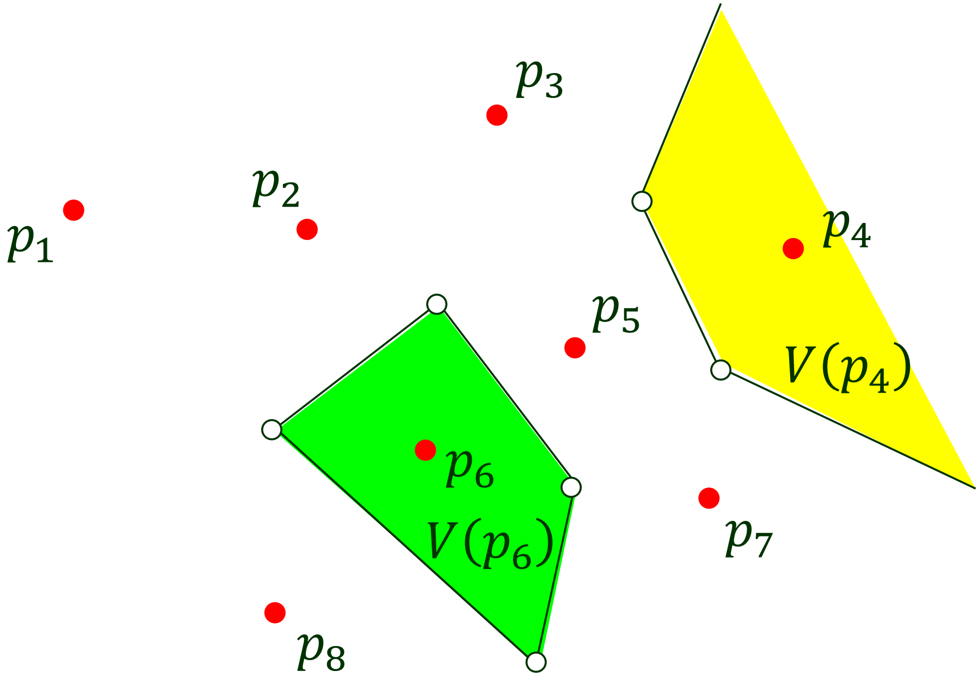

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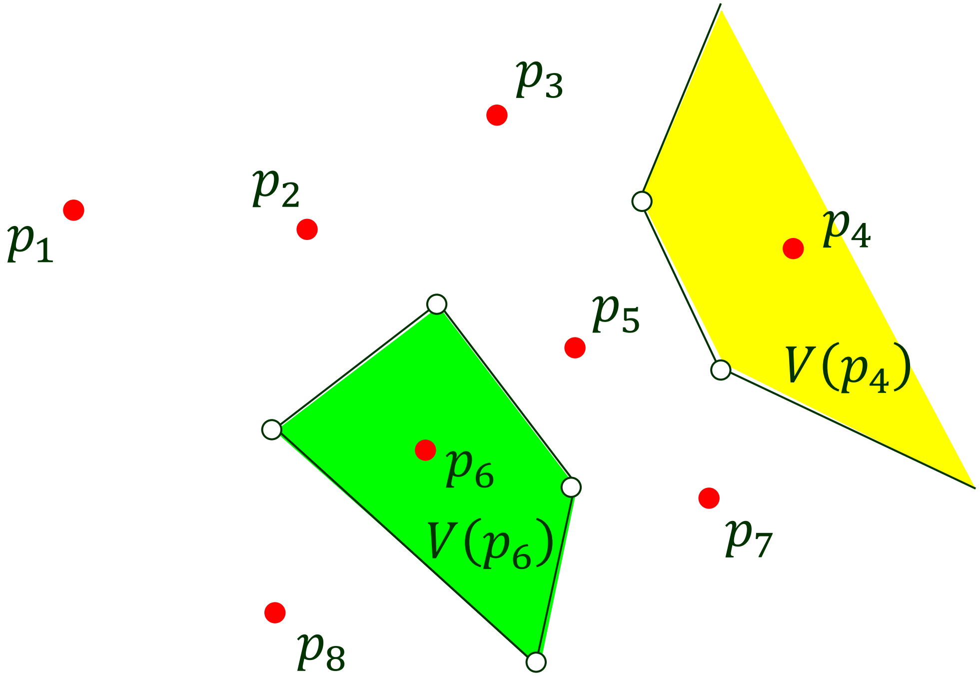

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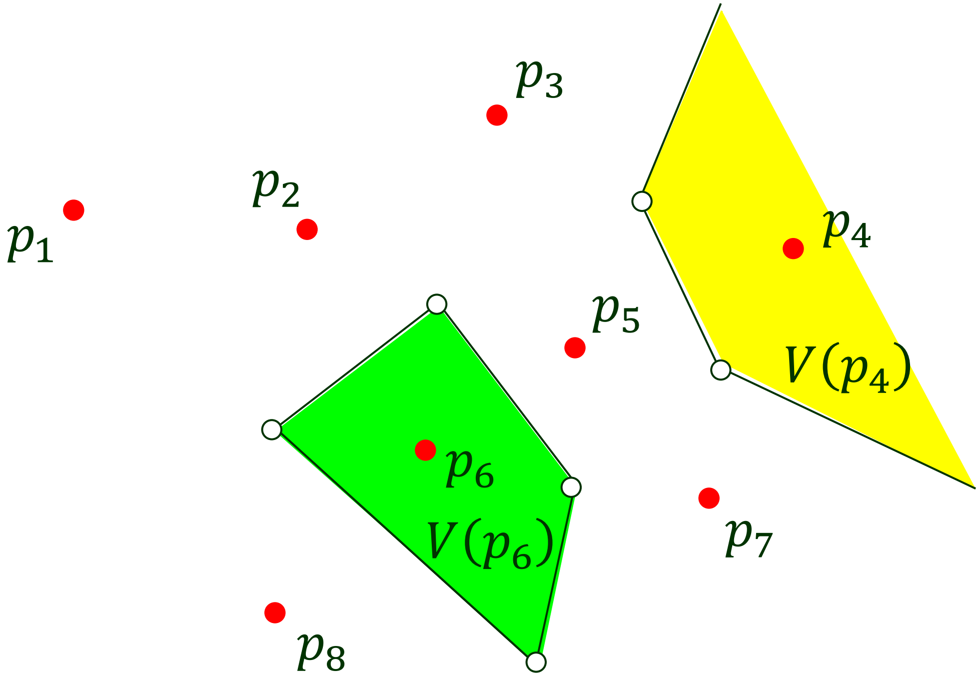

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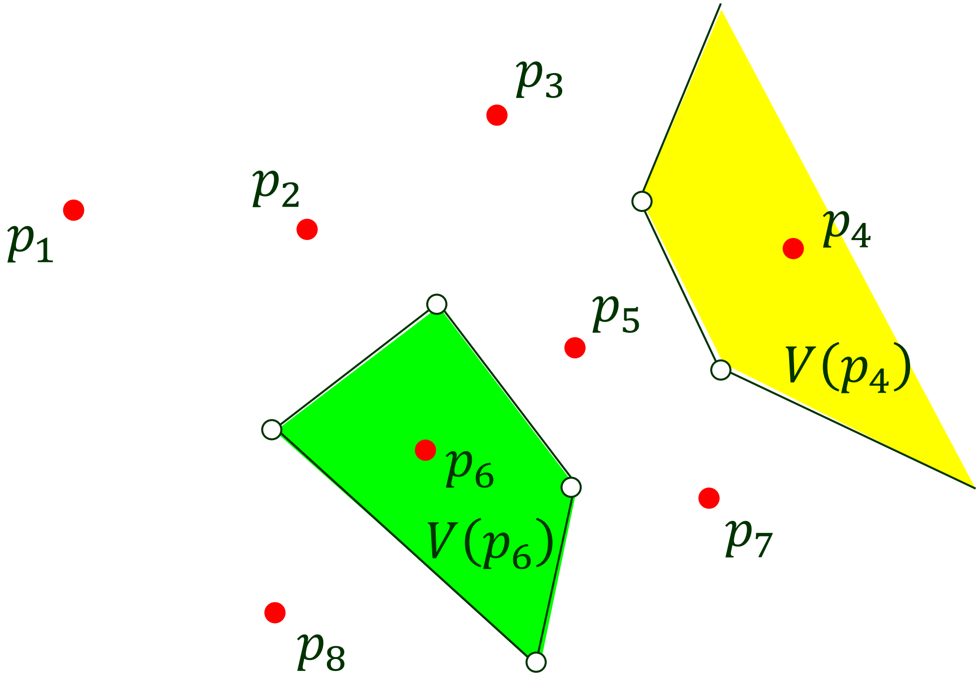
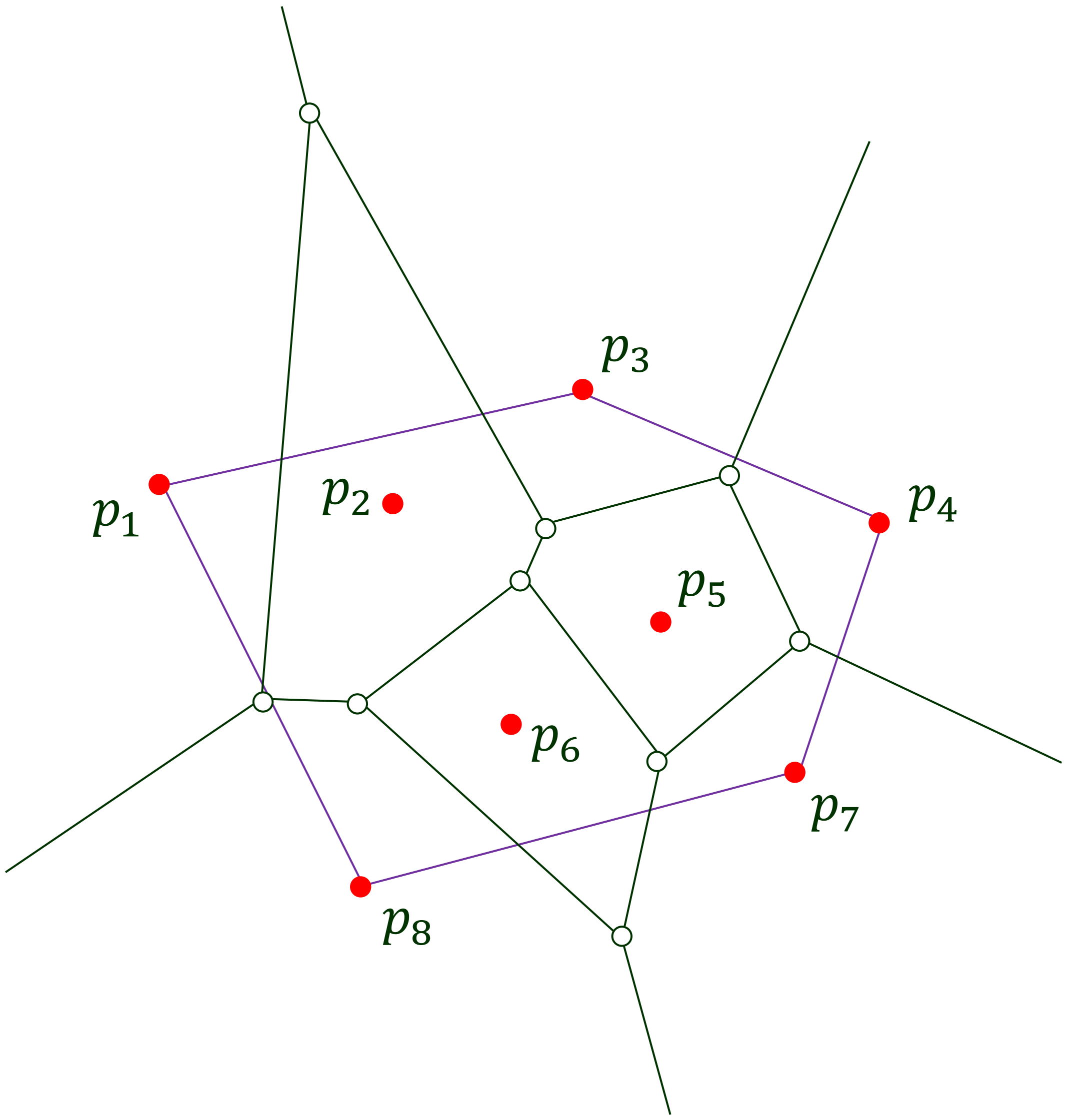


- Open convex region
  (open set not necessarily unbounded)
- Possibly unbounded
- $\le n - 1$ vertices
- $\le n - 1$ edges

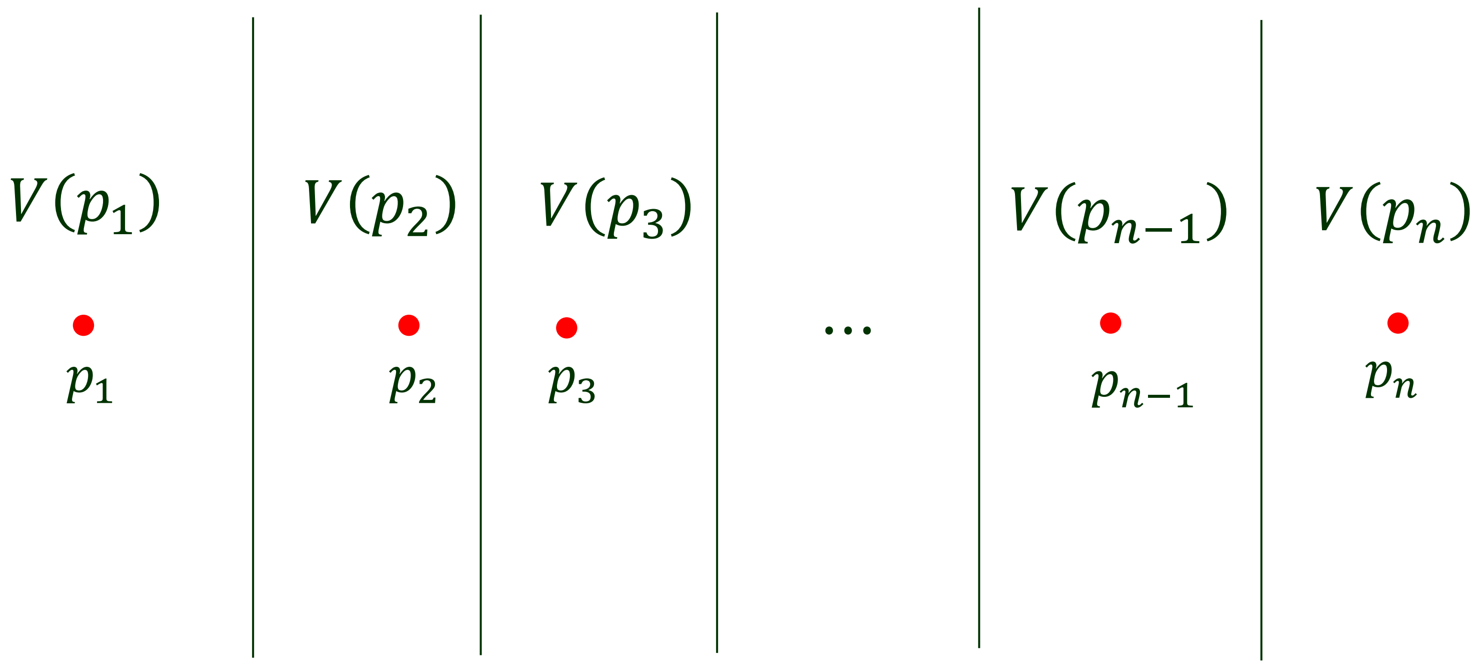
Unbounded Voronoi Cells



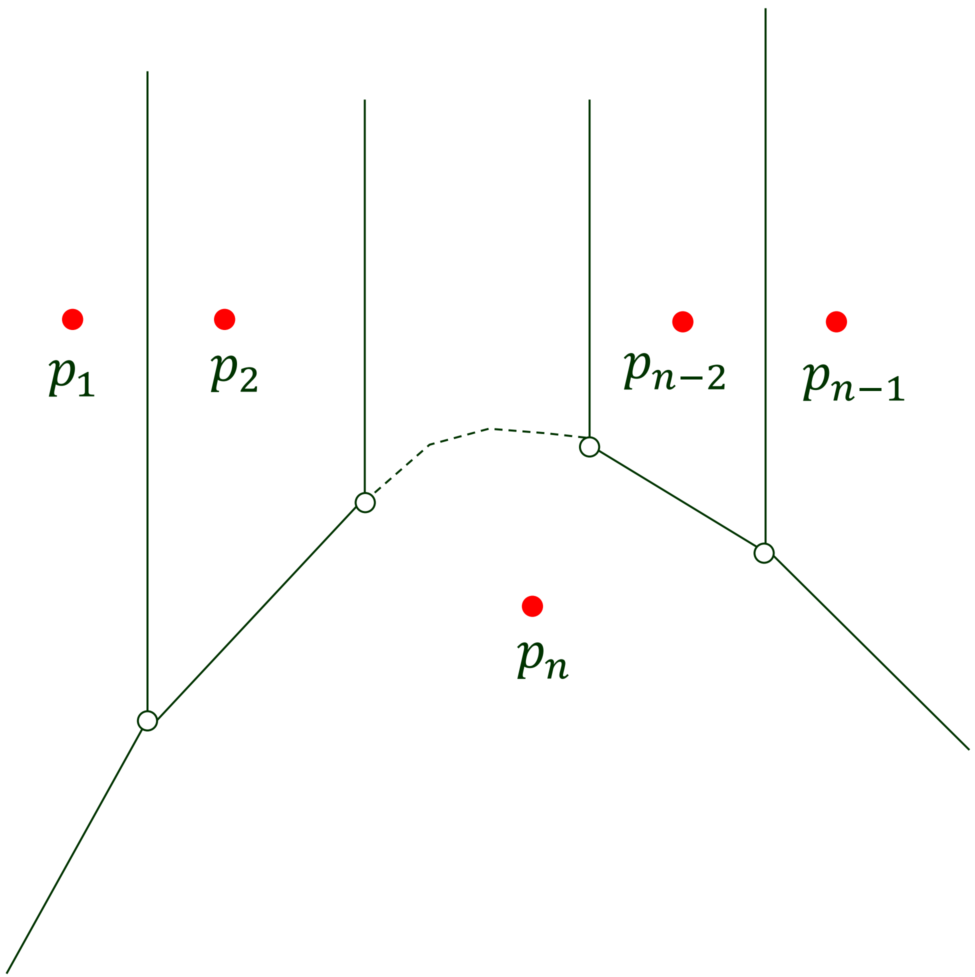
$V(p_i)$ is *unbounded* iff p_i lies on the boundary of the convex hull of P.

Only Case of Disconnected VD

All the sites are collinear.

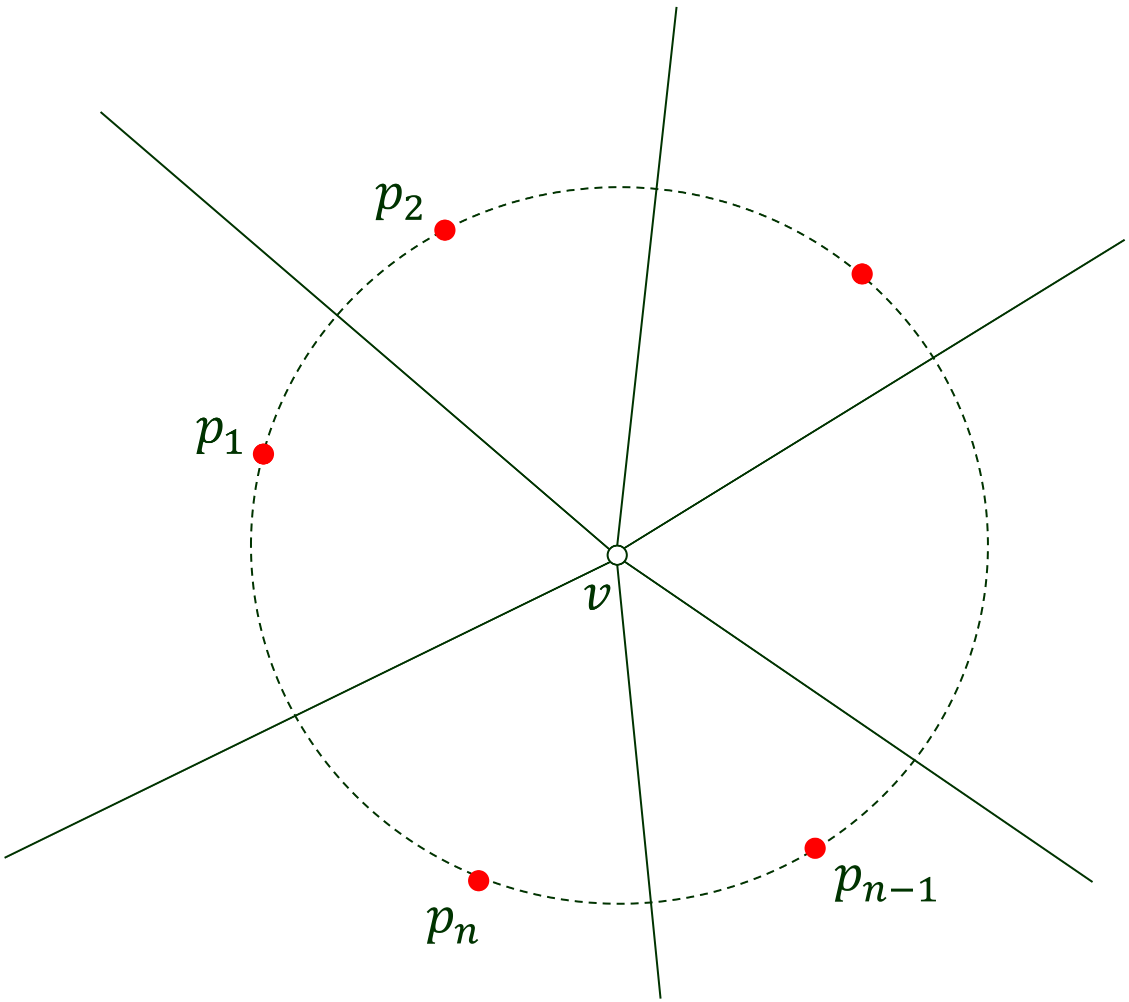


Only $n-$ 1 Sites Collinear

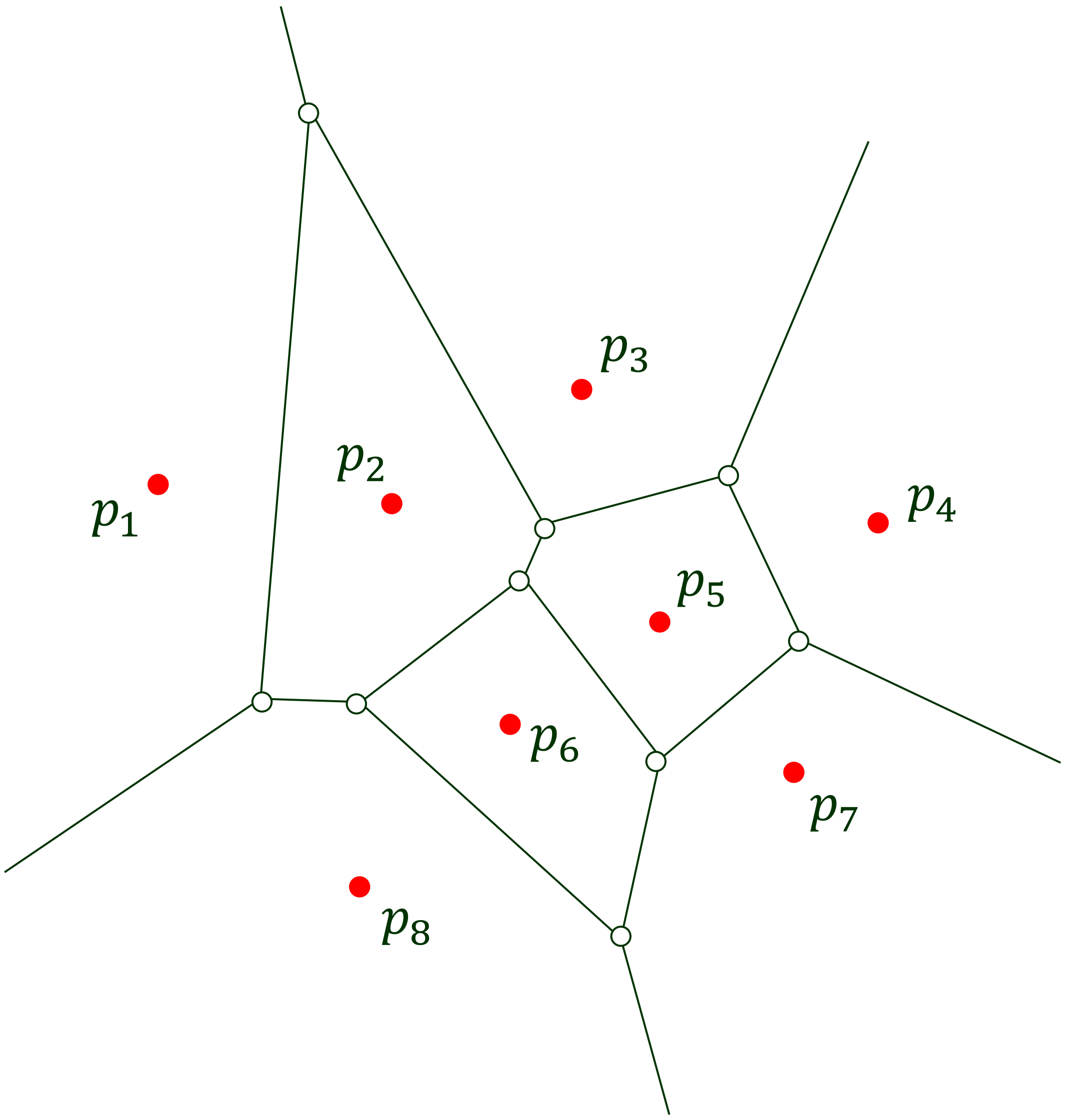


Only One Vertex

All the sites are on the same circle.



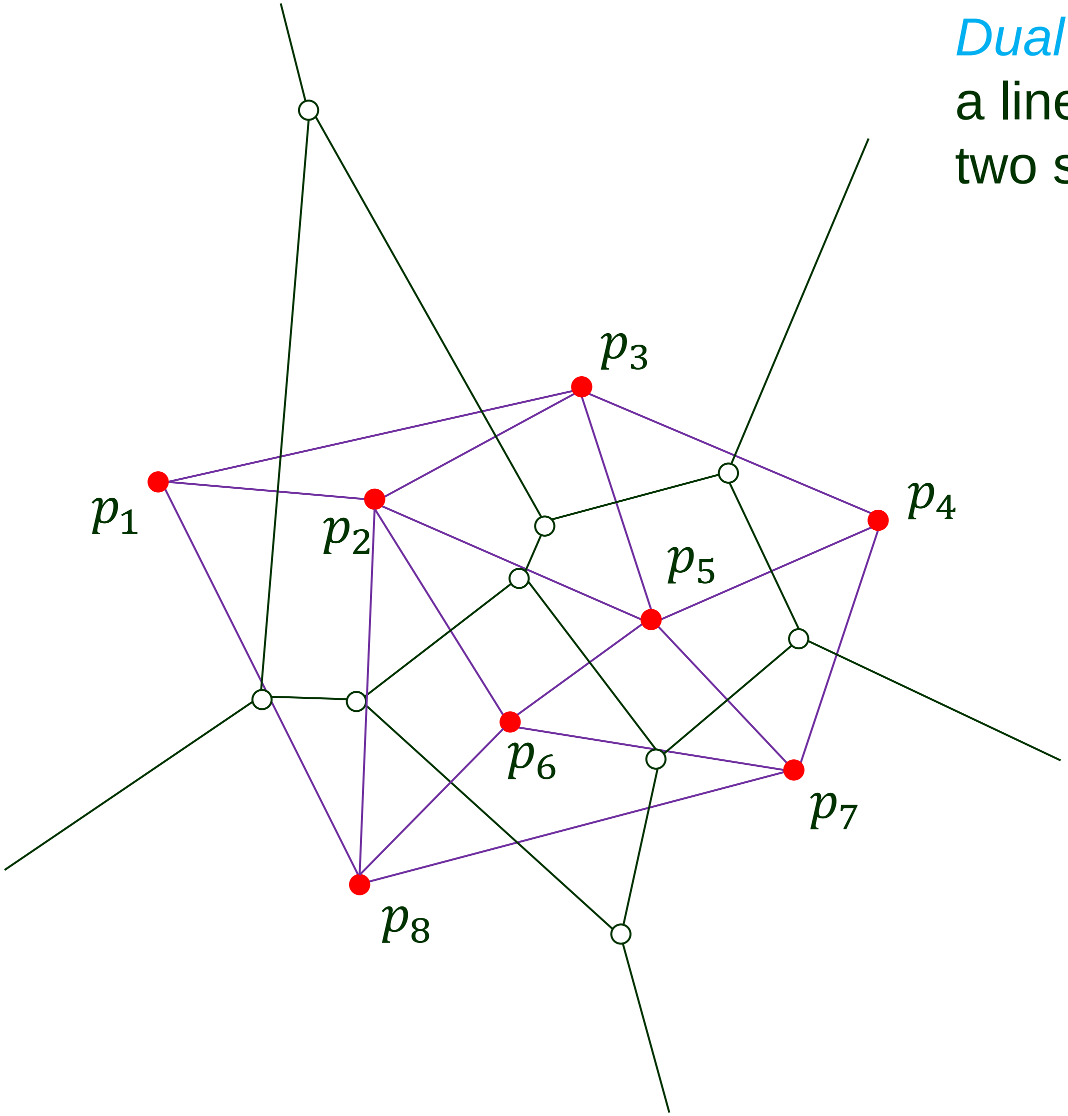
Not All Sites Collinear



$V(p_i)$ ***connected*** with two types of edges:

- Line segments
- Half-lines

III. Delaunay Triangulation



Dual graph obtained by adding a line segment between every two sites sharing a Voronoi edge.

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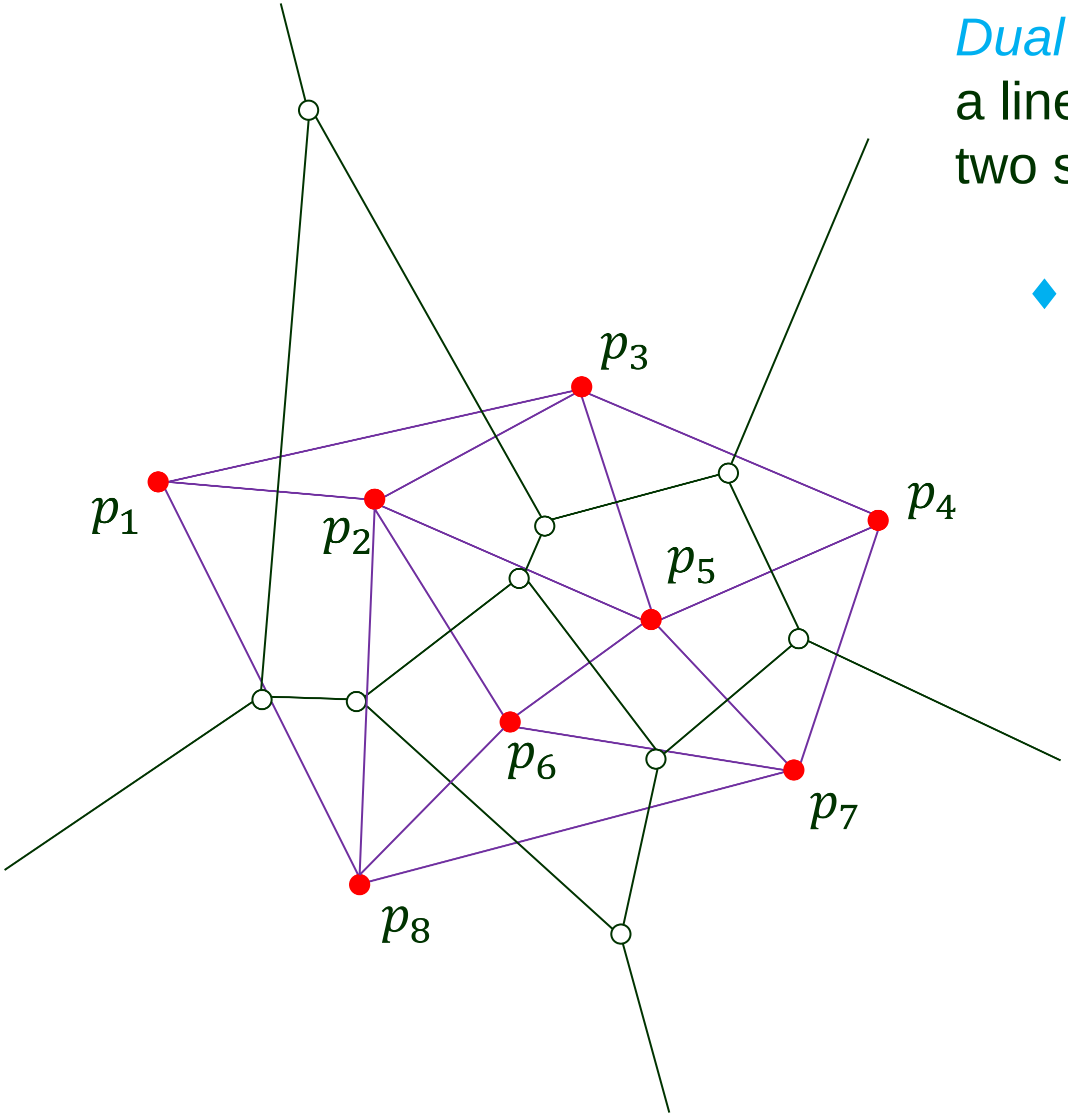

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- An edge and its dual may not even intersect.

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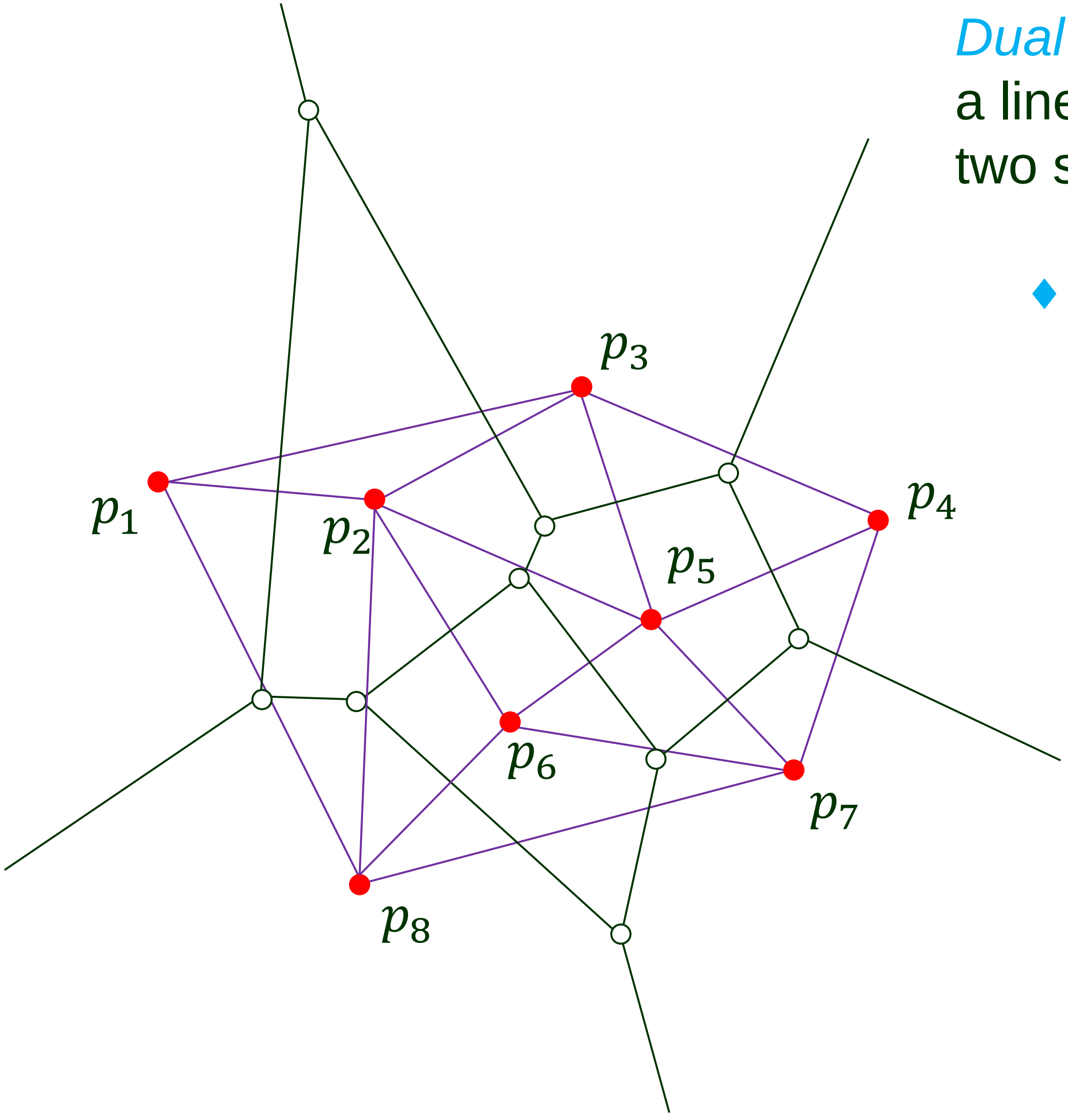

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E.g., $\overline{p_2 p_8}$, $\overline{p_1 p_3}$, $\overline{p_7 p_8}$, $\overline{p_6 p_7}$.

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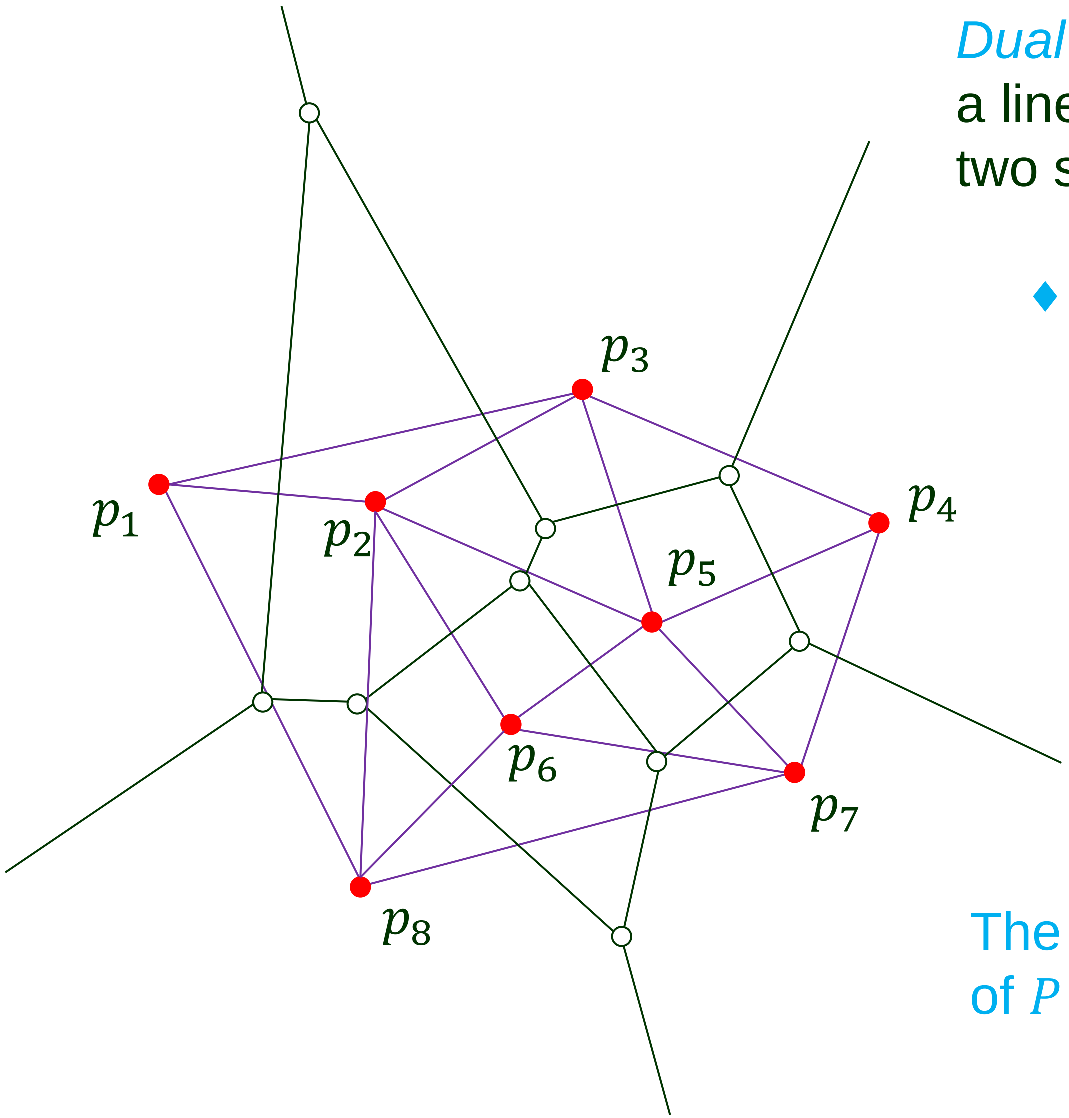

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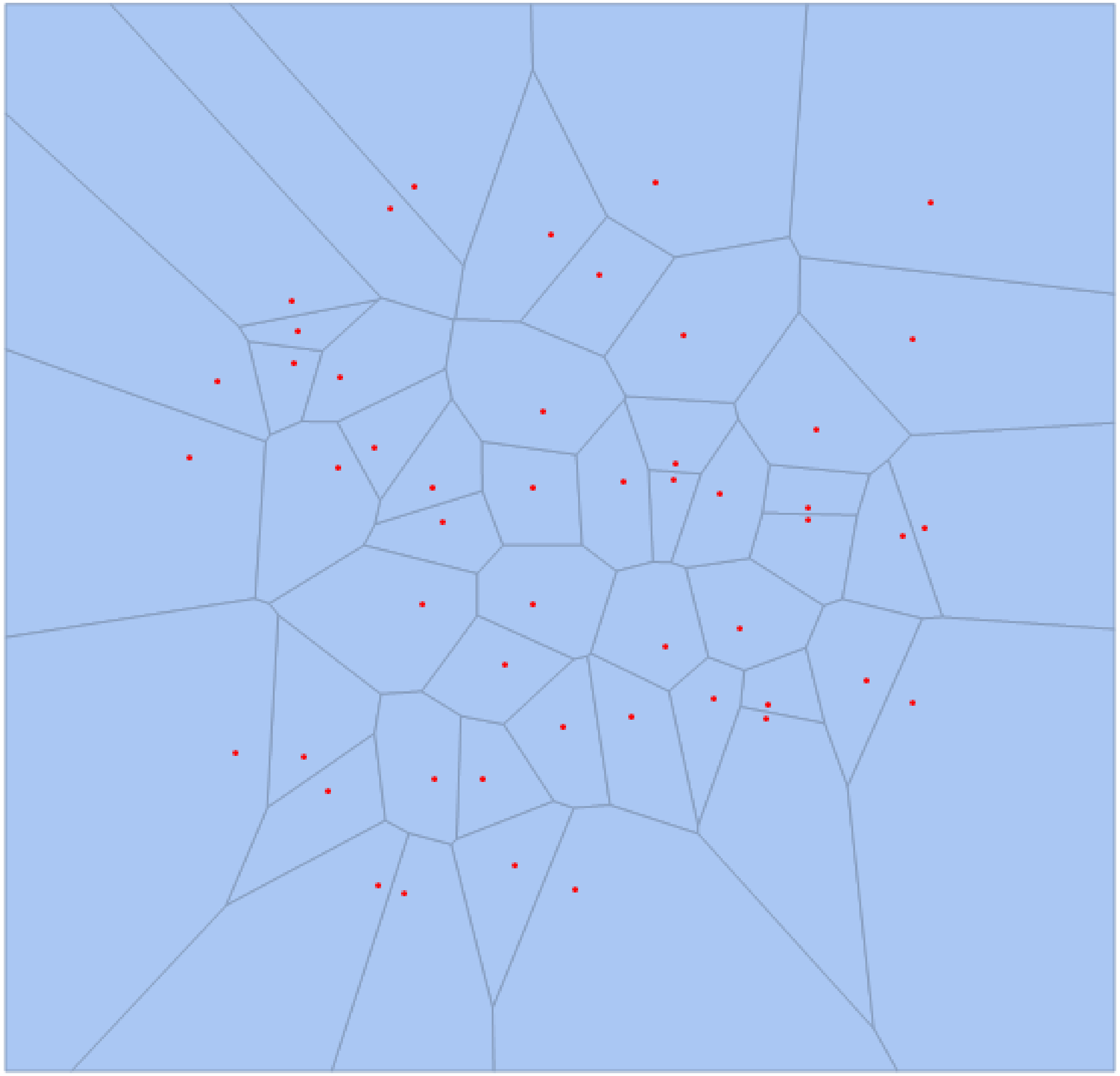
The dual graph is a triangulation of P (Delaunay 1934).

One More Example

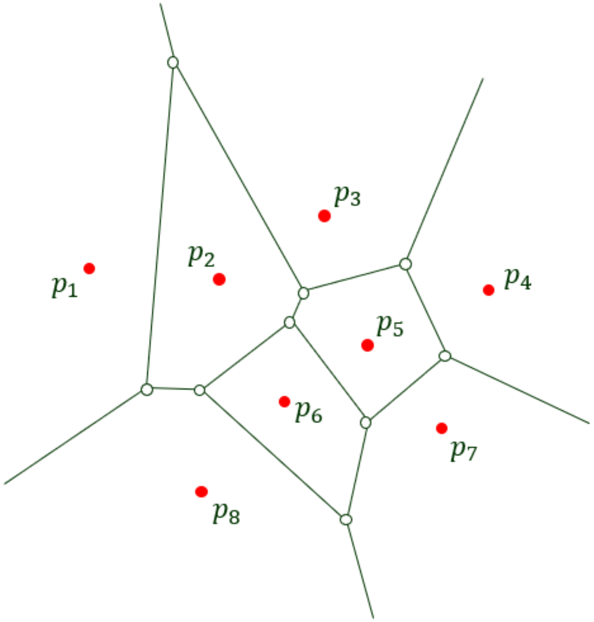
50 points
(generated using the Mathematica command `VoronoiMesh`)



IV. Complexity of Vor(P)

$\leq 2n - 5$ vertices

$\leq 3n - 6$ edges



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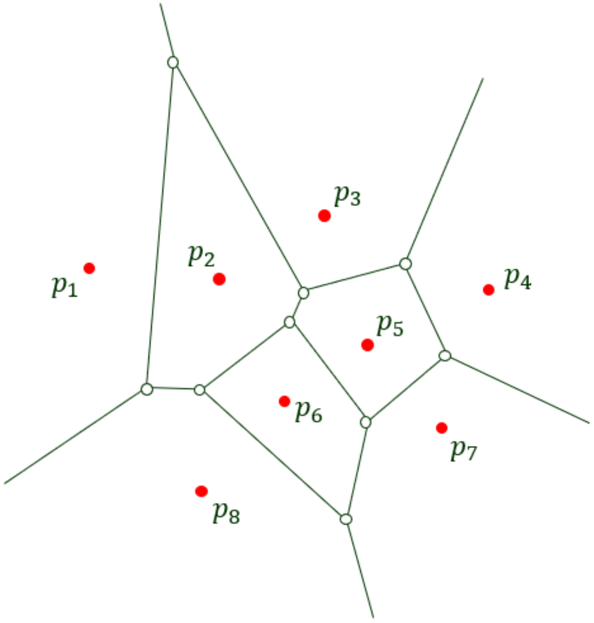

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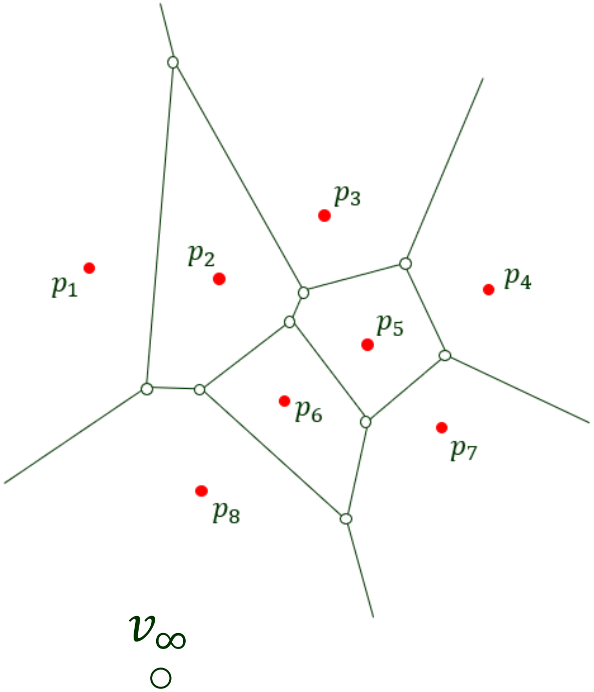
- Add vertex v_∞ far enough.



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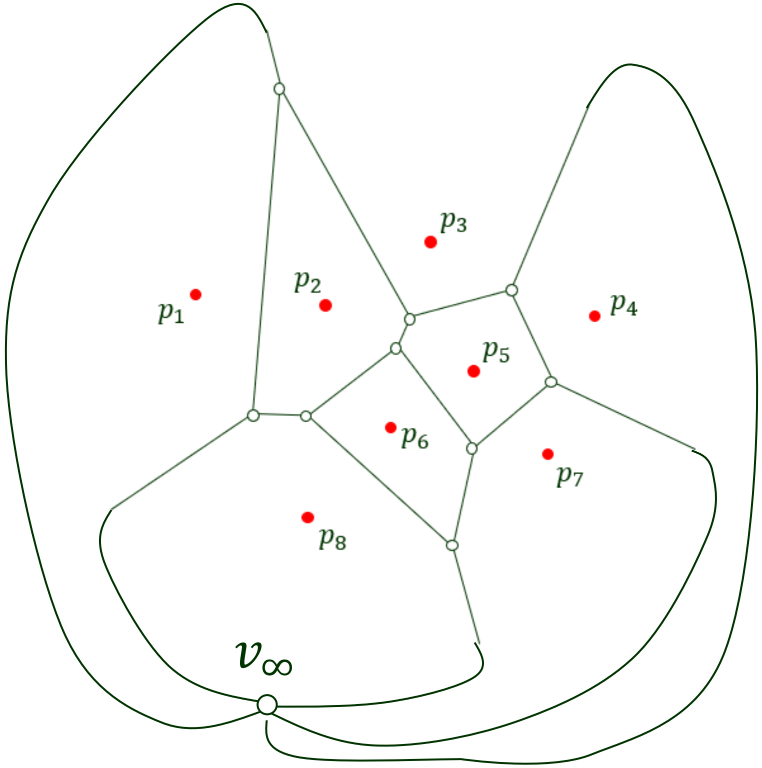

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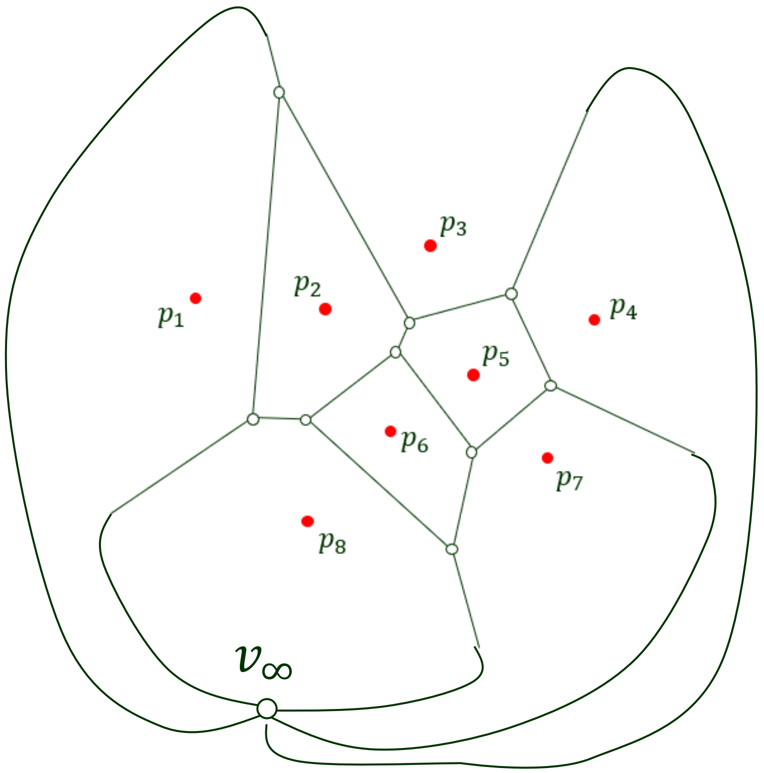
$\Downarrow$

a planar graph



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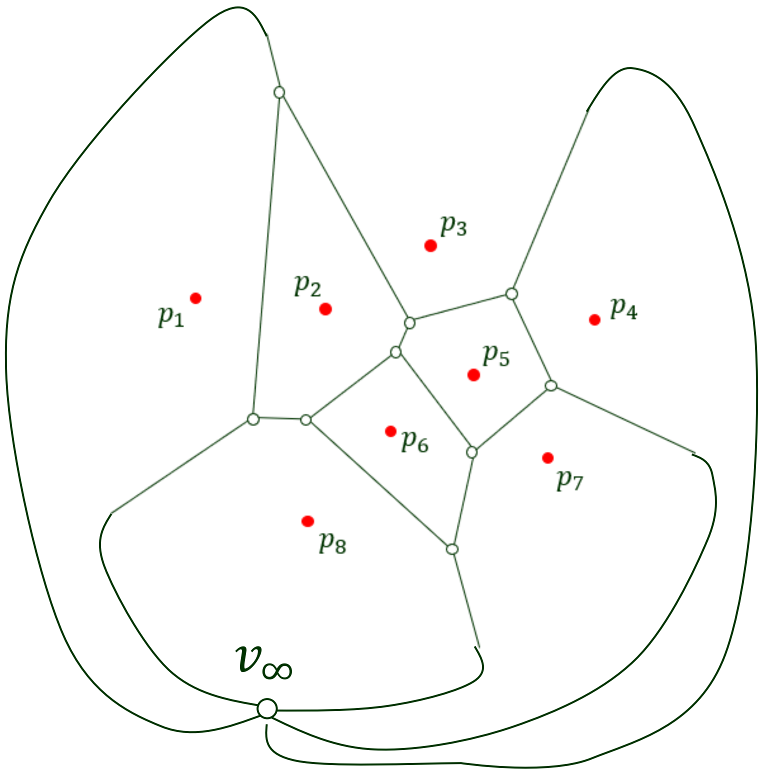
$(n_v + 1) - n_e + n = 2$ (Euler's formula)



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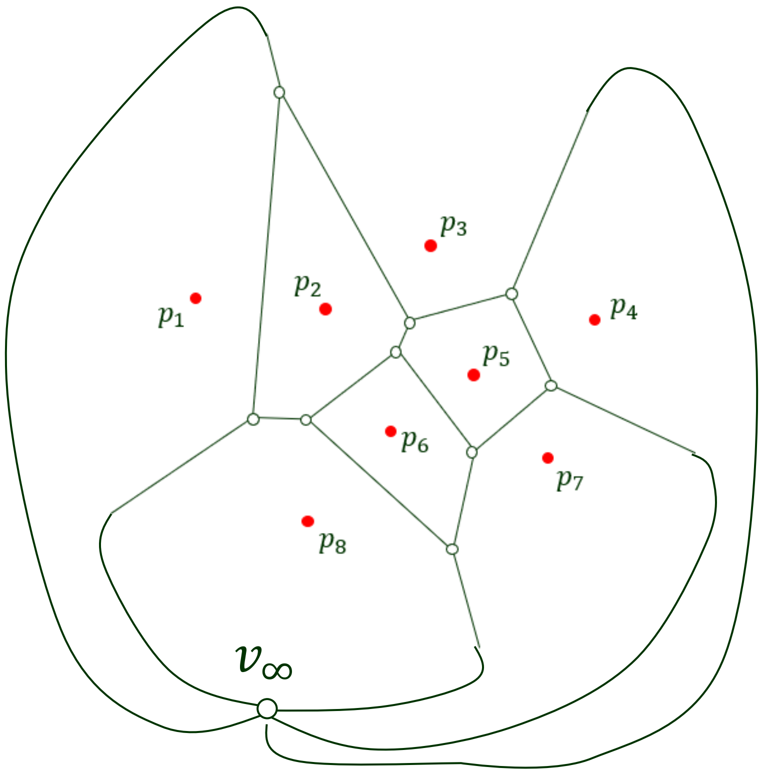
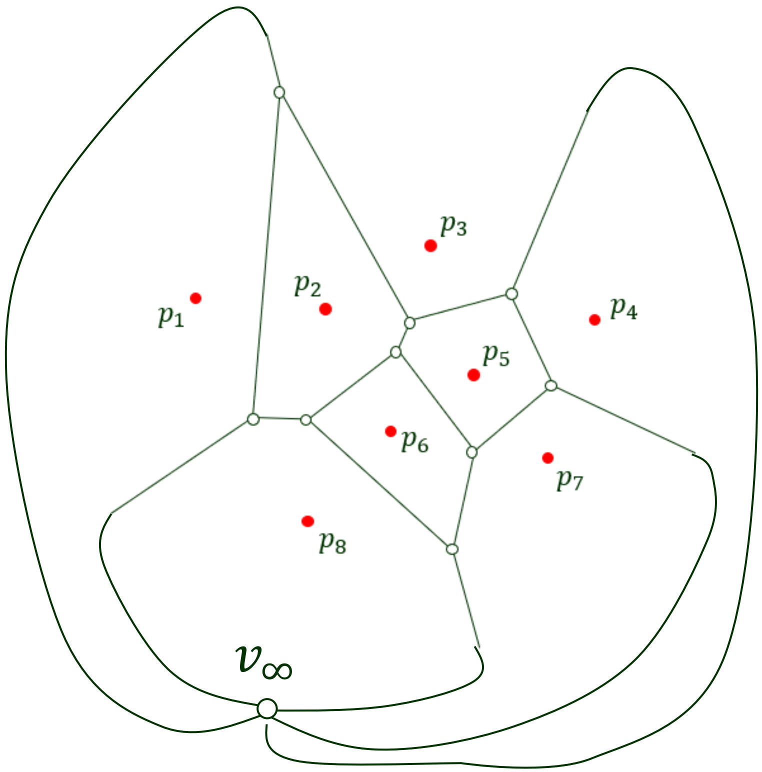
$\Downarrow$

$$n_e = n_v + n - 1$$
$$n_v = n_e - n + 1$$



Cont'd

- Every vertex has degree ≥ 3.



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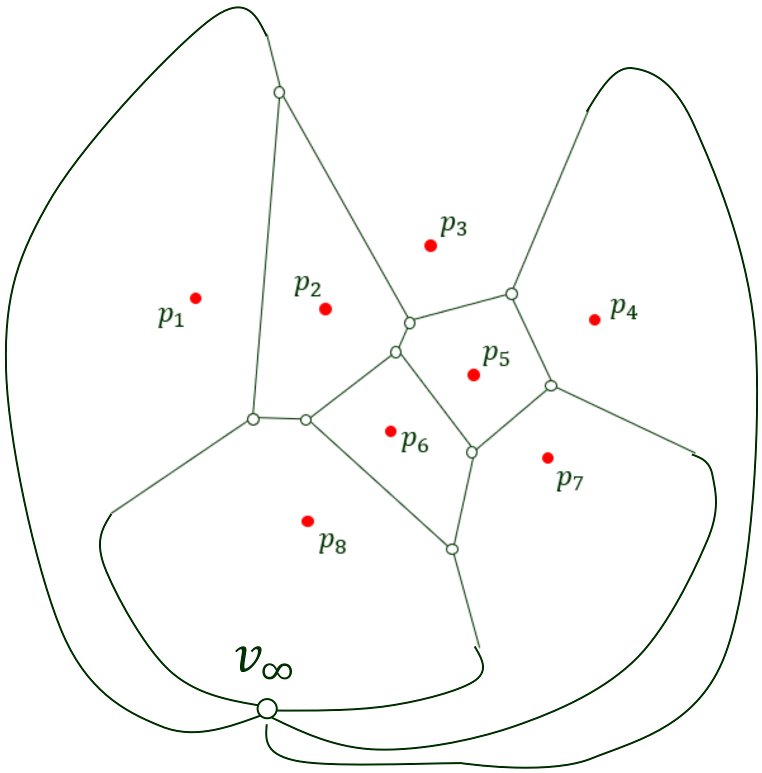
$$\Downarrow$$

Total degree $2n_e = \sum \deg(v) \geq 3(n_v + 1)$.



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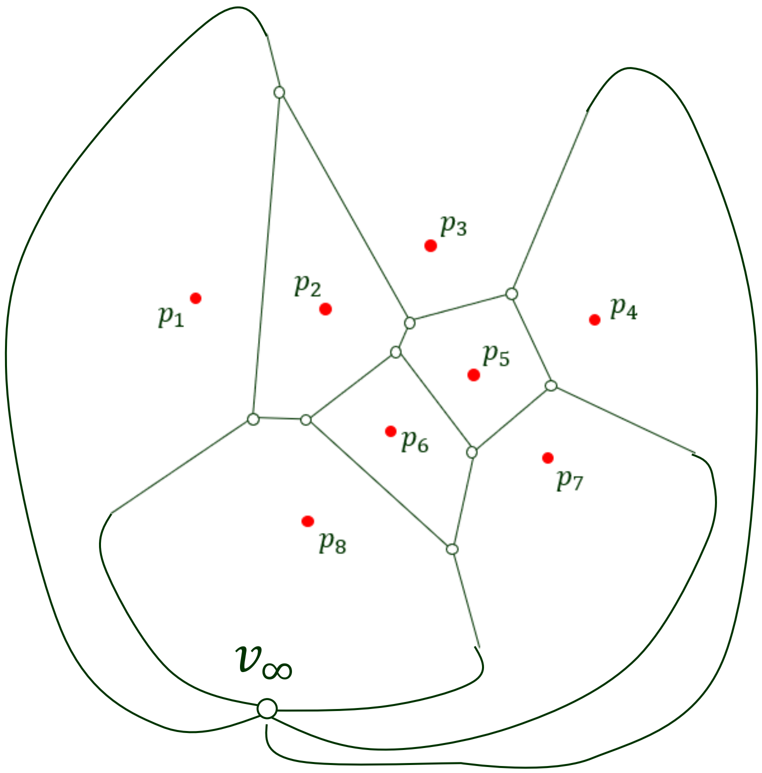

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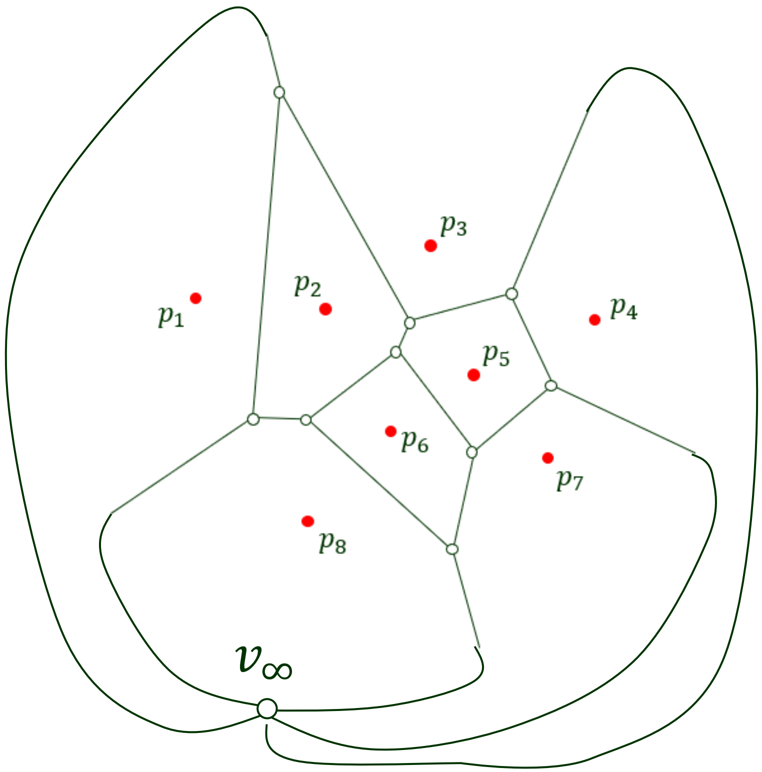
$$n_e = n_v + n - 1 \quad \Downarrow$$

$$n_v \leq 2n - 5$$



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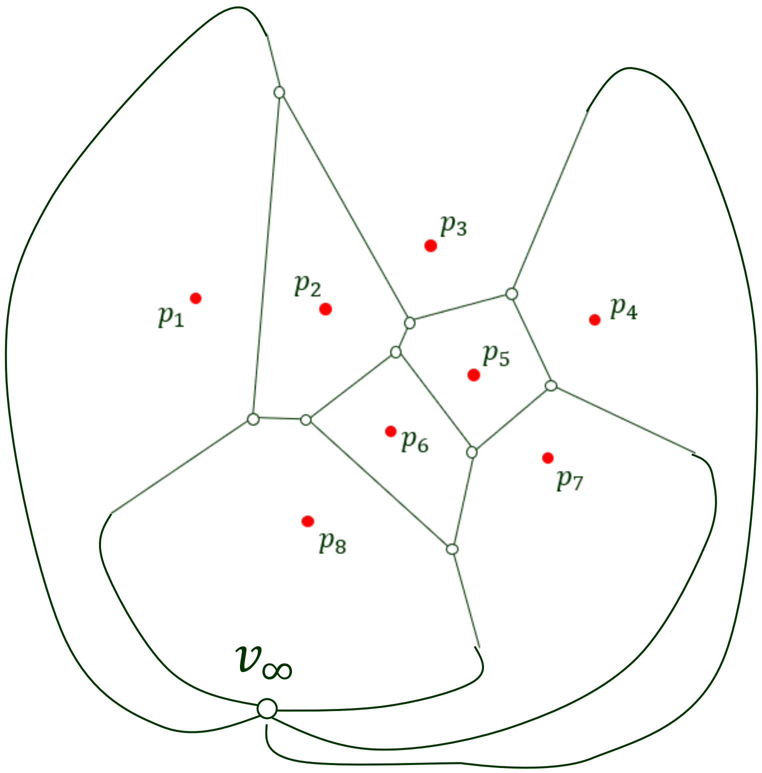
$\Downarrow$

$$n_v \leq 2n - 5$$



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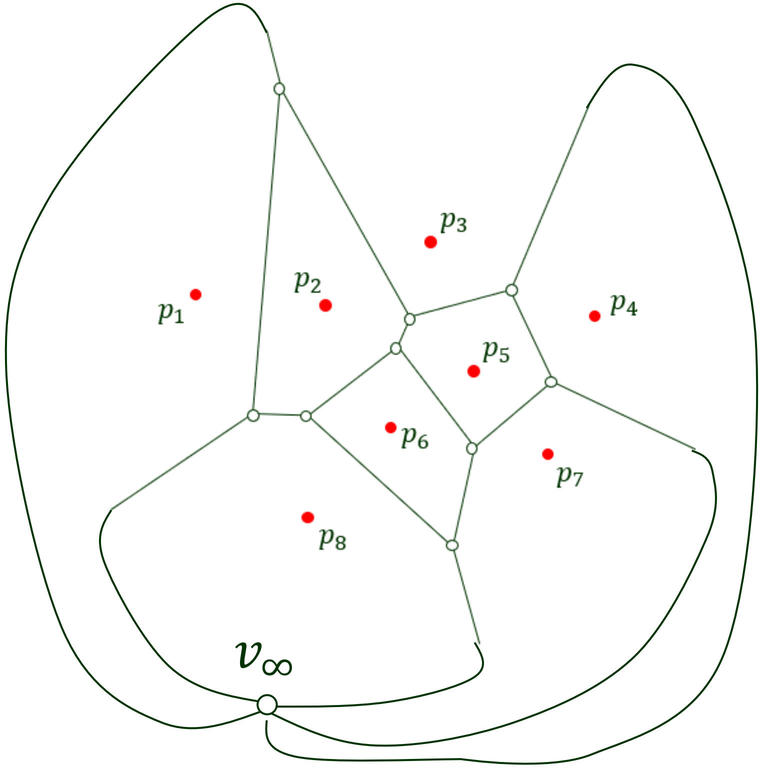
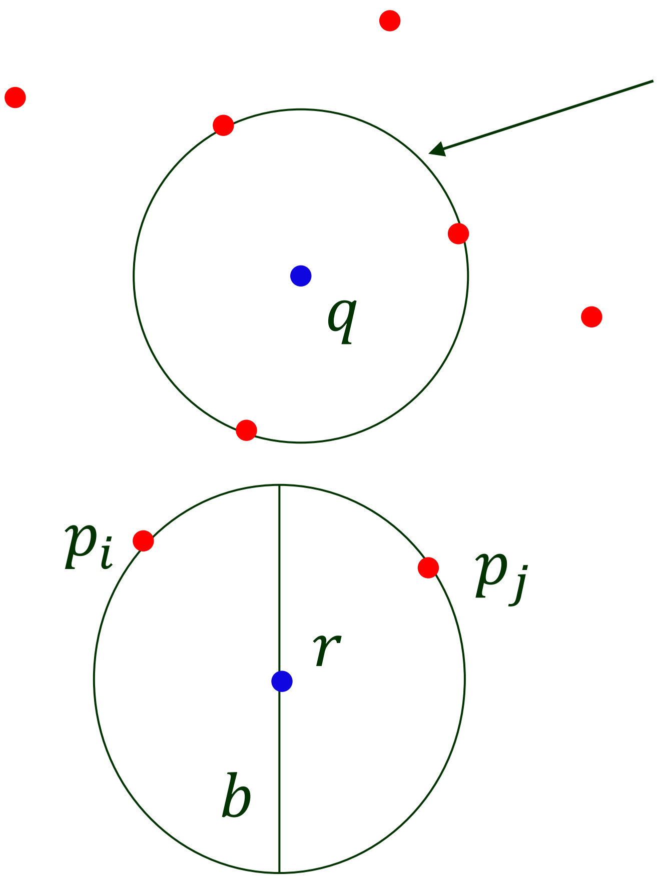


Vertex



$C_P(q)$: largest circle centered at q and not containing any site from P in its interior.

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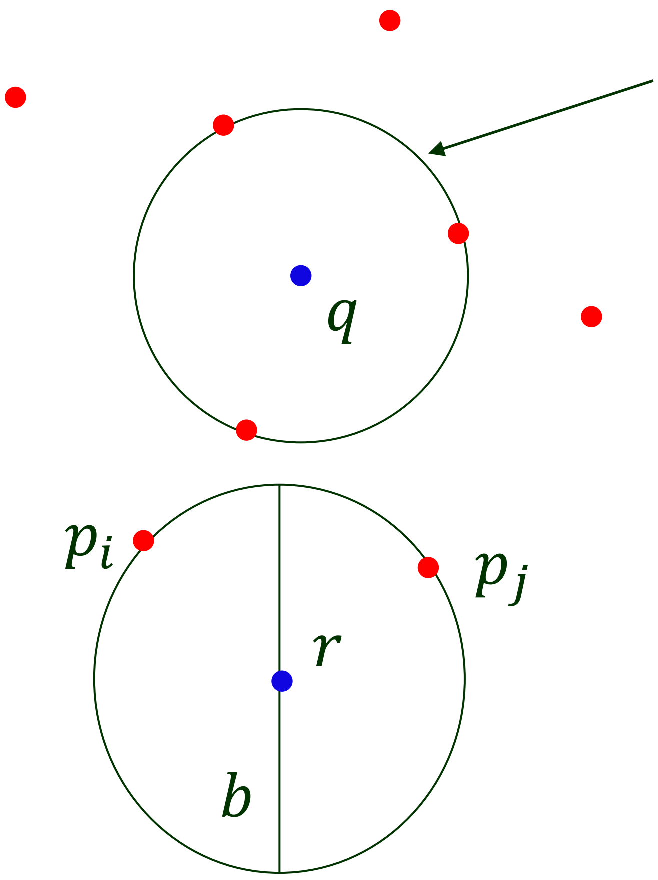


$C_P(q)$: largest circle centered at q and not containing any site from P in its interior.

Theorem

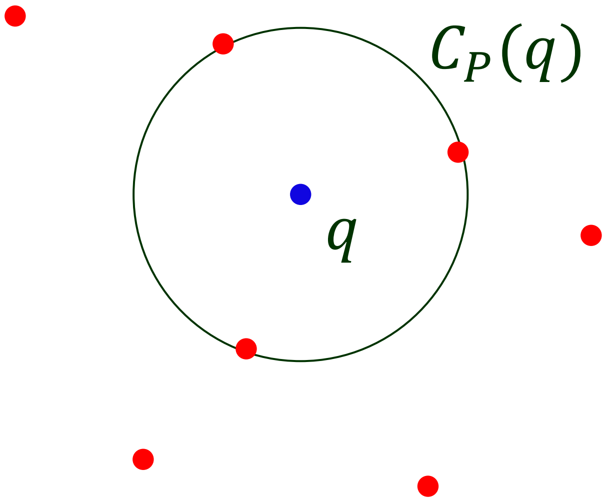
(i) q is a vertex of Vor(P) iff $C_P(q)$ passes through ≥ 3 sites.

(ii) Bisector b of p_i and p_j is an edge of Vor(P) iff for some point r on b, $C_P(r)$ passes through p_i and p_j but no other sites.

Proof of (i) Only

(i) q is a vertex of Vor(P) iff $C_P(q)$ passes through ≥ 3 sites.

$(\Leftarrow)$



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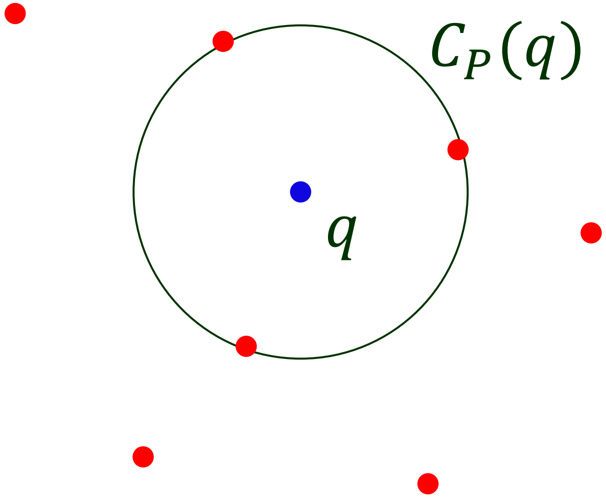

($\Leftarrow$) Suppose q exists such that $C_P(q)$ passes through ≥ 3 sites p_i, p_j, p_k.

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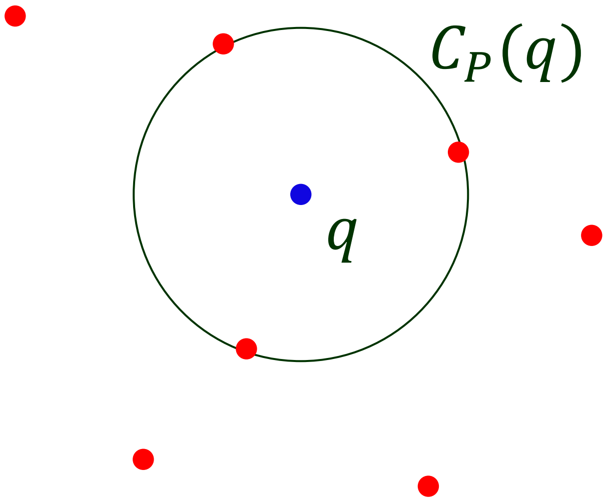

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$C_P(q)$ has no site in its interior.

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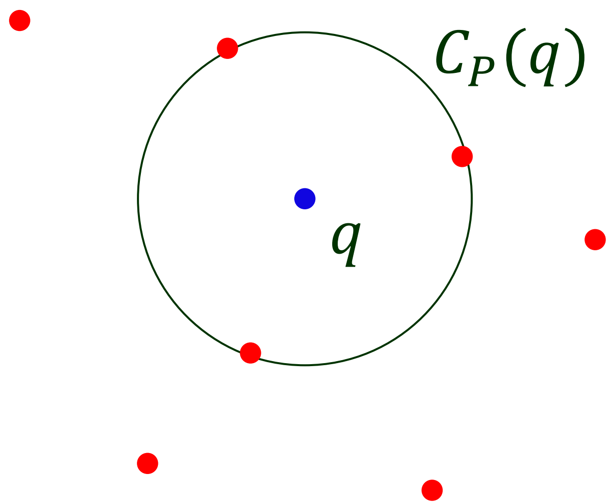

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$C_P(q)$ has no site in its interior.

$\Downarrow$

q must be on the boundary of $V(p_i)$, $V(p_j)$, and $V(p_k)$.

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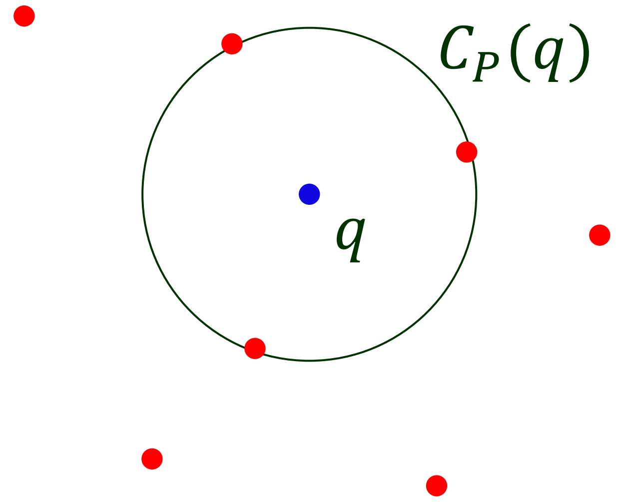

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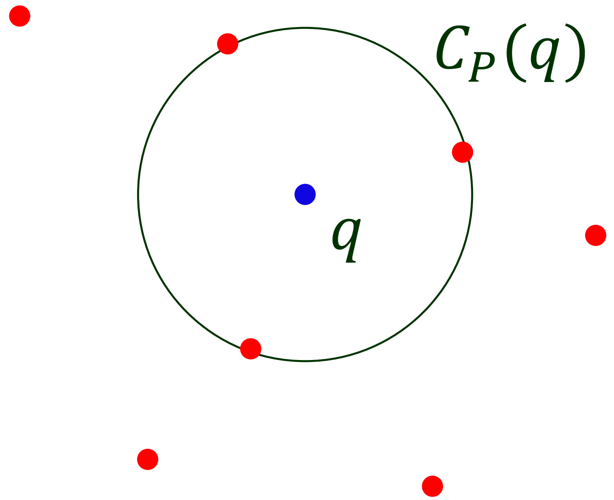

$(\Leftarrow)$ Suppose q exists such that $C_P(q)$ passes through ≥ 3 sites p_i, p_j, p_k.

$C_P(q)$ has no site in its interior.

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q must be on the boundary of $V(p_i)$, $V(p_j)$, and $V(p_k)$.

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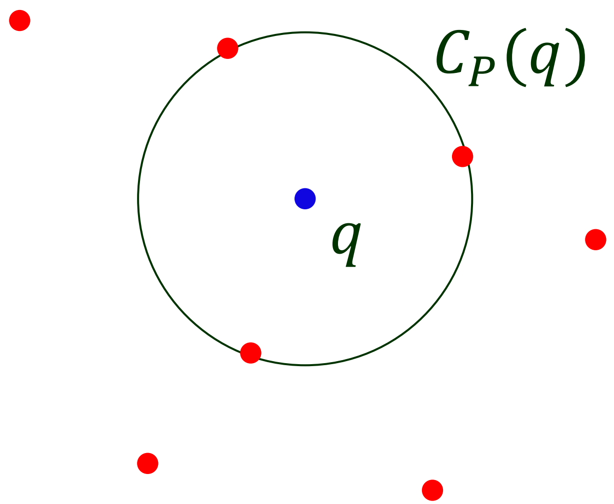

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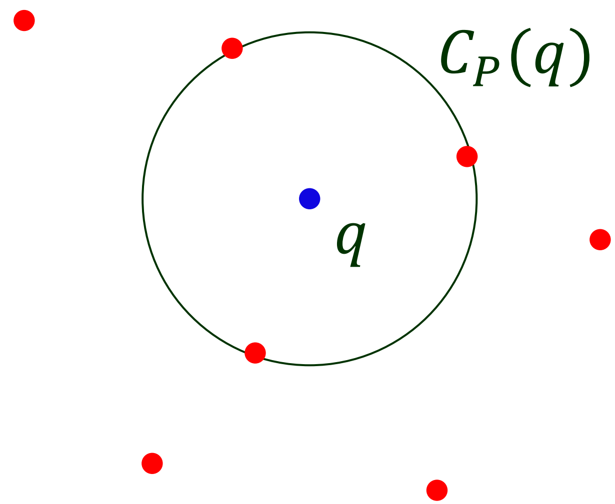

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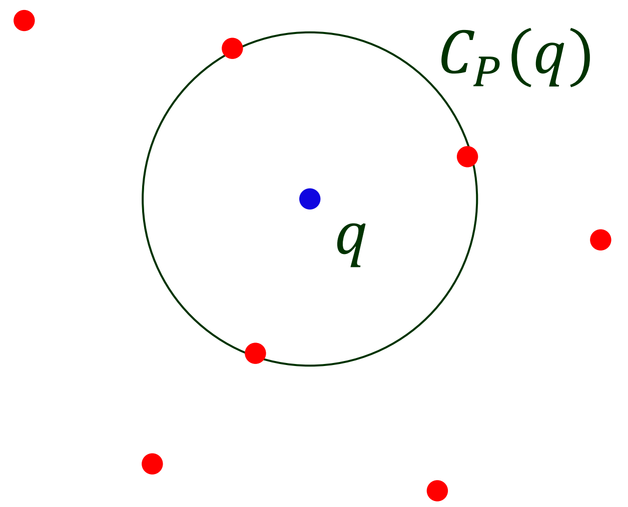

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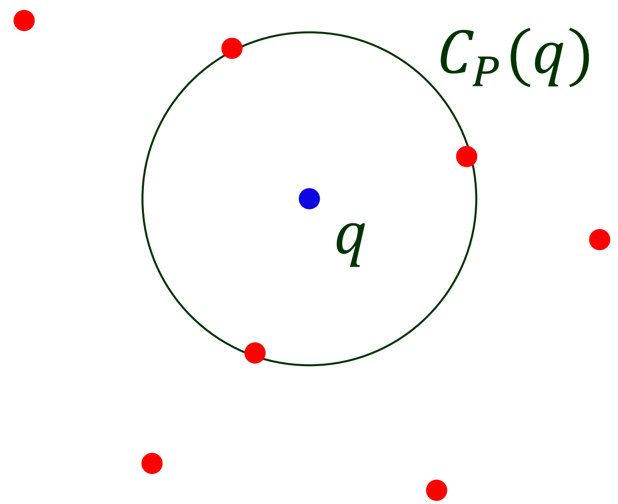

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V. Computing VD

Naive algorithm:

Compute every Voronoi cell $V(p_i)$.

- Half-plane intersection.

- n cells.

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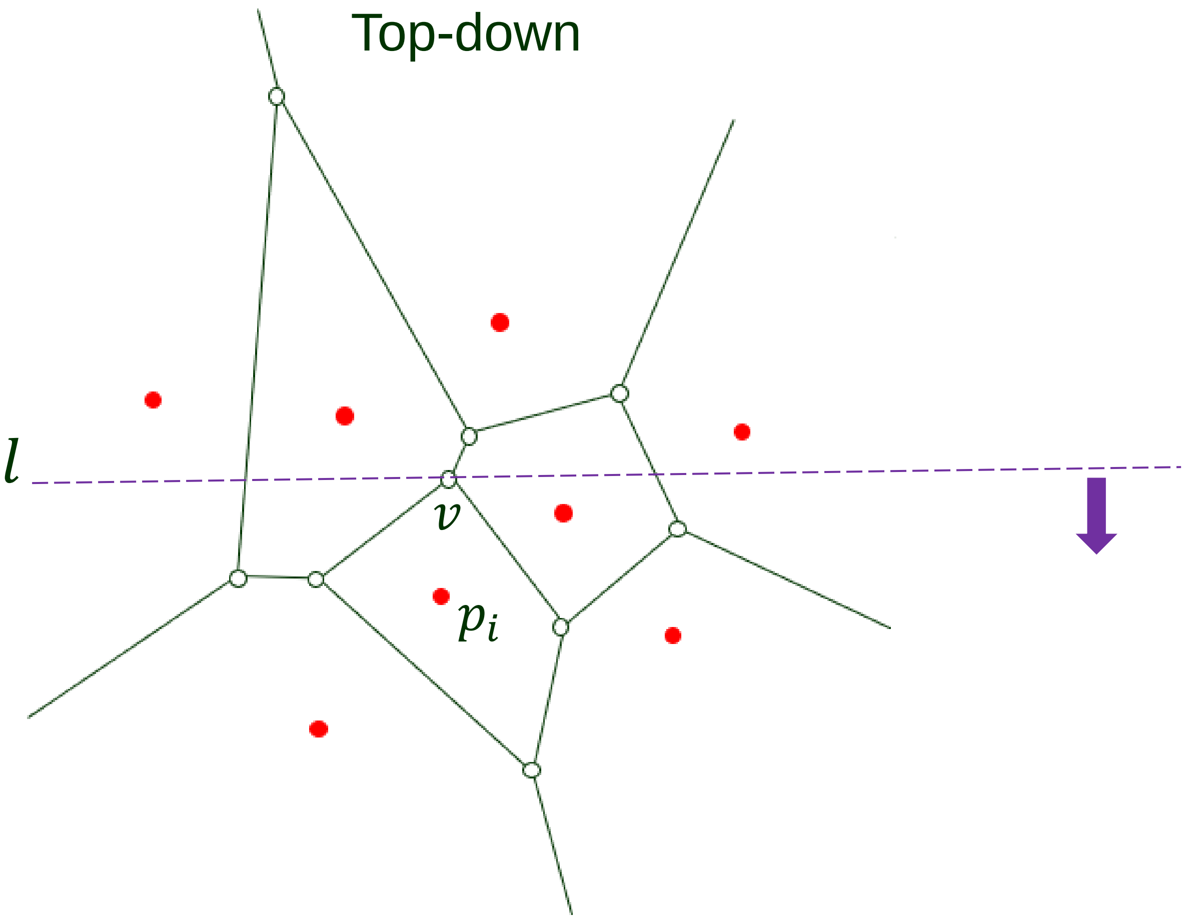
- ♦ n cells.

$$O(n^2 \log n)$$

Line Sweep Algorithm

Steve Fortune (1987)



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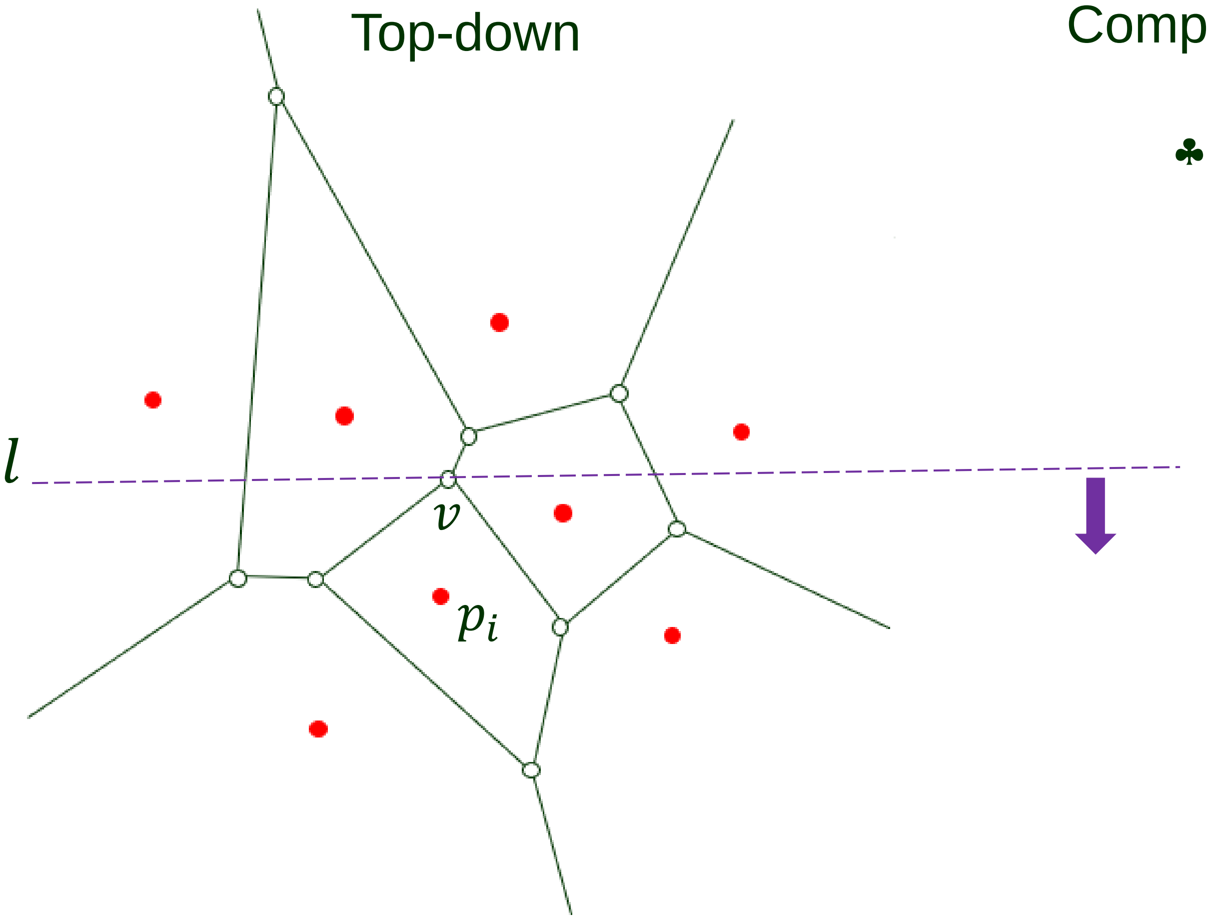

Complications:

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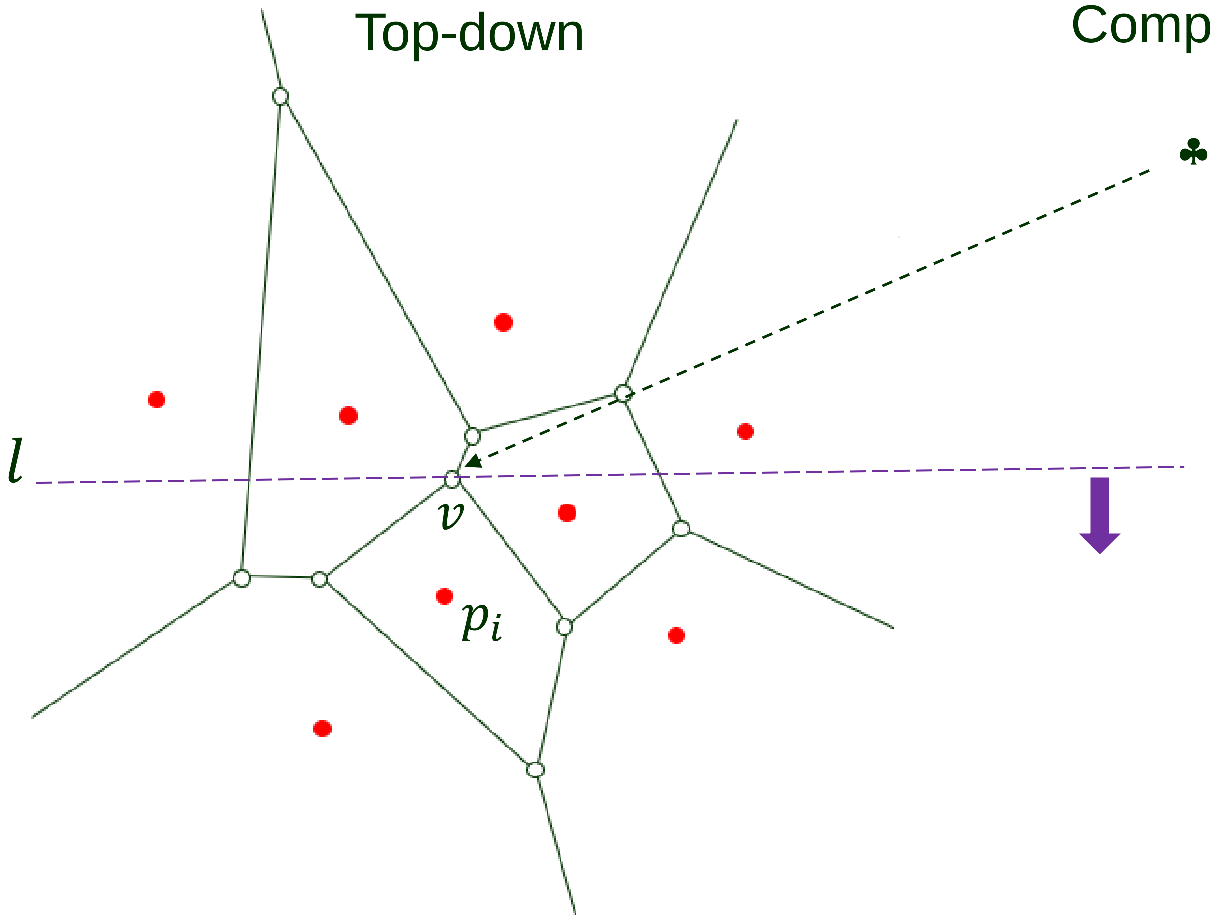

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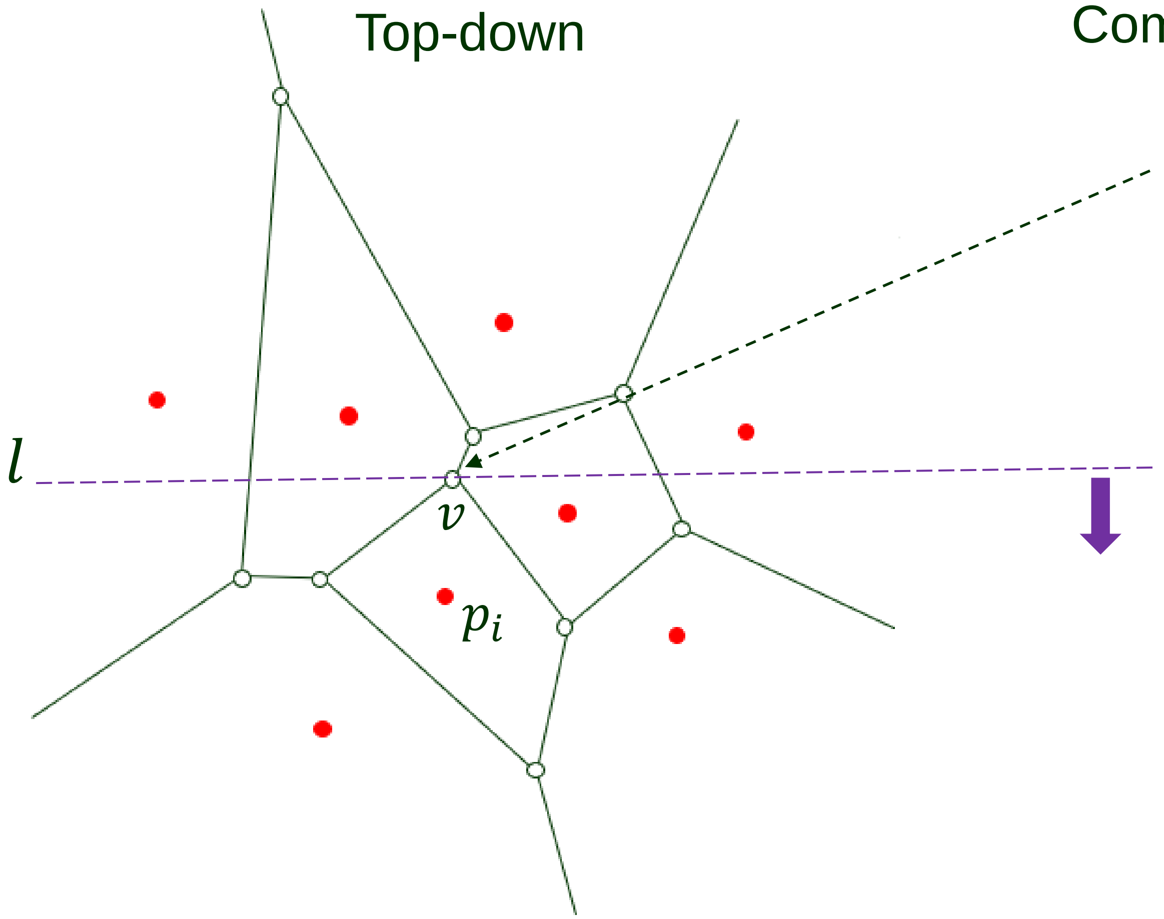

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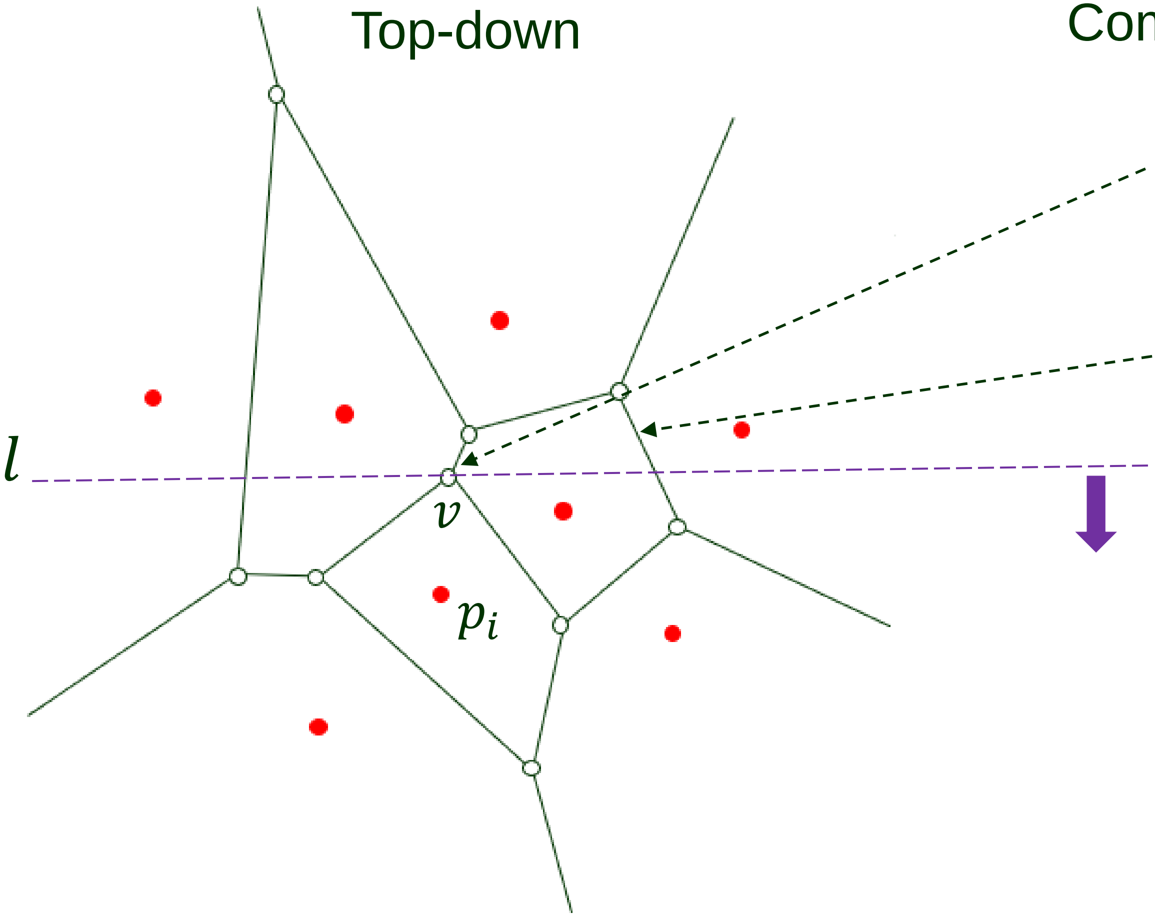

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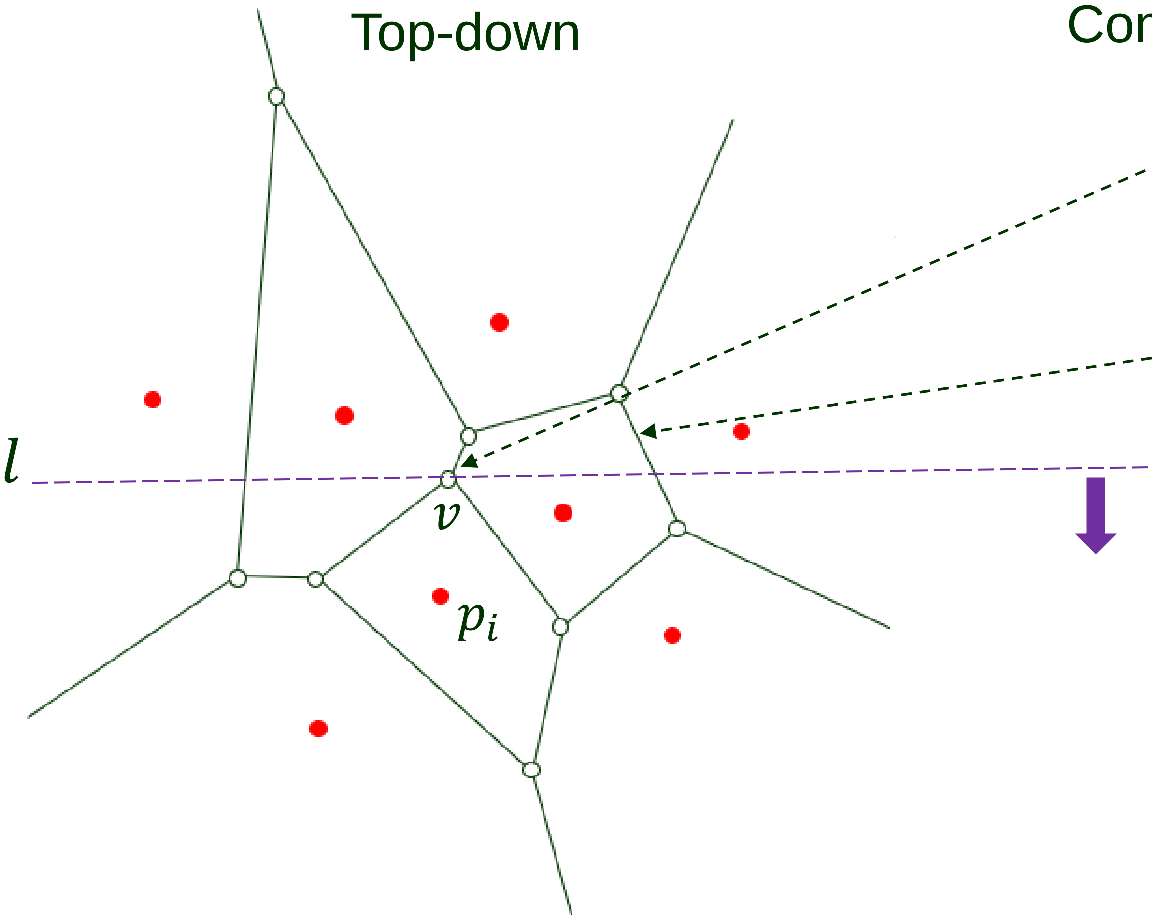
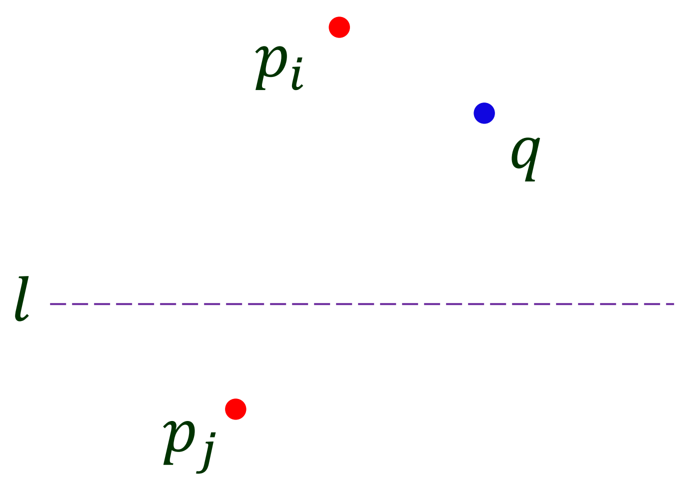


Complications:

- ♣ l may encounter the top vertex v of the cell $V(p_i)$ before the site p_i.
- ♣ Part of $\mathrm{Vor}(P)$ above l also depends on sites below l.
- ♣ We don't have all the information to compute v.

Sweep in a Different Fashion

- Do not maintain the intersection of Vor(P) with the half-plane above l.
- Maintain the part of Vor(P) of sites above l that will not change.



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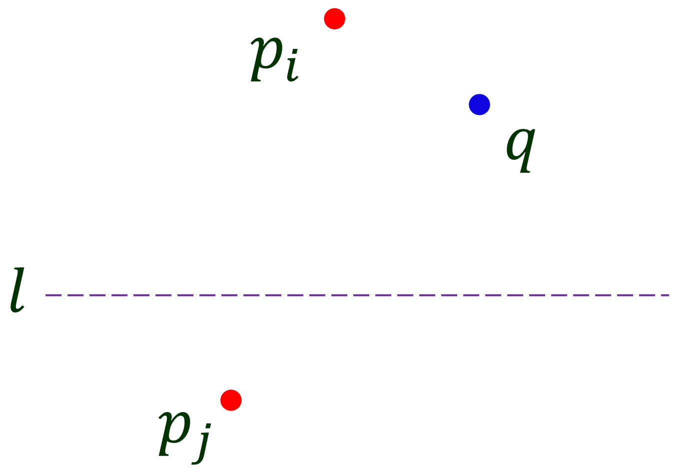

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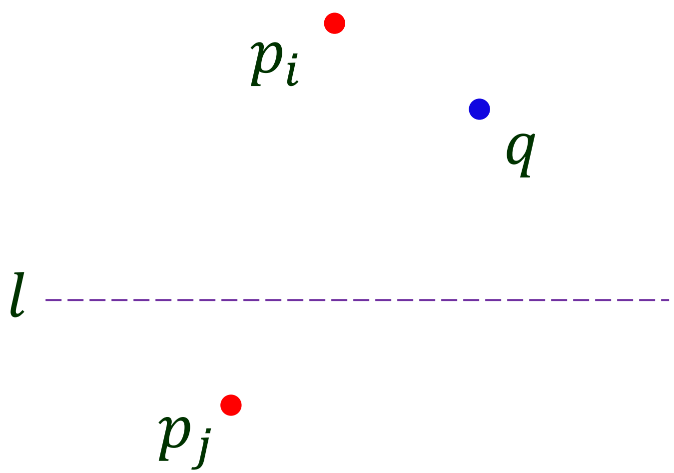

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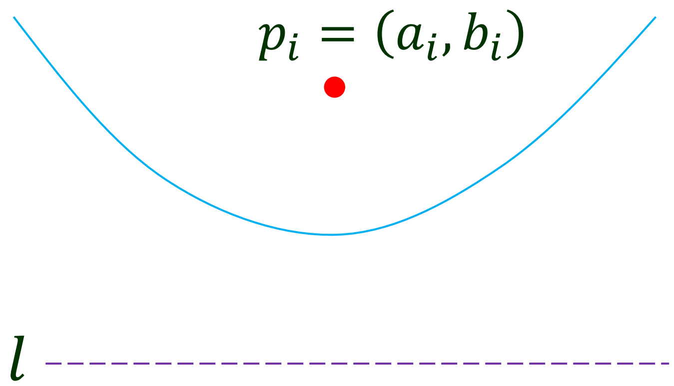
$\Downarrow$

If q is closer to p_i than to l, then
it must be closer to p_i than to p_j.

Point-Line Bisector

Locus of points equidistant to $p_i = (a_i, b_i)$ and $l: y = \ l_y$.



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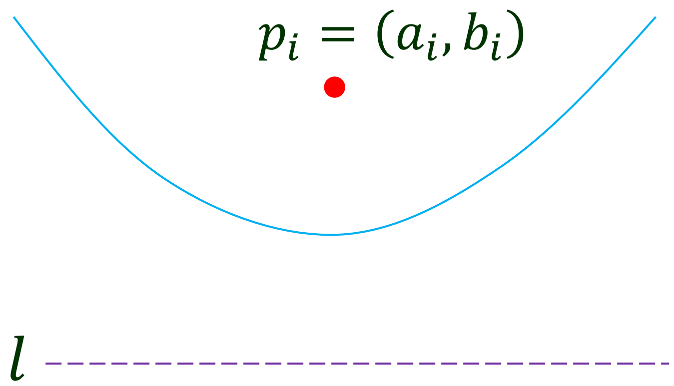

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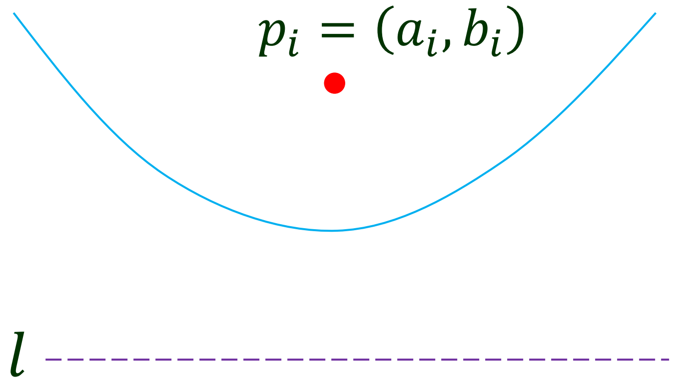

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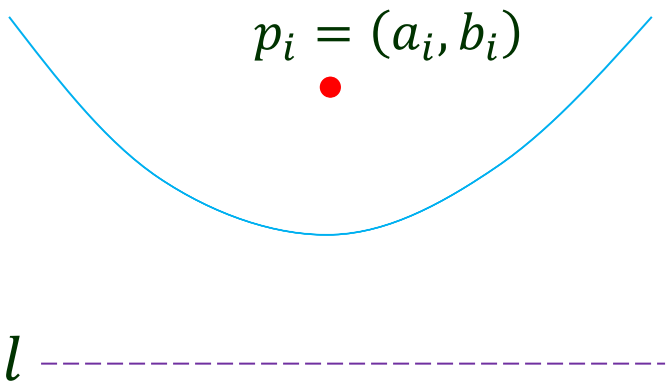

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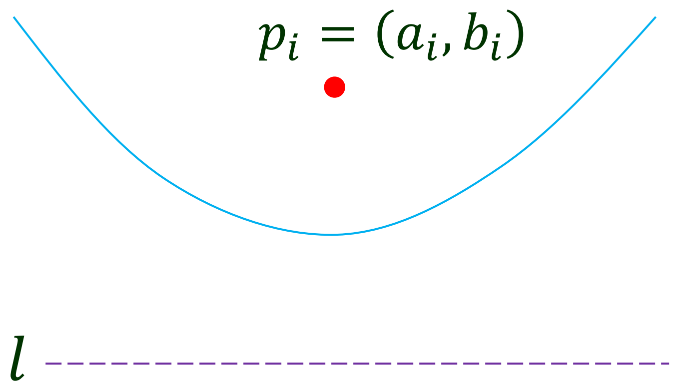

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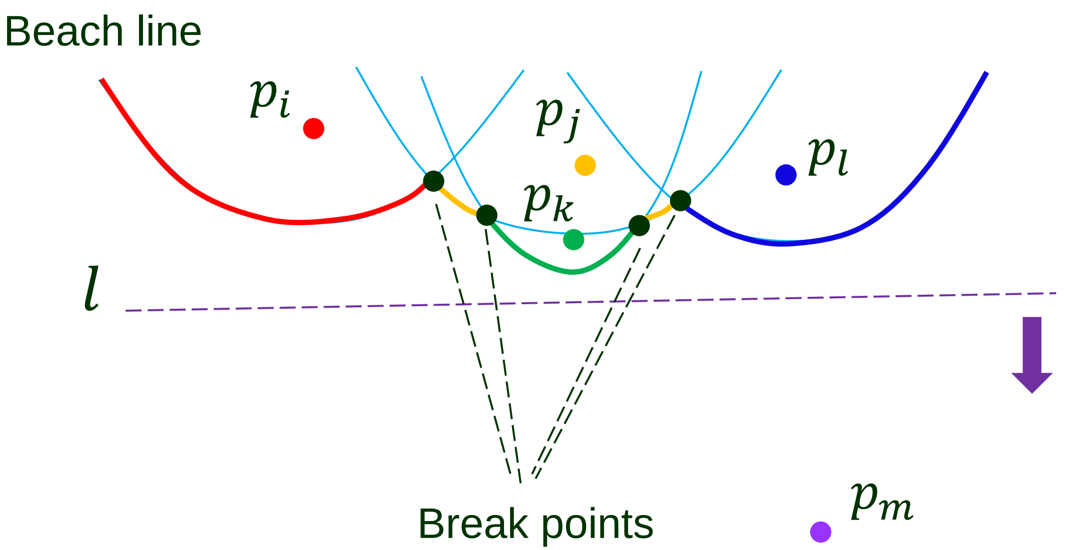
Parabola!

All the points above the parabola are closer to p_i than to l (and all the sites below l).

Beach Line

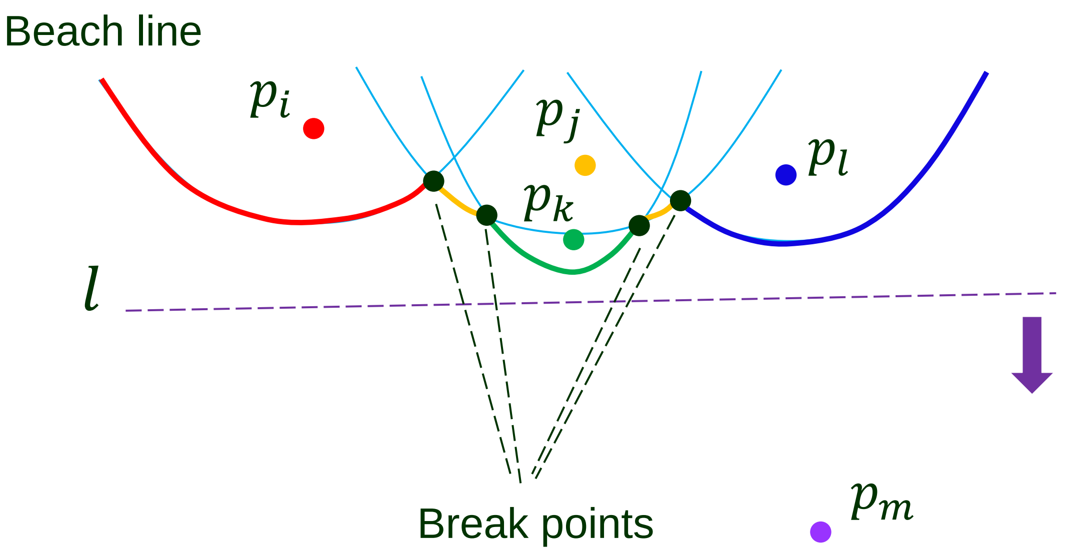
Parabolic arcs bounding the locus of points closer to some site above l than to l.



Beach Line

Parabolic arcs bounding the locus of points closer to some site above l than to l.

- Lower envelope of all the parabolas due to the sites above l.



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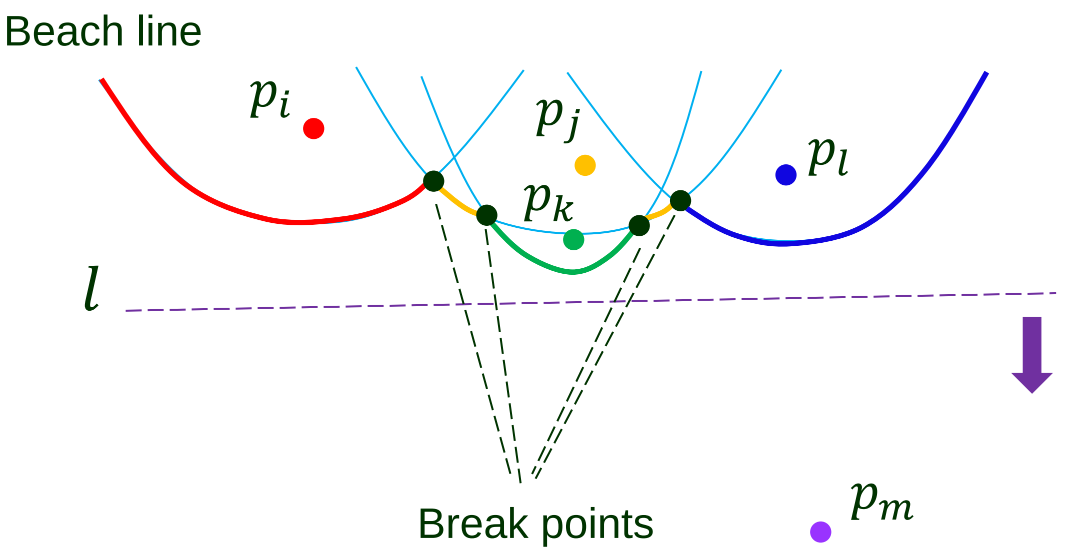

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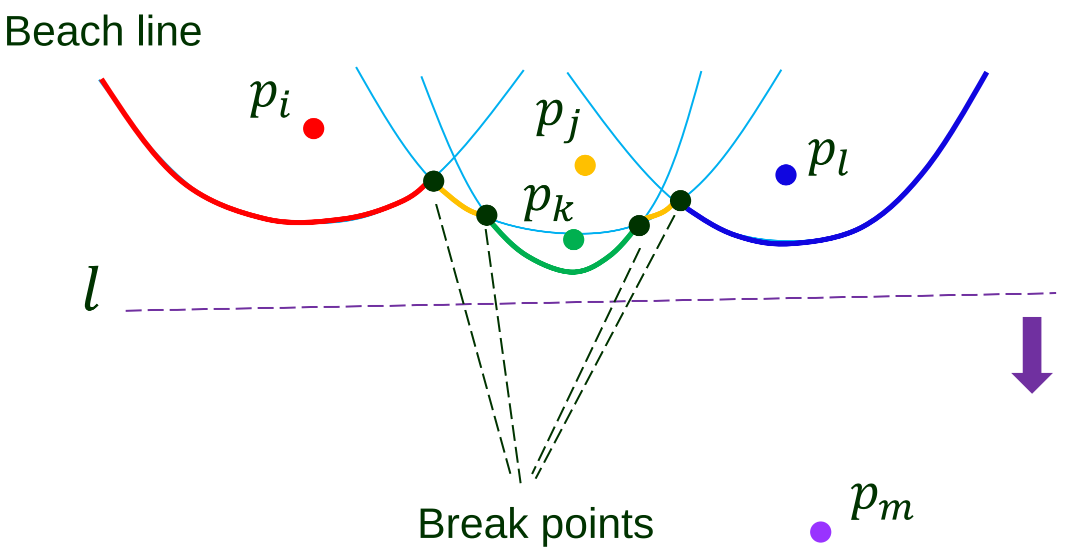

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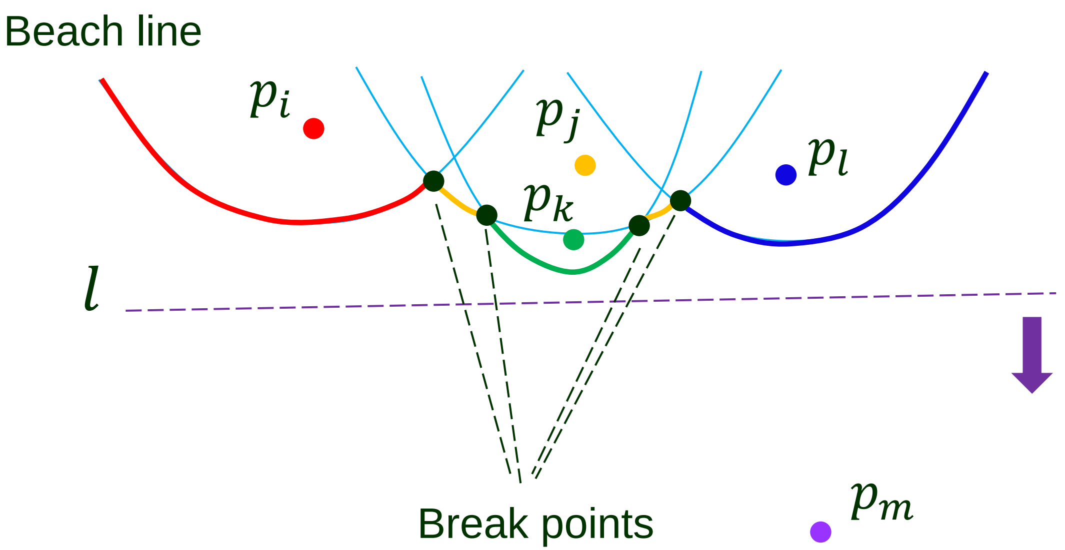

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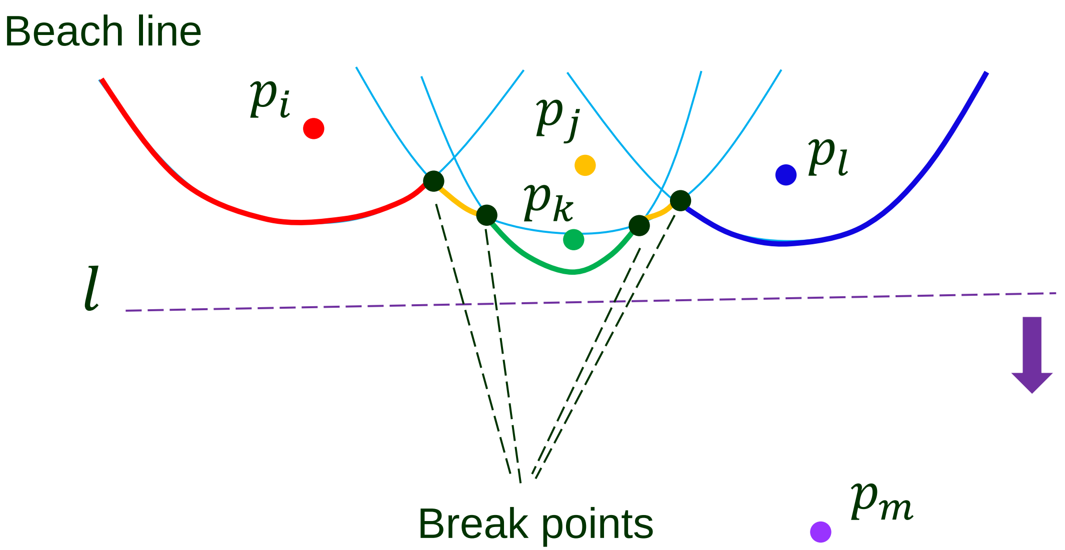

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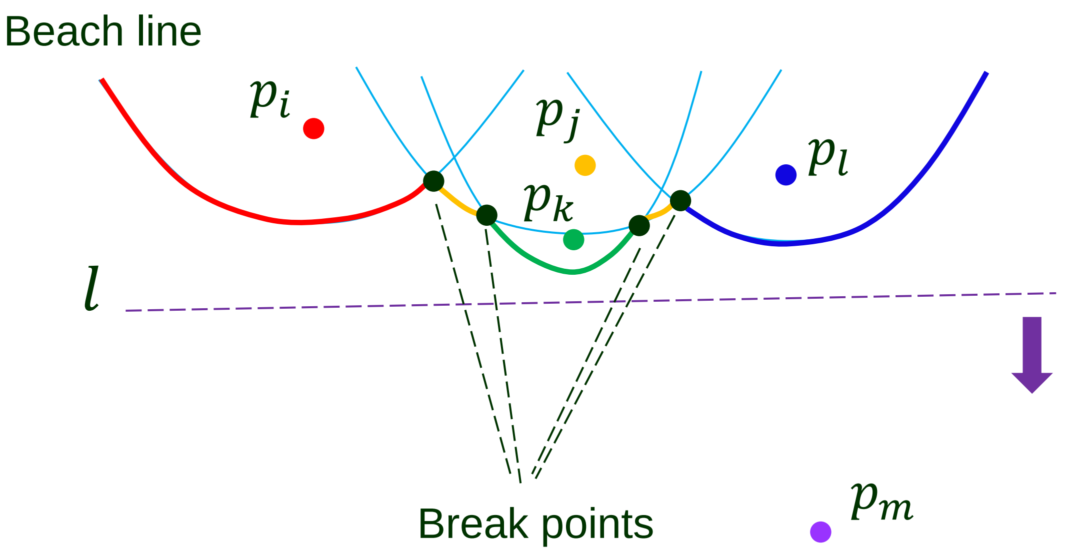

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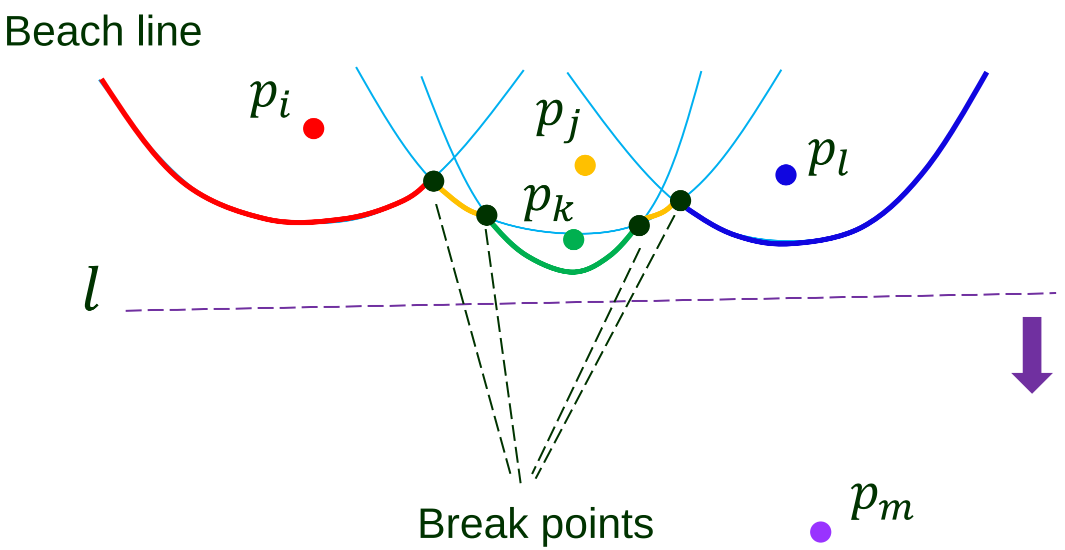
Maintain the beach line (not explicitly) during the sweep.

VI. Two Types of Events

As the sweep line moves downward, the beach line's topological structure changes when



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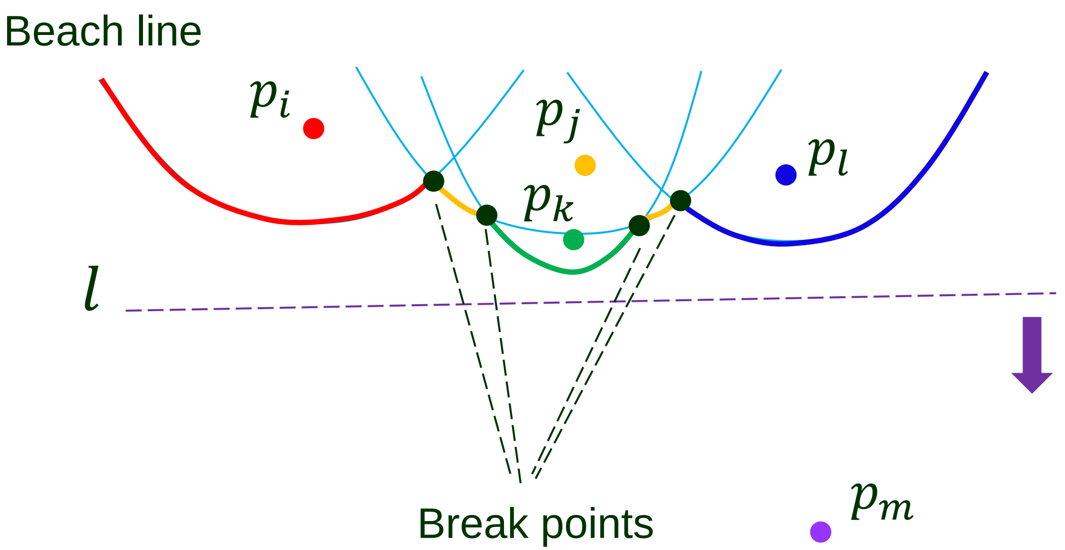

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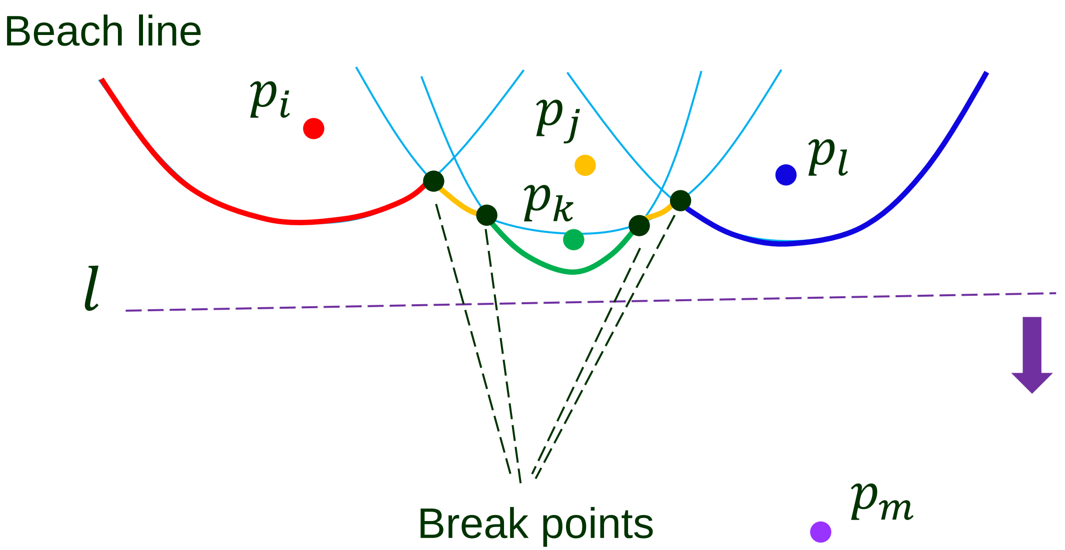
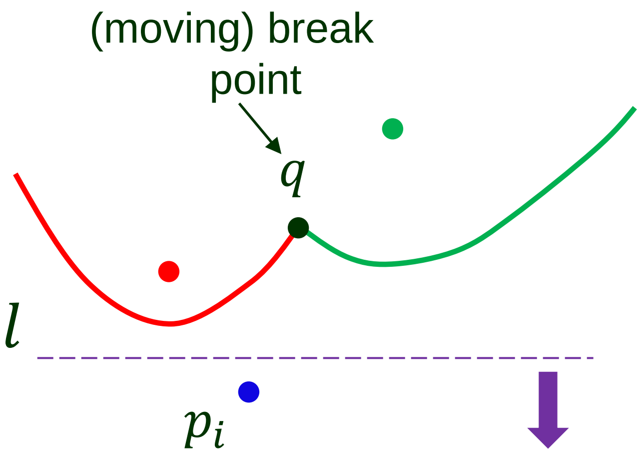


a) a new parabolic arc appears (a *site event*), or

b) a parabolic arc shrinks to a point and then vanishes (a *circle event*).

Site Event

The sweep line l reaches a new site.



(a) Before

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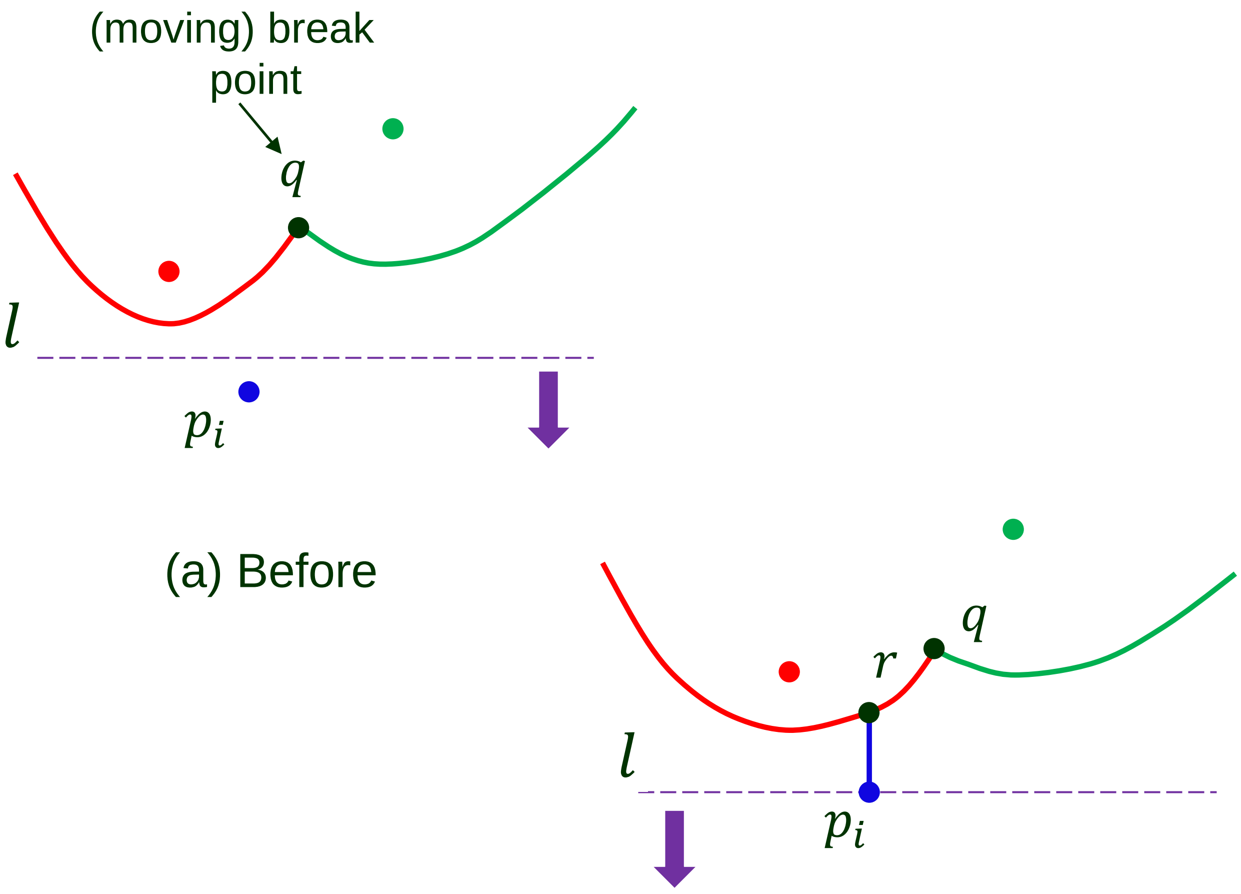

(a) Before

(b) At p_i

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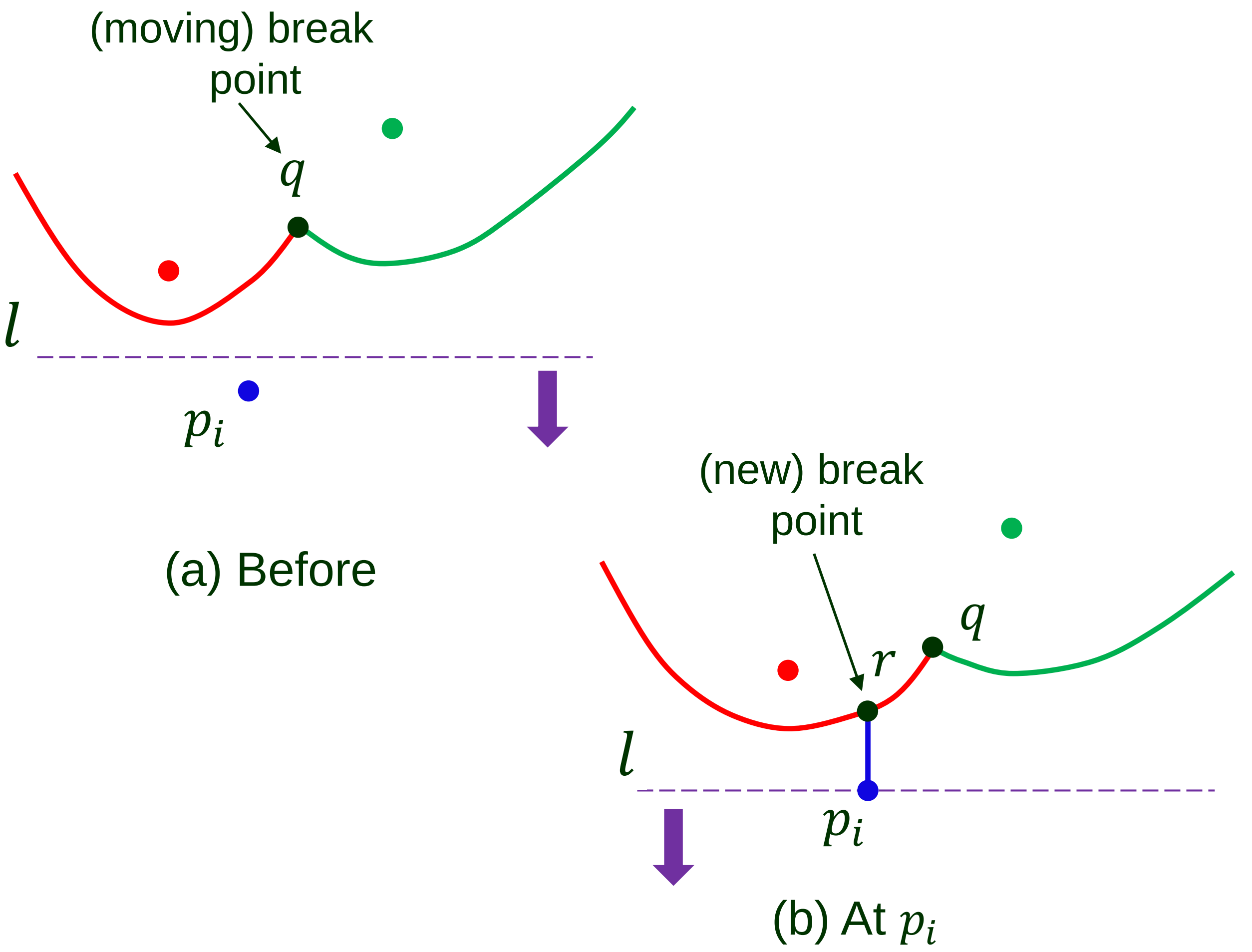

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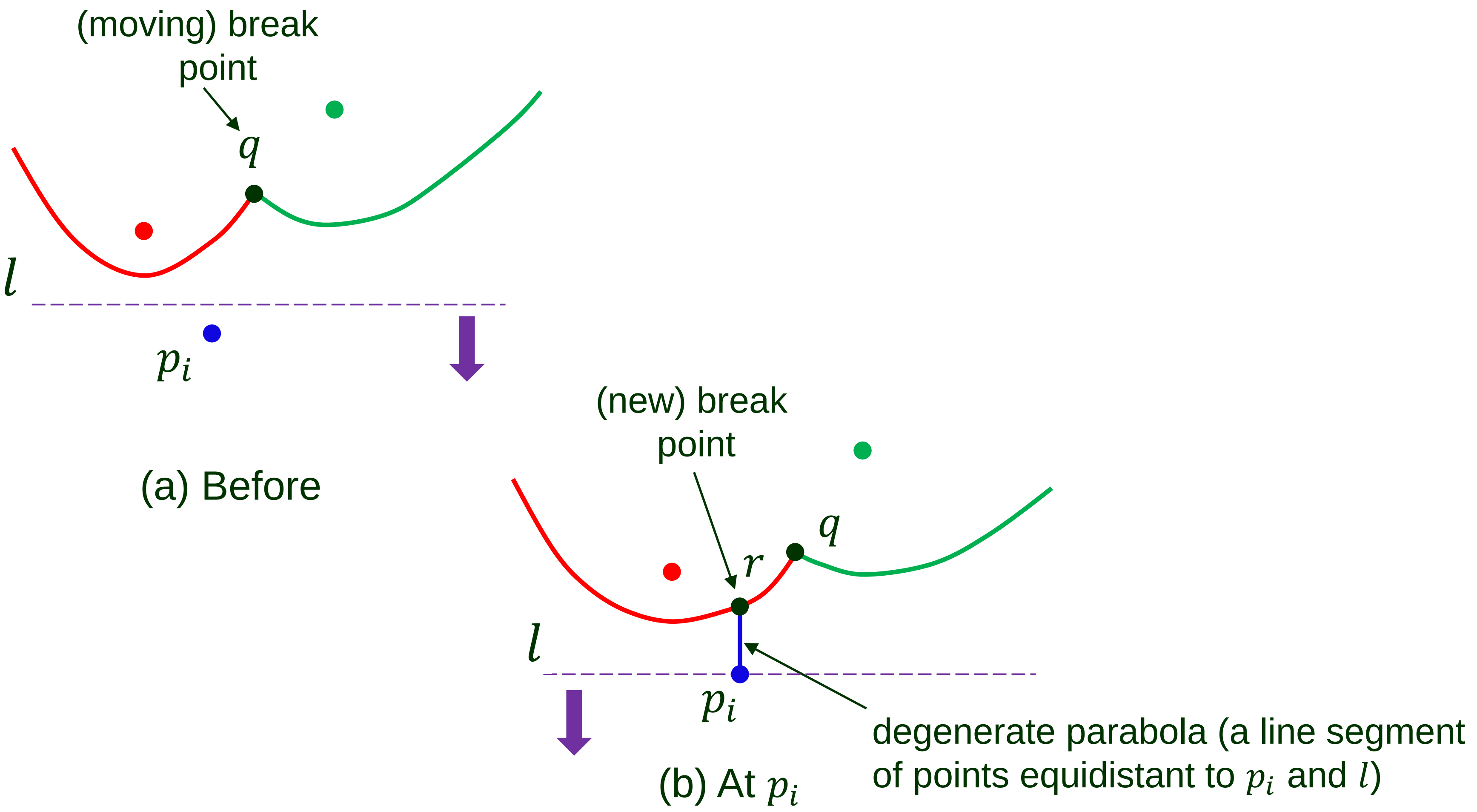

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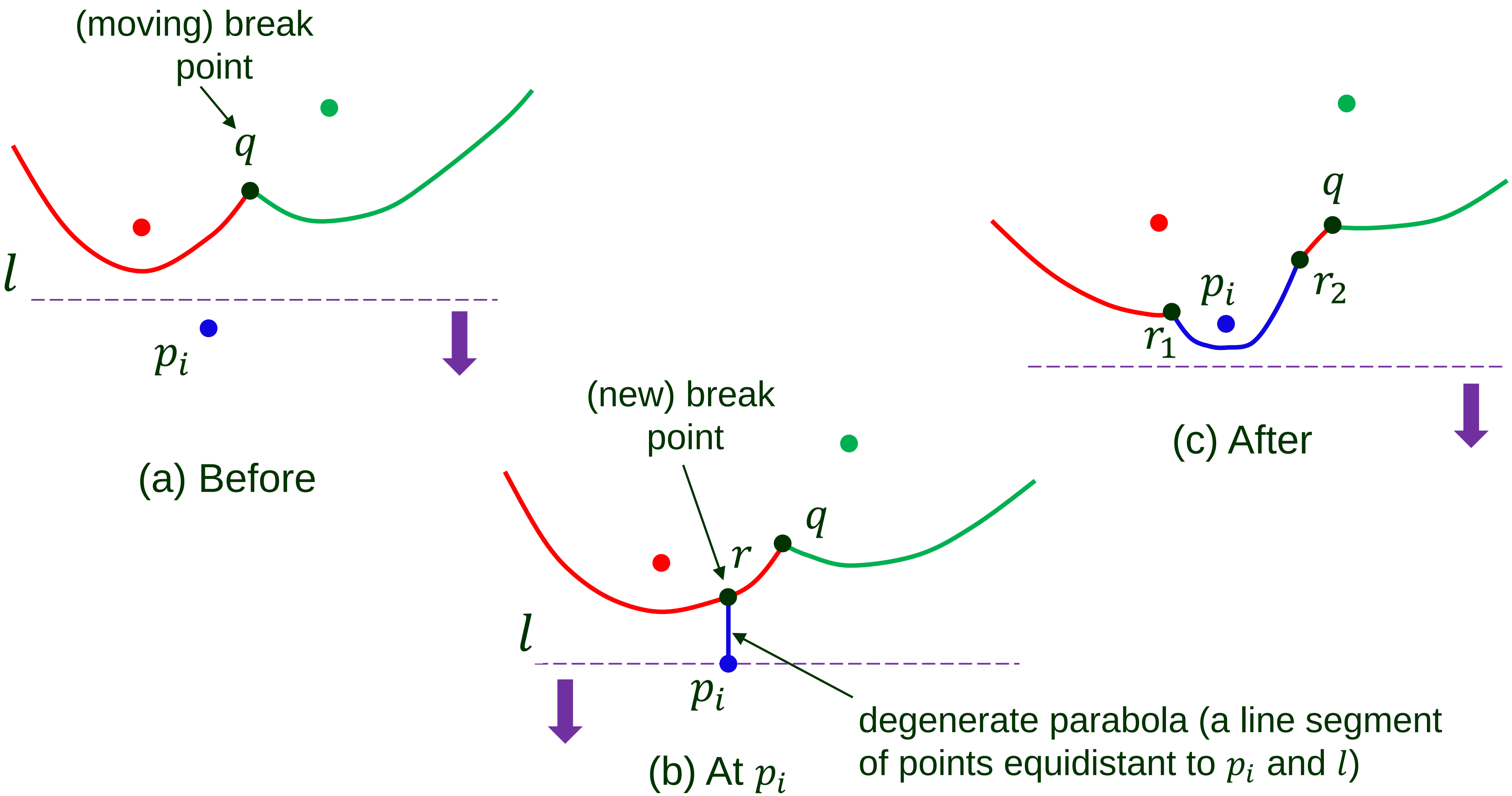

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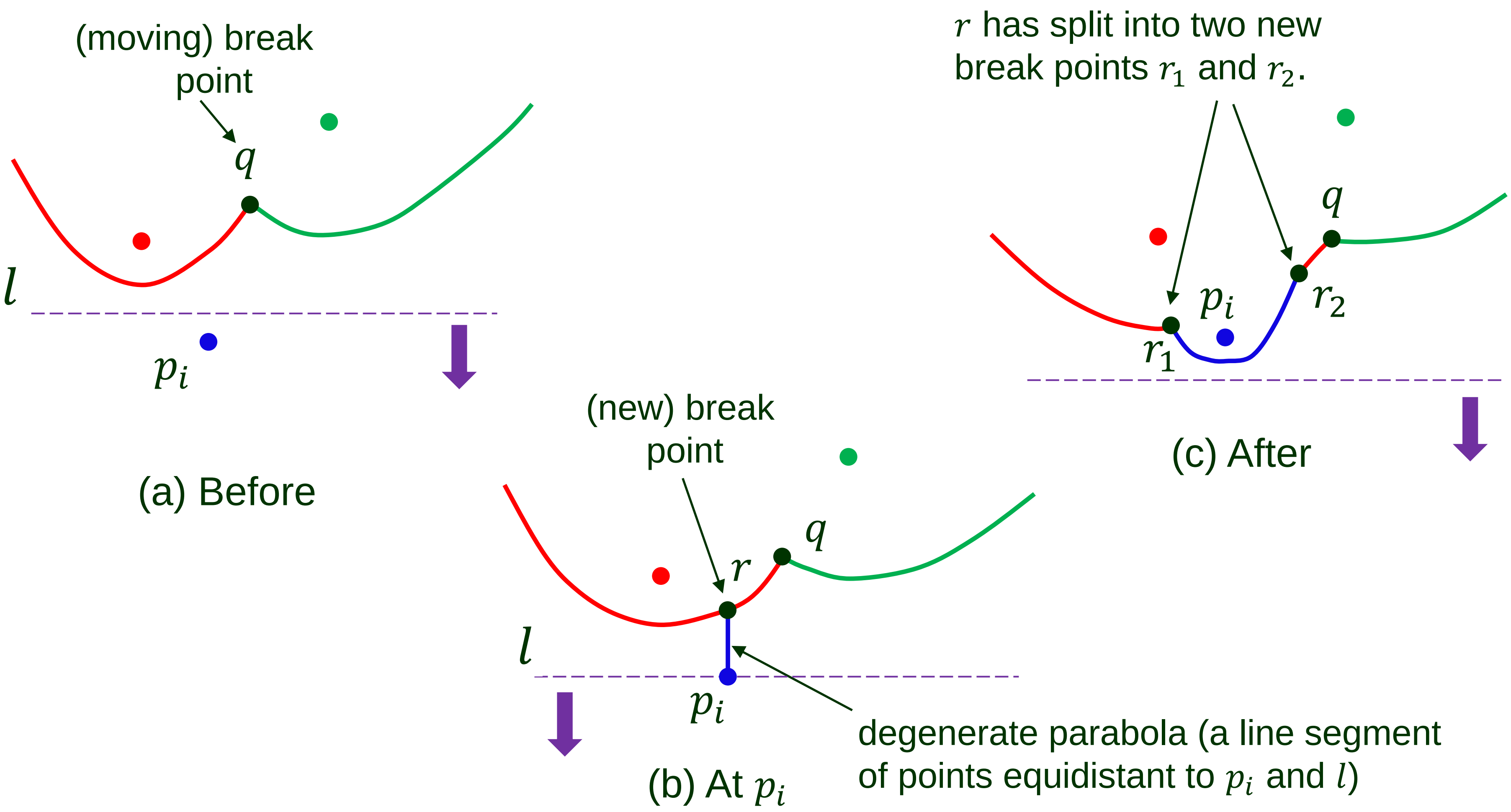
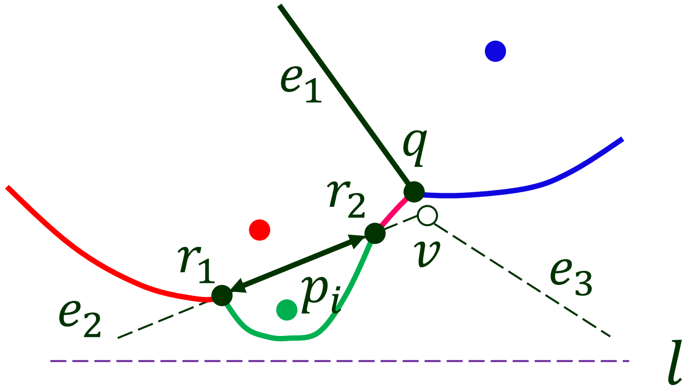


Tracing Out Voronoi Edges

- Two new break points emerge right after a site event.
- They trace out the same edge (e_2 below) in opposite directions.



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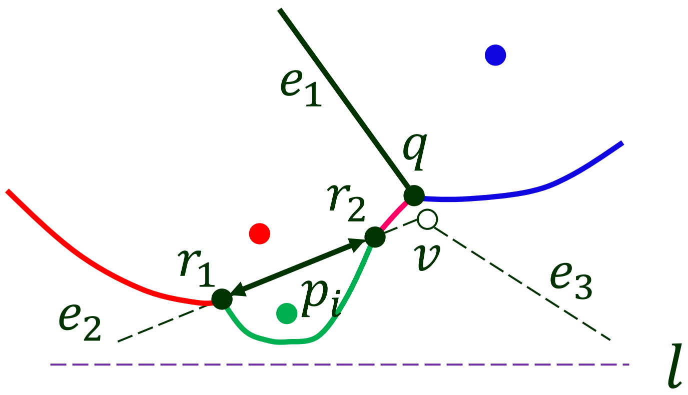

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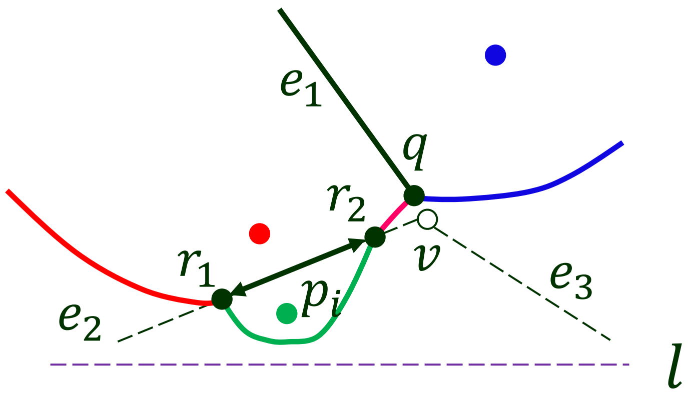

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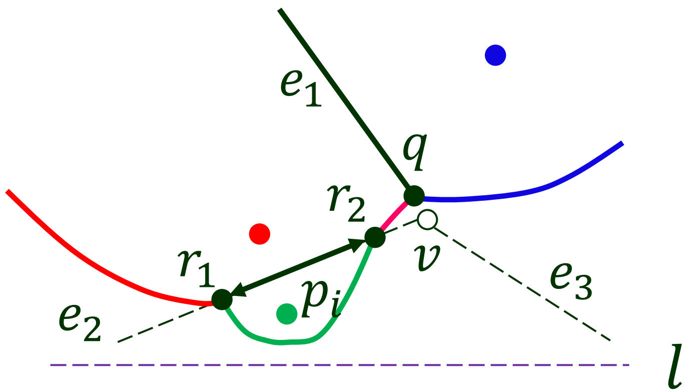

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It will grow and meet another edge and become connected.

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At a site event:

- A new arc appears on the beach line.
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$\Rightarrow$ # increase in arcs ≤ 2.

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- ♦ A new arc appears on the beach line.
- ♦ A new Voronoi edge starts to be traced out.

Lemma A new arc can only appear on the beach line via a site event.

Corollary The beach line consists of $\leq 2n - 1$ parabolic arcs.

Proof The first encountered site generates one arc.

Each newly encountered site

- yields one new arc, and
- splits at most one existing arc into two (i.e., adding one arc to the total count).

$\Rightarrow$ # increase in arcs ≤ 2.

arcs $\leq 1 + 2(n-1) = 2n - 1$.