Minkowski Sums

Outline:

I. Definition

II. C-obstacles

III. Complexity of the sum of two convex polygons

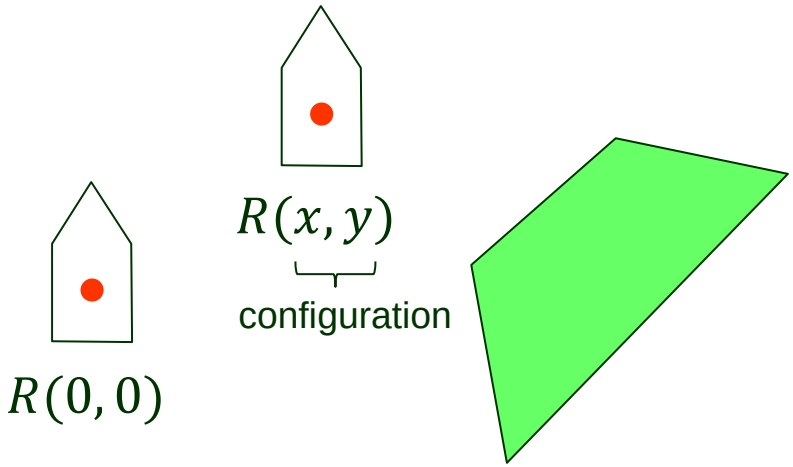
IV. Computation

V. Non-convex robot or obstacle

VI. Translational motion planning

C-obstacle for a Translational Robot

Robot R Obstacle P



C-obstacle for a Translational Robot

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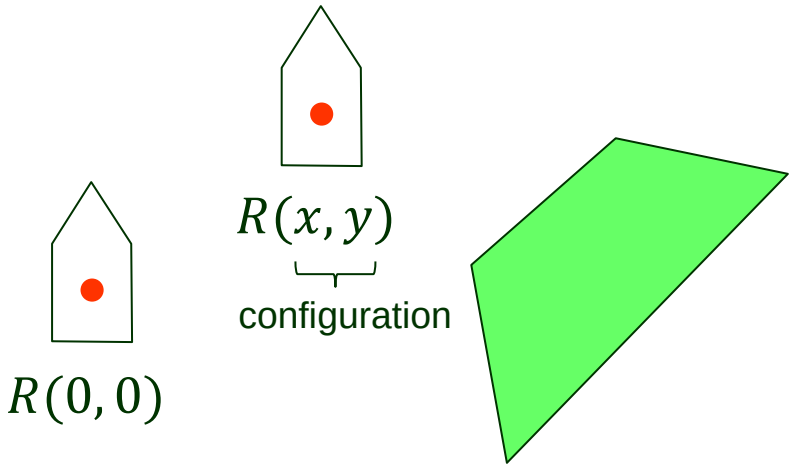

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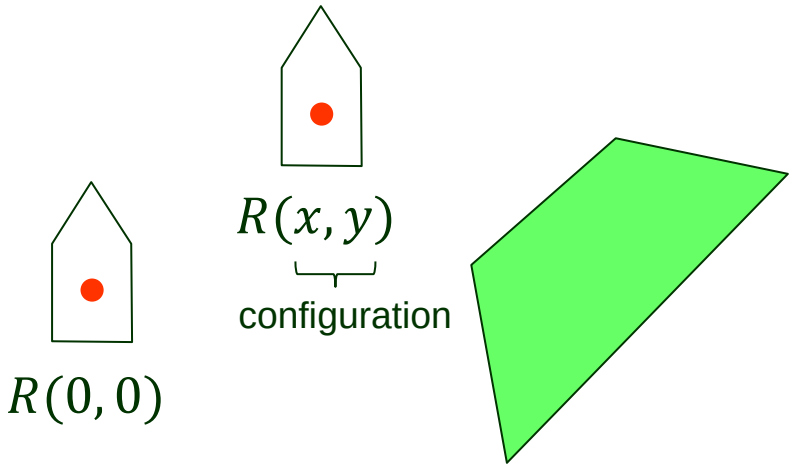

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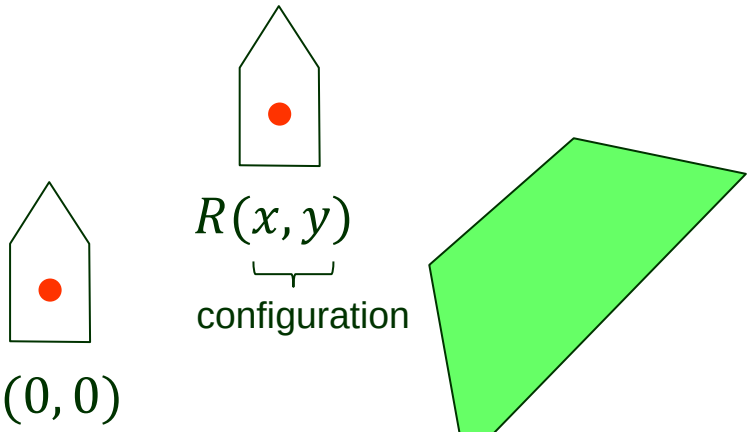

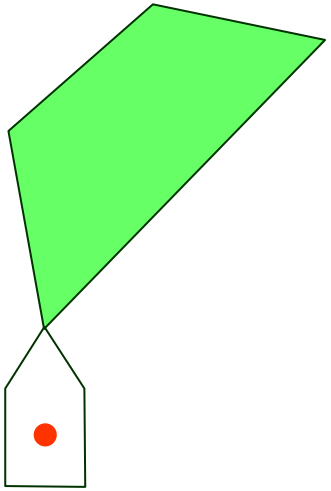

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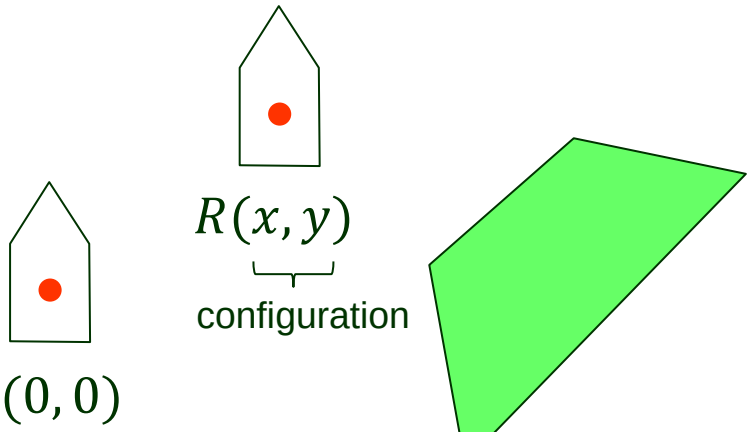

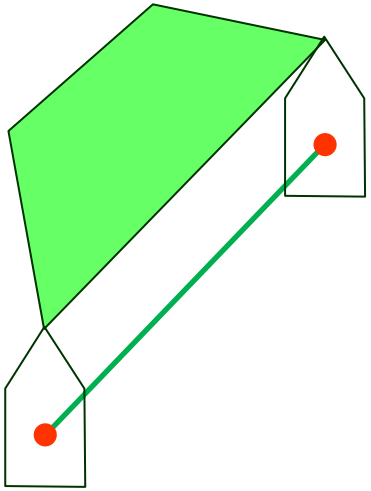

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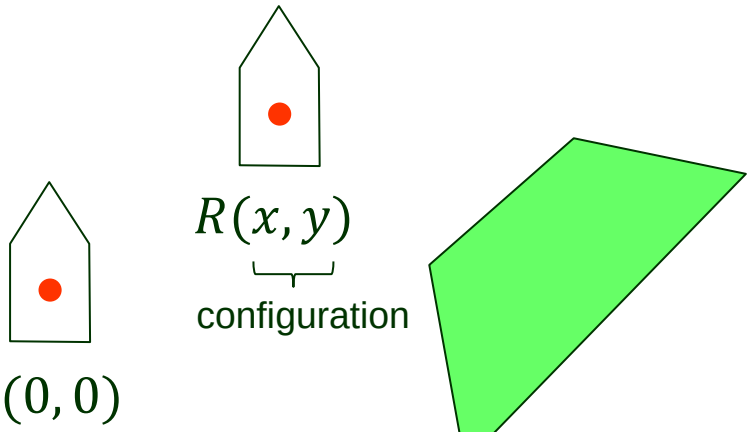

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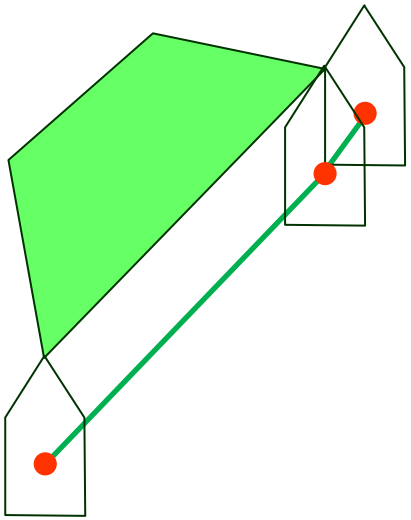

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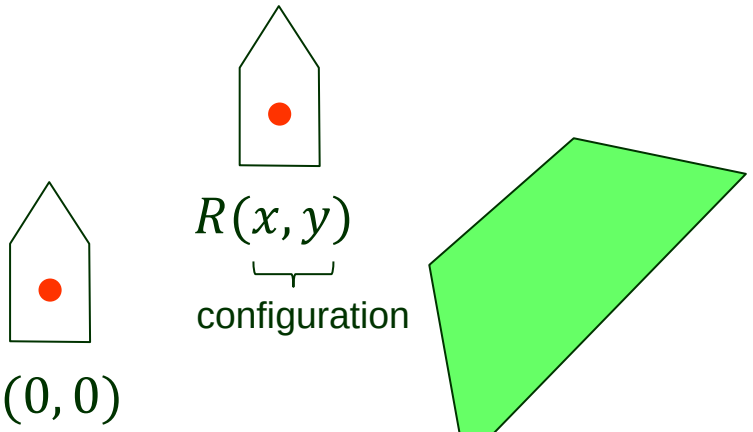

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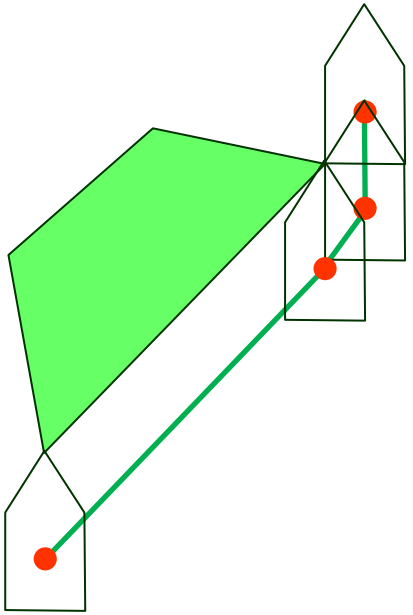

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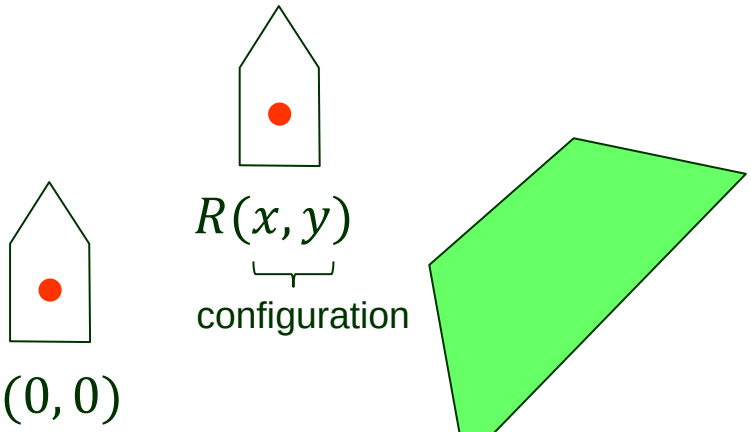

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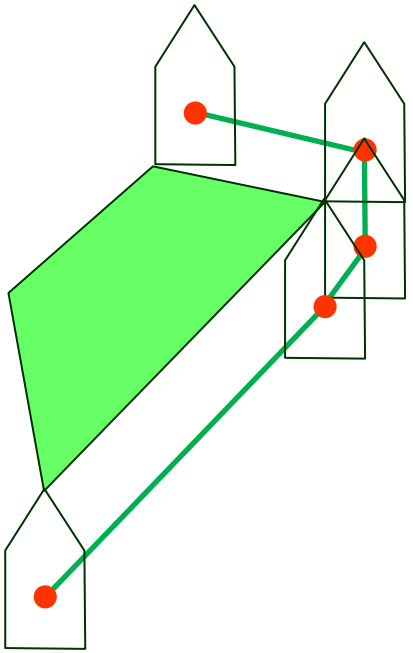

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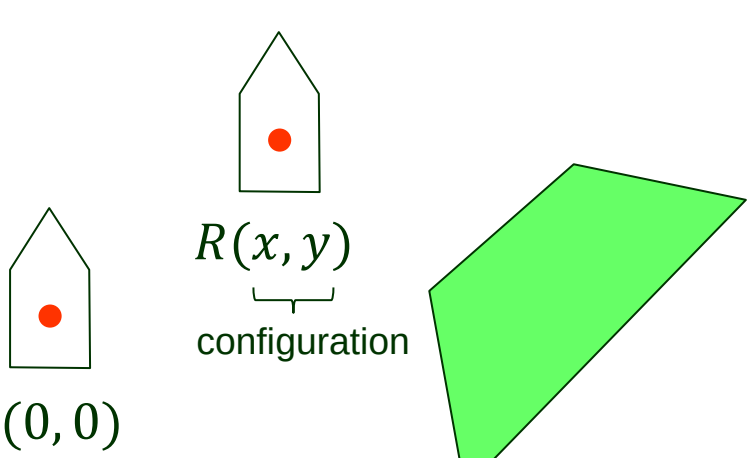

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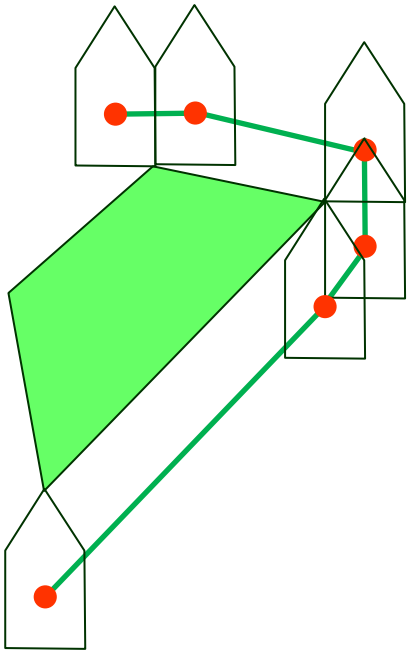

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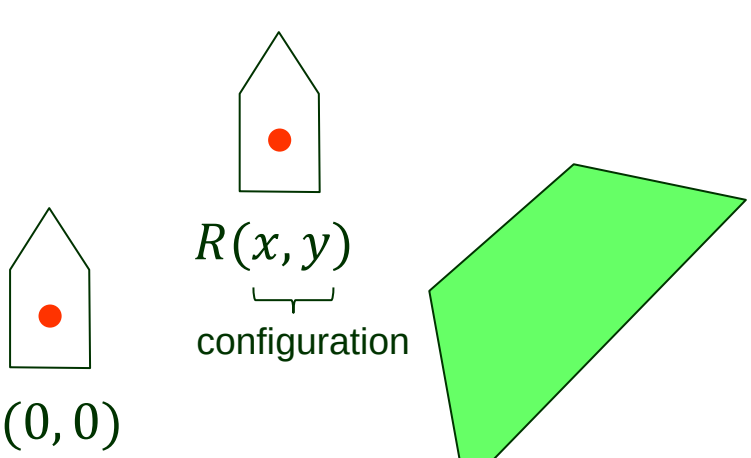

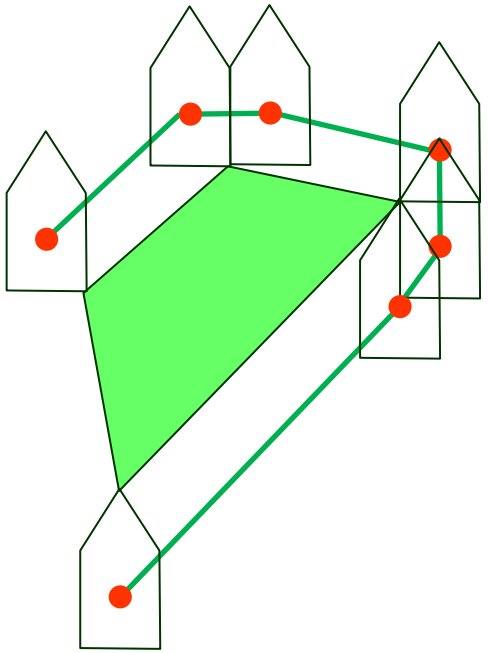

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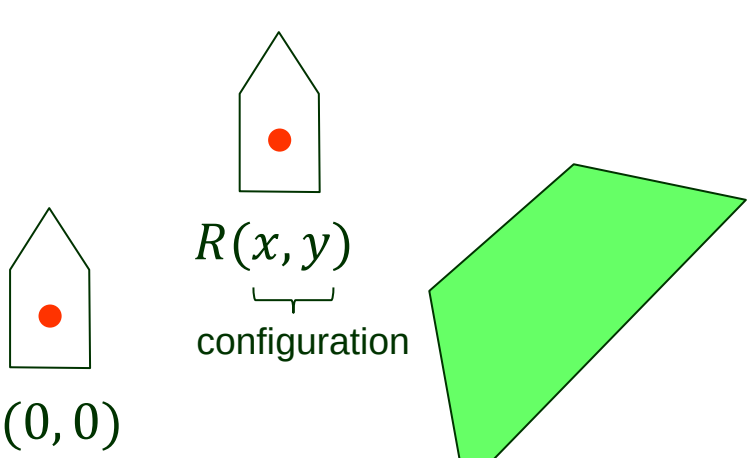

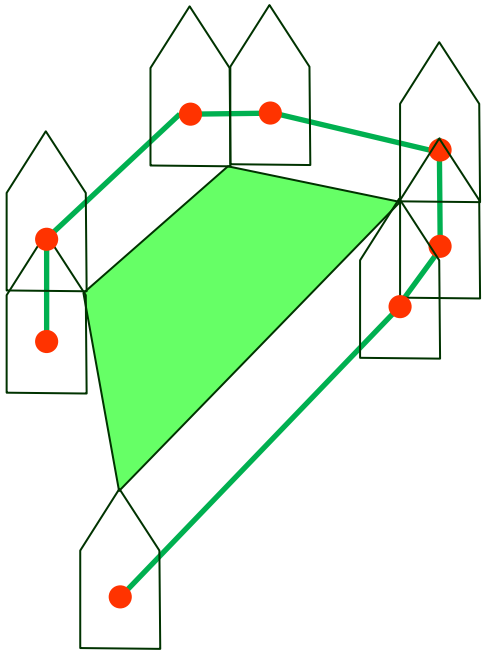

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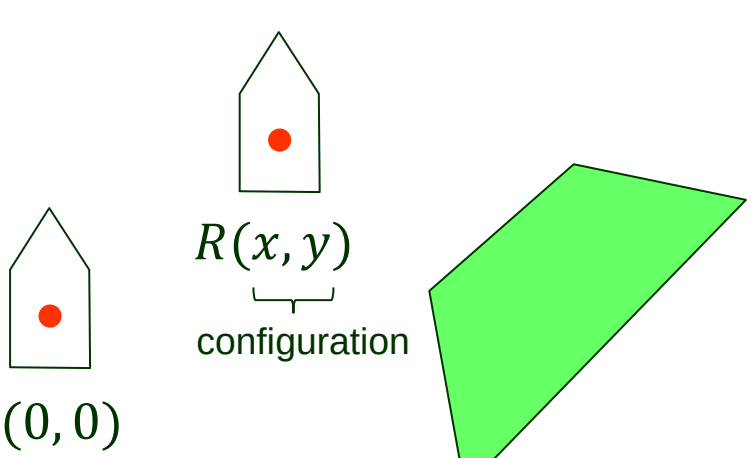

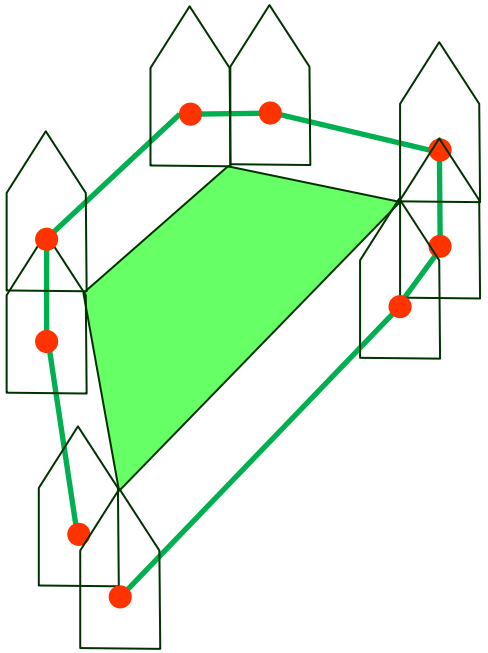

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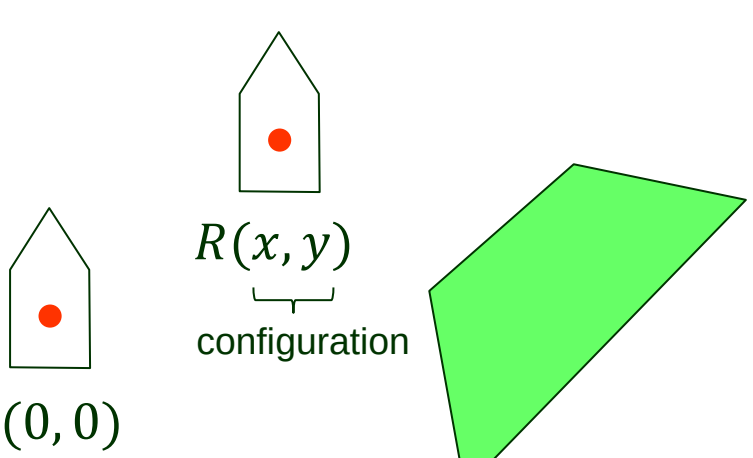

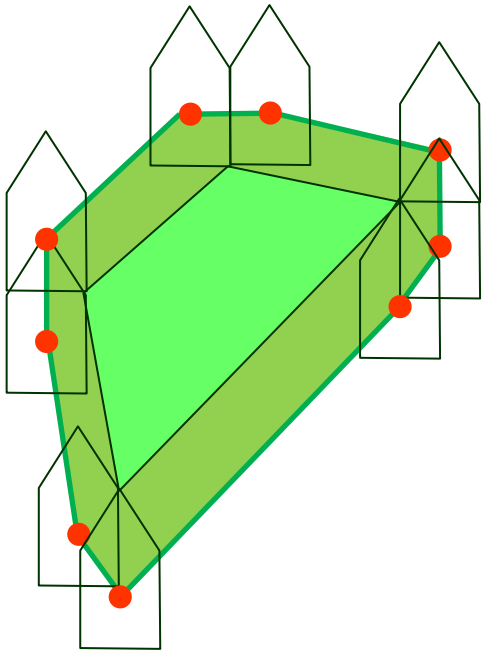

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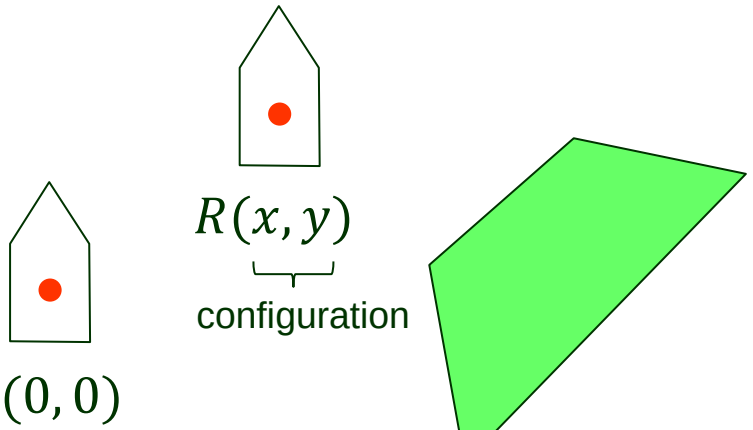

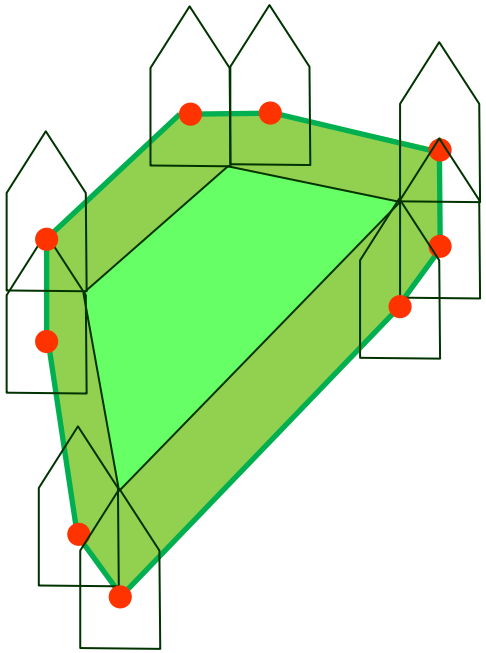

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The boundary of CP is traced out by the reference point of R as it slides along the boundary of P.

I. Minkowski Sum

Hermann Minkowski (1864-1909) – Albert Einstein was his former student.

Two sets $S_1, S_2 \in \mathbb{R}^2$

$$S_1 \oplus S_2 = \{p + q \mid p \in S_1, q \in S_2\}$$

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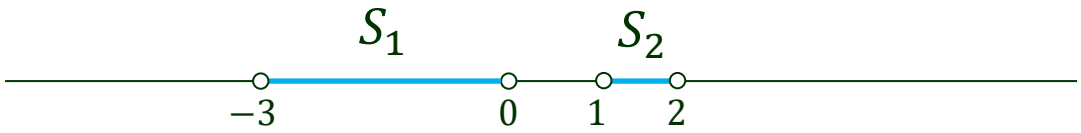

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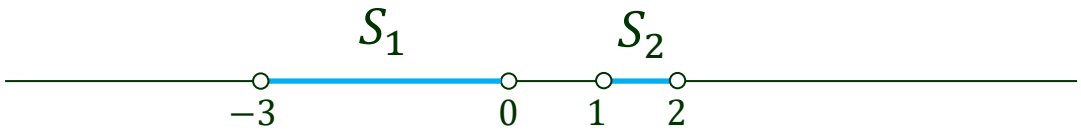

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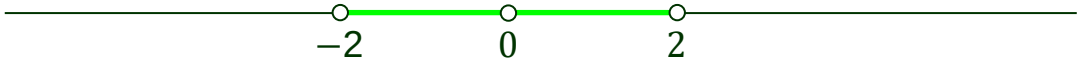

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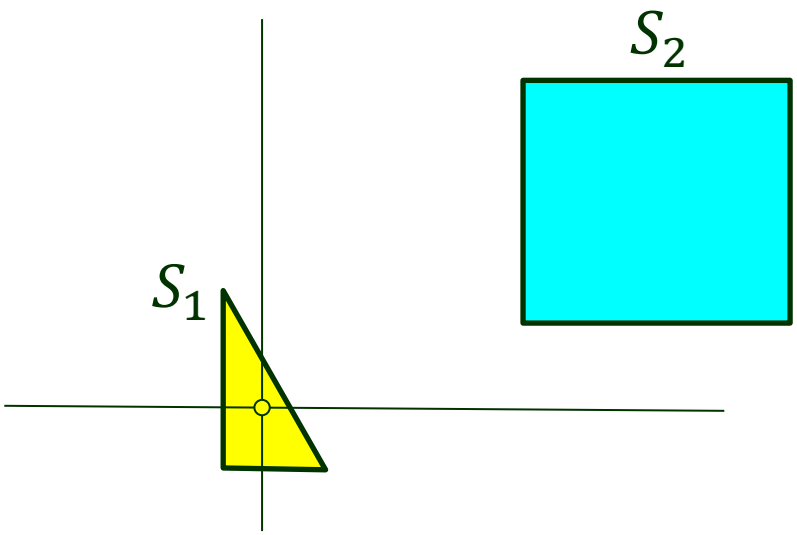

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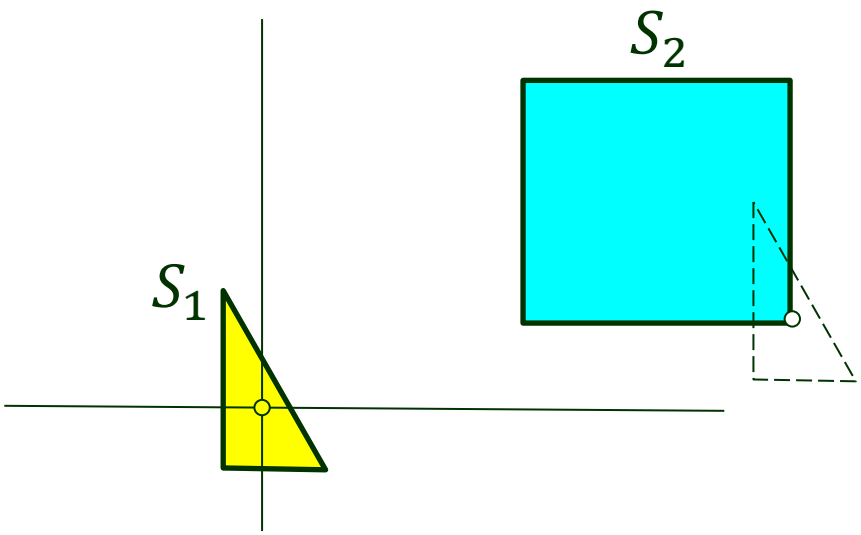

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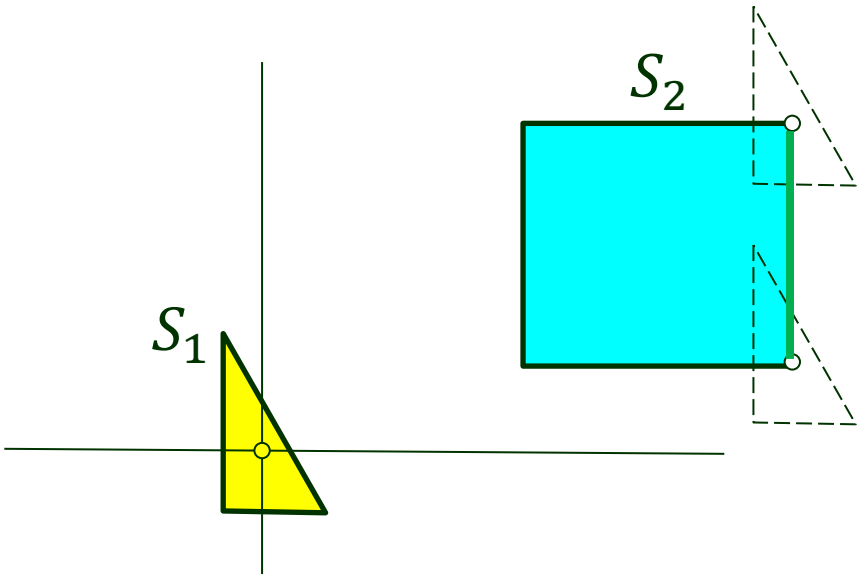

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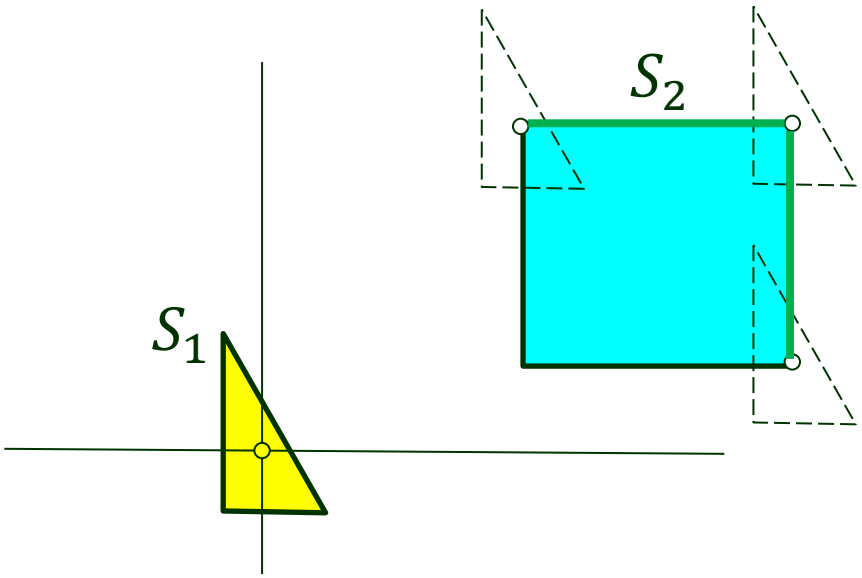

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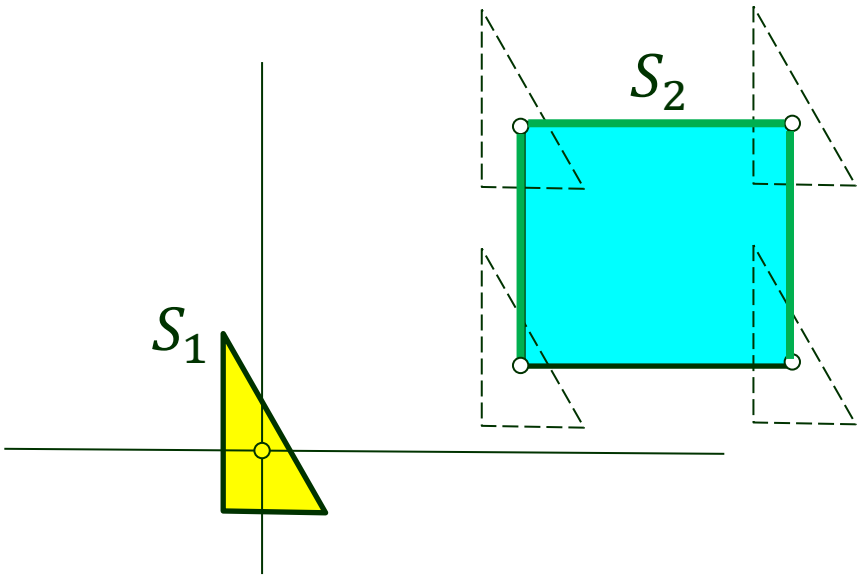

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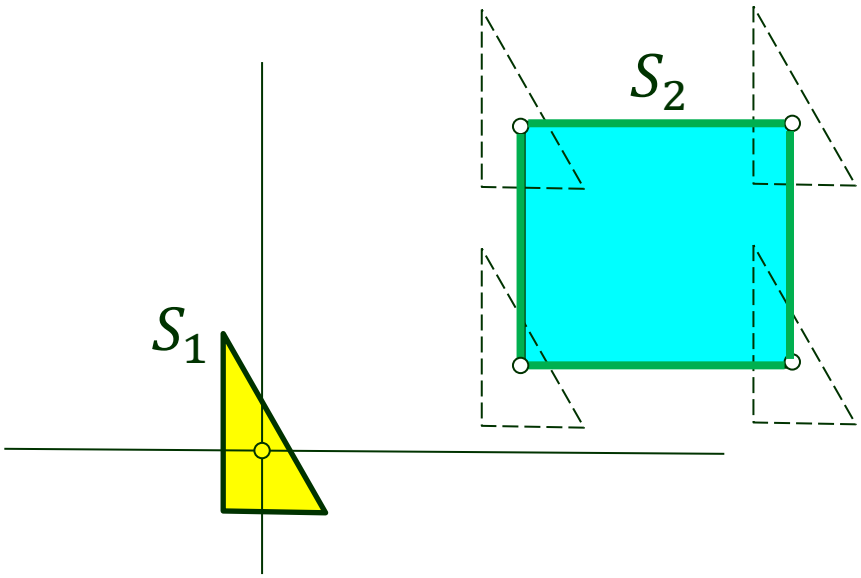

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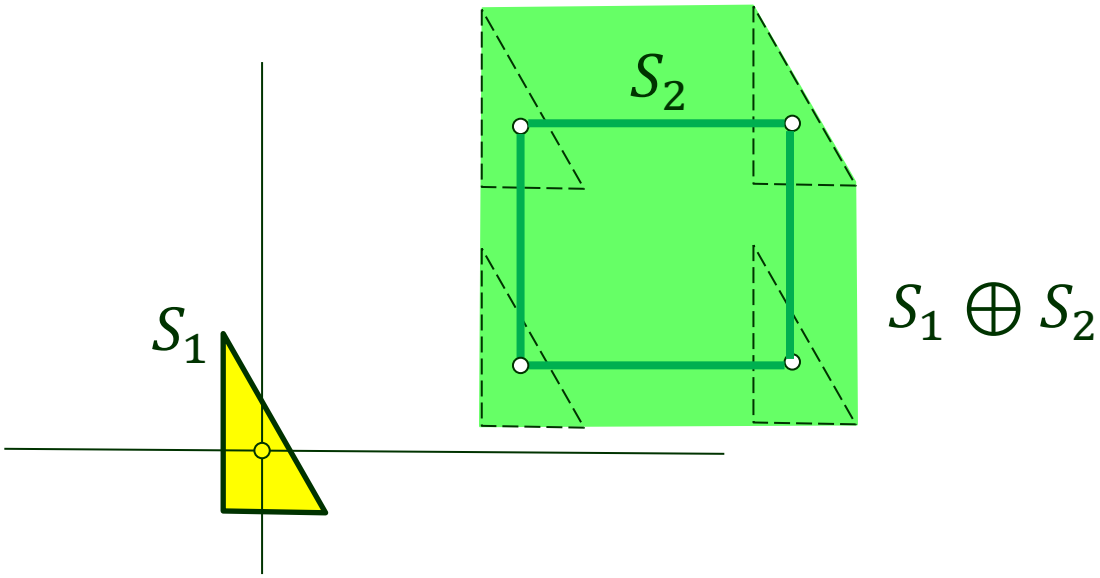

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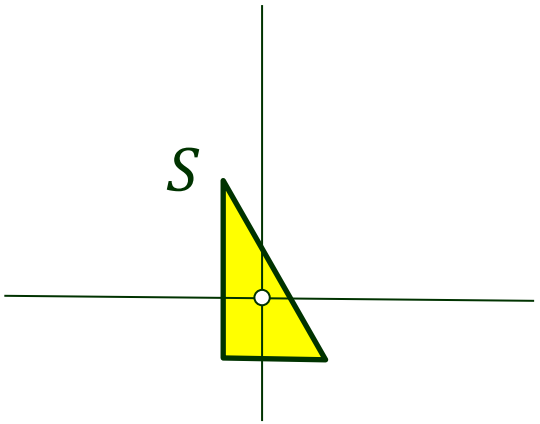

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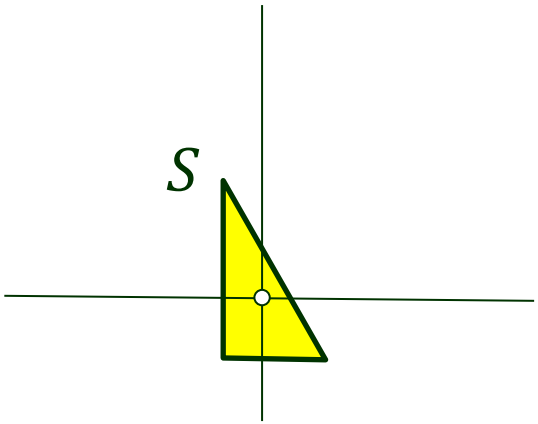

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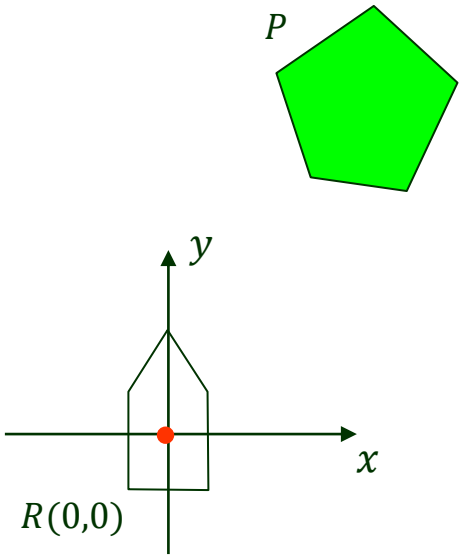

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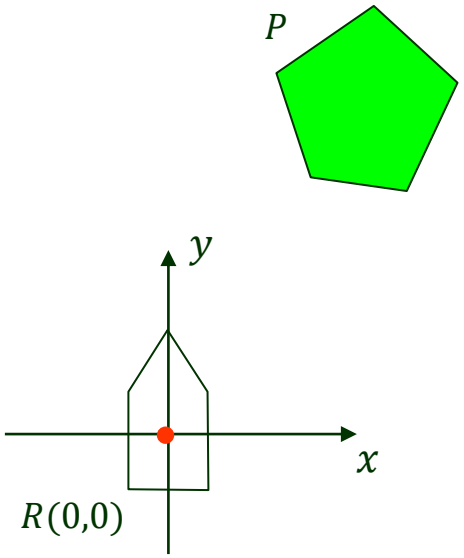

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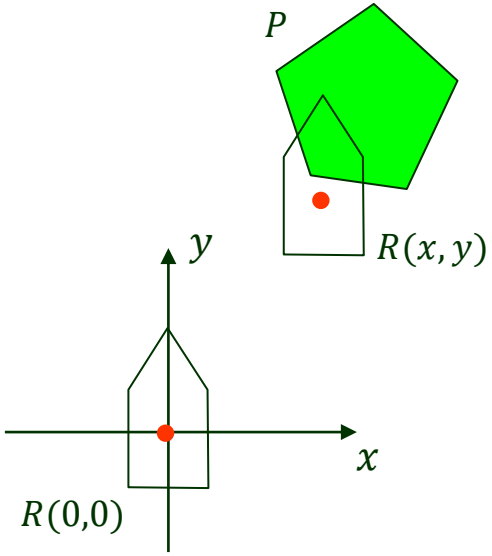

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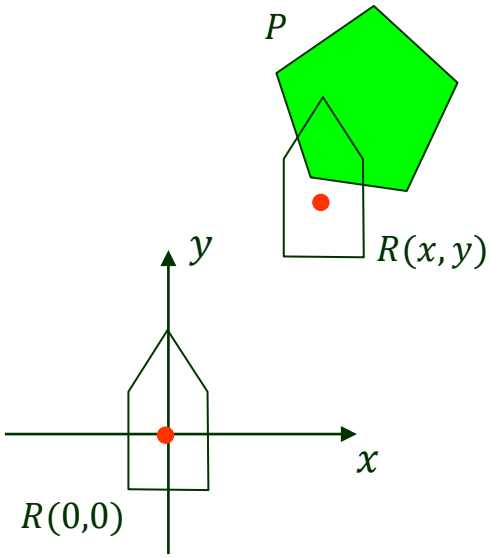

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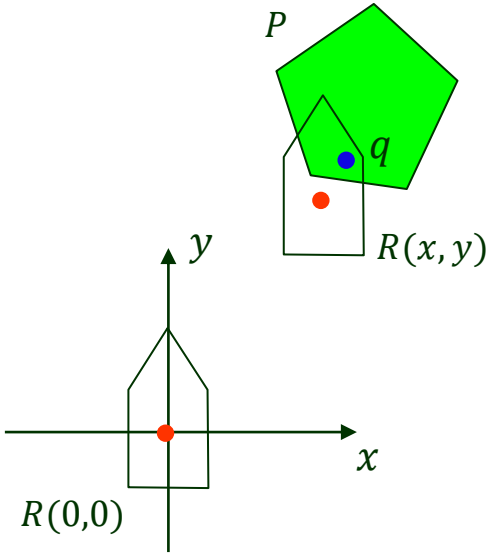

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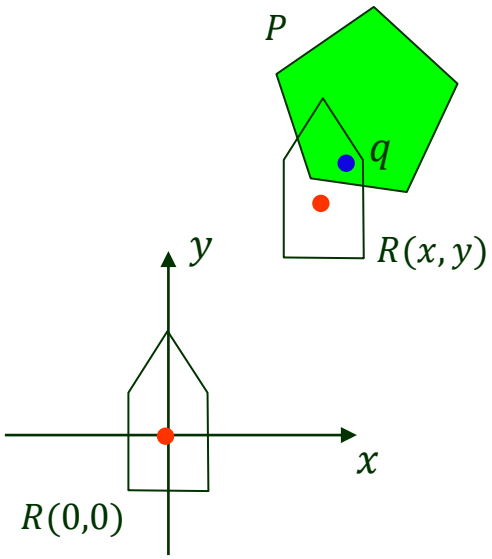

II. Formula for C-obstacle

Theorem 1 The C-obstacle CP of P is $P \oplus \underline{(-R(0,0))}$.

$$= \{(-x,-y) |\ (x,y) \in R(0,0)\}$$

Proof ($\Rightarrow$) Suppose $(x,y) \in CP$, i.e., $R(x,y)$ intersects P.

We need to show $(x,y) \in P \oplus (-R(0,0))$.

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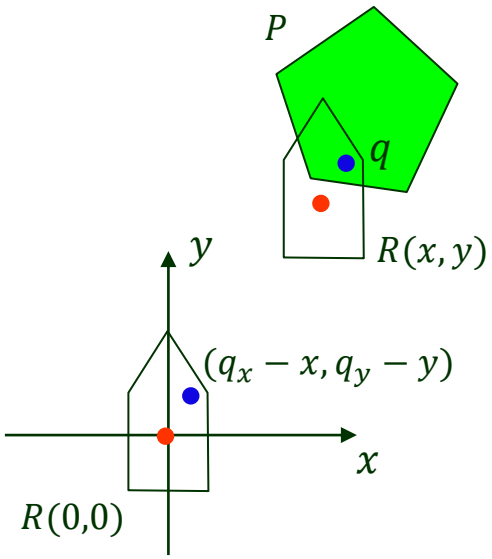

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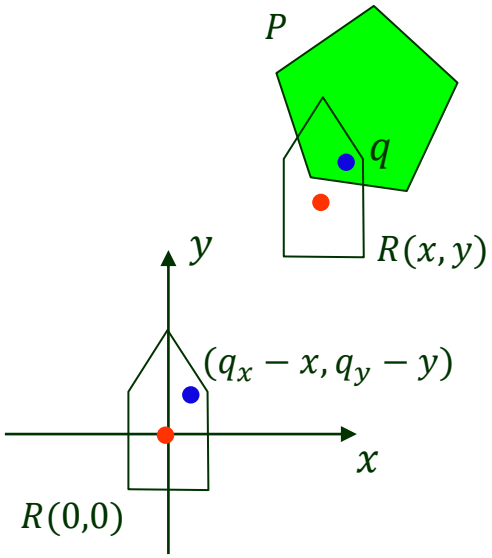

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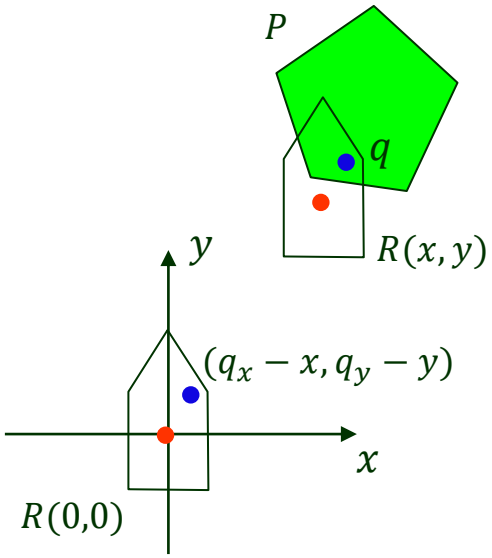

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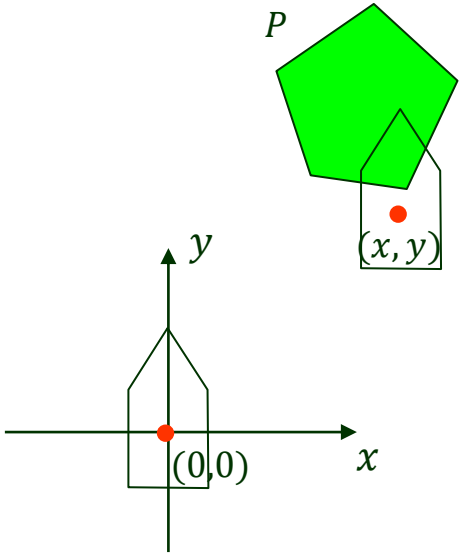
$$q \in P \ \Downarrow$$

$$q + \big(-q_x + x, -q_y + y\big) = (x,y) \in P \oplus (-R(0,0))$$

Proof (cont'd)

($\Leftarrow$) Let $(x, y) \in P \oplus (-R(0,0))$.



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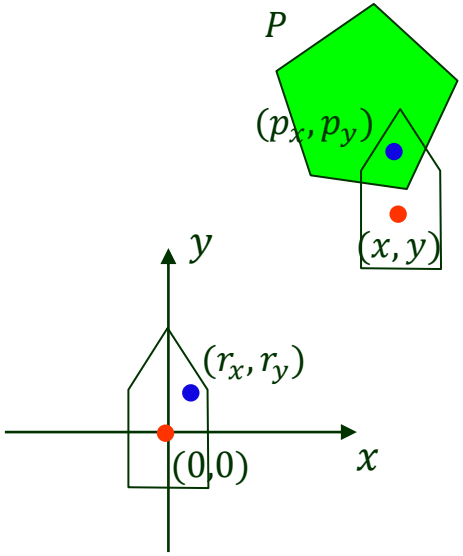

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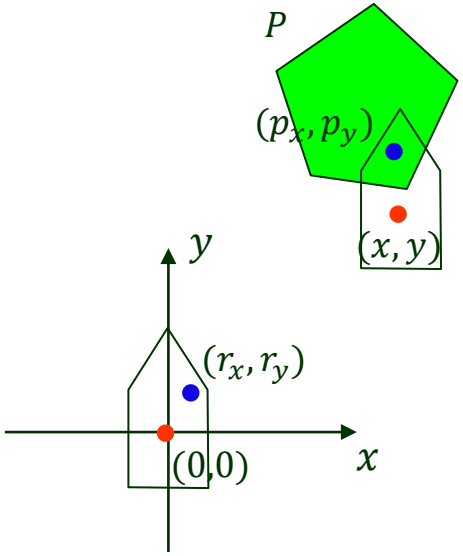

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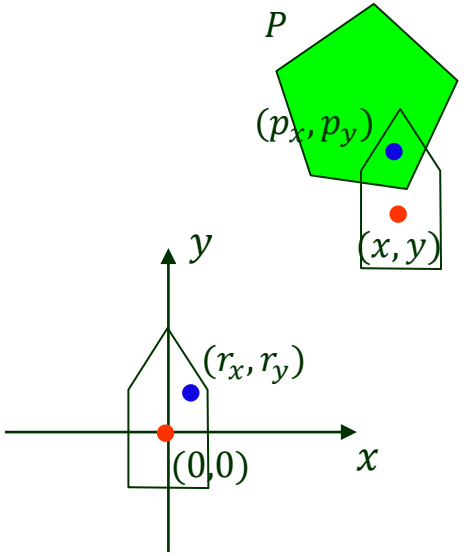

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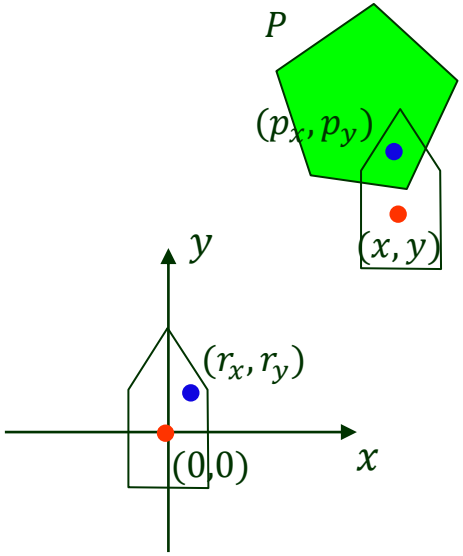
$$(p_x, p_y) \in P \cap R(x, y)$$



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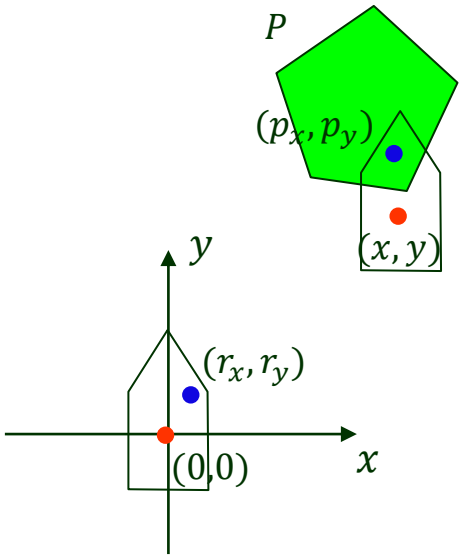

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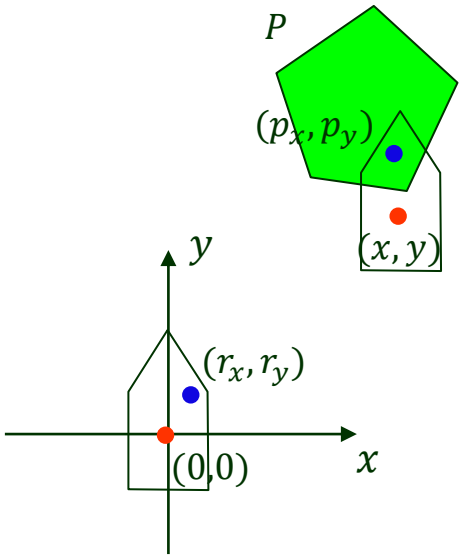
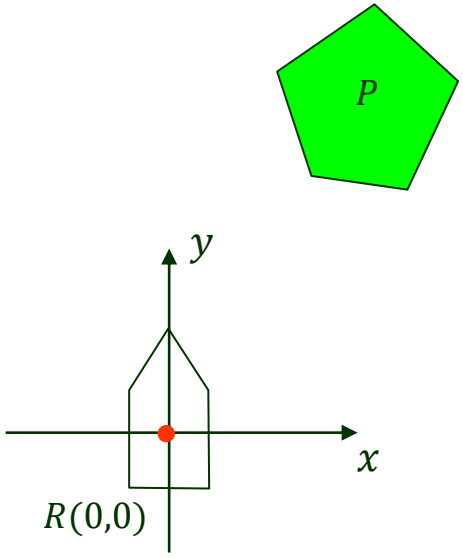


□

Verification via an Example

Two equivalent ways of C-obstacle construction:

Straightforward



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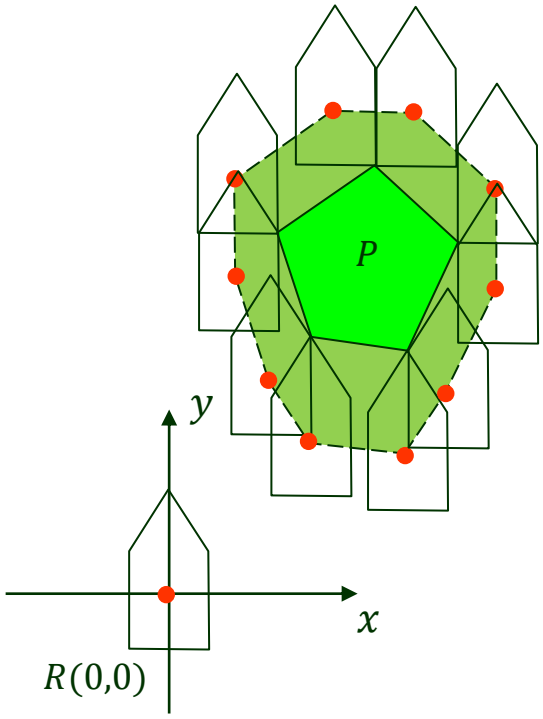

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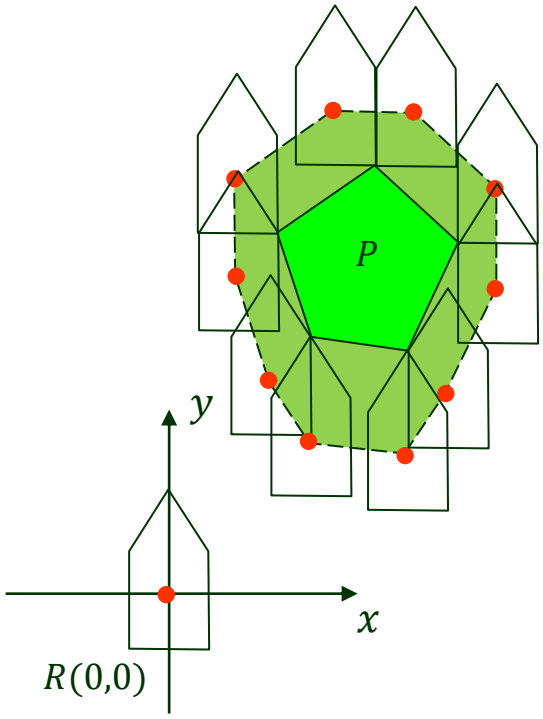

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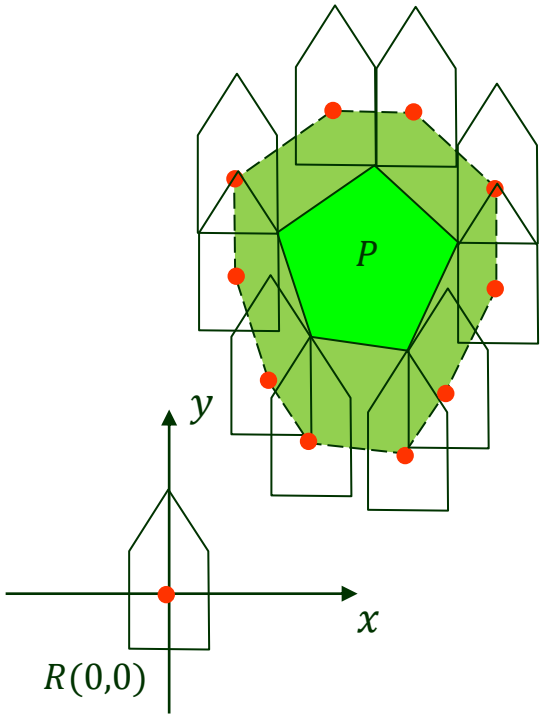

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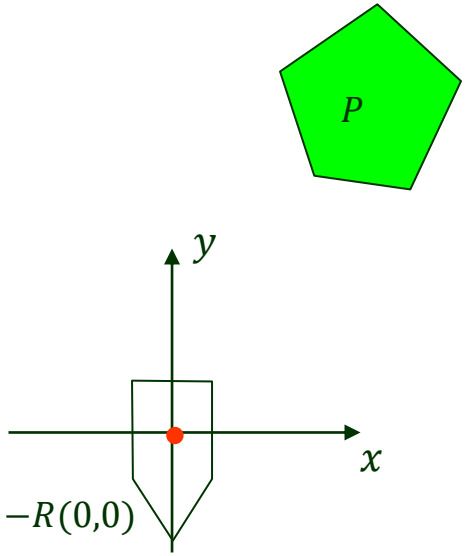

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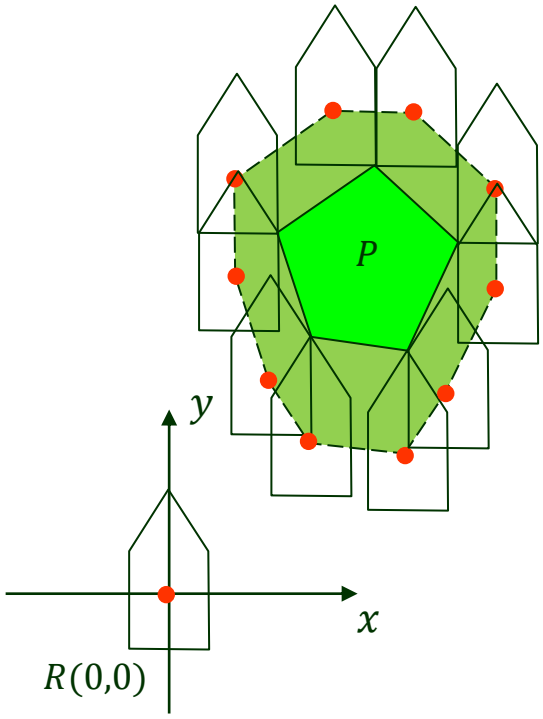

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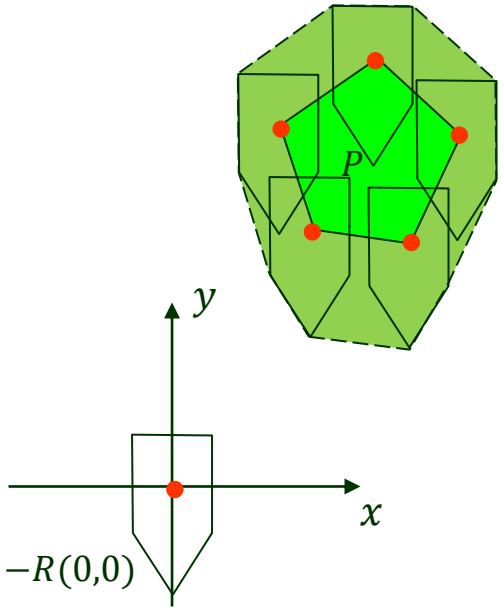
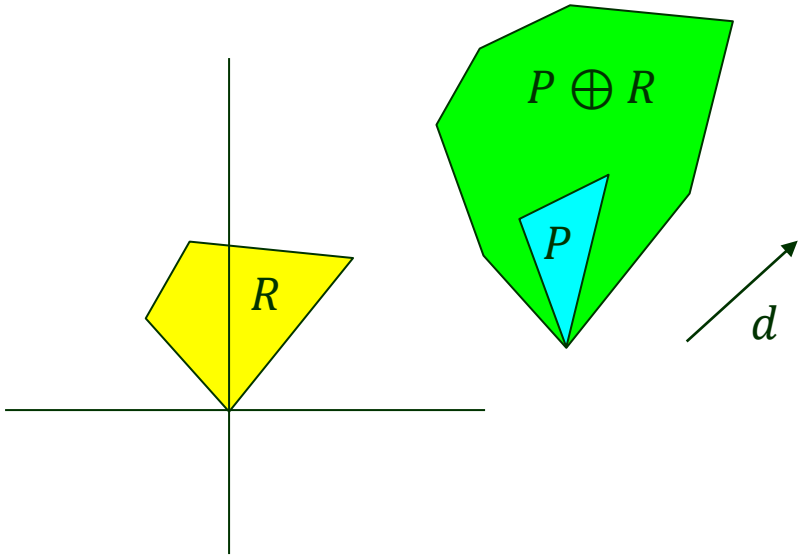


III. Extreme Points

Two polygons P and R.



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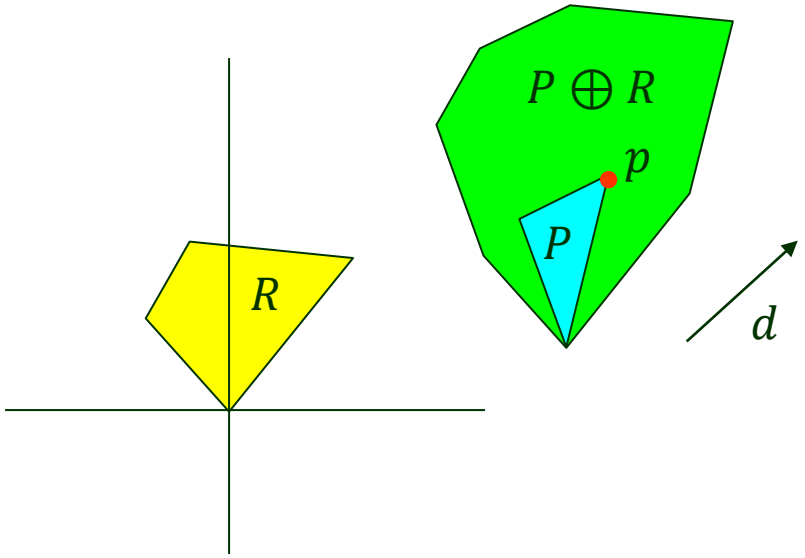

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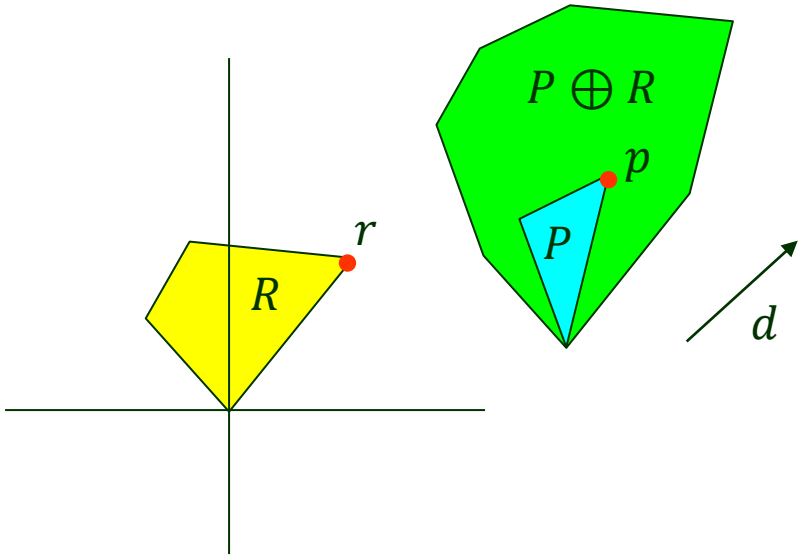

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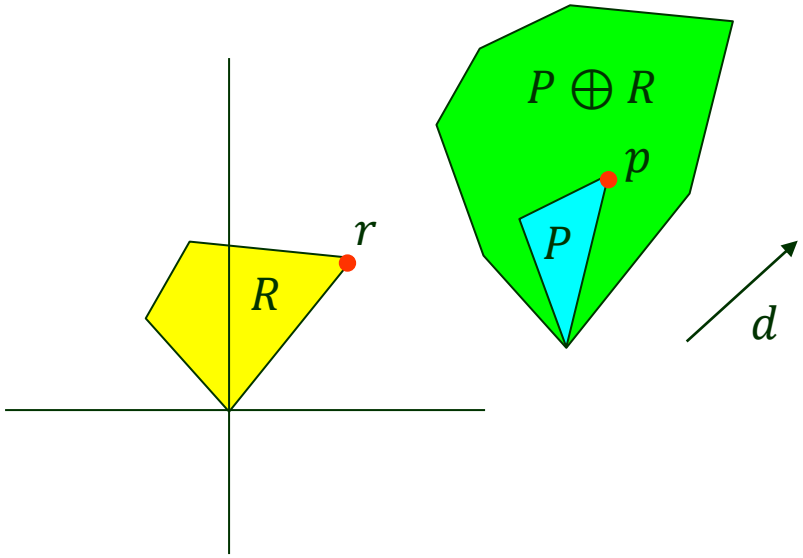

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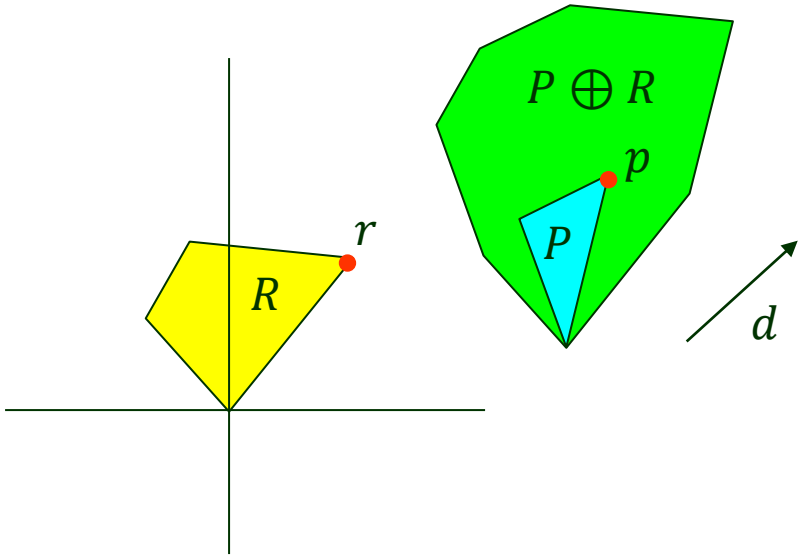

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$$s \cdot d = (q + t) \cdot d \leq (p + r) \cdot d$$

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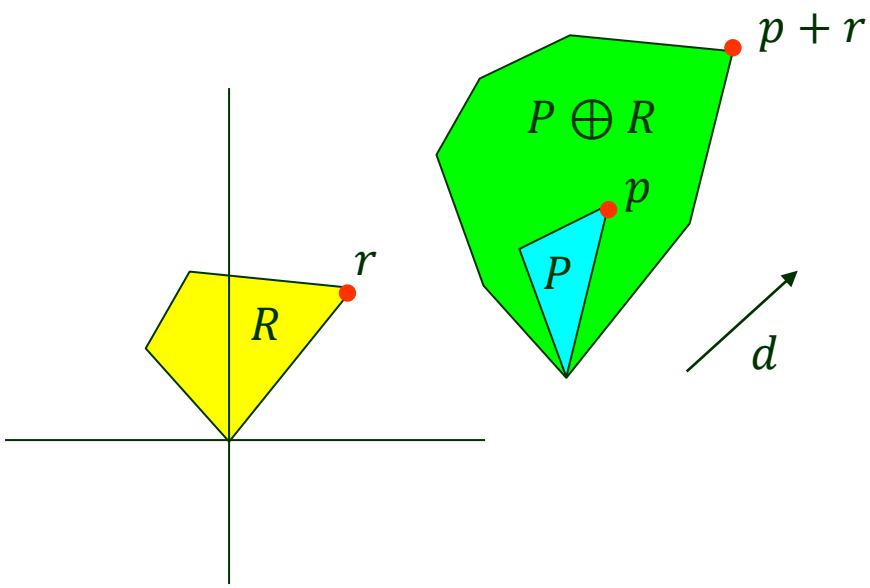

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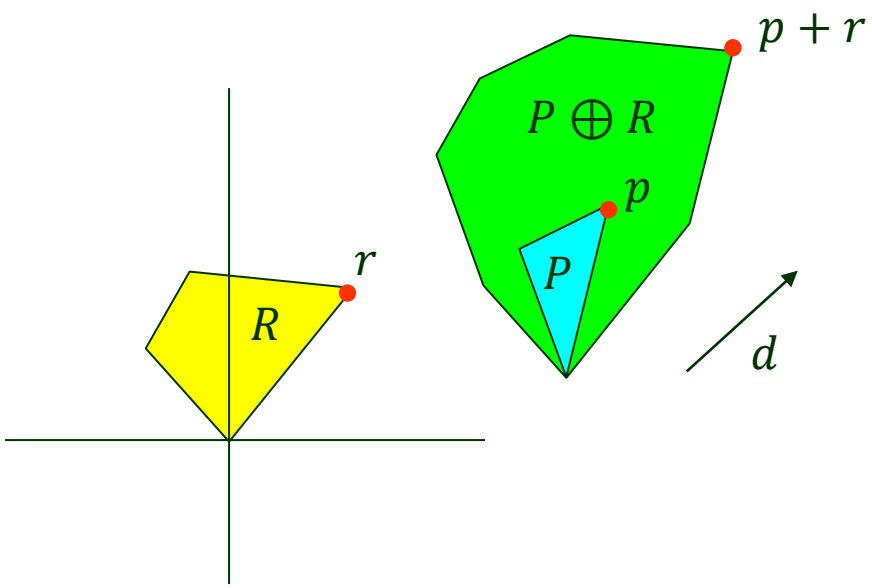

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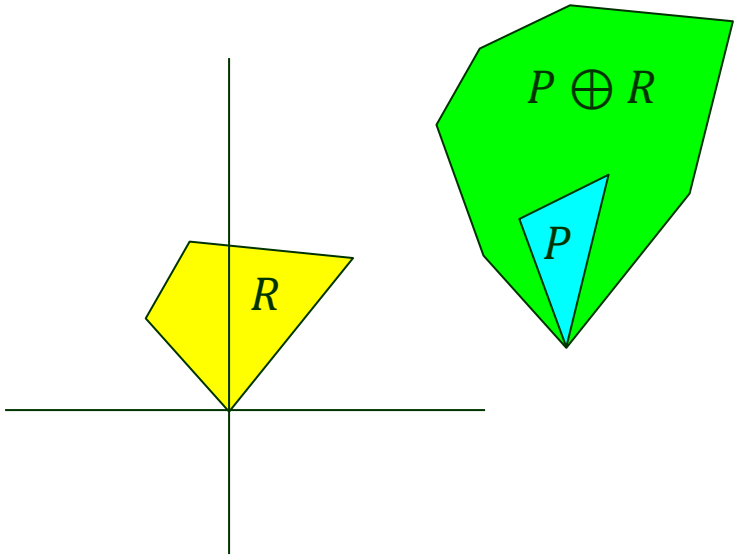
$p + r$ is an extreme point in the direction d on $P \oplus R$.

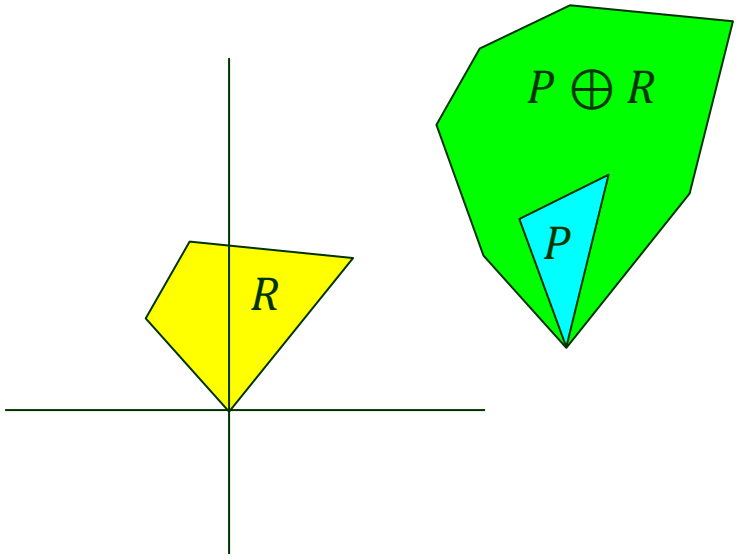
An extreme point in the direction d on $P \oplus R$ is the sum of two extreme points in d on P and R, respectively.

Minkowski Sum of Convex Polygons



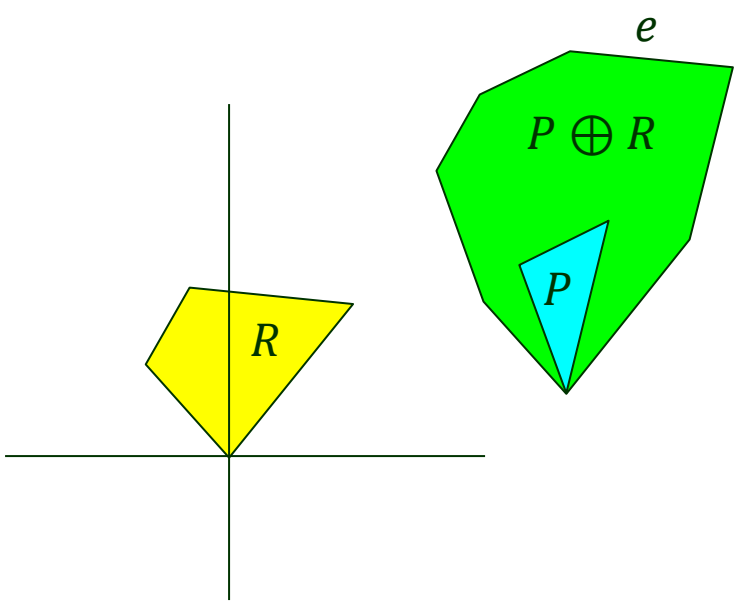
Theorem 2 Let P and R be two convex polygons with n and m edges, respectively. Then $P \oplus R$ is a convex polygon with $\leq n + m$ edges.

Complexity of Minkowski Sum



Proof Convexity of the Minkowski sum of two convex sets follows from the definition.

Complexity of Minkowski Sum



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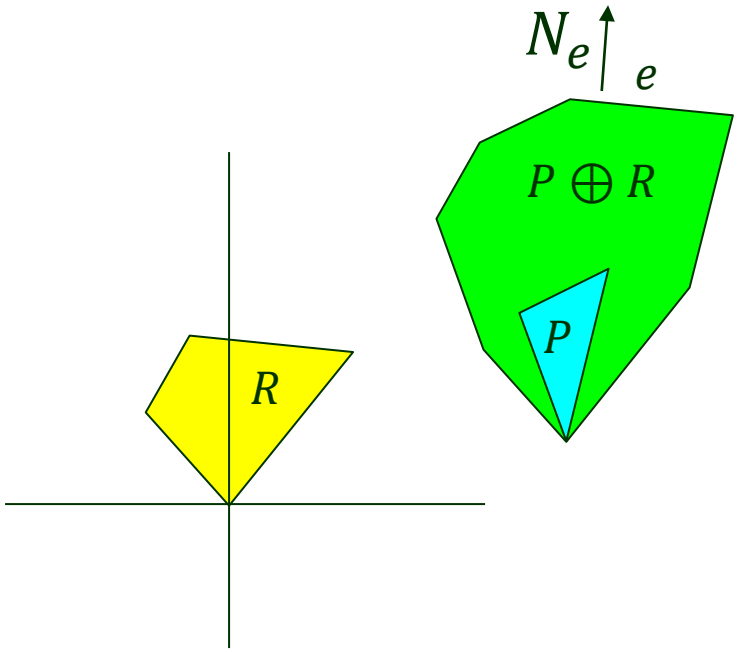
To bound the complexity, consider an edge e of $P \oplus R$.

Complexity of Minkowski Sum



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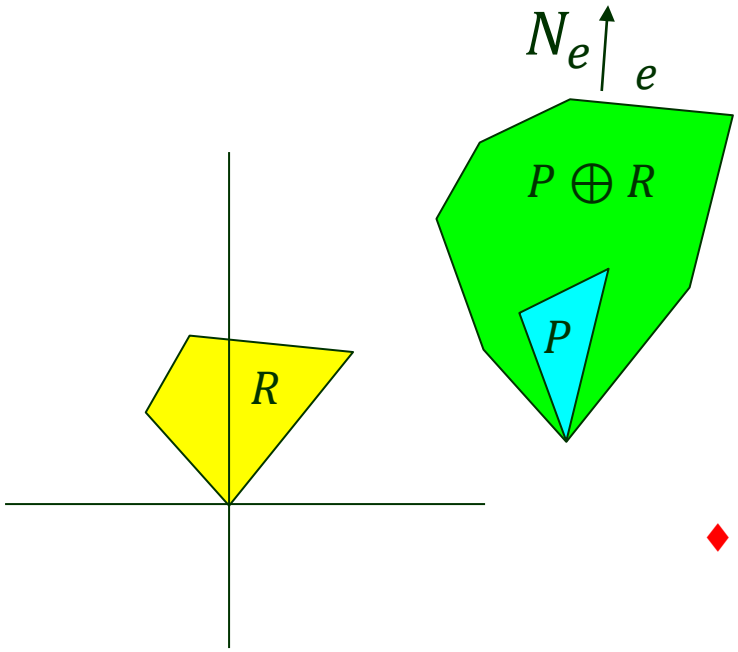
N_e: outward normal of e.

Complexity of Minkowski Sum



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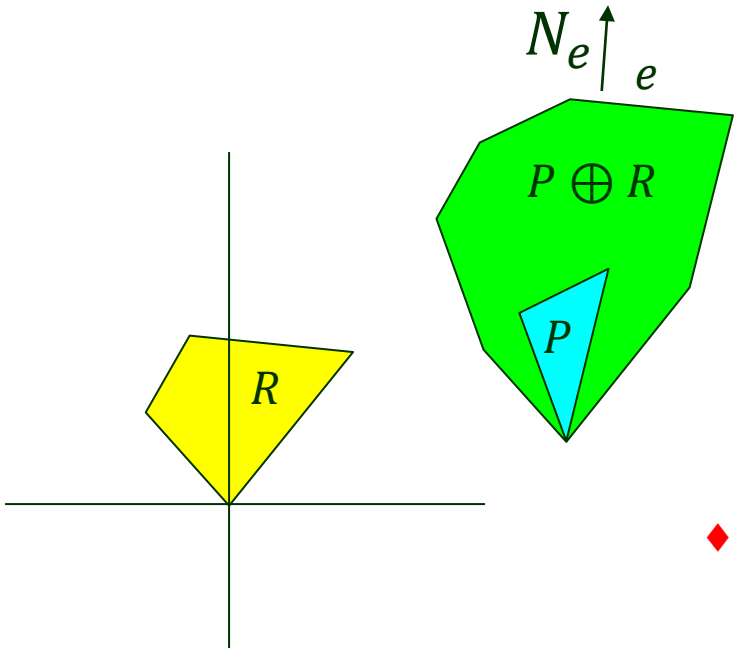
Complexity of Minkowski Sum



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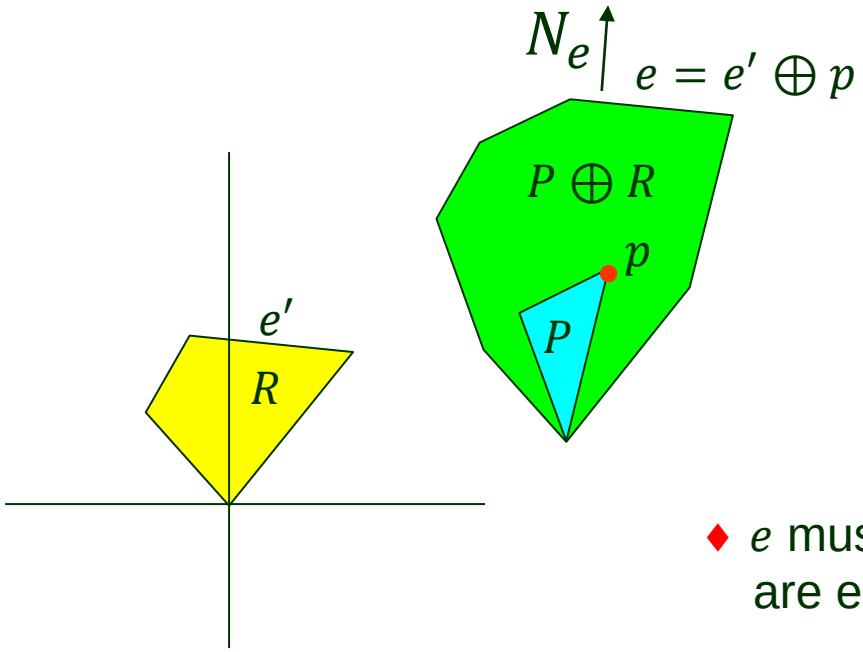
Complexity of Minkowski Sum



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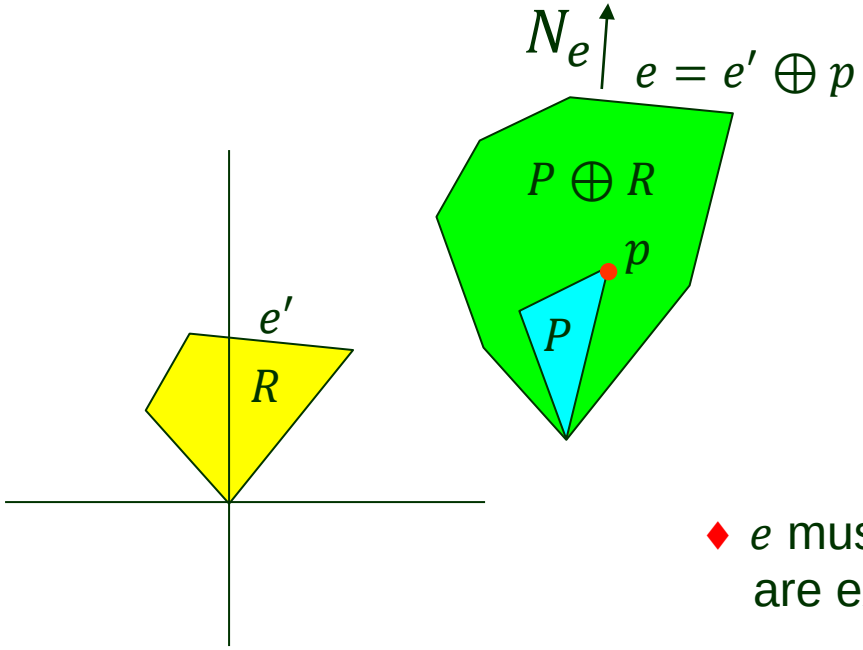
- Without loss of generality, this edge is, say, e' on R.

Complexity of Minkowski Sum



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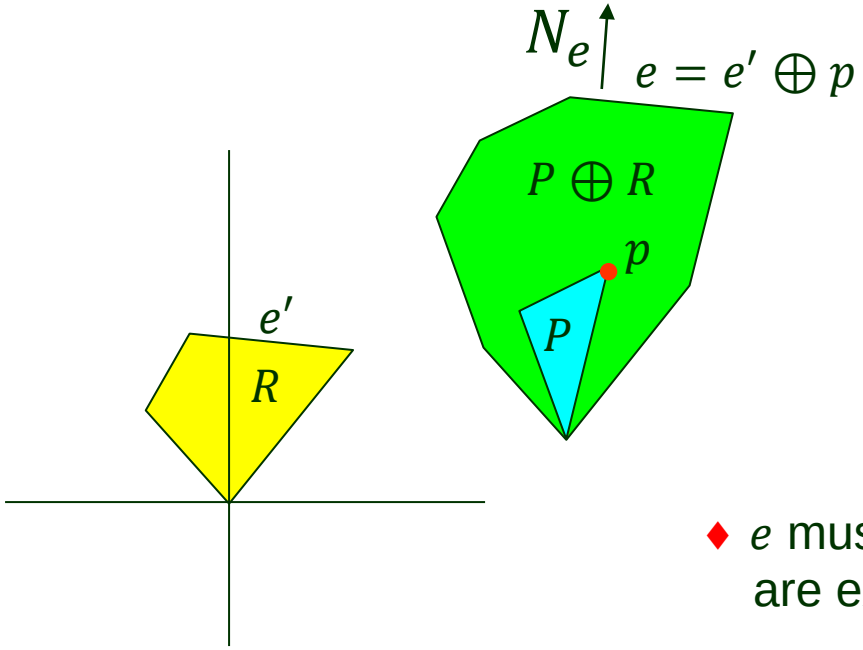
- Charge e to e'.

Complexity of Minkowski Sum



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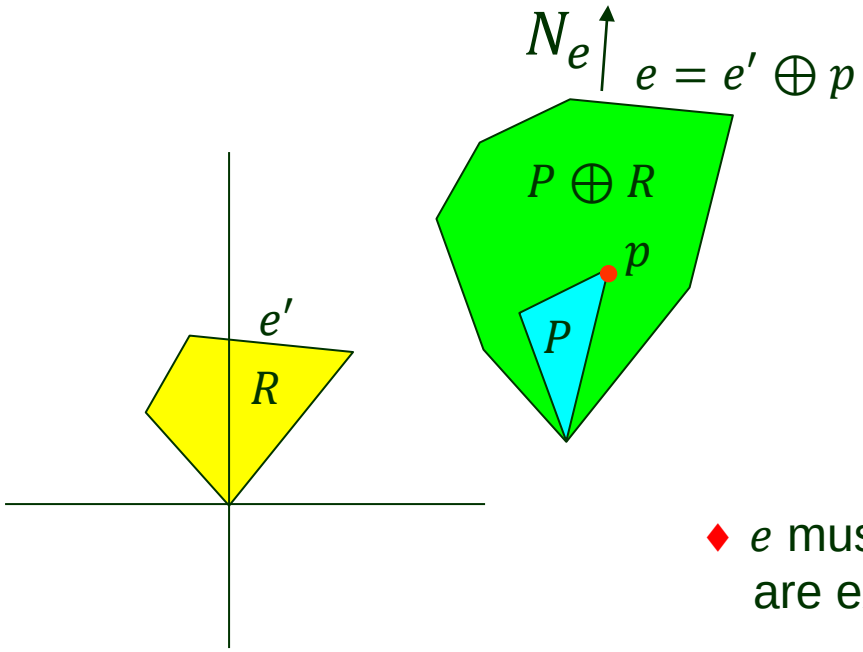
- Without loss of generality, this edge is, say, e' on R.
- Charge e to e'.

Every edge of P and R is charged at most once.

Complexity of Minkowski Sum



Proof Convexity of the Minkowski sum of two convex sets follows from the definition.

To bound the complexity, consider an edge e of $P \oplus R$.

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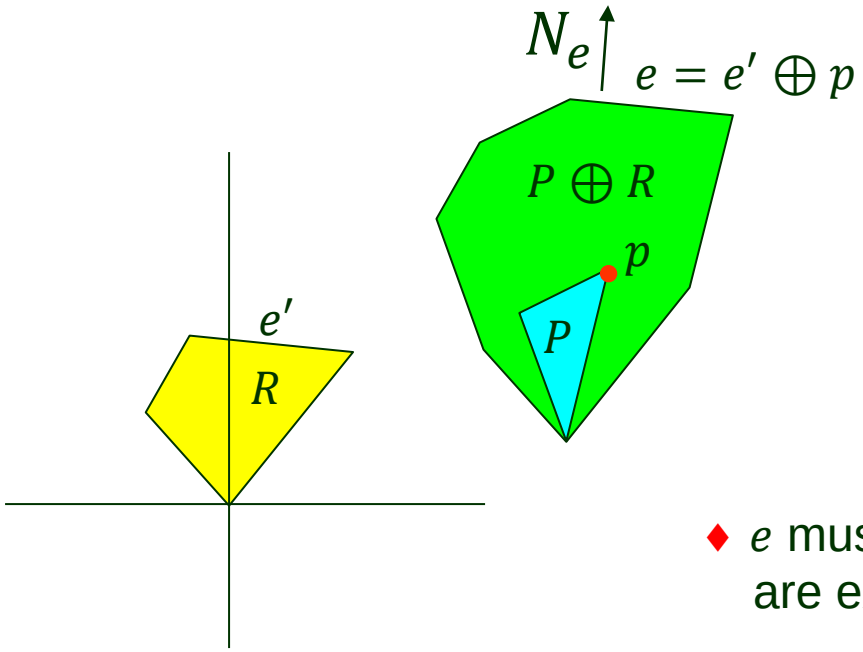

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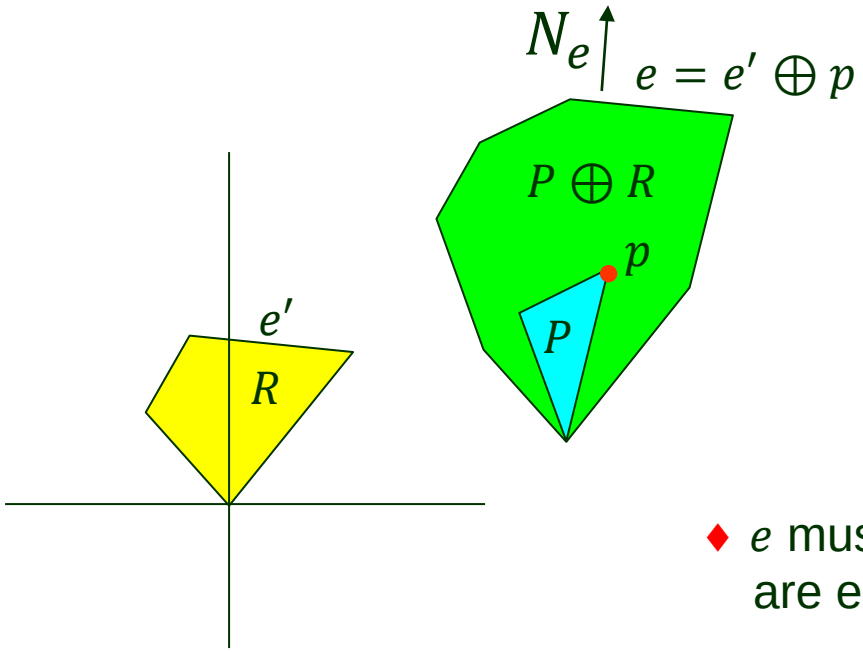

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The upper bound $n + m$ is achieved if P and R have no parallel edges.

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Compute $P \oplus R$ when P and R are convex.

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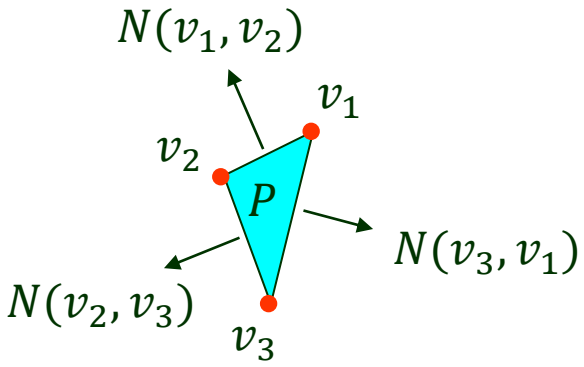
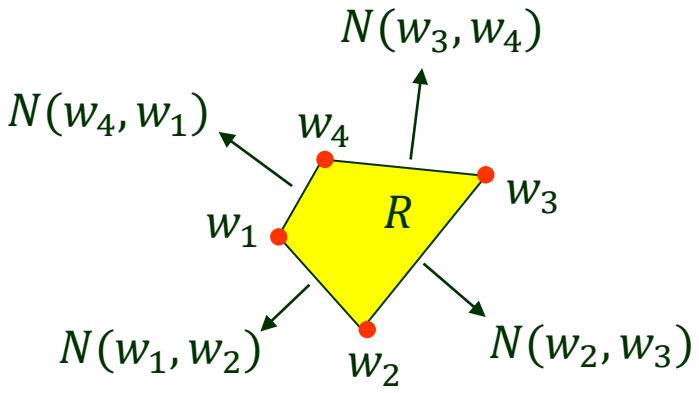
$$O(nm(\log n + \log m))$$

Faster Computation

Idea: Look at a pair of vertices that are extreme in the same direction.





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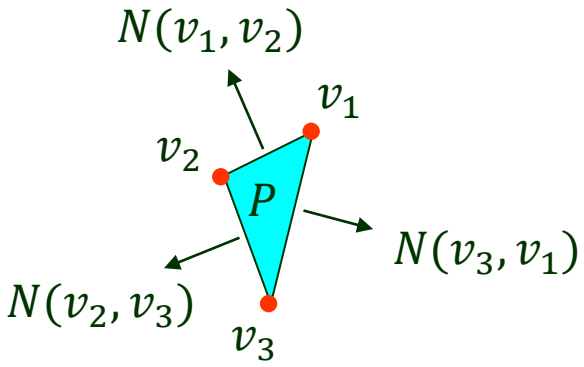

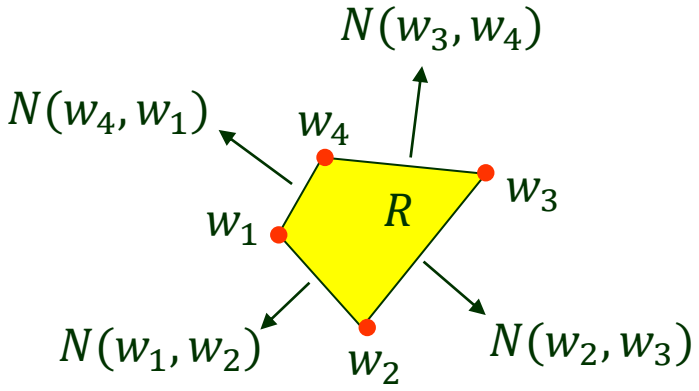

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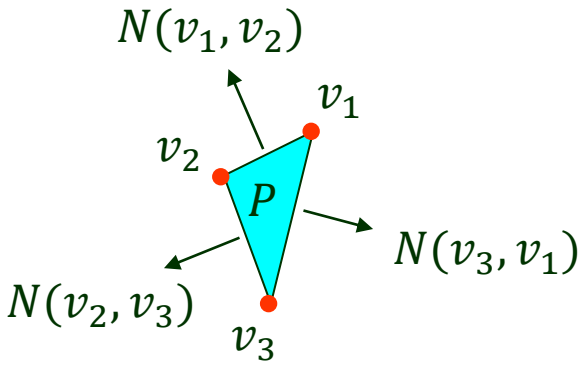

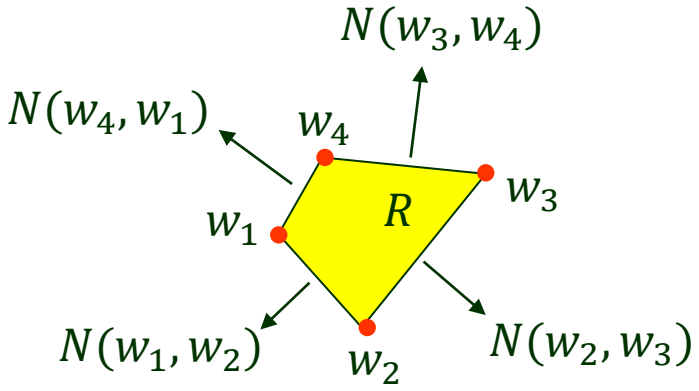

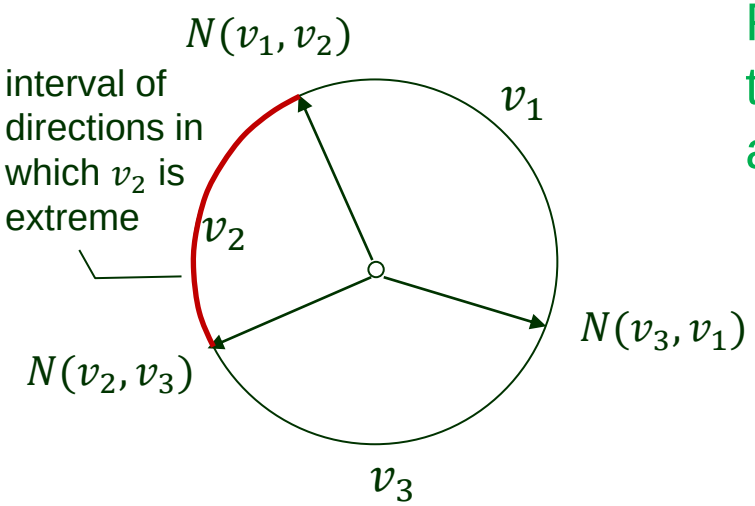

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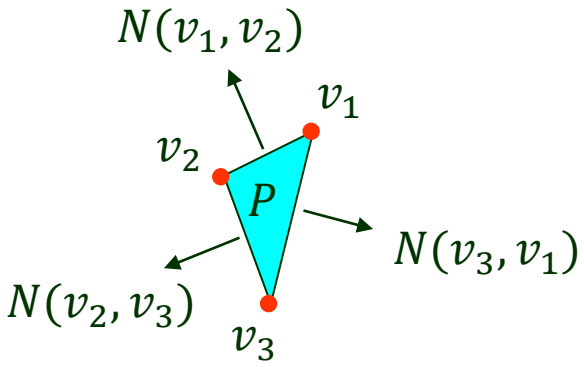

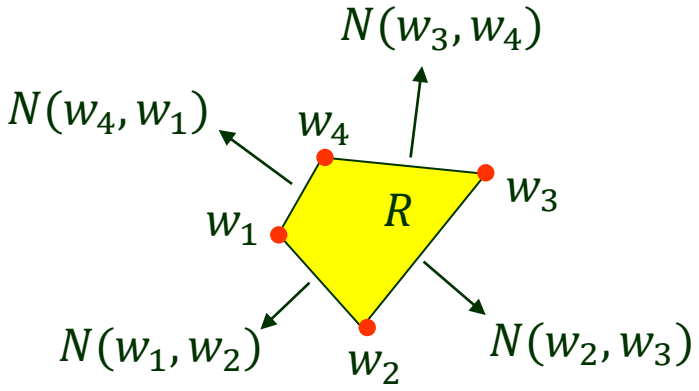

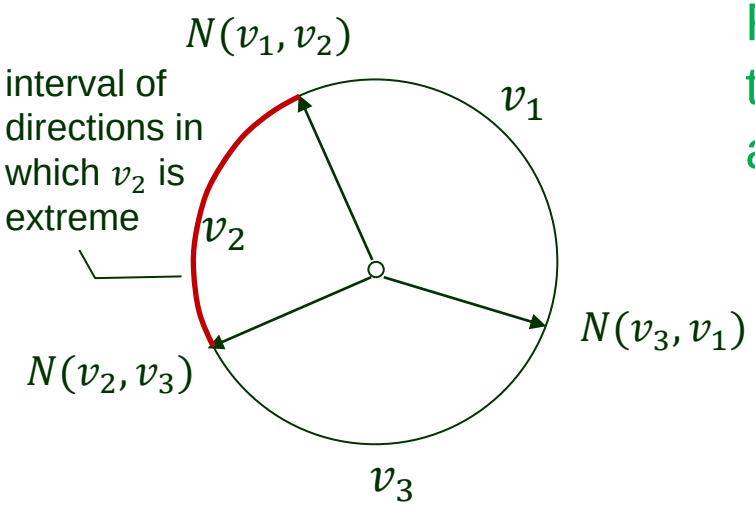

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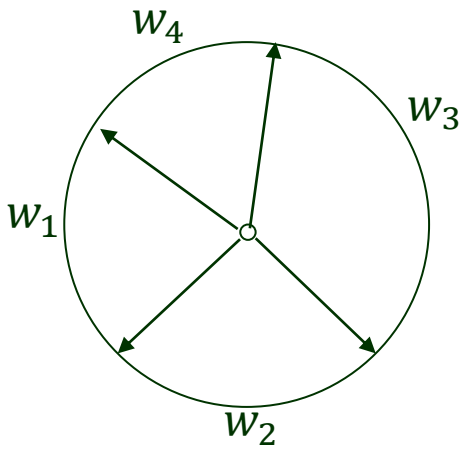
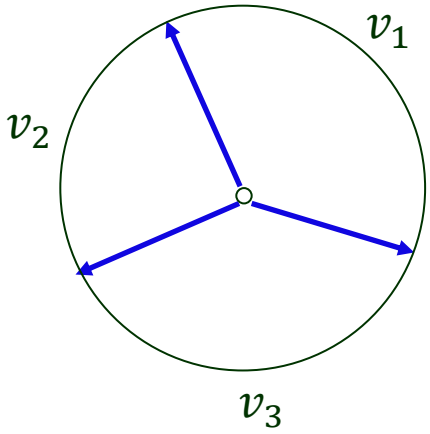
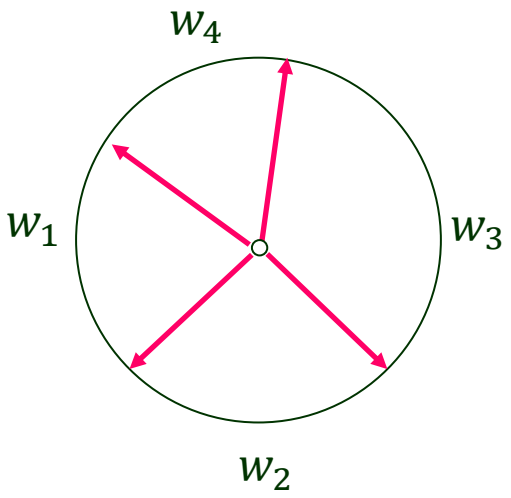


Extreme Pairs

Superpose the two partitioning.





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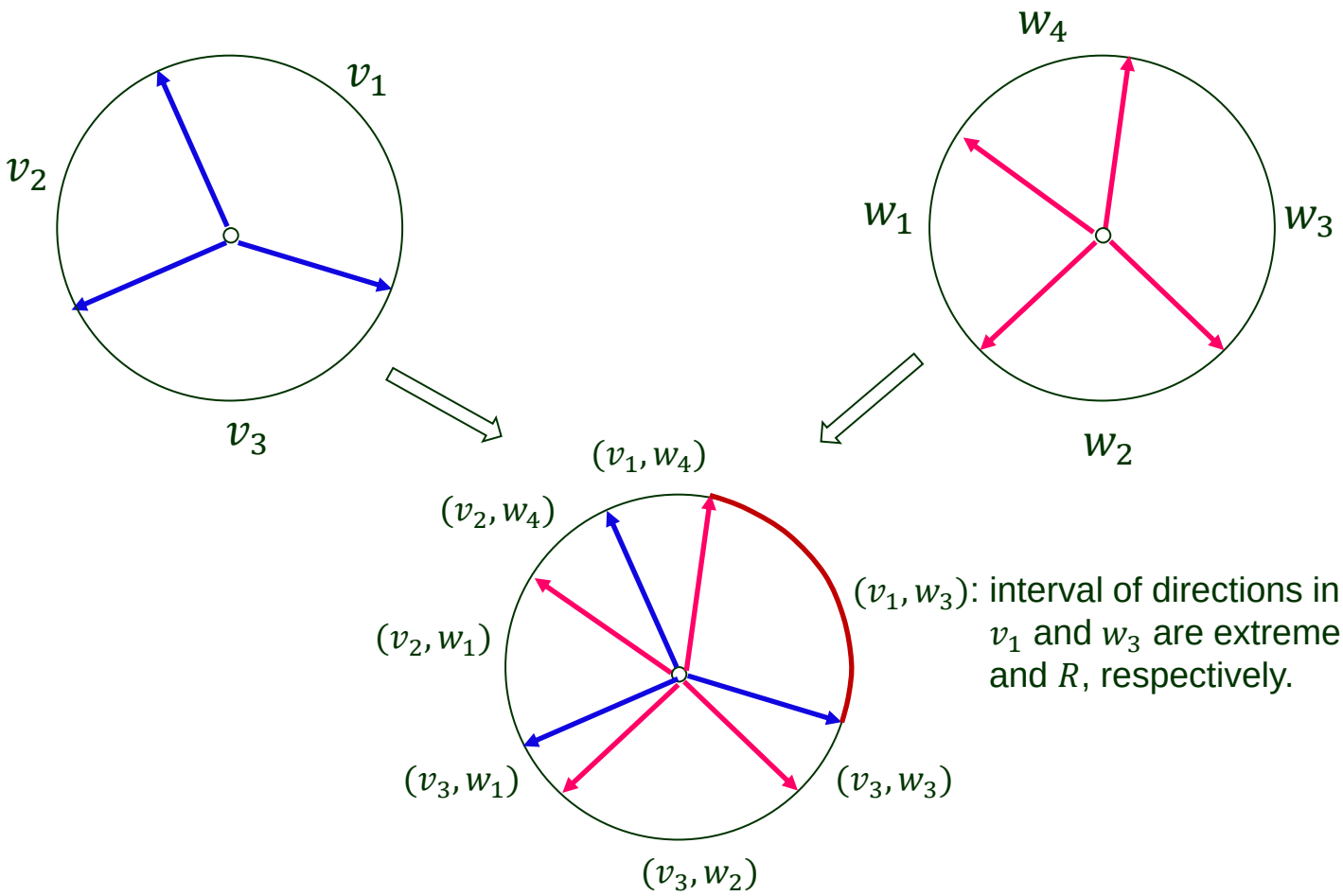

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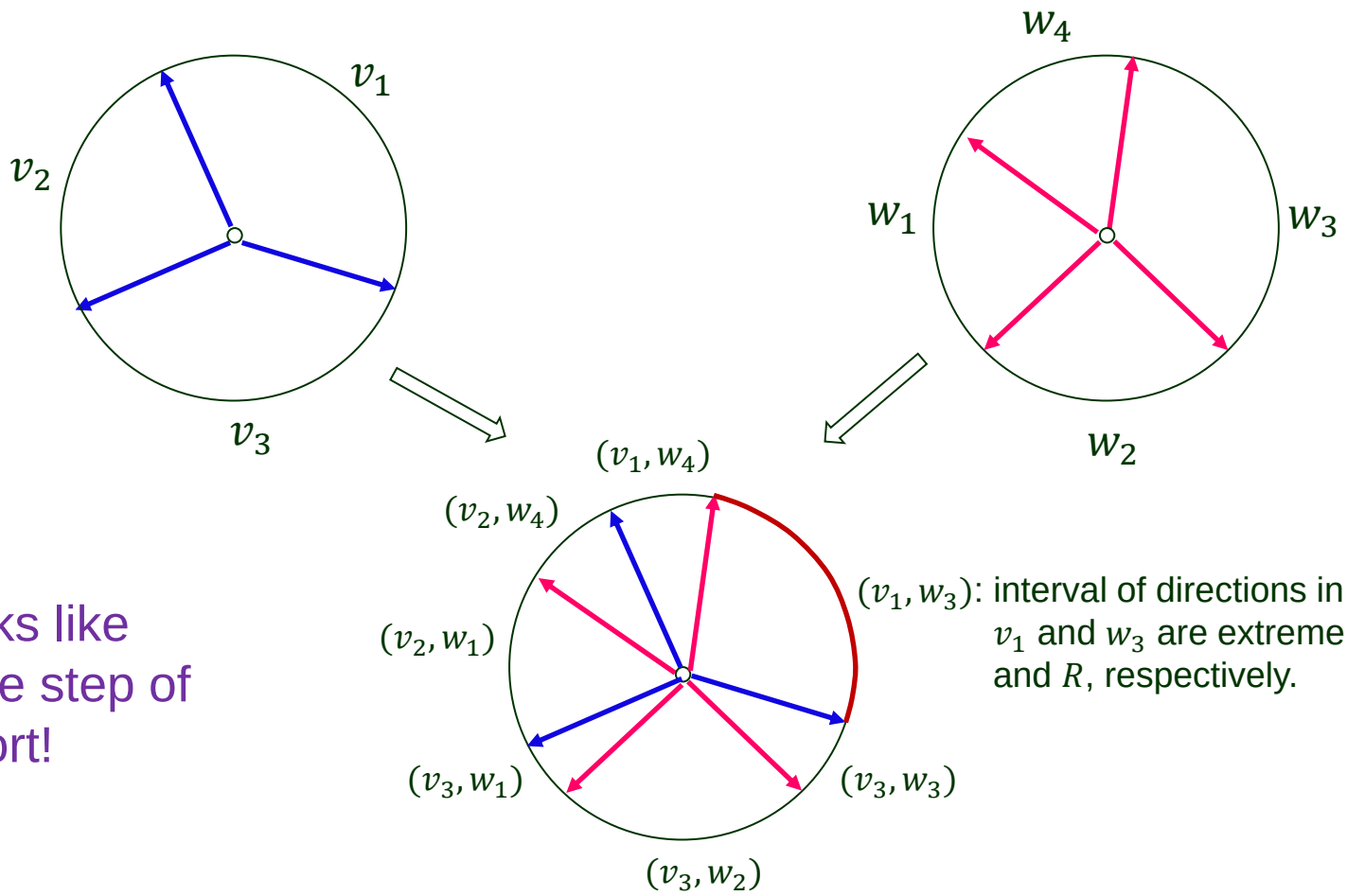

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This works like the merge step of merge sort!

The Algorithm

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MinkowskiSum(P, R)
// v_1,…,v_n and w_1,…,w_m in counterclock-
// wise order with v_1 and w_1 having the
// smallest y-coordinate
1.  i ← 1; j ← 1
2.  v_{n+1} ← v_1; v_{n+2} ← v_2; w_{m+1} ← w_1; w_{m+2} ← w_2
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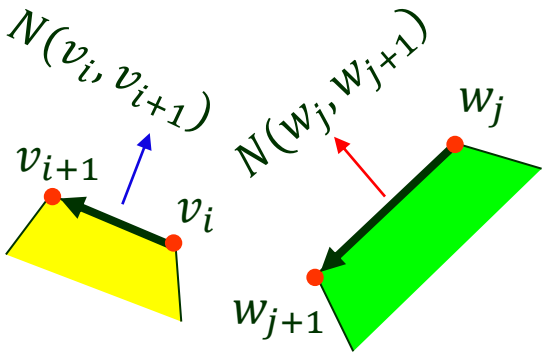

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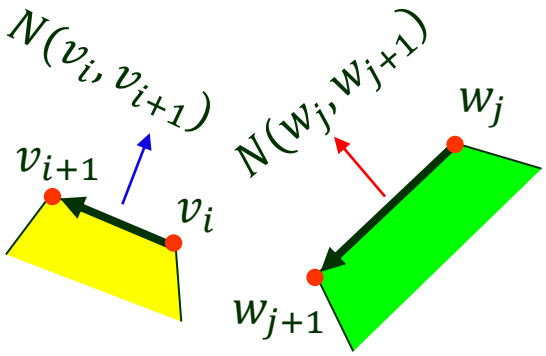

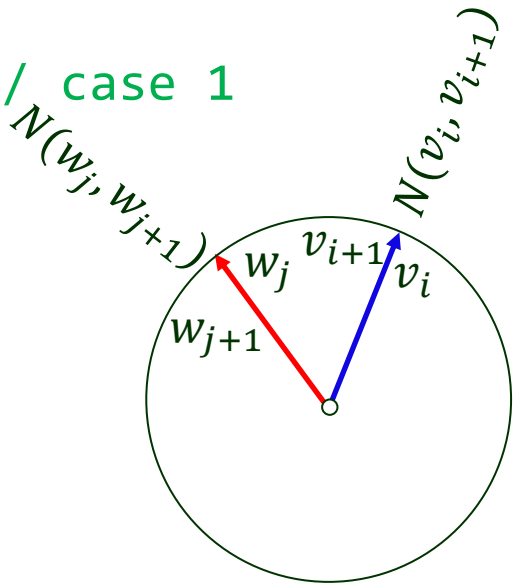

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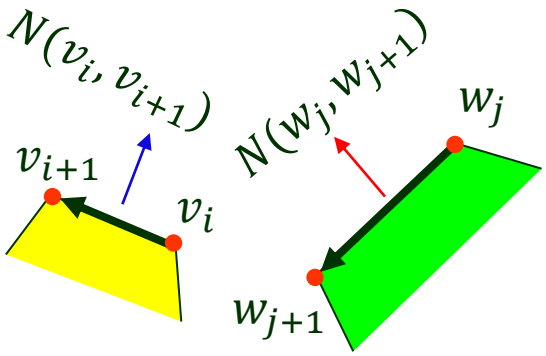

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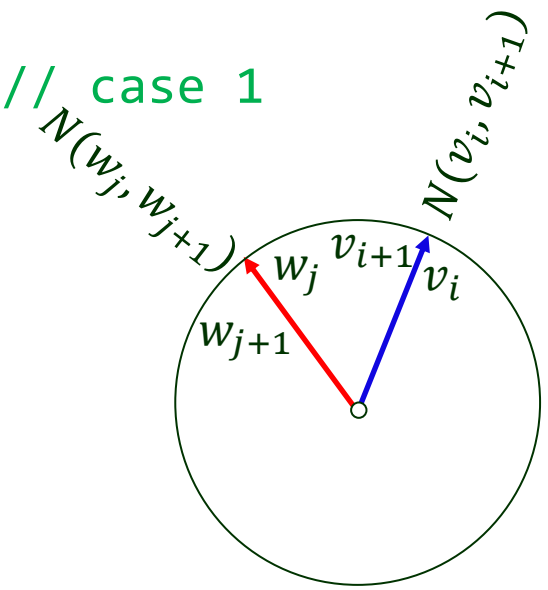

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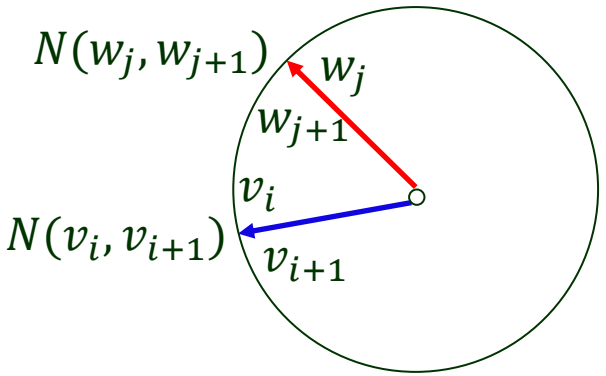
9. ...



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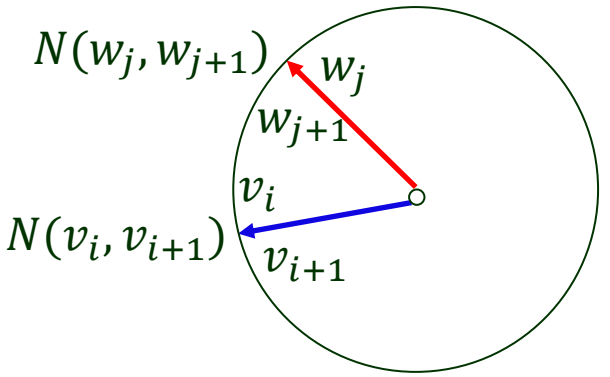

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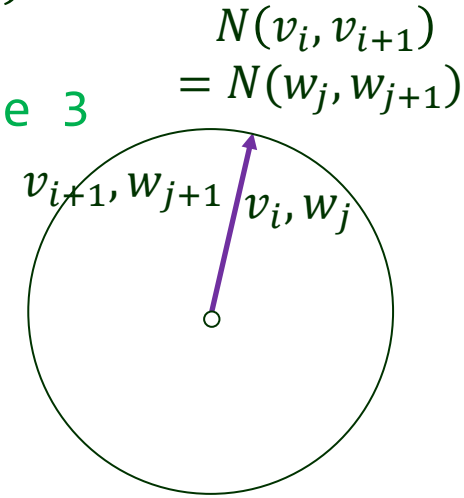

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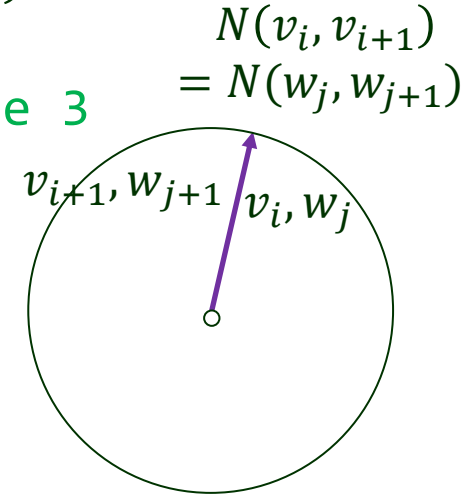

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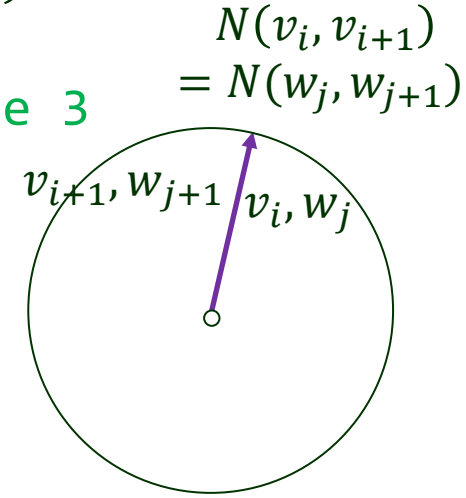

Running time $O(n + m)$

V. Nonconvex Robot or Obstacle

Triangulate whichever is nonconvex.

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Make use of the following equality for three sets S_1, S_2 and S_3:

$$S_1 \oplus (S_2 \cup S_3) = (S_1 \oplus S_2) \cup (S_1 \oplus S_3)$$

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Theorem 4 Translational motion planning for a convex robot of $O(1)$ complexity among polygonal obstacles of $O(n)$ total complexity can be solved in $O(n)$ time with preprocessing in $O(n \log^2 n)$ expected time.