Yan-Bin Jia, Iowa State University

Solving CSPs

Outline

I. Binary CSPs

II. Constraint Propagation

* Figures are from the textbook site (or drawn by the instructor) unless the source is specifically cited.

I. Binary CSP

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A *global constraint* involves an arbitrary number of variables (but not necessarily all the variables).

Alldiff$(v_1, \ldots, v_k)$: variables $v_1, \ldots, v_k$ must have different values.

e.g., Sudoku (all variables in a row, column, or 3×3 box).

Cryptarithmetic Puzzle

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\begin{array}{cccc}
 & T & W & O \\
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Addition constraints on the four columns:

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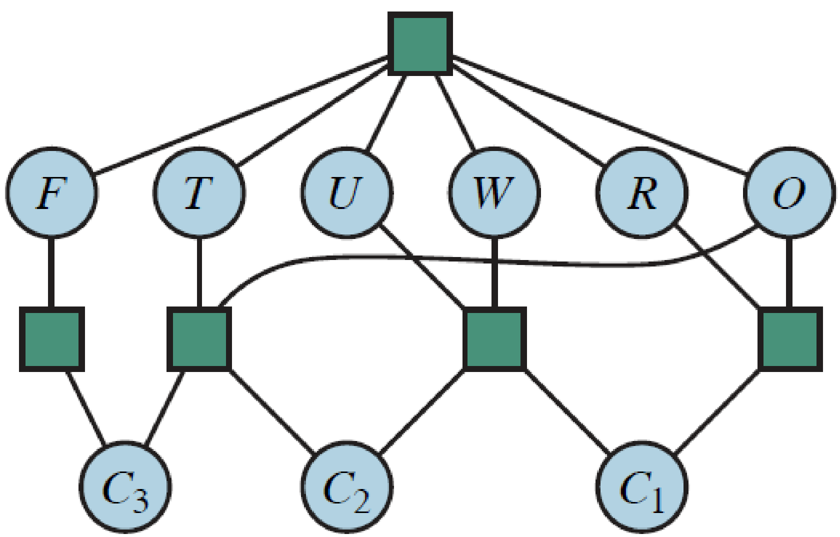

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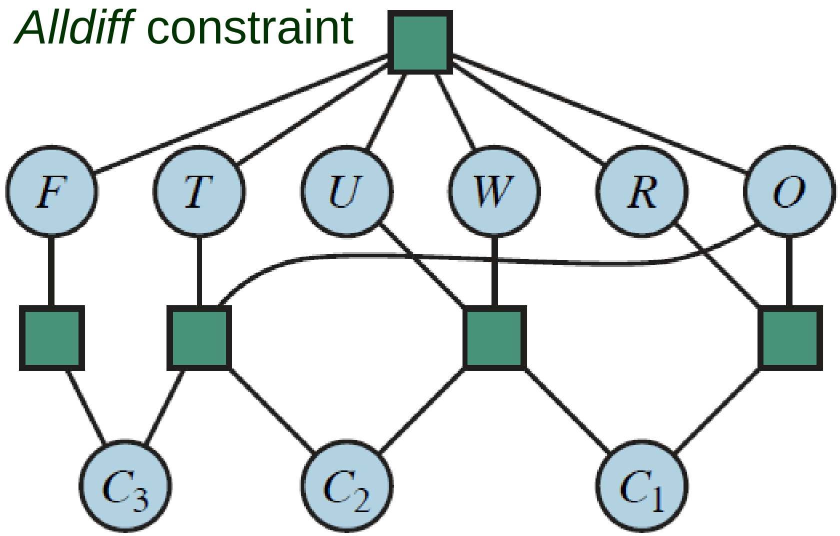

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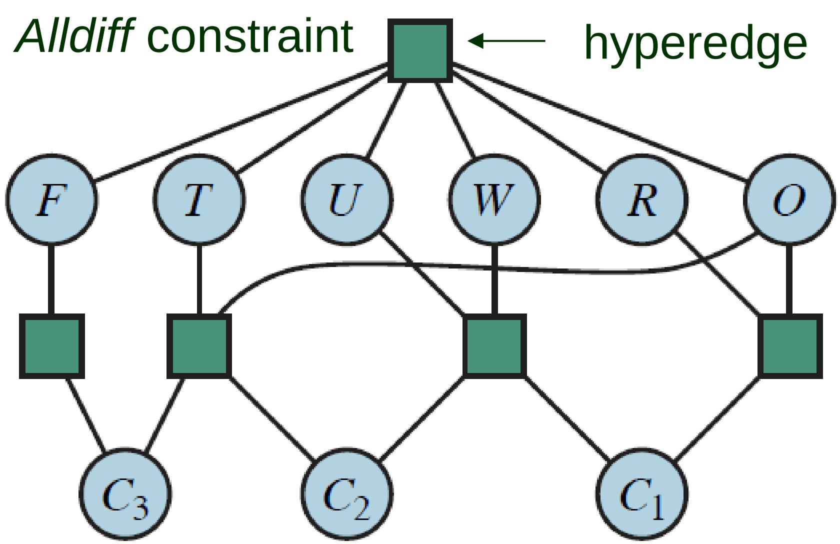

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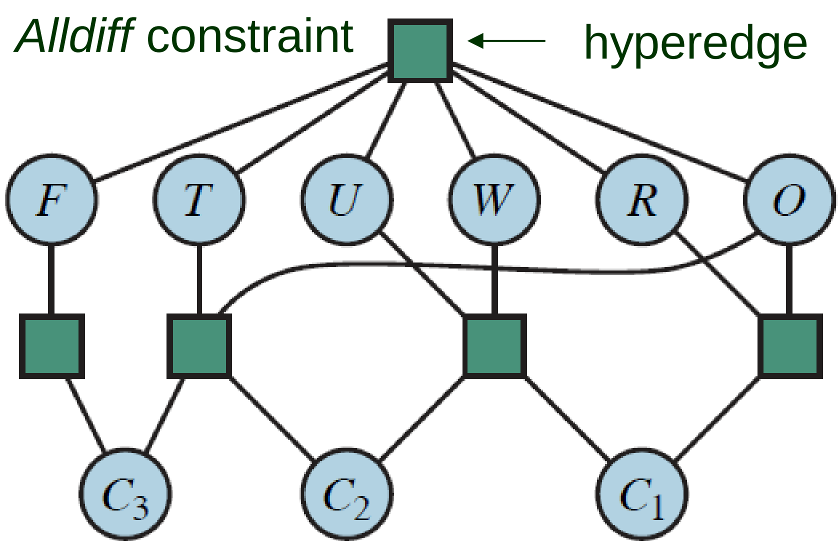

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A *hypergraph* generalizes a graph in that an edge (called an *hyperedge*) can connect more than two vertices.

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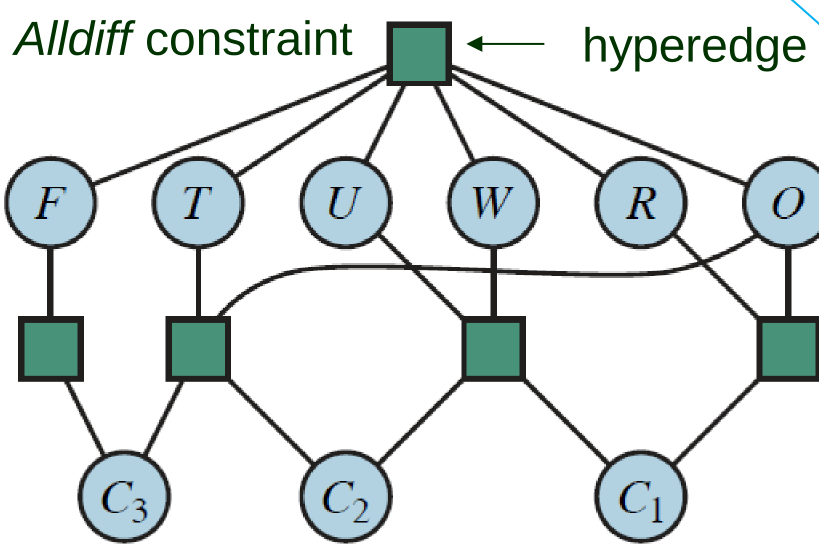

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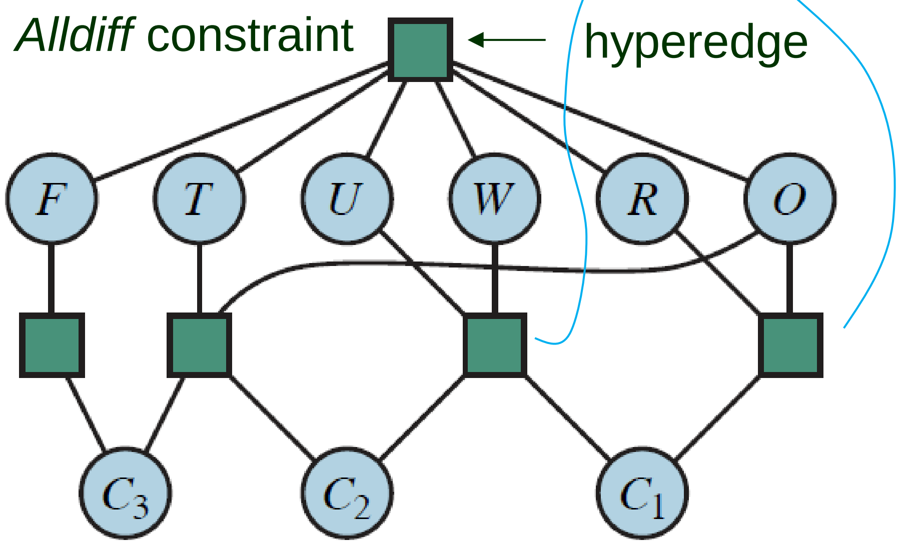

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Transformation to Binary Constraints

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Advantages of a global constraint such as *Alldiff*:

- Easier and less error-prone to write.
- Allowing efficient special-purpose inference algorithms.

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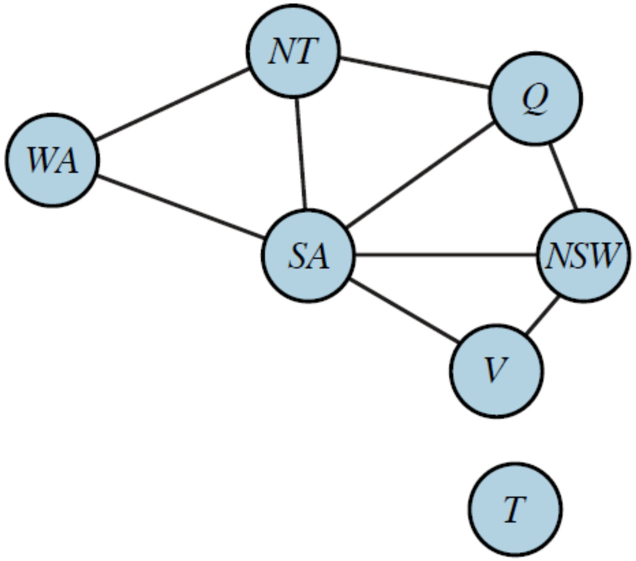

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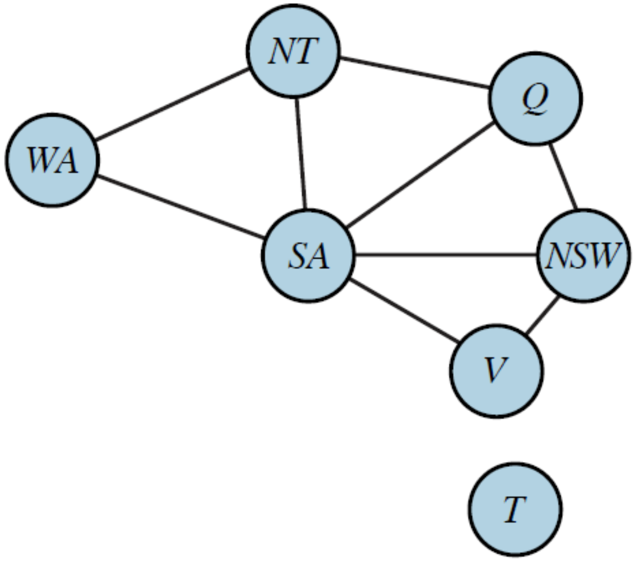

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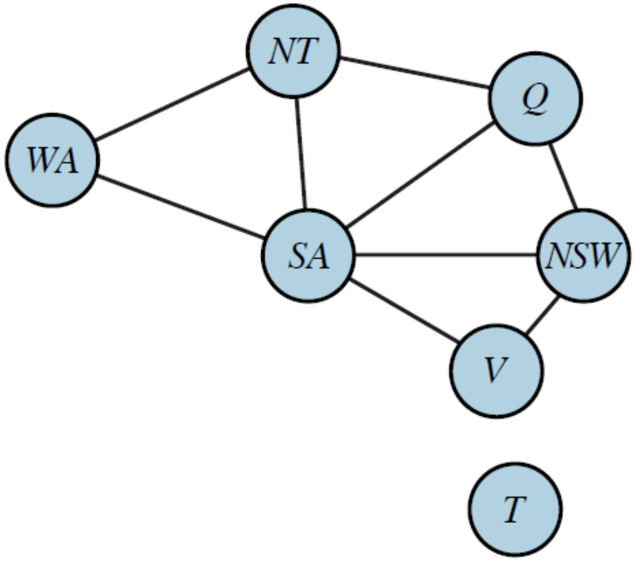

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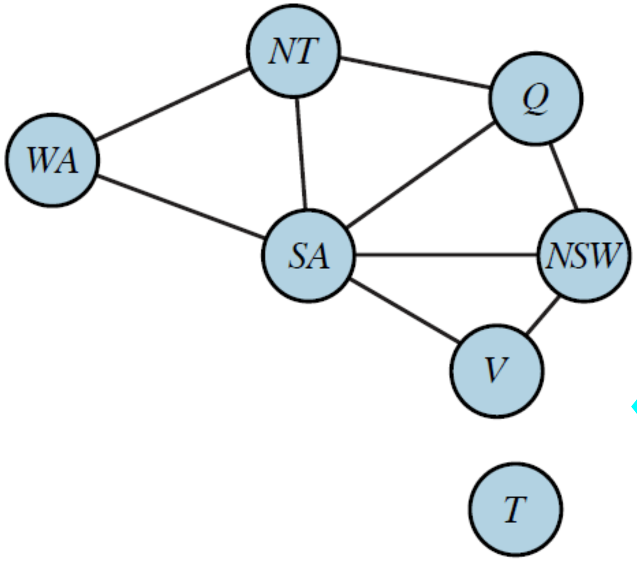

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Domain for *SA*: {red, blue}

♦ Eliminate all the unary constraints by reducing the domains of the involved variables at the start.

♦ A variable is *node-consistent* if all the values in its domain satisfy its unary constraints.

Arc Consistency

A variable X_i is *arc-consistent* with respect to (w.r.t.) another variable X_j if for *every* value in D_i there exists a value in D_j that satisfies the binary constraint on the arc (i.e., edge (X_i, X_j)).

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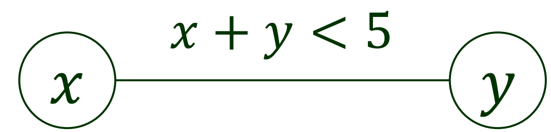

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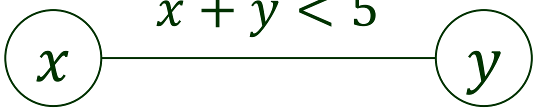

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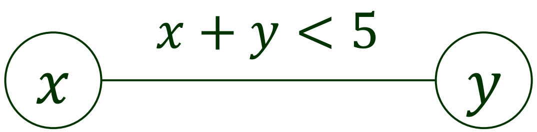

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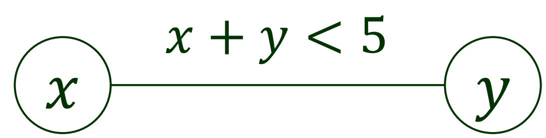

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The constraint graph is *arc-consistent* if every variable is arc consistent with every other variable.

Application of Arc Consistency

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$$Y = X^2 \quad \text{where } X, Y \in \{0, 1, \dots, 9\}$$

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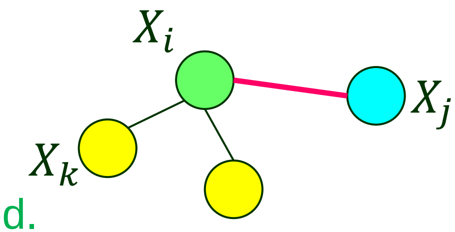
No effect on the domain {*red*, *green*, *blue*} of each variable.

Arc-Consistency Algorithm AC-3



function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
    queue ← a queue of arcs, initially all the arcs in *csp*

    while *queue* is not empty **do**
        (X_i, X_j) ← POP(*queue*)
        if REVISE(*csp*, X_i, X_j) **then** // domain of X_i has been reduced.
            if size of $D_i = 0$ **then return** *false*
            for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do** // propagate to other variables
                add (X_k, X_i) to *queue* // sharing a constraint with X_i
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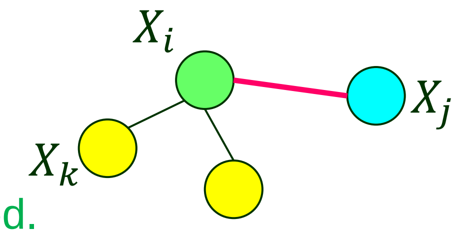

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Domain size ≤ d & c binary constraints.

- Each arc inserted into the queue ≤ d times (X_i has ≤ d values to delete).
- Each checking takes $O(d^2)$ time.

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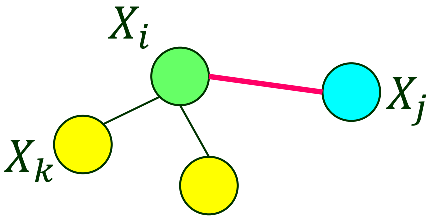

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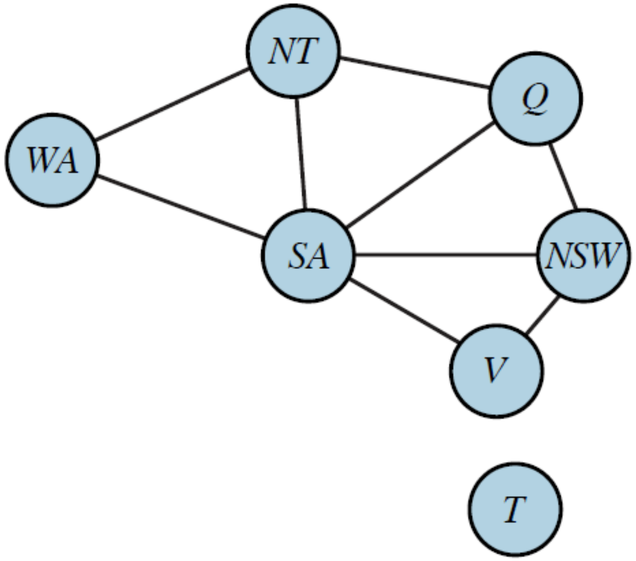
$$O(cd^3)$$

Path Consistency

Arc consistency reduces the domains.



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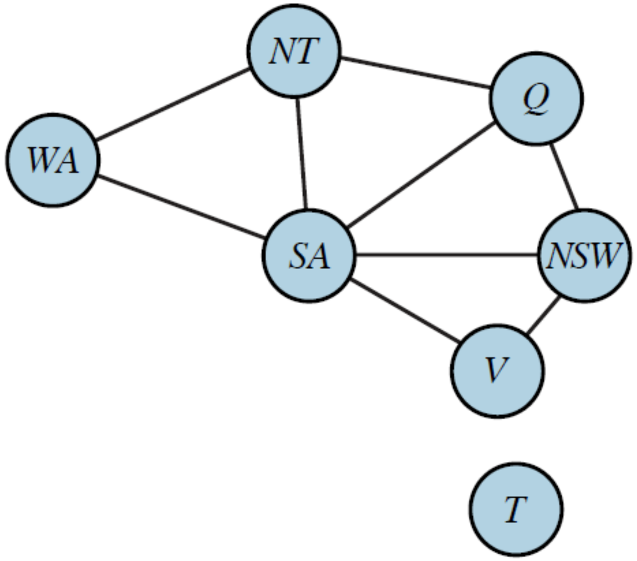

No solution if using two colors even though every constraint can be satisfied individually.

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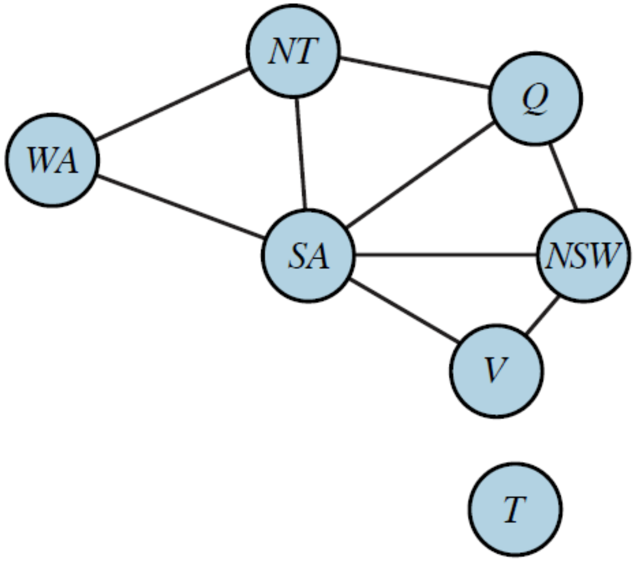

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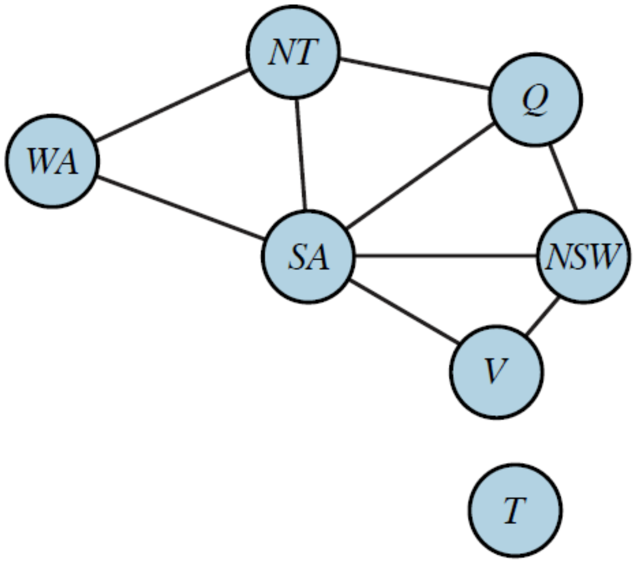

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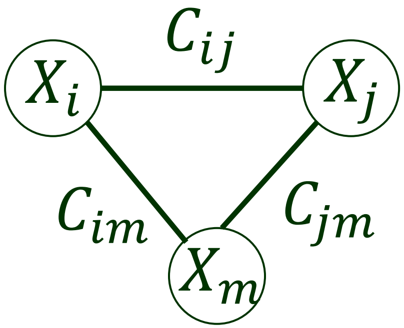
$\{X_i, X_j\}$ is *path-consistent* w.r.t. X_m if for every assignment to X_i, X_j consistent with their constraint C_{ij} (if exists), there exists an assignment to X_m that satisfies the constraints C_{im} and C_{jm} between them and X_m.



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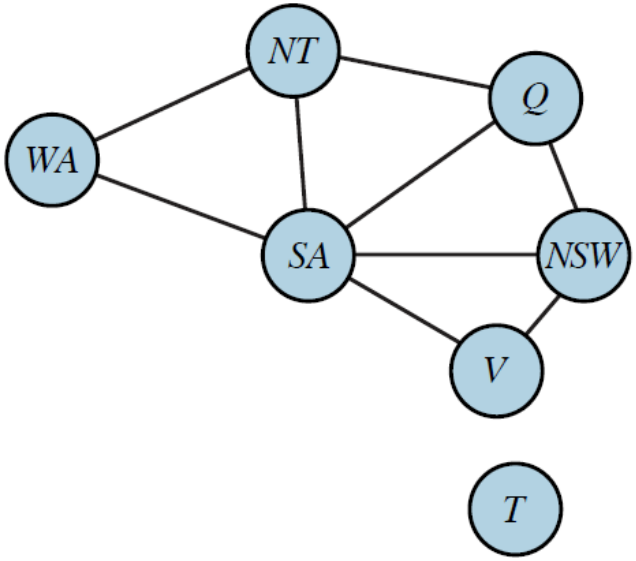

Make {*WA*, *SA*} path-consistent w.r.t. *NT*.

Assume two colors only: *red*, *blue*.

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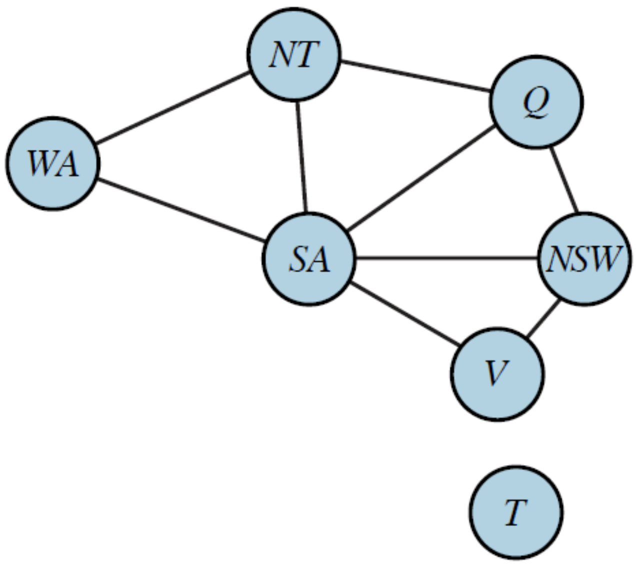

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{*WA* = *red*, *SA* = *blue*}

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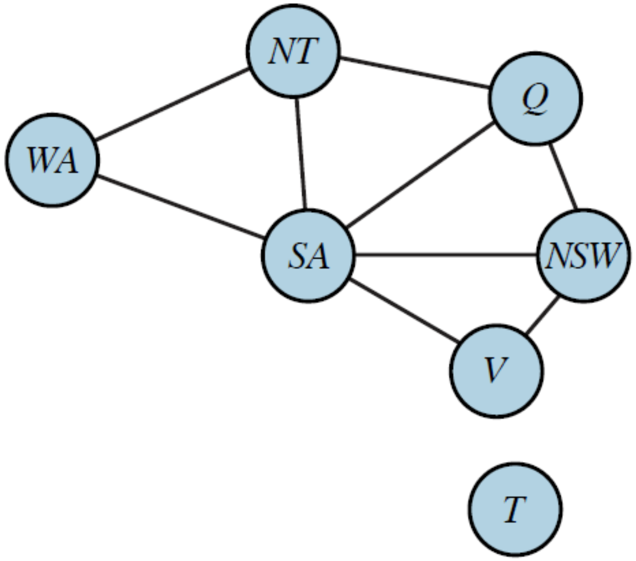

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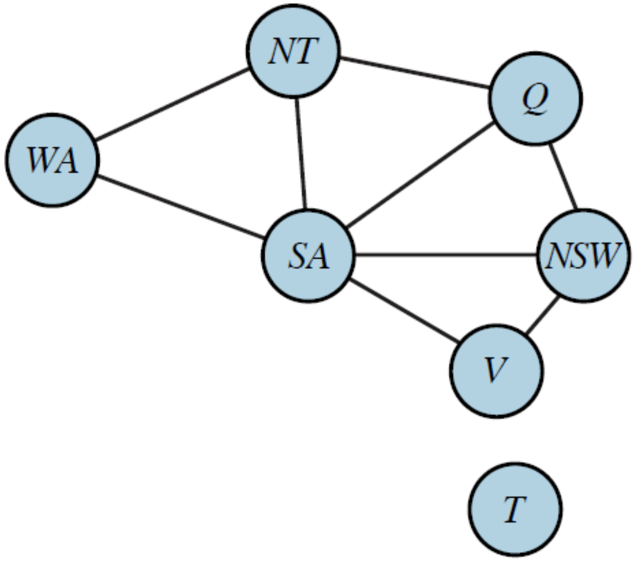
No assignment for *NT* in either case!

Application of Path Consistency



Make {*WA*, *SA*} path-consistent w.r.t. *NT*.

Assume two colors only: *red*, *blue*.

Only two possible assignments:

{*WA* = *red*, *SA* = *blue*}

{*WA* = *blue*, *SA* = *red*}

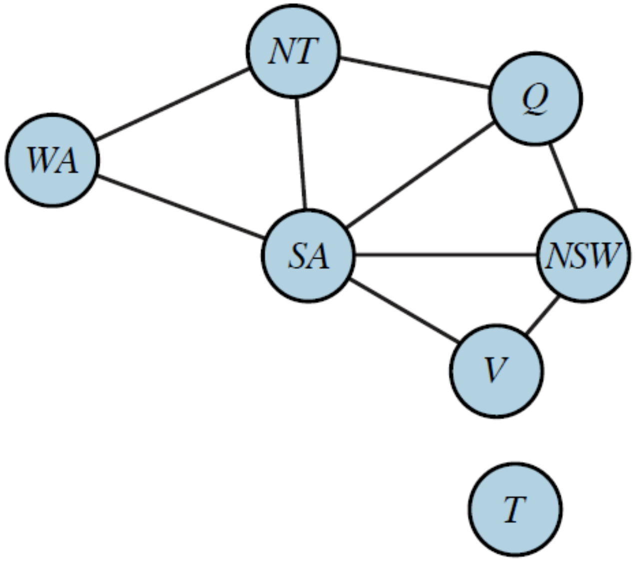
⇩

No assignment for *NT* in either case!

⇩

Eliminate both assignments to *WA* and *SA*.

Application of Path Consistency



Make {*WA*, *SA*} path-consistent w.r.t. *NT*.

Assume two colors only: *red*, *blue*.

Only two possible assignments:

{*WA* = *red*, *SA* = *blue*}

{*WA* = *blue*, *SA* = *red*}

⇩

No assignment for *NT* in either case!

⇩

Eliminate both assignments to *WA* and *SA*.

⇩

No solution to the problem.

Bounds Propagation

Two flights F_1 and F_2 have domains:

$$D_1 = [0,165] \text{ and } D_2 = [0,385]$$

max capacity

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⇓ domain reduction

$$D_1 = [420 - 385,165] = [35,165]$$
$$D_2 = [420 - 165,385] = [255,385]$$

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Bounds-consistent if for any two variables X, Y, whether X takes on its lower- or upper-bound values, there exists some value of Y that satisfies the constraint between X and Y.

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Two flights F_1 and F_2 have domains:

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Constraint: the two flights together carry 420 people.

⇓ domain reduction

$$D_1 = [420 - 385,165] = [35,165]$$
$$D_2 = [420 - 165,385] = [255,385]$$

Bounds-consistent if for any two variables X, Y, whether X takes on its lower- or upper-bound values, there exists some value of Y that satisfies the constraint between X and Y.

* By the above definition, for some value of X between its lower and upper bounds, there may not exist a value of Y to meet the constraint (e.g., a nonlinear constraint). e.g., for $X \in [0,\pi], Y \in \left[0,\frac{1}{2}\right]$ the constraint $Y = \sin X$ is not satisfied at $X = \pi/2$.

Sudoku

Fill the digits 1 to 9 in a 9×9 grid such that no digit appears twice in any row, column, or 3×3 box.

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Sudoku

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E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \ldots, A9, B1, \ldots, I9$

Domain: $D = \{1,2, \ldots, 9\}$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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F			6	7		8	2		
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27 *Alldiff* constraints:

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \ldots, A9, B1, \ldots, I9$

Domain: $D = \{1,2,\ldots,9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
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27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

⋮

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \dots, A9, B1, \dots, I9$

Domain: $D = \{1,2, \dots, 9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

⋮

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

⋮

Alldiff$(G7, G8, G9, H7, H8, H9, I7, I8, I9)$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \ldots, A9, B1, \ldots, I9$

Domain: $D = \{1,2, \ldots, 9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

⋮

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

⋮

Alldiff$(G7, G8, G9, H7, H8, H9, I7, I8, I9)$

A CSP solver can handle thousands of puzzles per second!

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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F			6	7		8	2		
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H	8			2		3			9
I			5		1		3		

Variables: $A1, \ldots, A9, B1, \ldots, I9$

Domain: $D = \{1,2, \ldots, 9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

⋮

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

⋮

Alldiff$(G7, G8, G9, H7, H8, H9, I7, I8, I9)$

A CSP solver can handle thousands of puzzles per second!
Only the simplest ones can be solved by AC-3.

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
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	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
D			8	1		2	9		
E	7					?			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Consider E6:

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
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Consider E6:

$D_{E6} \leftarrow \{1,2,\dots,9\}$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

$$= \{3,4,5,6,9\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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$$= \{3,4,5,6,9\}$$

$\Downarrow$ constraints in the column

$$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$$

$$= \{4\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
D			8	1		2	9		
E	7					4			8
F			6	7		8	2		
G			2	6		9	5		
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Consider E6:

$$D_{E6} \leftarrow \{1,2,\dots,9\}$$

⇓ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	?	3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\dots,9\}$$

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$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

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Consider I6:

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	?	3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

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$\Downarrow$ constraints in the column

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$$= \{4\}$$

Consider I6: $D_{I6} \leftarrow \{1,2,\dots,9\}$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$
$$= \{3,4,5,6,9\}$$

$\Downarrow$ constraints in the column

$$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$$
$$= \{4\}$$

Consider I6: $D_{I6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7					4			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	?	3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the box

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$$= \{4\}$$

Consider I6:

$$D_{I6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

$\Downarrow$ constraints in the box

$$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider E6:

$D_{E6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the box

$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$

$= \{3,4,5,6,9\}$

$\Downarrow$ constraints in the column

$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$

$= \{4\}$

Consider I6: $D_{I6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$

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Example of CSP Solution

	1	2	3	4	5	6	7	8	9
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I			5		1	7	3		

Consider E6:

$D_{E6} \leftarrow \{1,2,\ldots,9\}$

⇓ constraints in the box

$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$

$= \{3,4,5,6,9\}$

⇓ constraints in the column

$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$

$= \{4\}$

Consider I6: $D_{I6} \leftarrow \{1,2,\ldots,9\}$

⇓ constraints in the column

$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$

⇓ constraints in the box

$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$

Consider A6:

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2	?	6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7					4			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider E6:

$D_{E6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the box

$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$

$= \{3,4,5,6,9\}$

$\Downarrow$ constraints in the column

$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$

$= \{4\}$

Consider A6:

$D_{A6} \leftarrow \{1,2,\ldots,9\}$

Consider I6:

$D_{I6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$

$\Downarrow$ constraints in the box

$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2	?	6		
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

⇩ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

$$= \{3,4,5,6,9\}$$

⇩ constraints in the column

$$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$$

$$= \{4\}$$

Consider I6:

$$D_{I6} \leftarrow \{1,2,\ldots,9\}$$

⇩ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

⇩ constraints in the box

$$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$$

Consider A6:

$$D_{A6} \leftarrow \{1,2,\ldots,9\}$$

⇩ constraints in the column

$$D_{A6} \leftarrow \{1\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2	1	6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7					4			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider E6:

$D_{E6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the box

$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$

$= \{3,4,5,6,9\}$

$\Downarrow$ constraints in the column

$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$

$= \{4\}$

Consider I6:

$D_{I6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$

$\Downarrow$ constraints in the box

$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$

Consider A6:

$D_{A6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$D_{A6} \leftarrow \{1\}$