Minkowski Sums

Outline:

I. Definition

II. C-obstacles

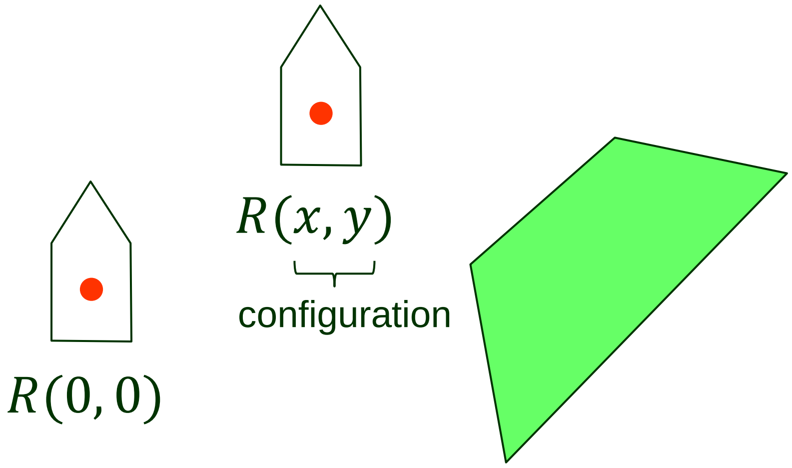
III. Complexity of the sum of two convex polygons

IV. Computation

V. Translational motion planning

C-obstacle for a Translational Robot

Robot R Obstacle P



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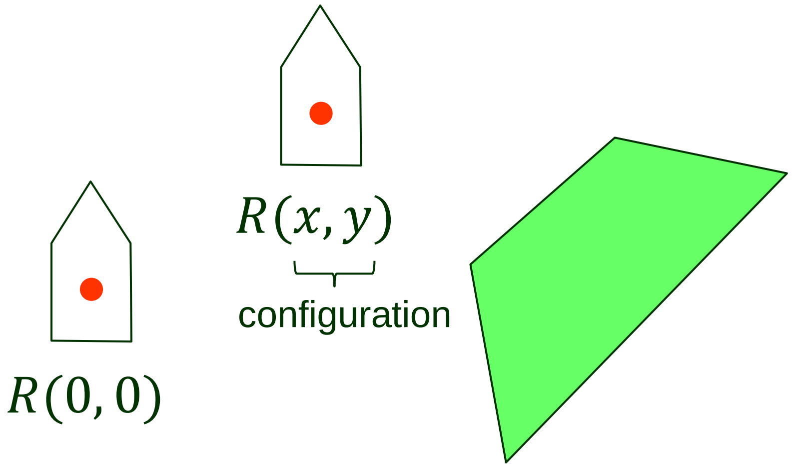

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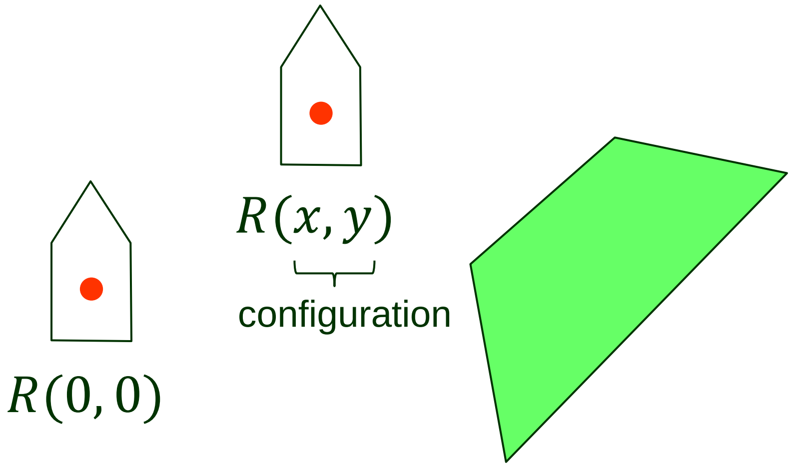

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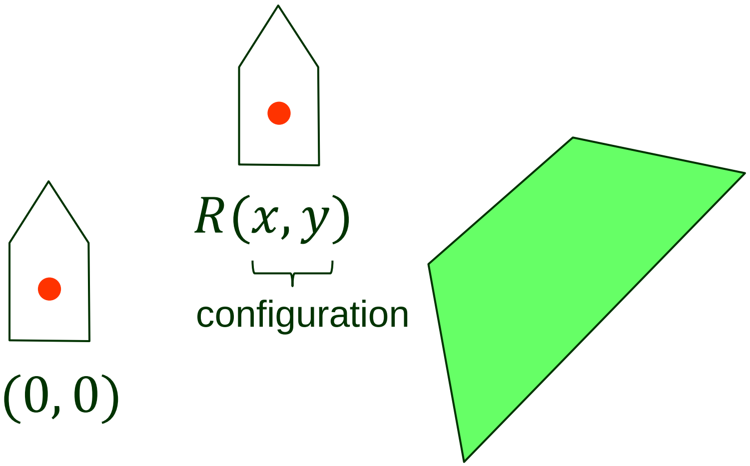

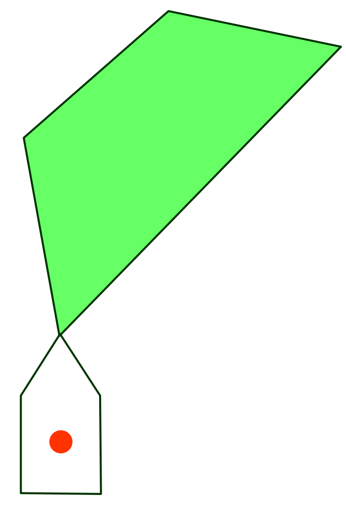

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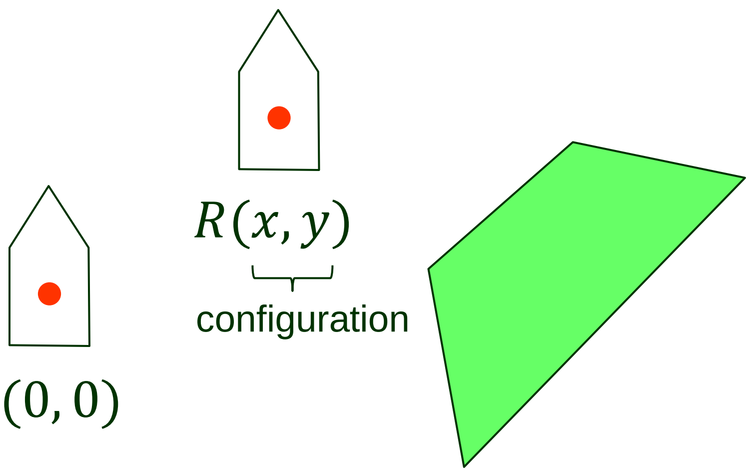

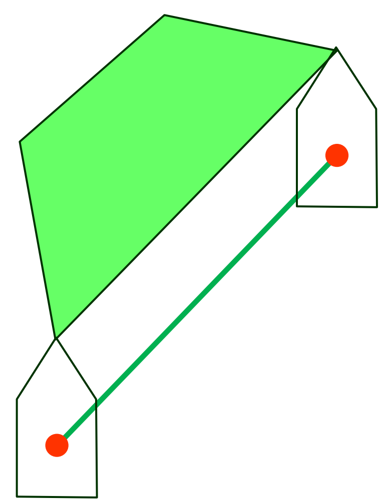

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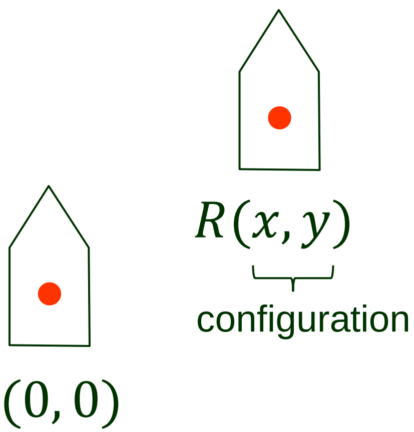

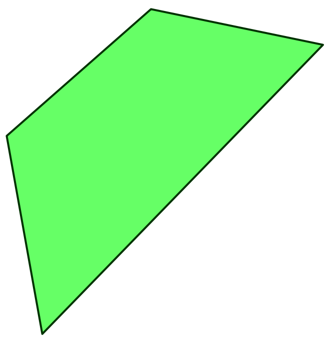

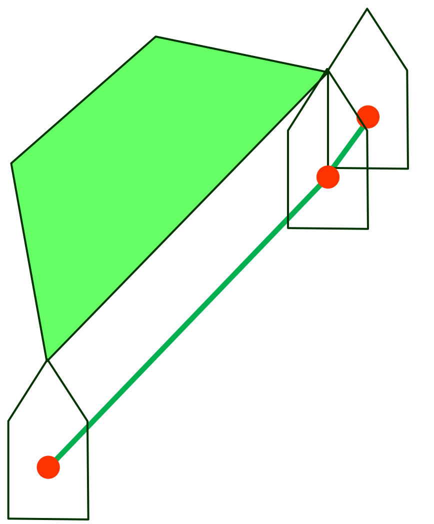

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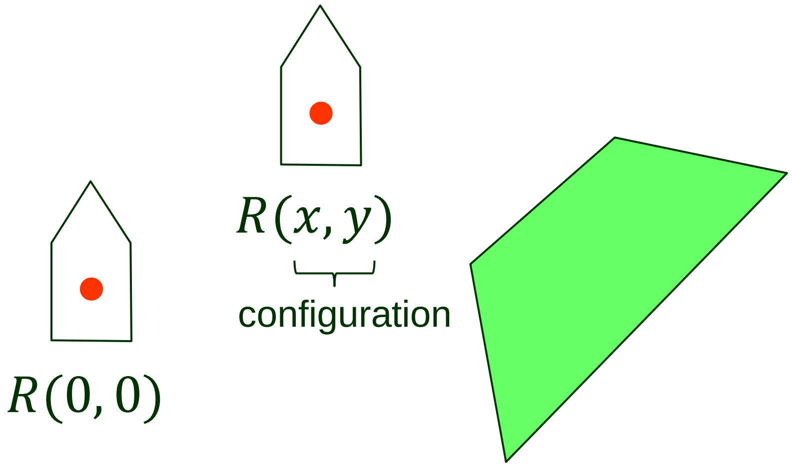

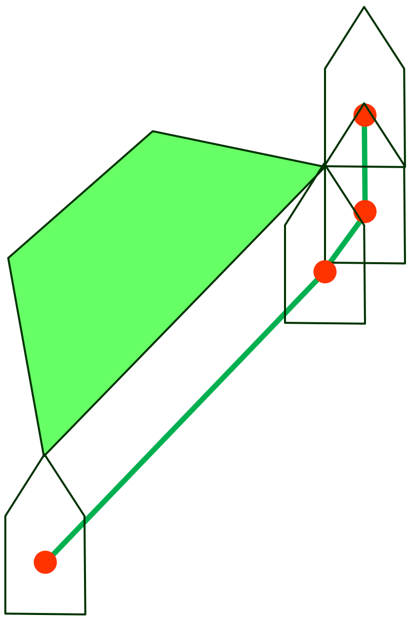

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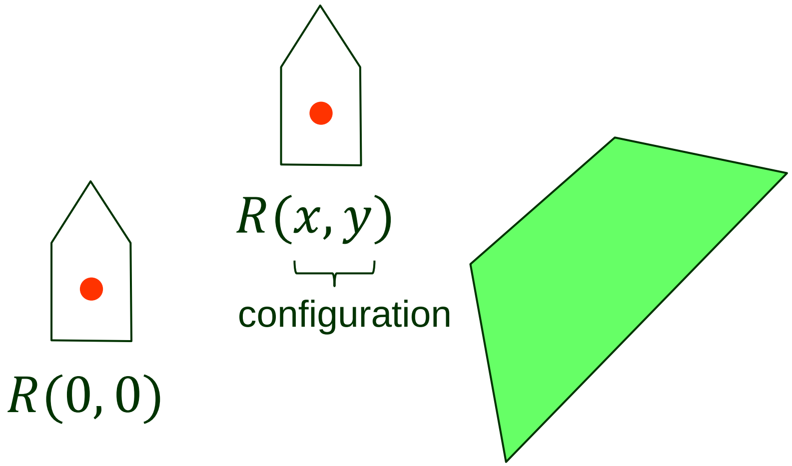

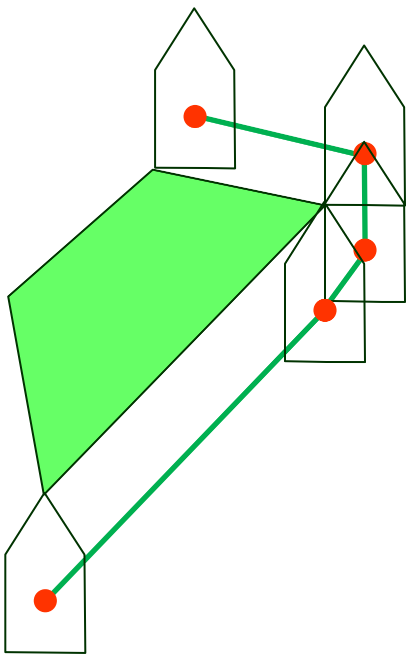

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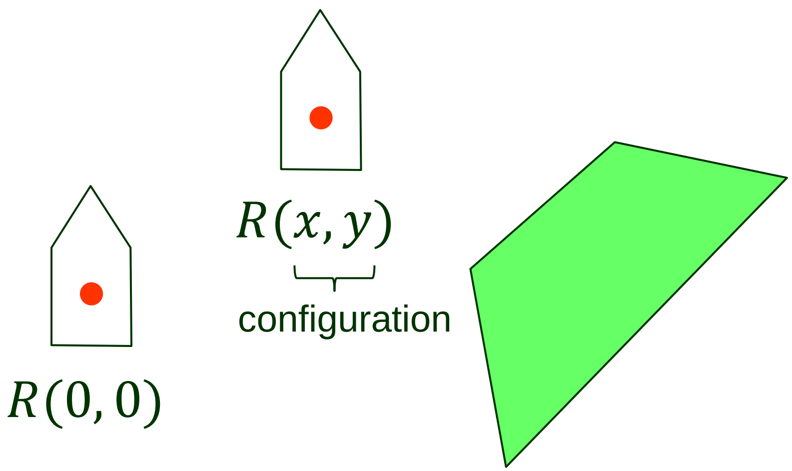

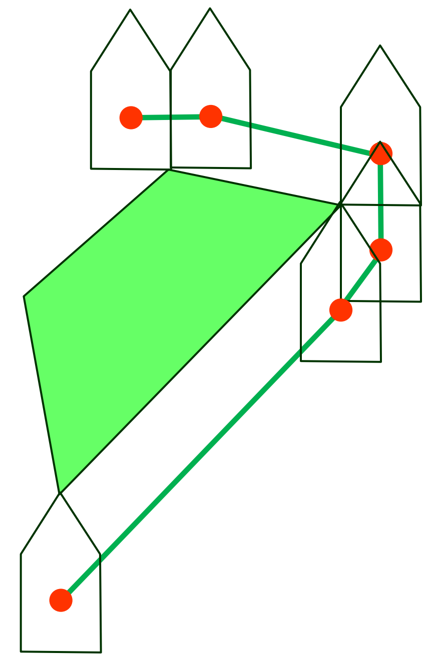

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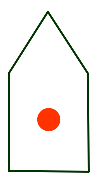
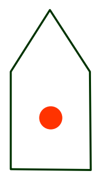
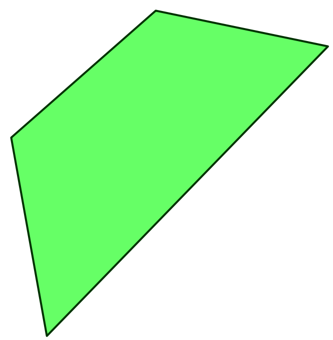
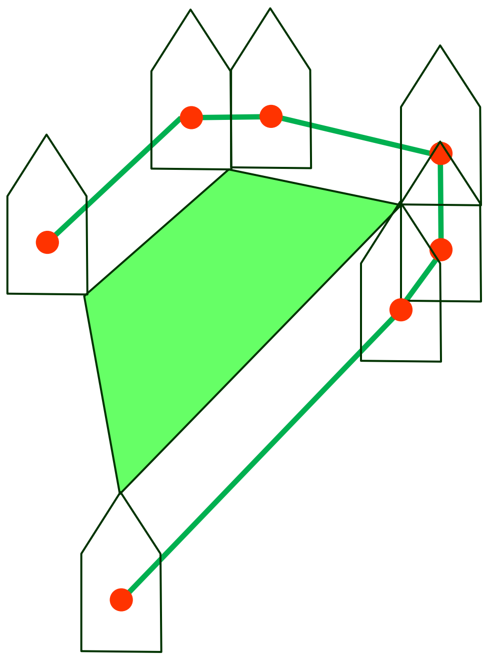




$R\underbrace{(x,y)}_{\text{configuration}}$

$R(0,0)$





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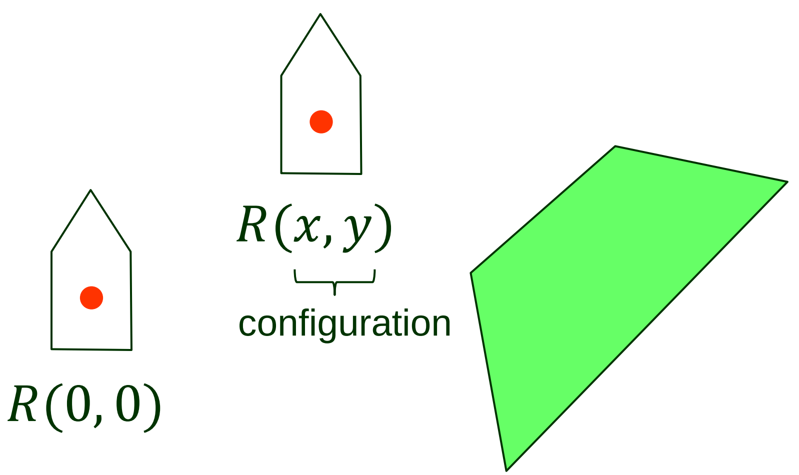

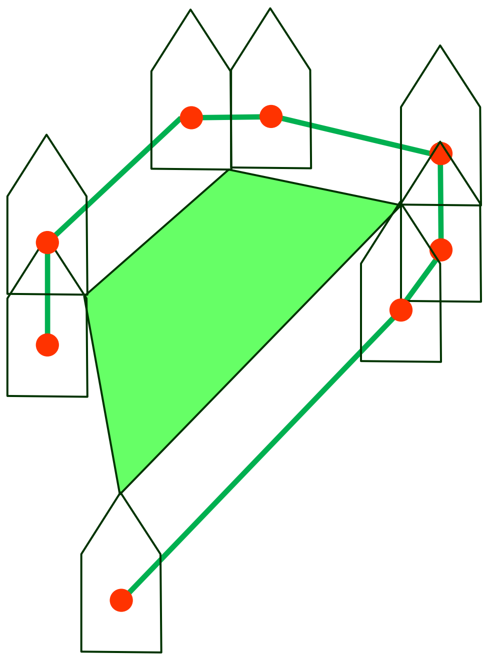

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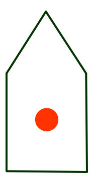

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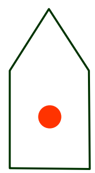

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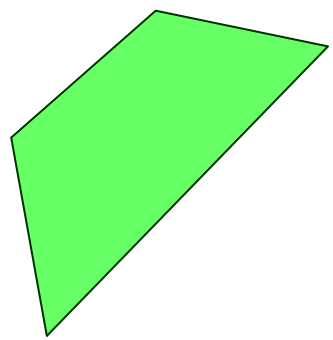
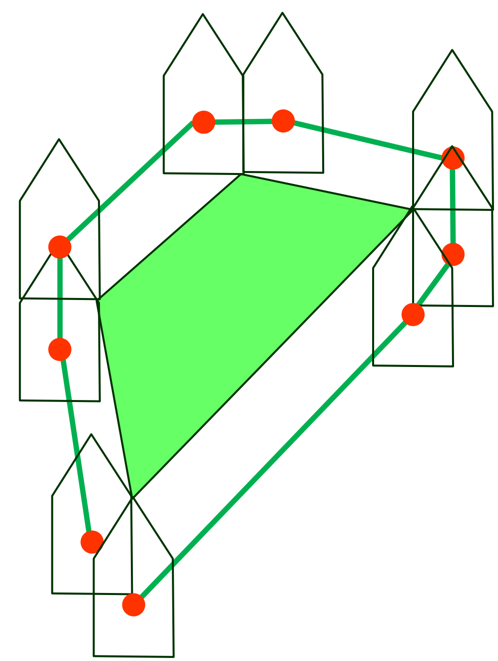


C-obstacle CP



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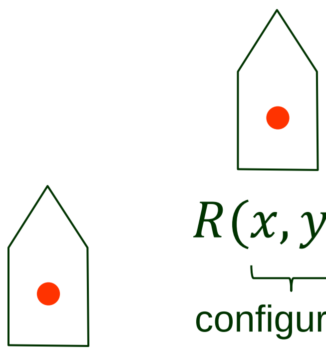

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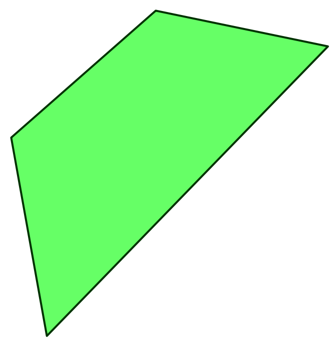

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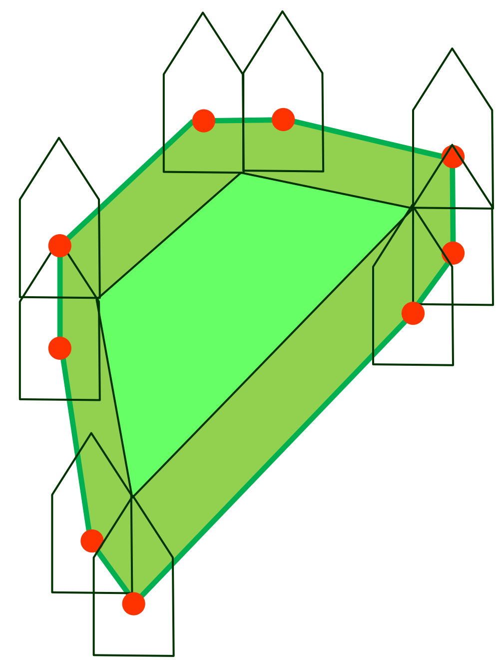

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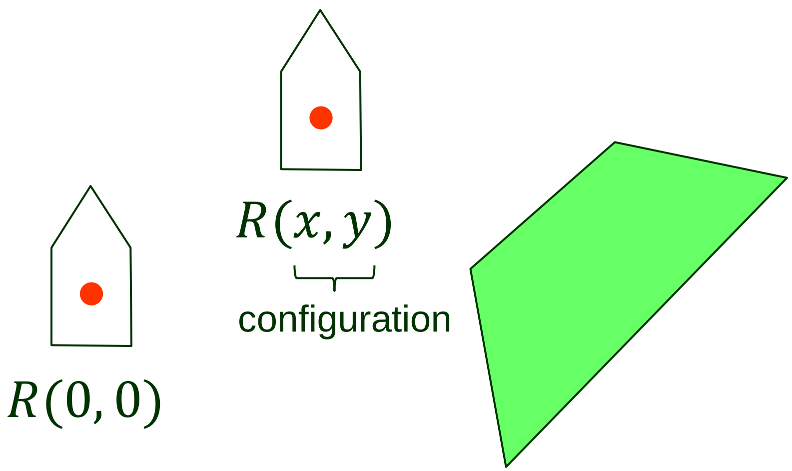

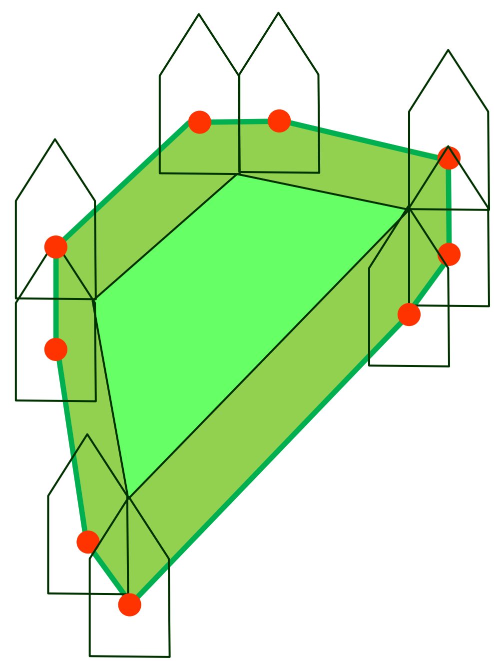

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The boundary of CP is traced out by the reference point of R as it slides along the boundary of P.

I. Minkowski Sum

Hermann Minkowski (1864-1909) – Albert Einstein was his former student.

Two sets $S_1, S_2 \in \mathbb{R}^2$

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Minkowski Sum of 1D Sets

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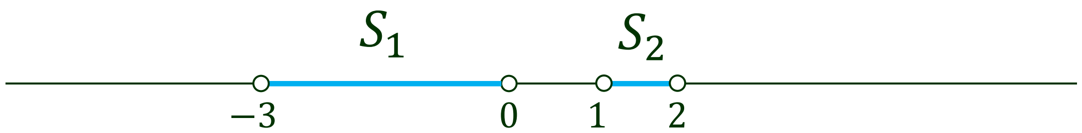

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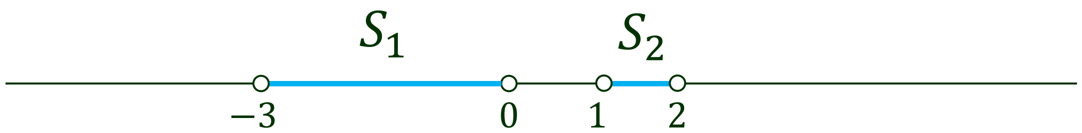

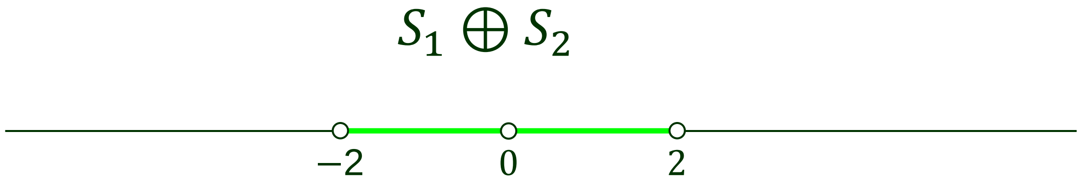

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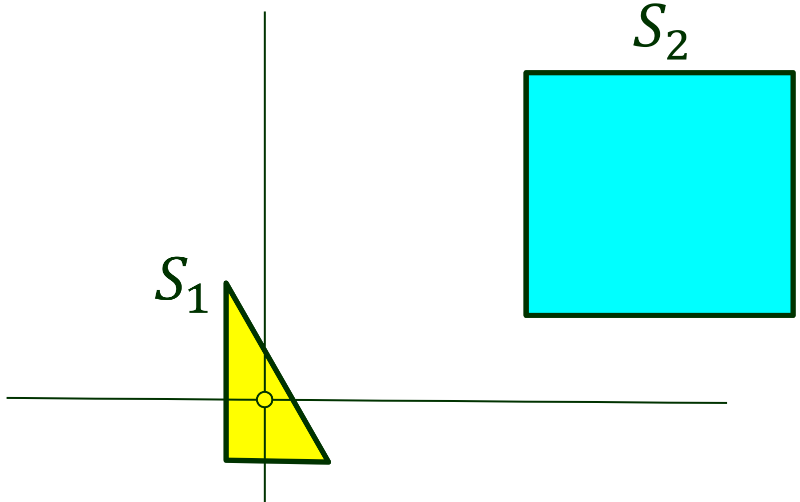

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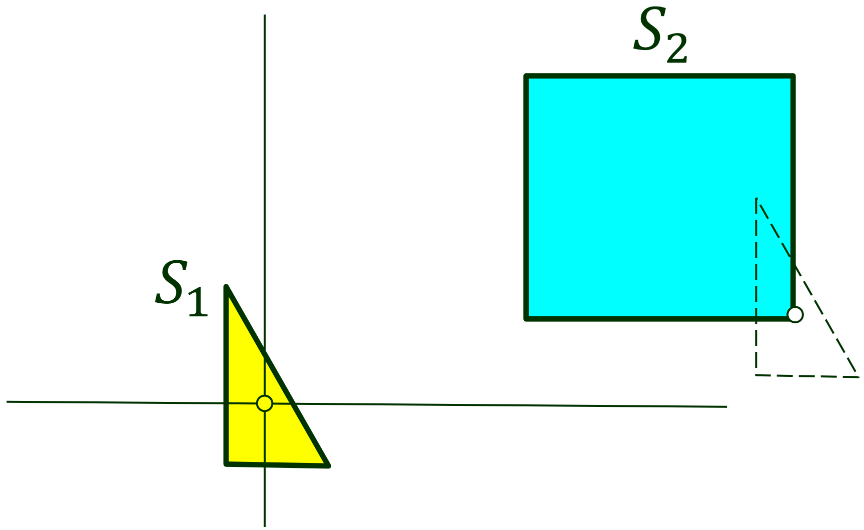

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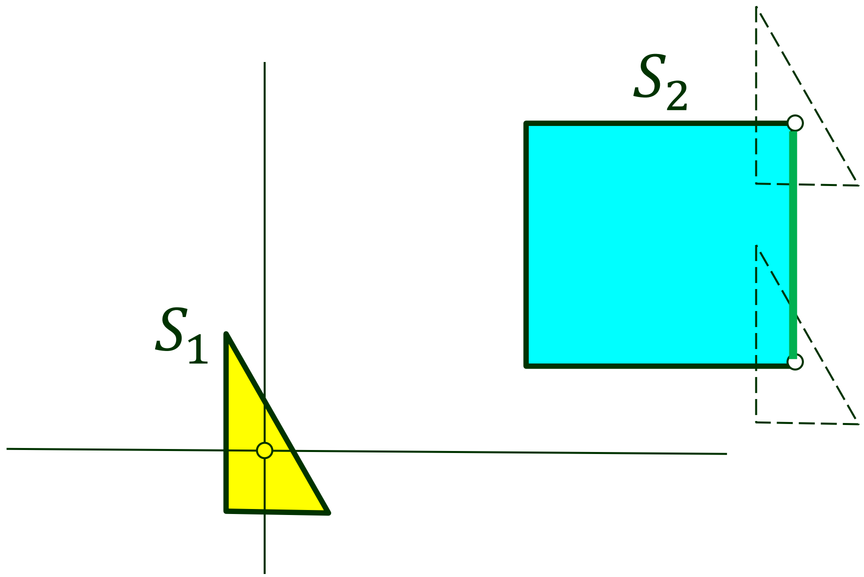

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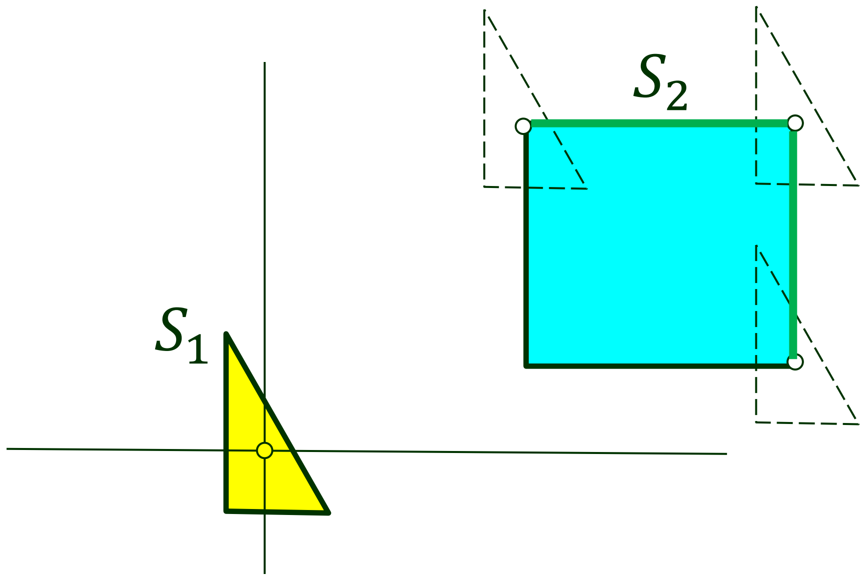

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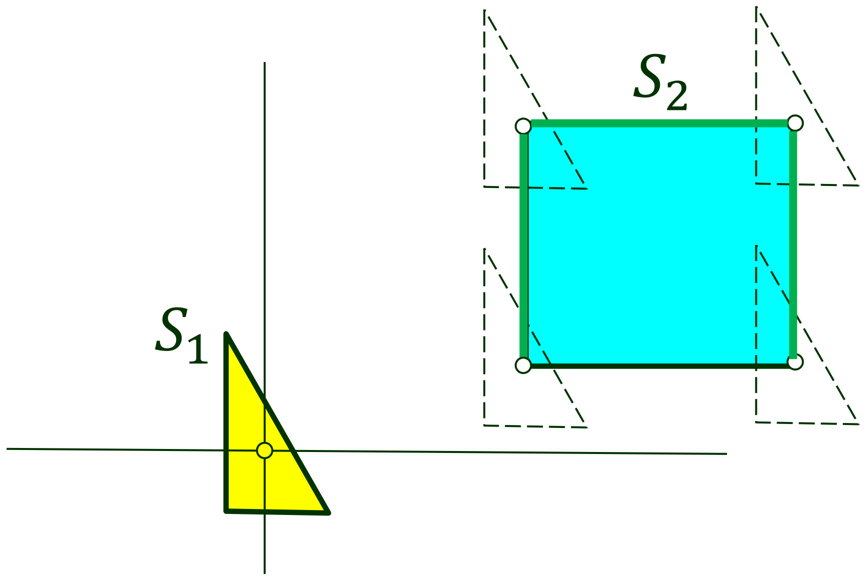

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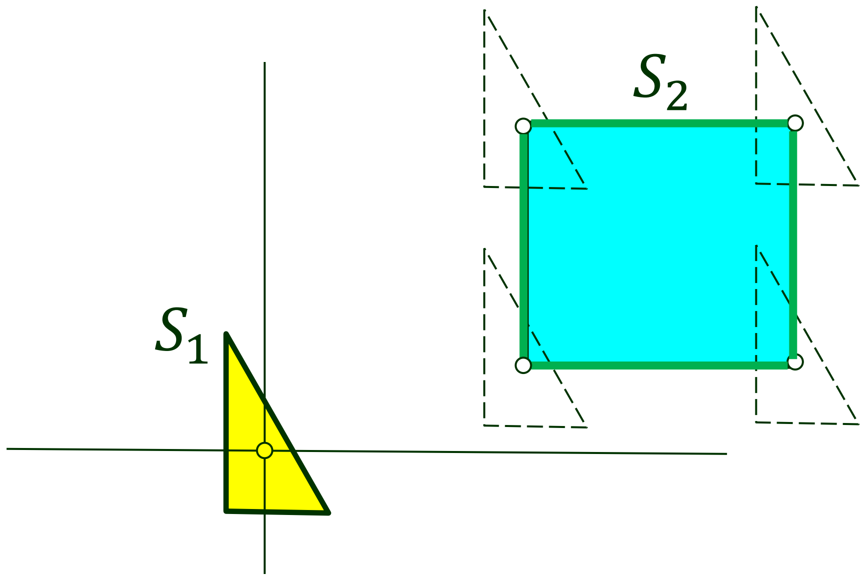

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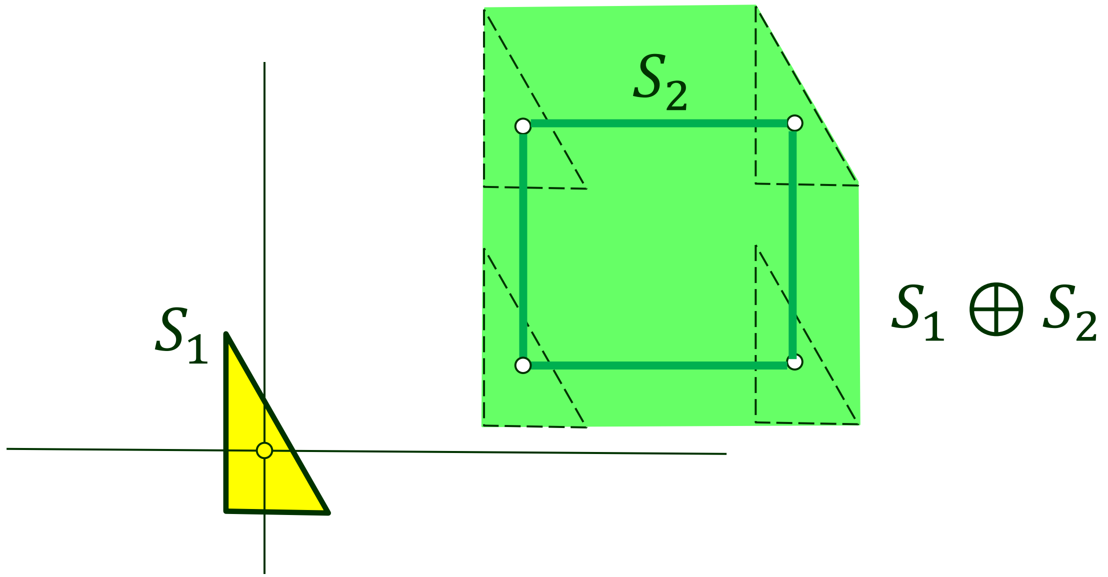

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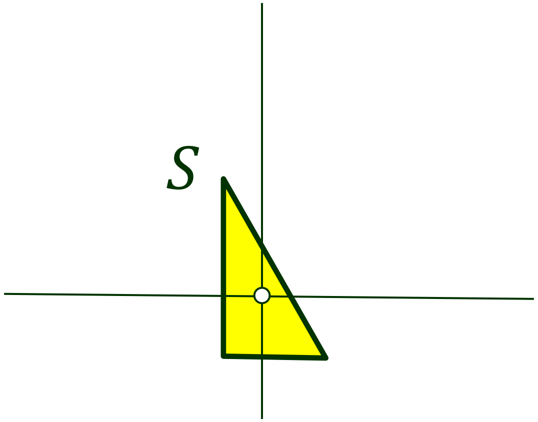

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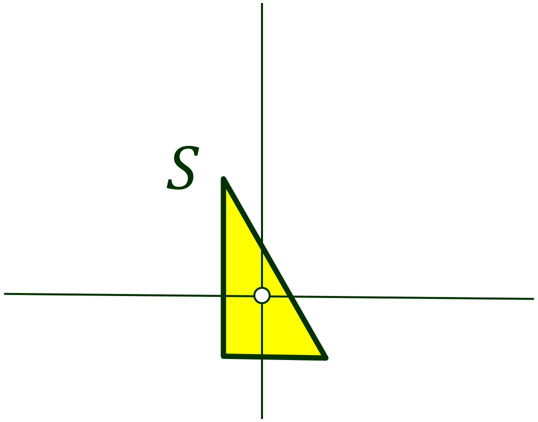

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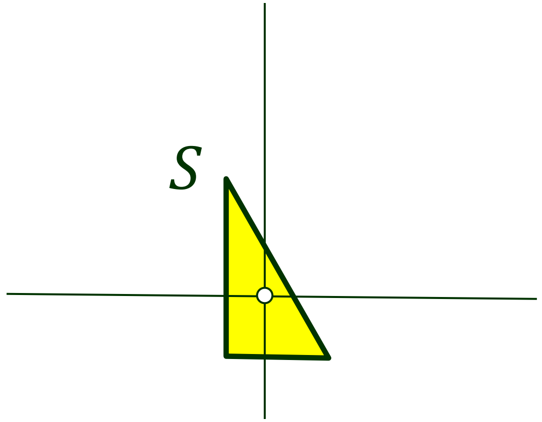

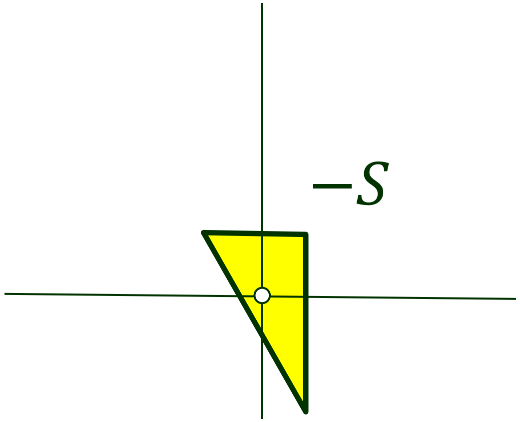

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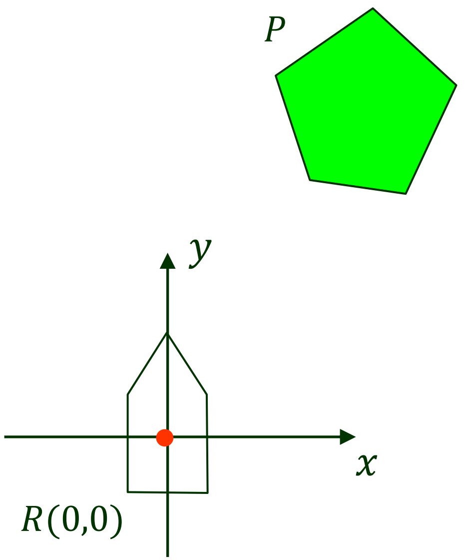
Theorem 1 The C-obstacle CP of P is $P \oplus \left(-R(0,0)\right)$.



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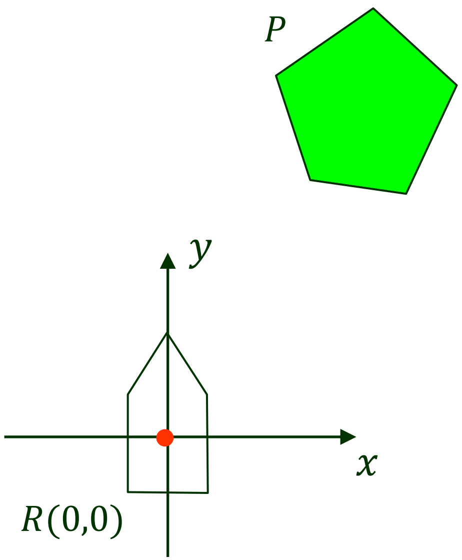

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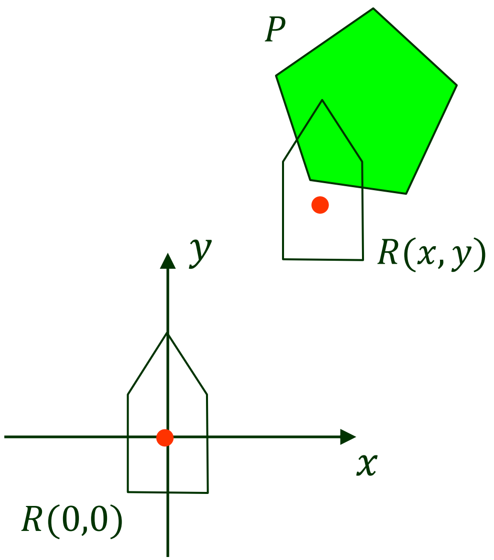

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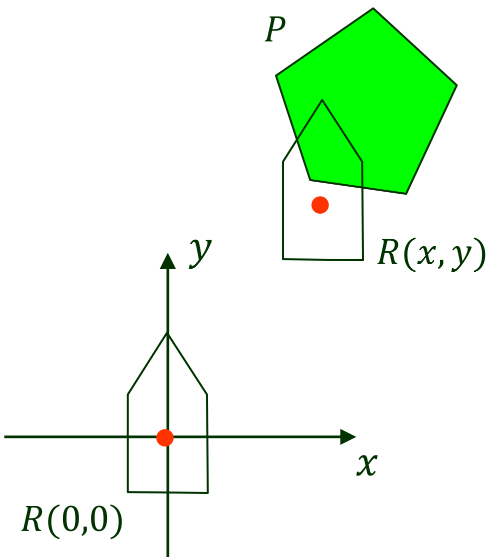

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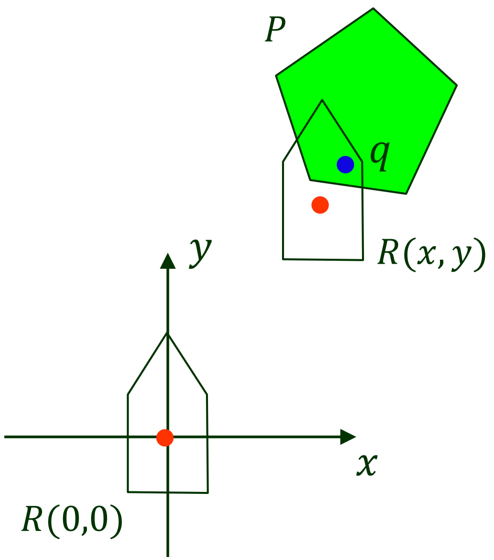

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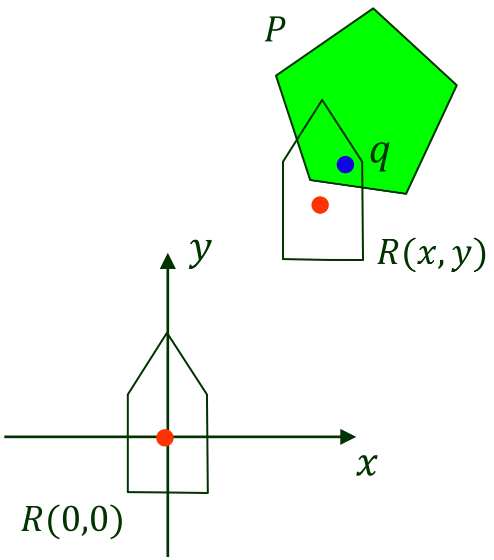

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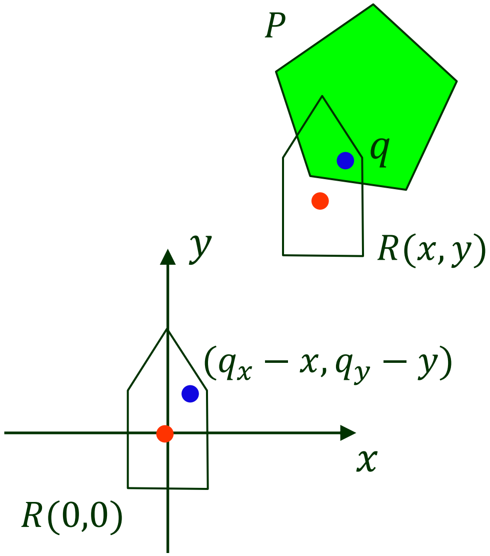

II. Formula for C-obstacle

Theorem 1 The C-obstacle CP of P is $P \oplus \left(\underline{-R(0,0)}\right)$.

$= \{(-x, -y) \mid (x, y) \in R(0, 0)\}$

Proof $(\Rightarrow)$ Suppose $(x, y) \in CP$, i.e., $R(x, y)$ intersects P.

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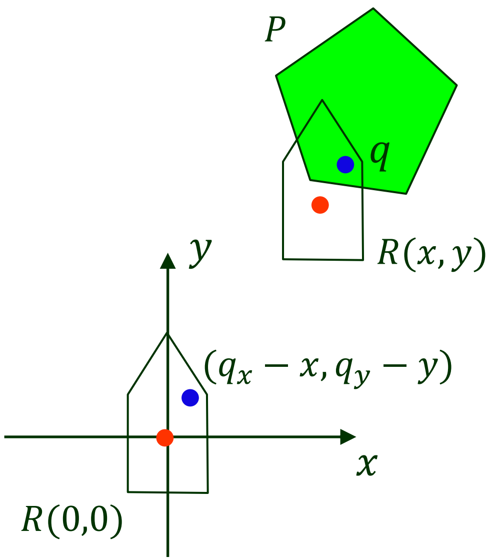

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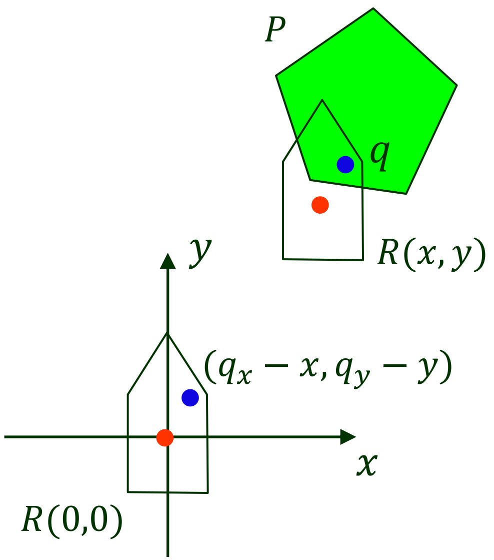

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$$q \in P \quad \Downarrow$$

$$q + \left(-q_x + x, -q_y + y\right) = (x, y) \in P \oplus (-R(0,0))$$

Proof (cont'd)

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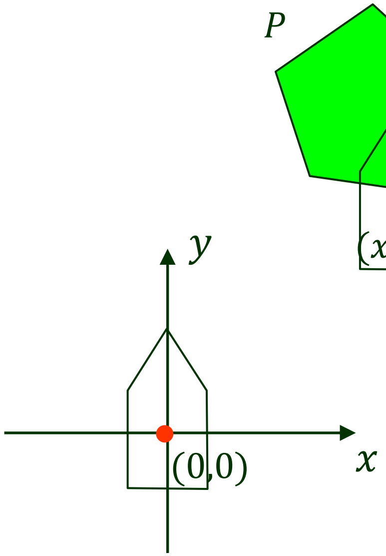

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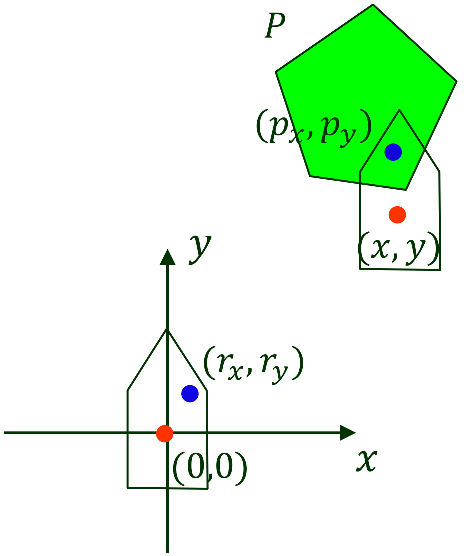

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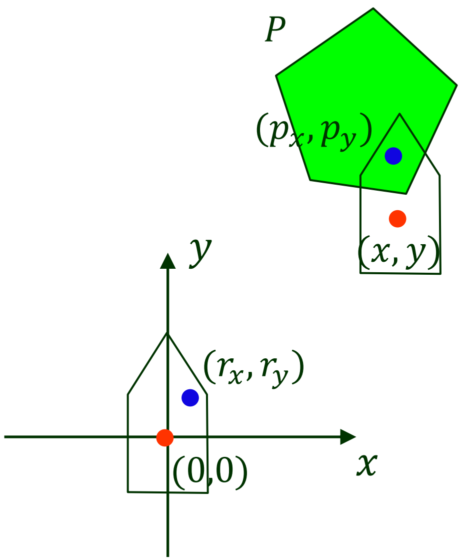

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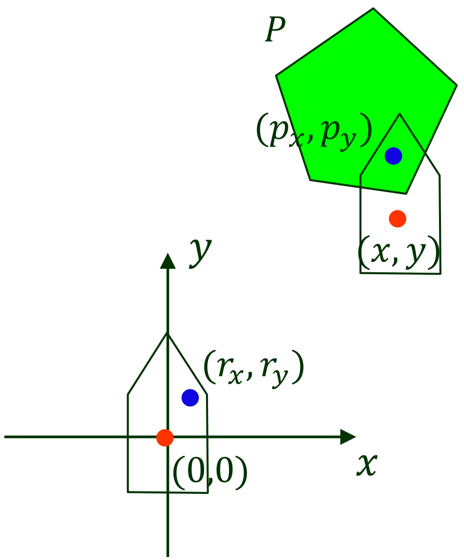

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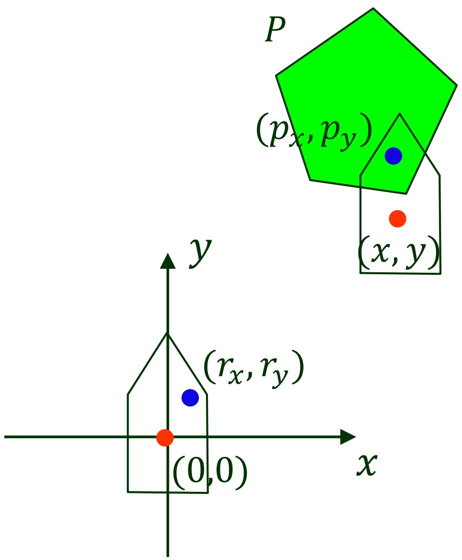

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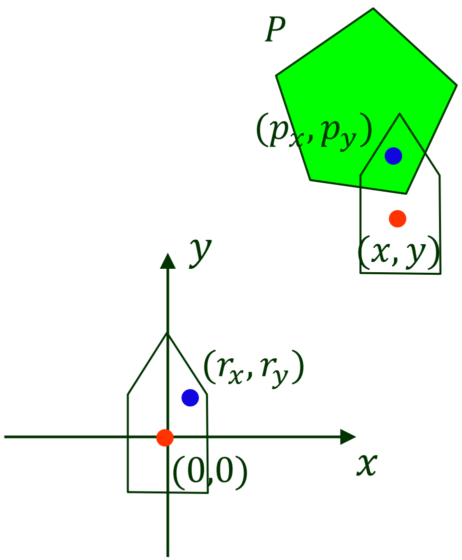
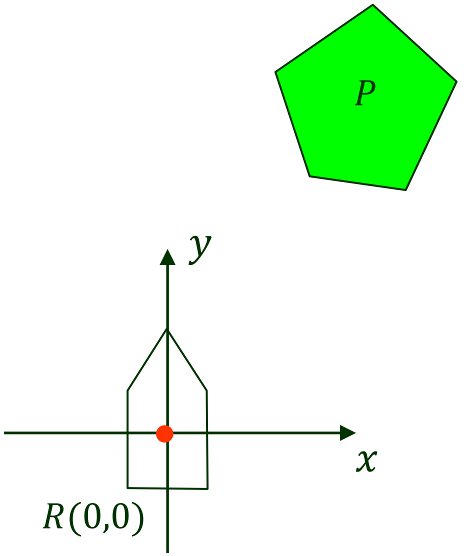
$R(x, y)$ intersects P, i.e., $(x, y) \in CP$.



Verification via an Example

Two equivalent ways of C-obstacle construction:

Straightforward



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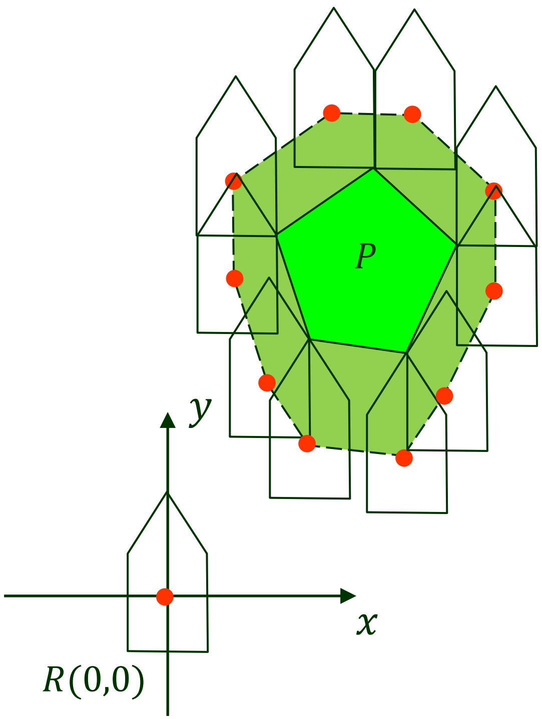

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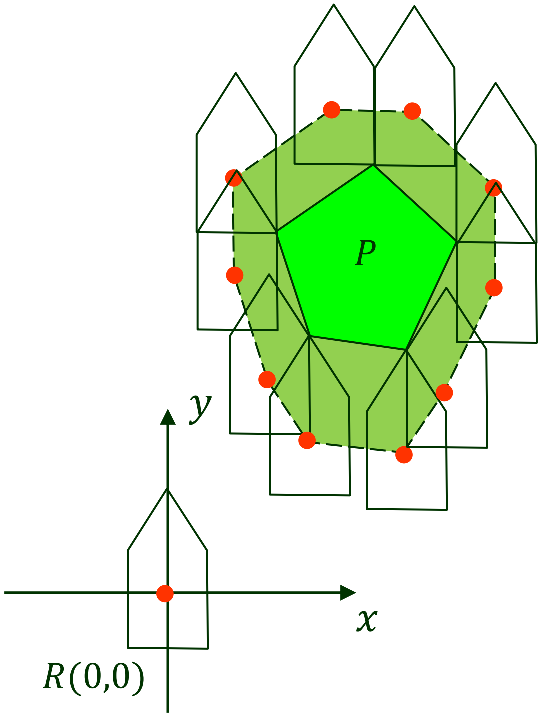

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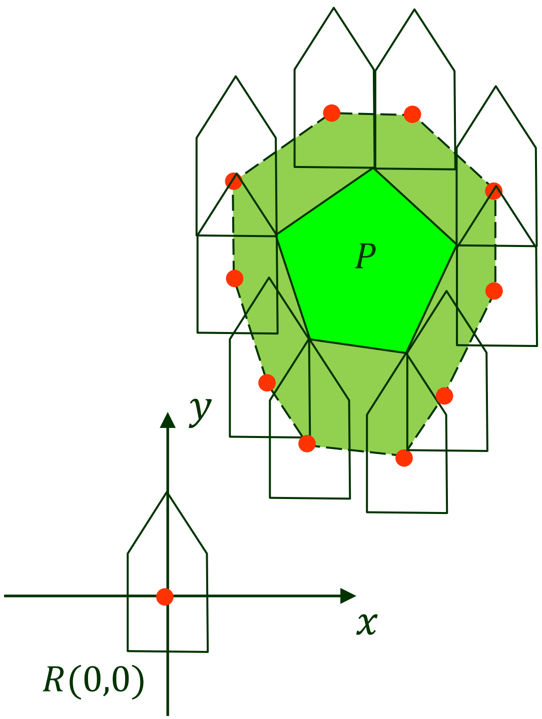

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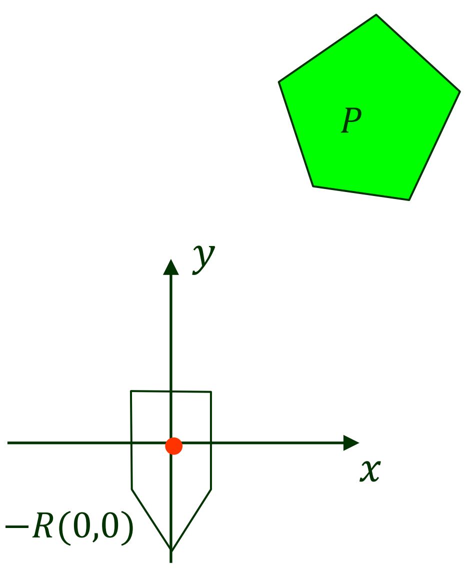

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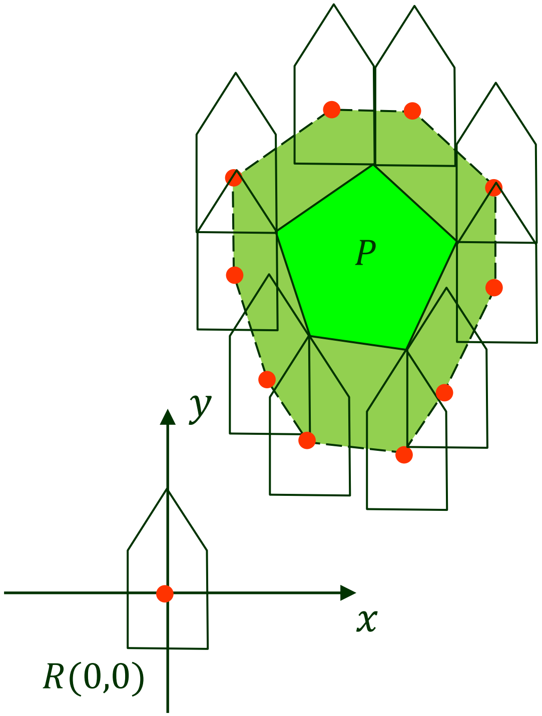

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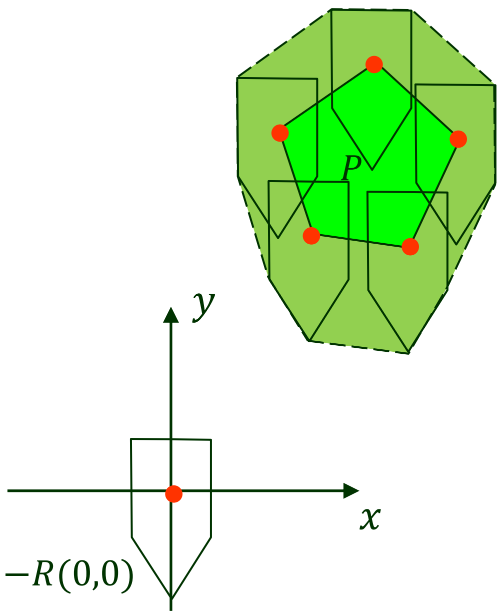
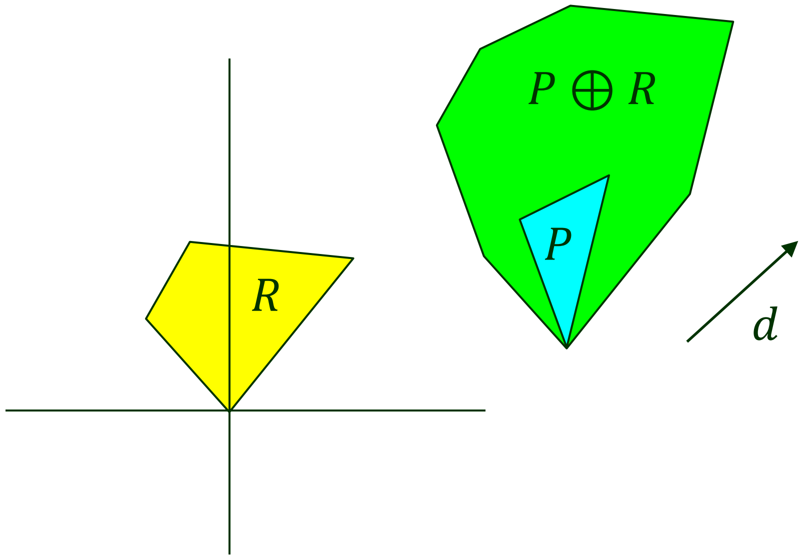


III. Extreme Points

Two polygons P and R.



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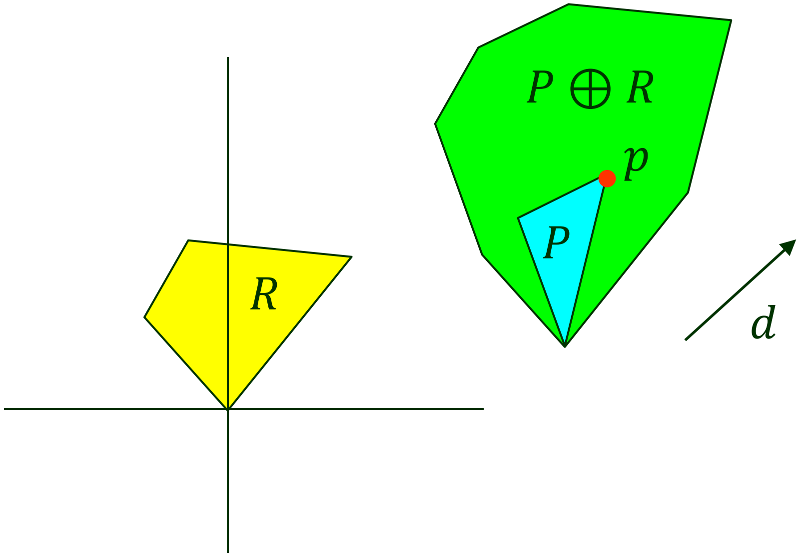

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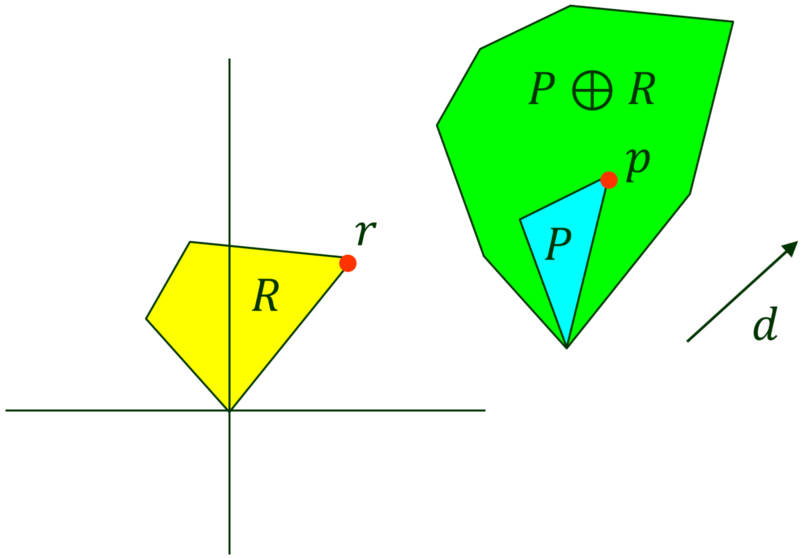

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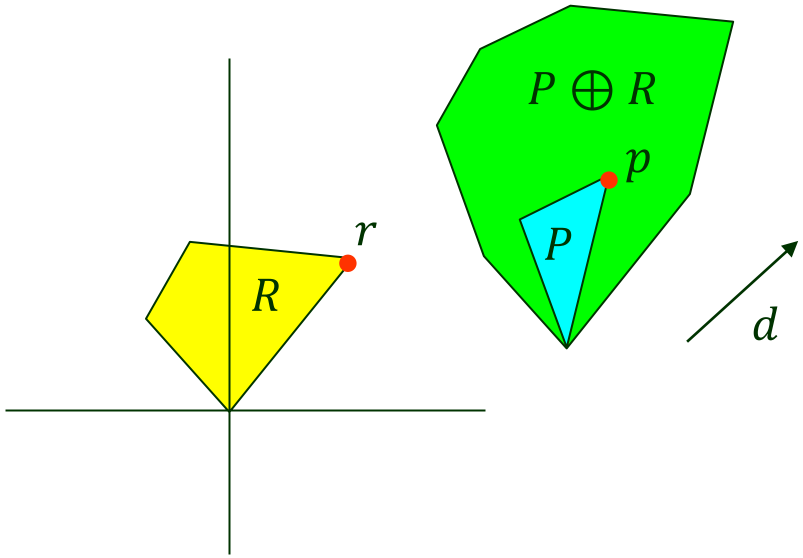

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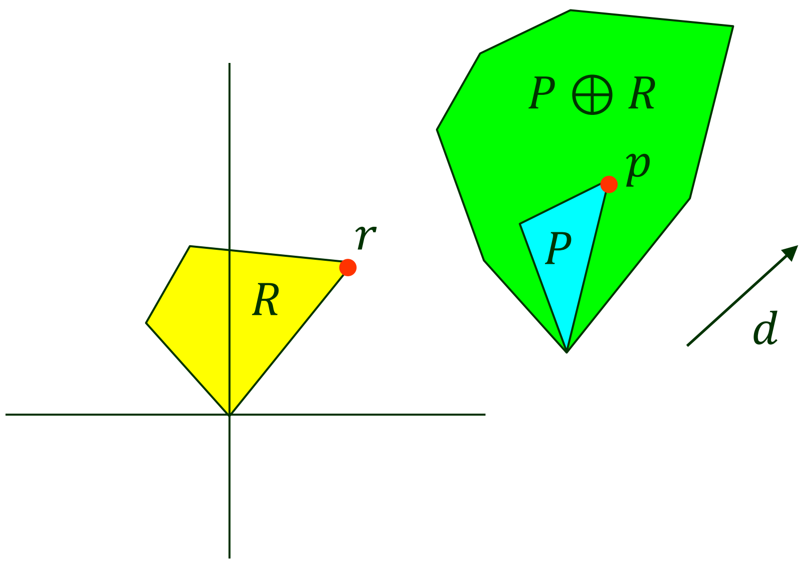

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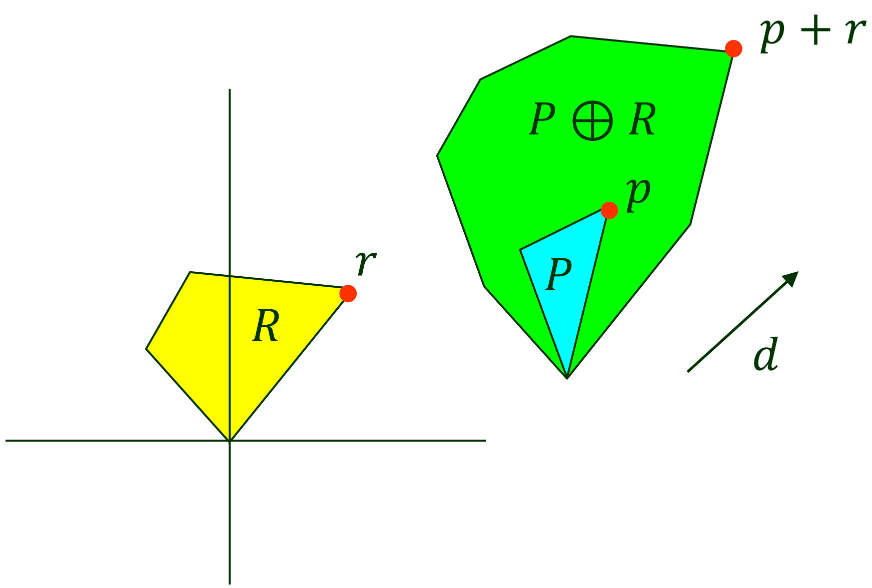

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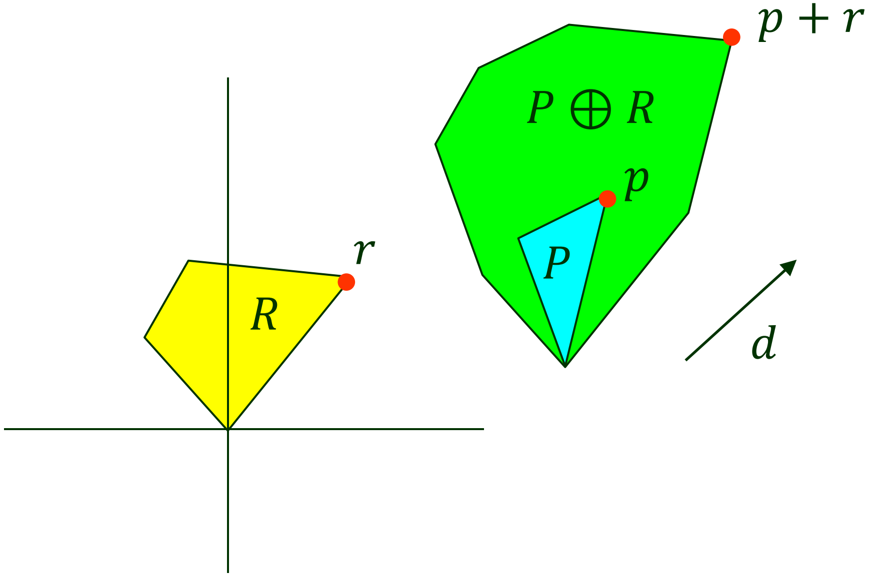

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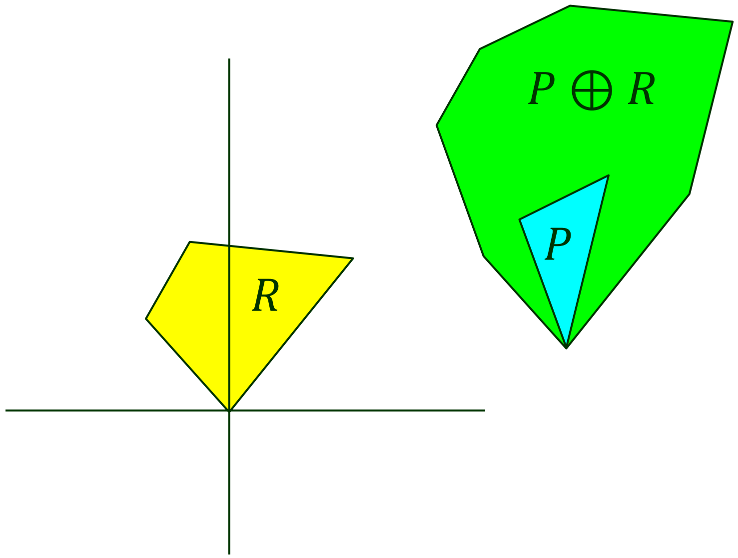
$p + r$ is an extreme point in the direction d on $P \oplus R$.

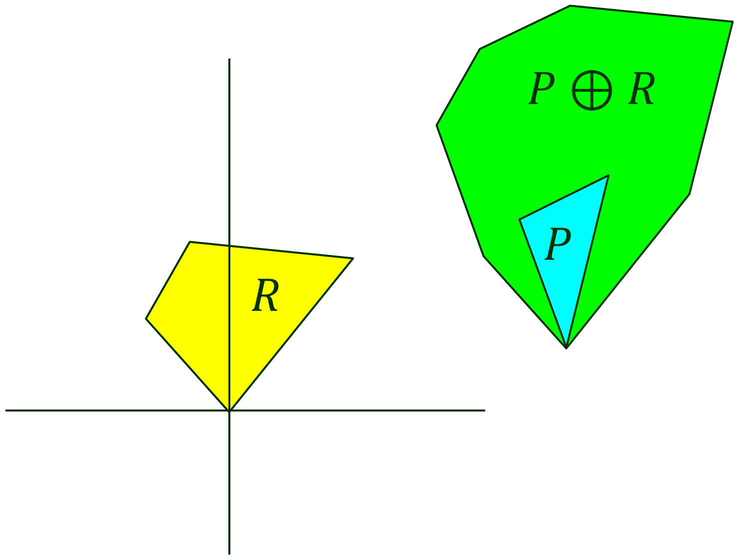
An extreme point in the direction d on $P \oplus R$ is the sum of two extreme points in d on P and R, respectively.

Minkowski Sum of Convex Polygons



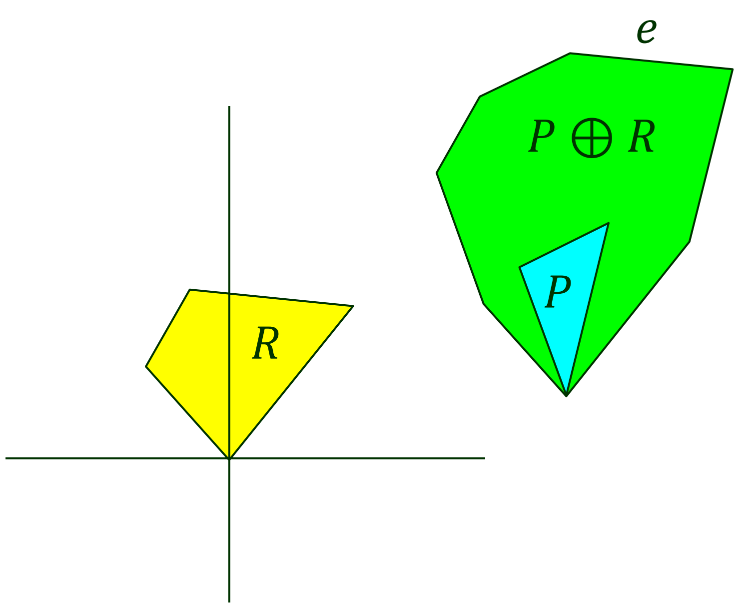
Theorem 2 Let P and R be two convex polygons with n and m edges, respectively. Then $P \oplus R$ is a convex polygon with $\leq n + m$ edges.

Complexity of Minkowski Sum



Proof Convexity of the Minkowski sum of two convex sets follows from the definition.

Complexity of Minkowski Sum



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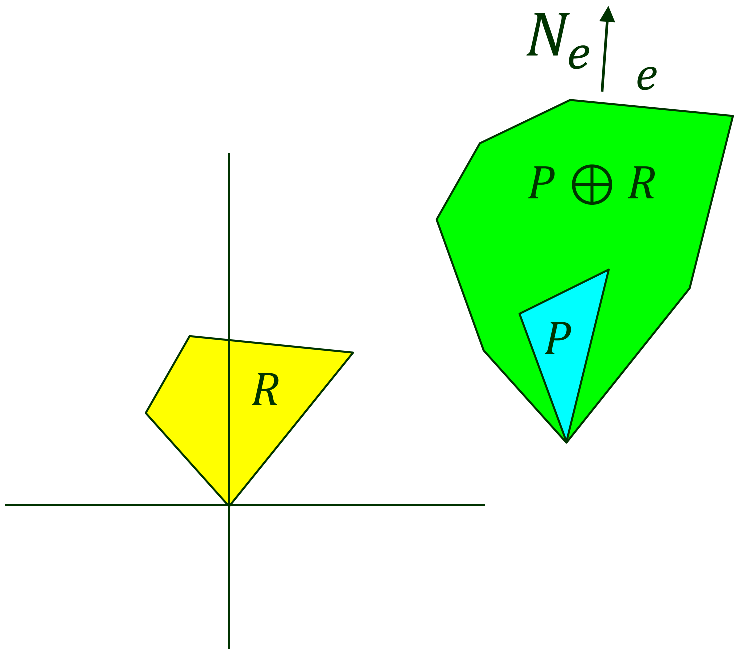
To bound the complexity, consider an edge e of $P \oplus R$.

Complexity of Minkowski Sum



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N_e: outward normal of e.

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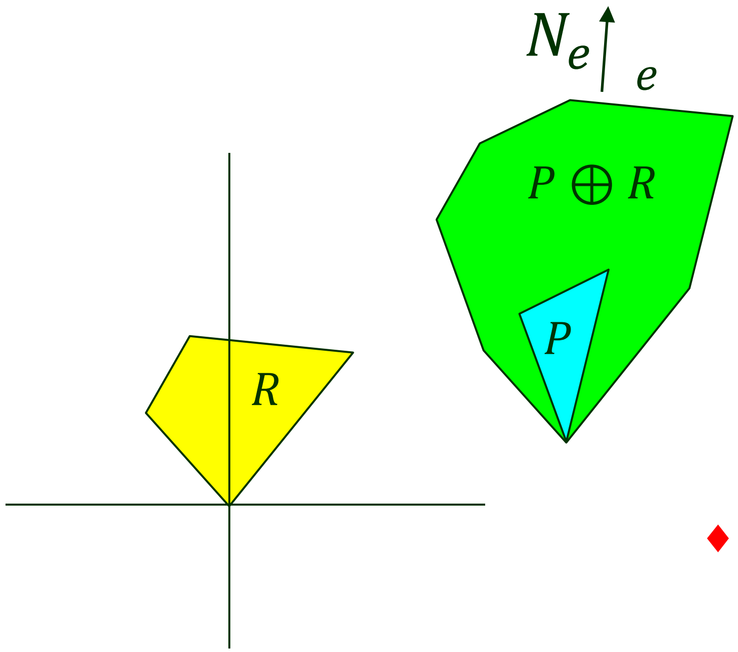

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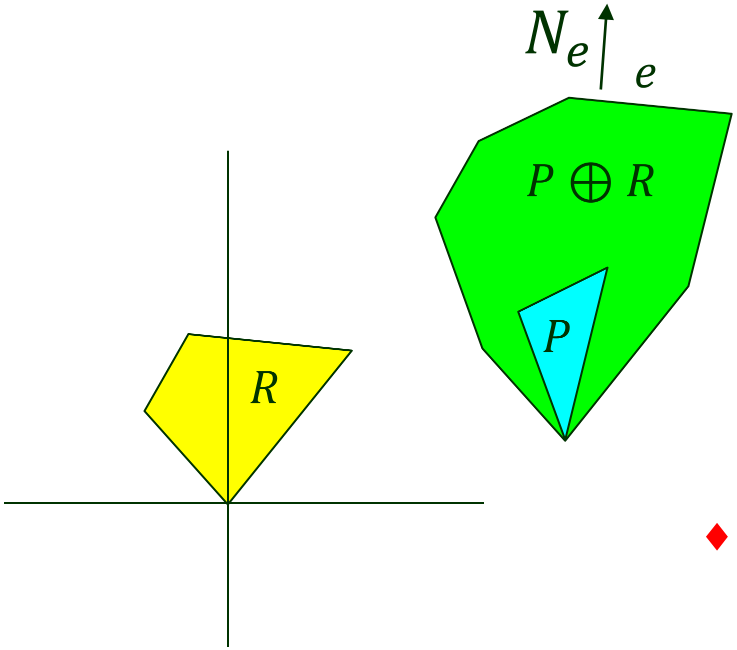
Complexity of Minkowski Sum



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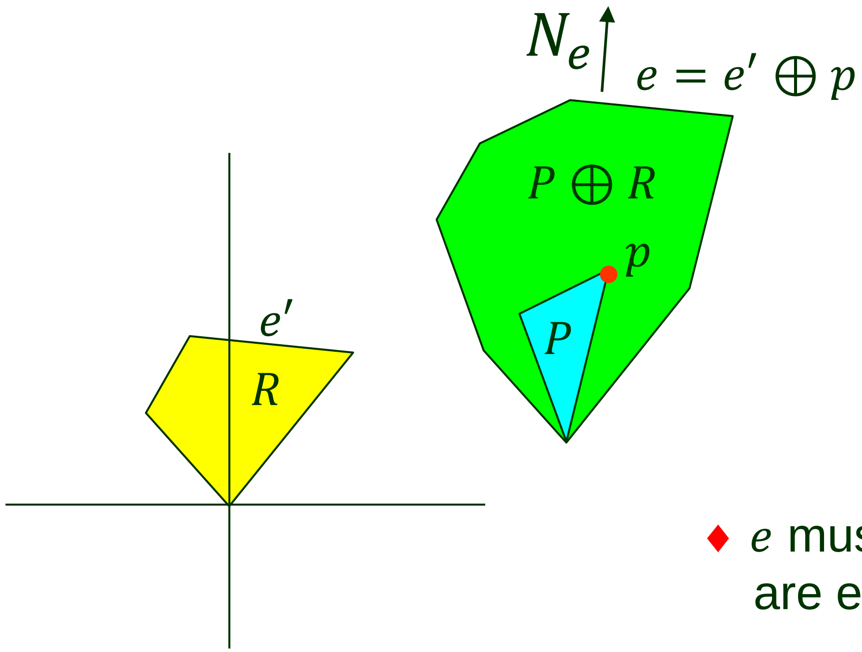

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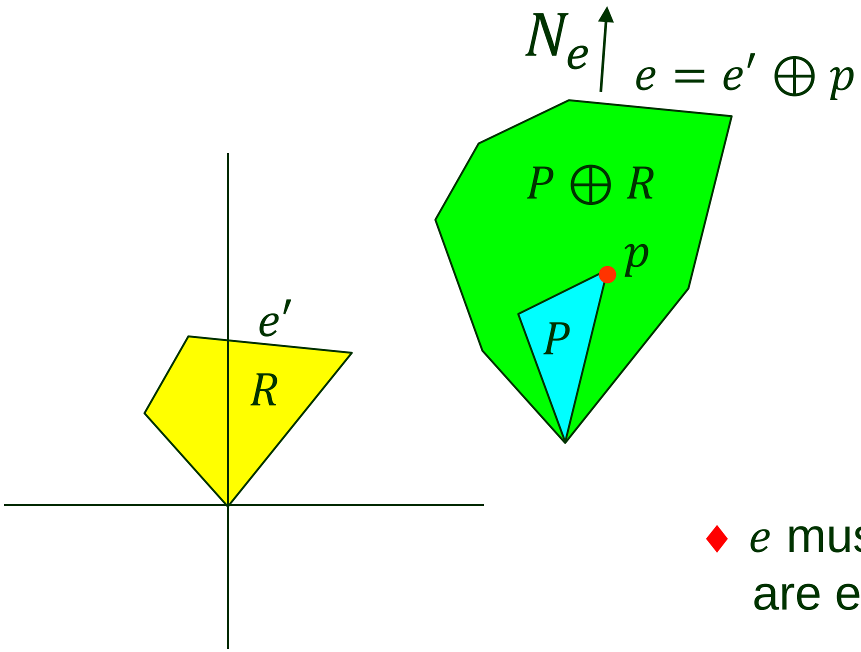

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- Charge e to e'.

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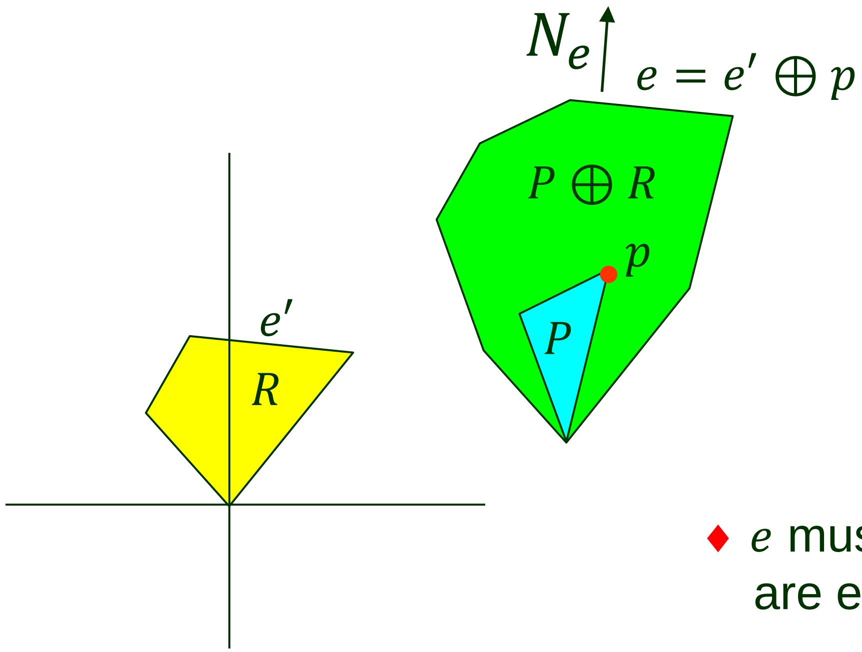

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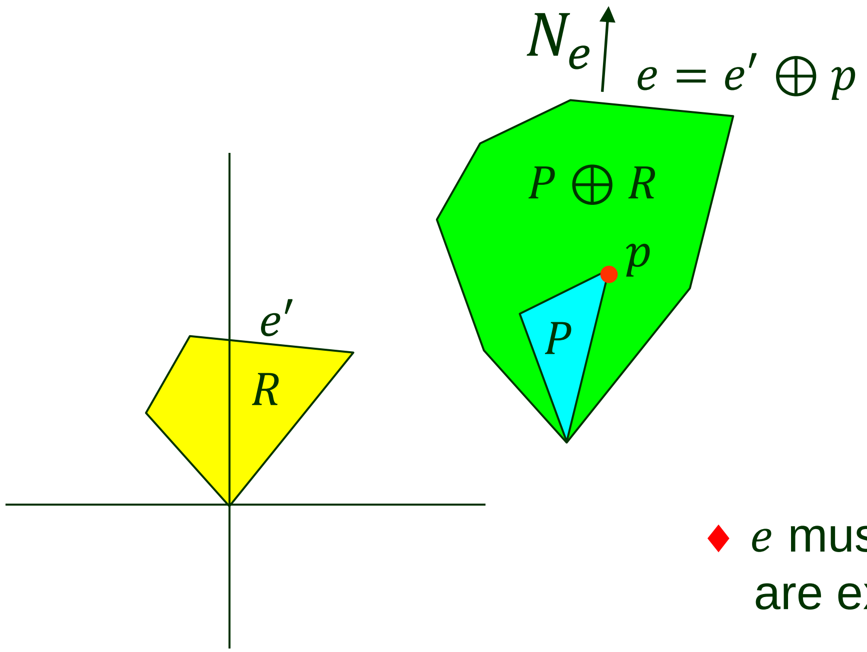
Every edge of P and R is charged at most once.

Complexity of Minkowski Sum



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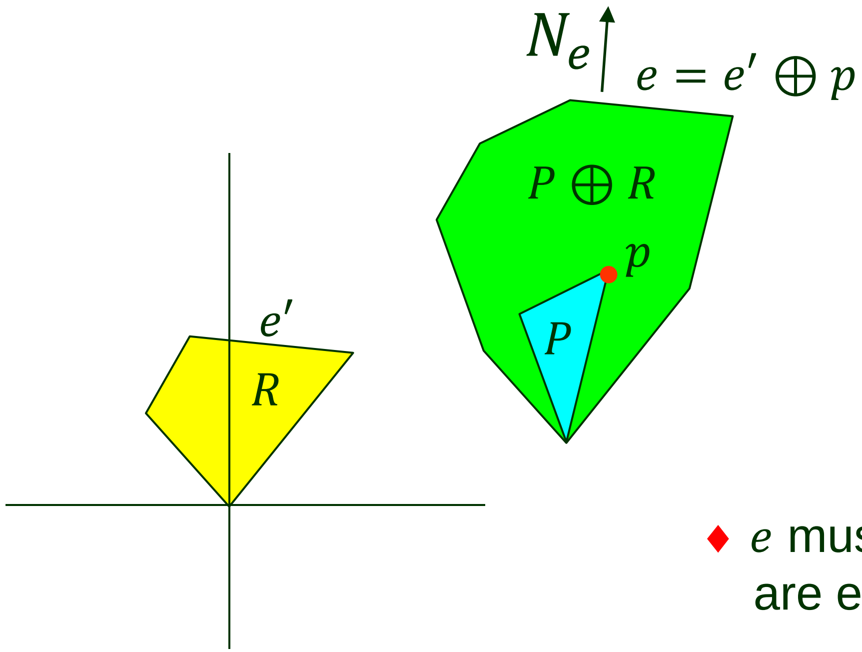
Complexity of Minkowski Sum



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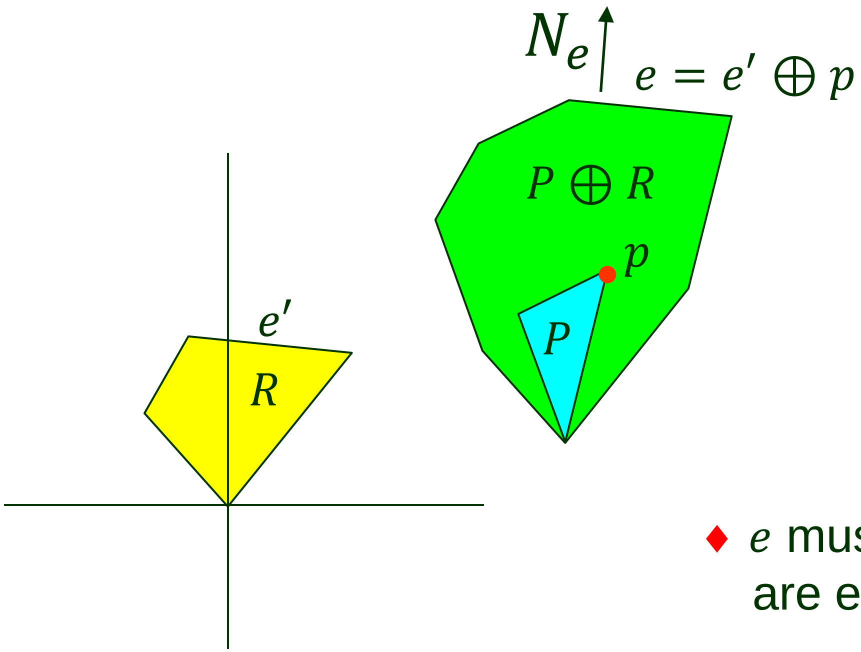

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The upper bound $n + m$ is achieved if P and R have no parallel edges.

IV. Computation of the Minkowski Sum

Compute $P \oplus R$ when P and R are convex.

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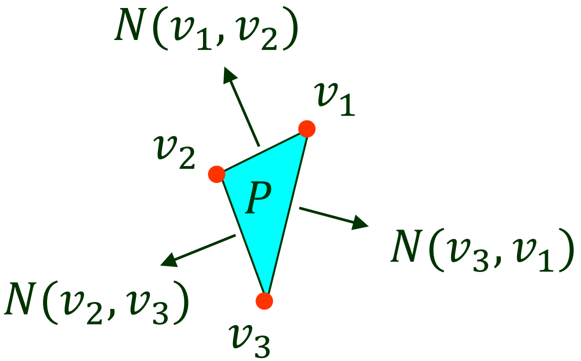
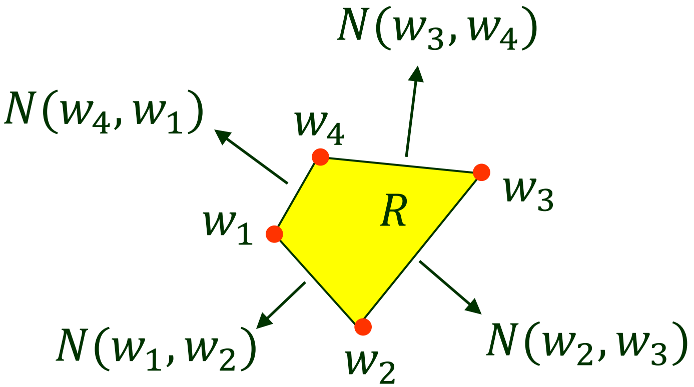
$$O(nm(\log n + \log m))$$

Faster Computation

Idea: Look at a pair of vertices that are extreme in the same direction.





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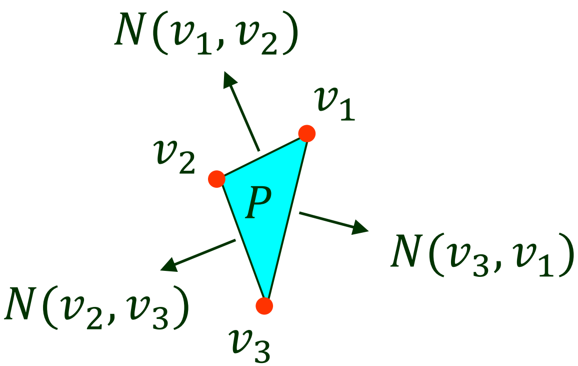

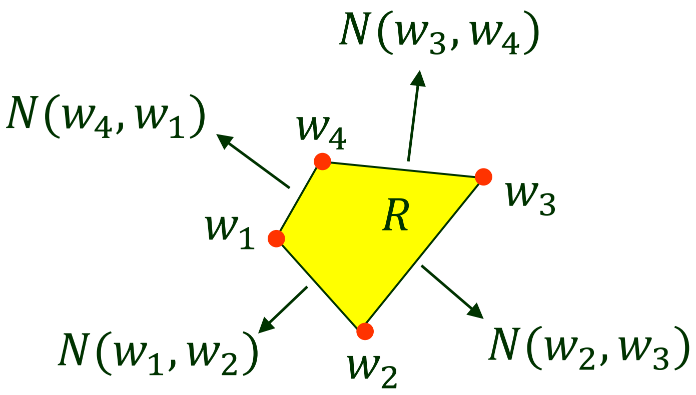

Represent all the directions by a unit circle.

Faster Computation

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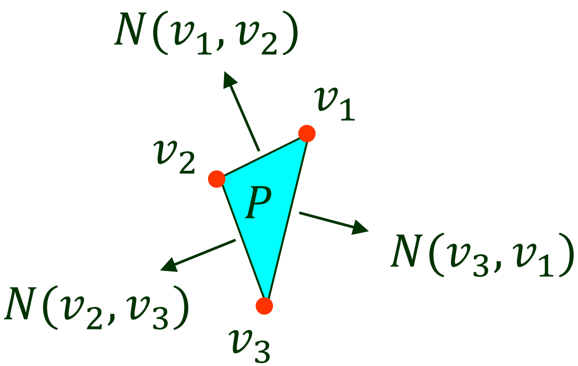

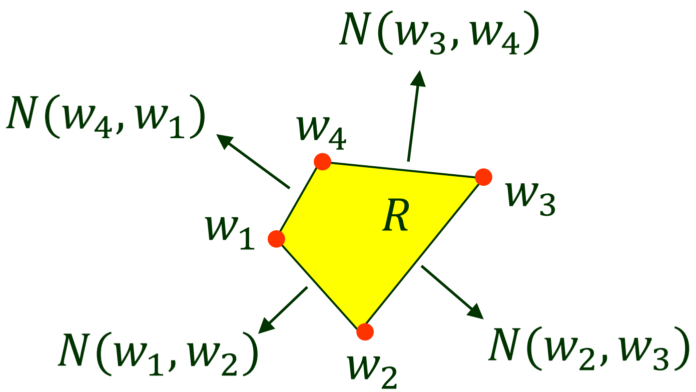

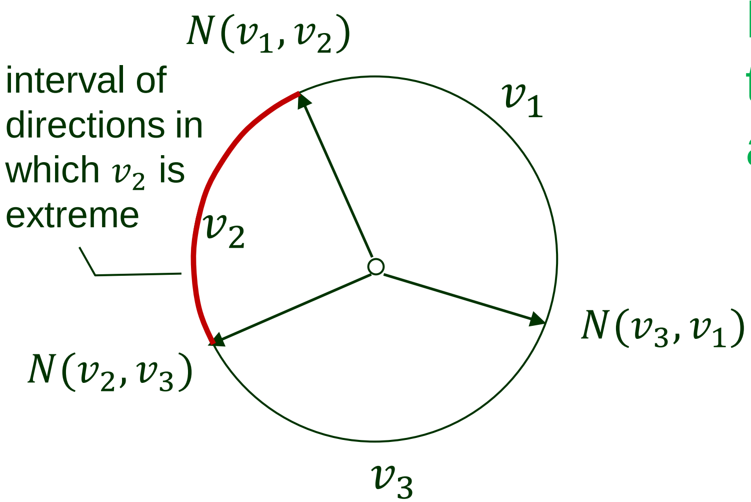

Represent all the directions by a unit circle.

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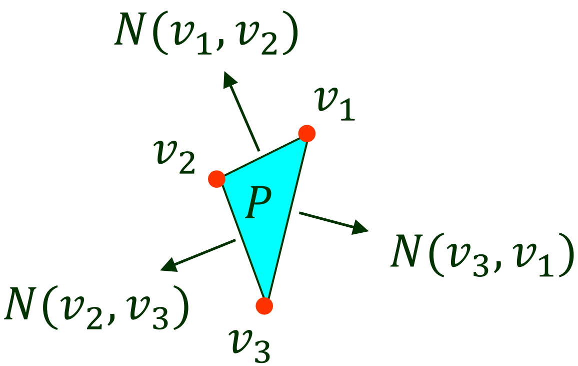

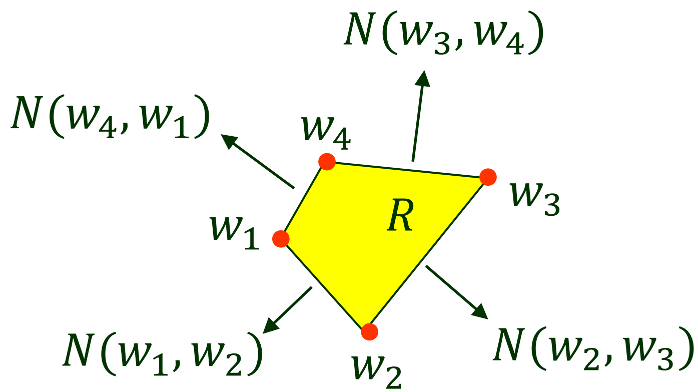

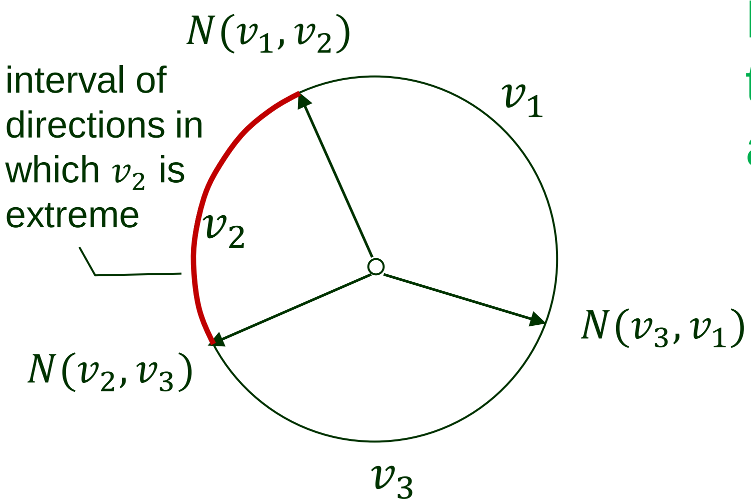

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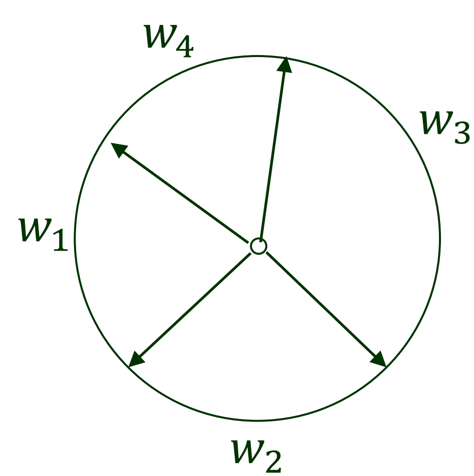
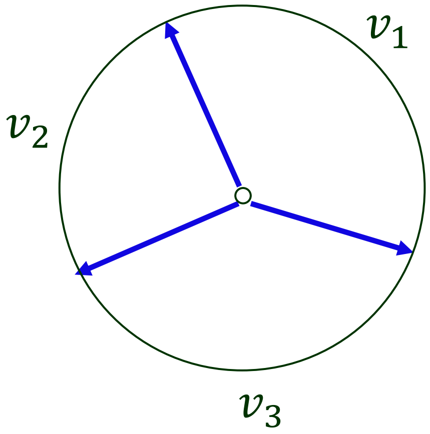
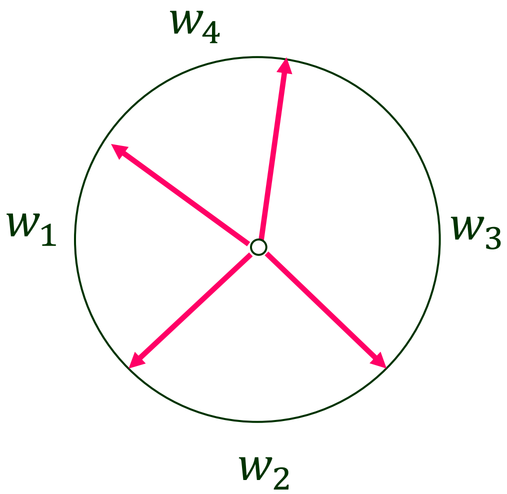


Extreme Pairs

Superpose the two partitioning.





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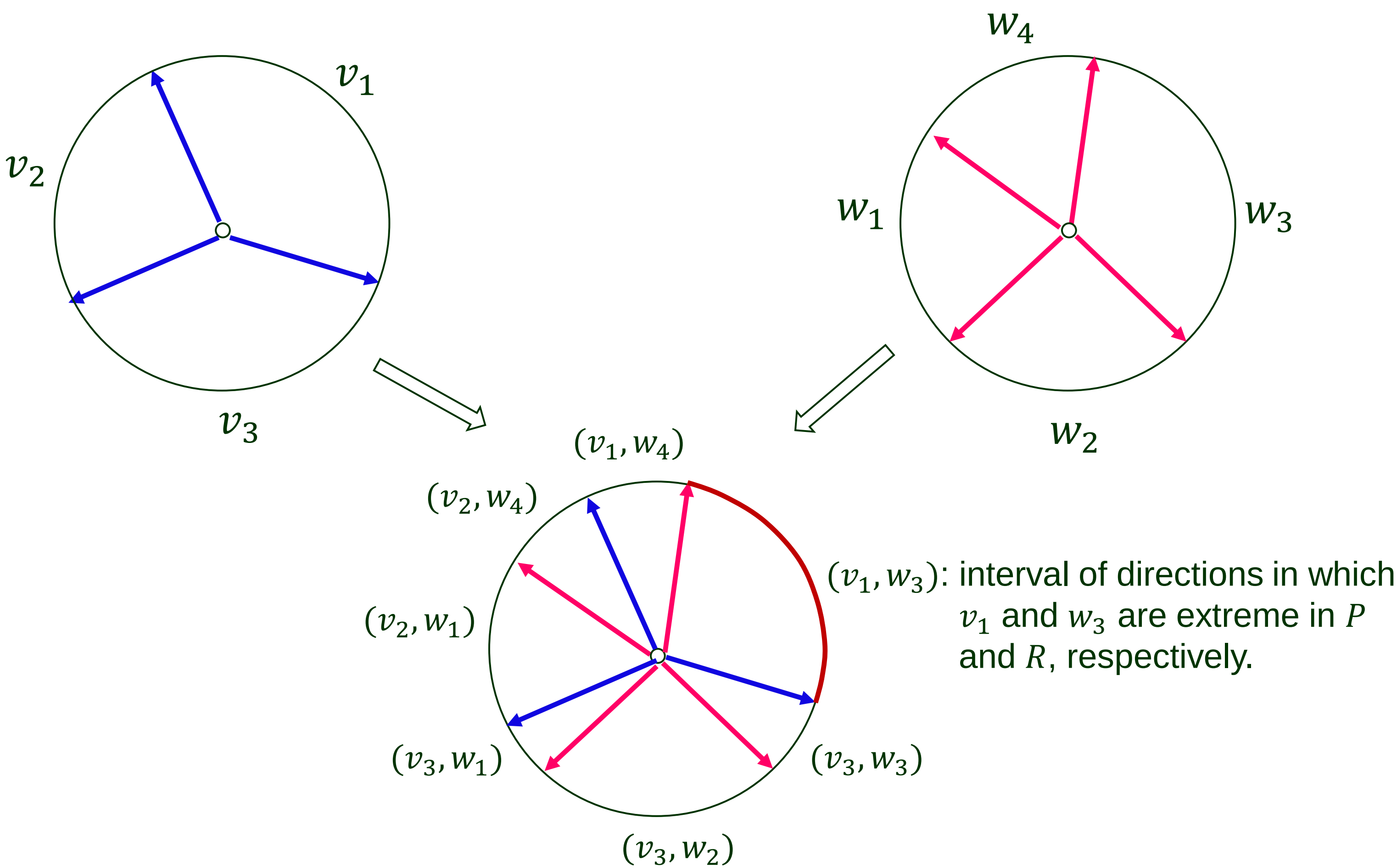

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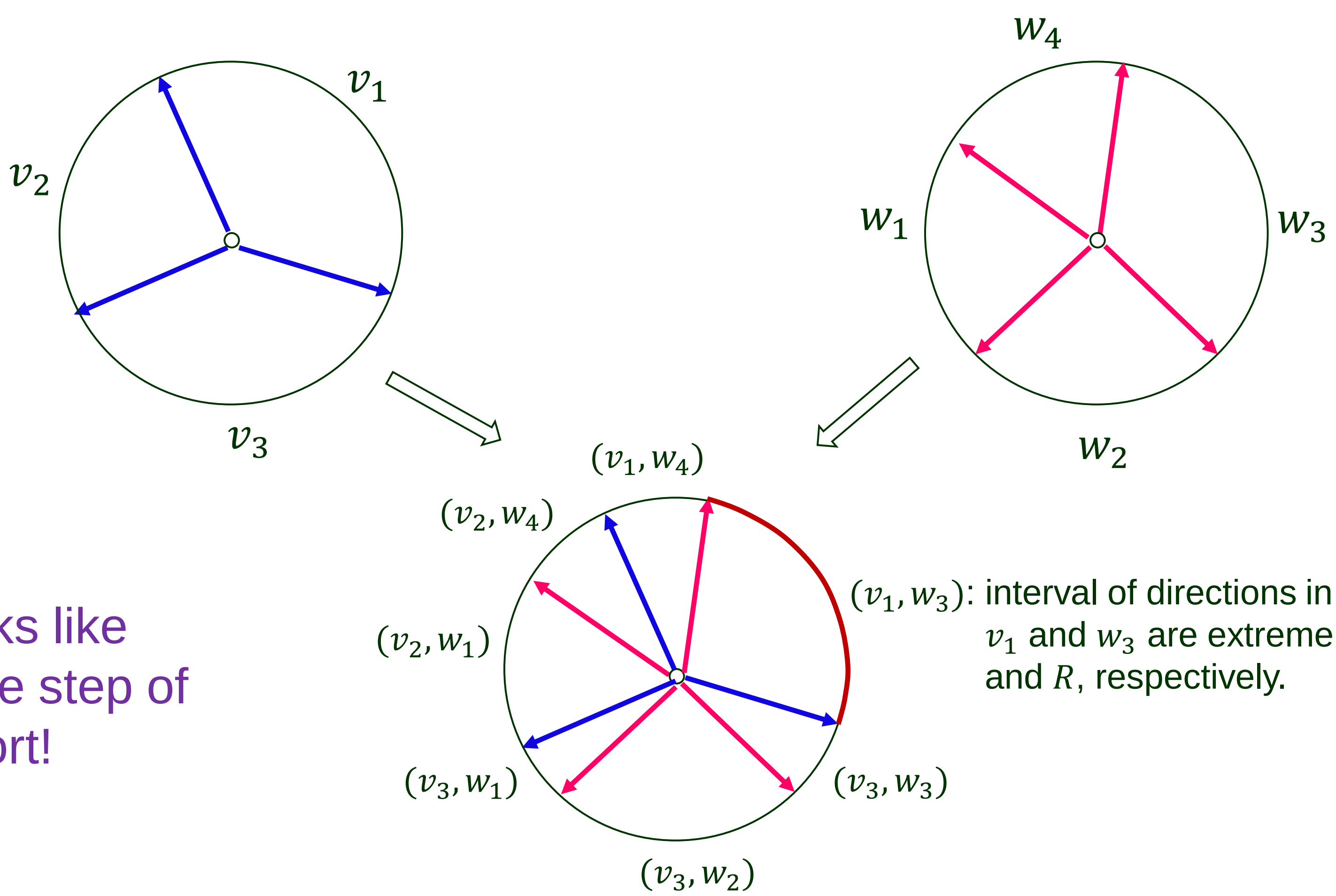

(v_1, w_3): interval of directions in which v_1 and w_3 are extreme in P and R, respectively.

This works like the merge step of merge sort!

The Algorithm

```
MinkowskiSum(P, R)

1.  i ← 1; j ← 1
2.  v_{n+1} ← v_1; v_{n+2} ← v_2; w_{m+1} ← w_1; w_{m+2} ← w_2
3.  repeat
4.      add v_i + w_j as a vertex to P ⊕ R
5.      if angle(v_i, v_{i+1}) < angle(w_j, w_{j+1}) // case 1
6.         then i ← i + 1
7.         ...
```

The Algorithm

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1. $i \leftarrow 1; j \leftarrow 1$
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angle made by $\overrightarrow{v_i v_{i+1}}$ with the x-axis

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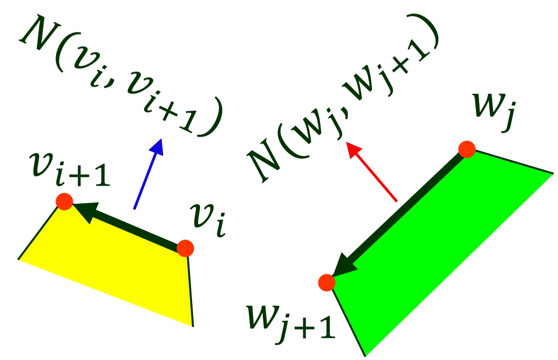

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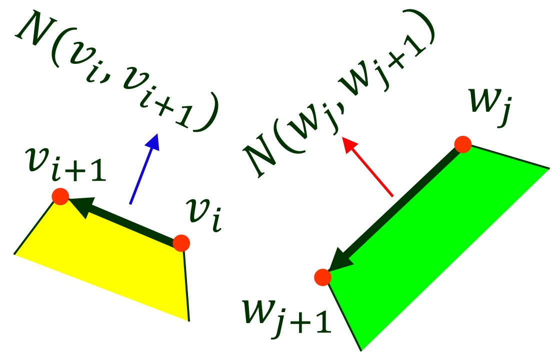

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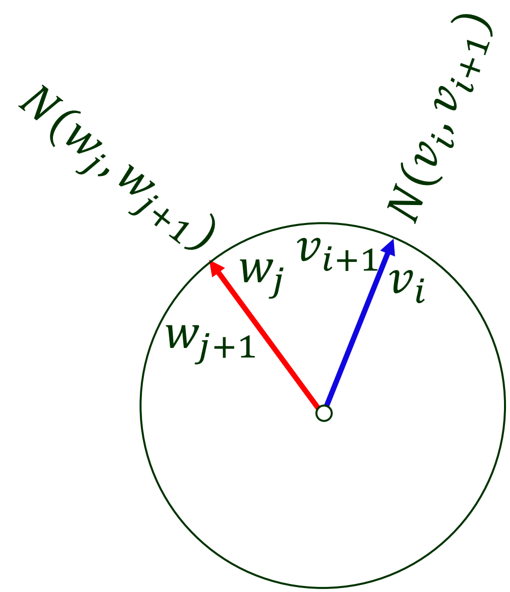

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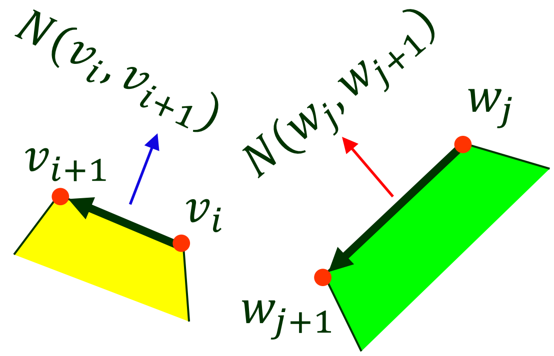

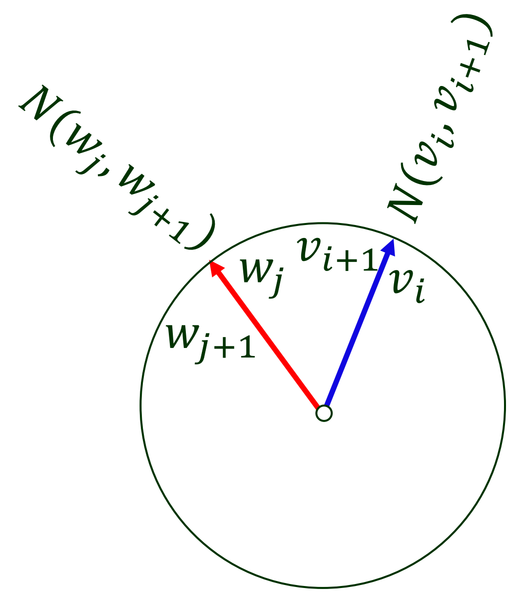

$$\cdots \to (v_i, w_j) \to (v_{i+1}, w_j) \to \cdots$$

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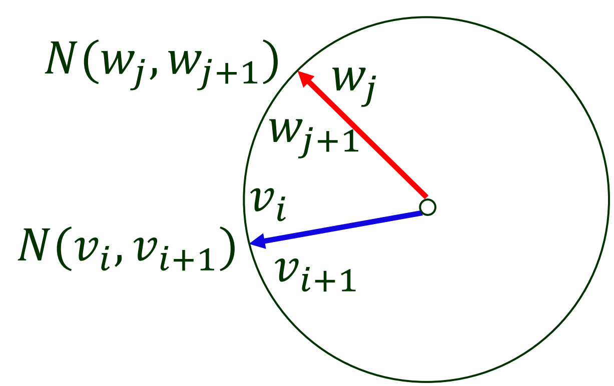

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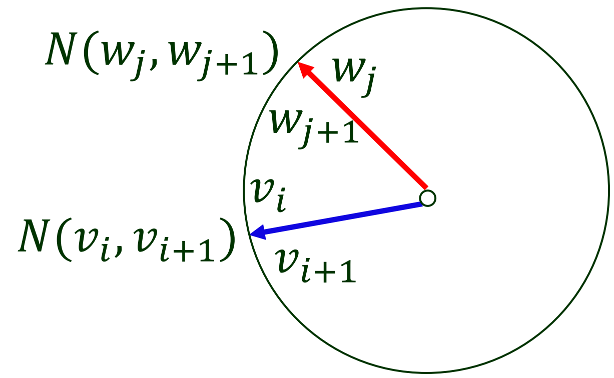

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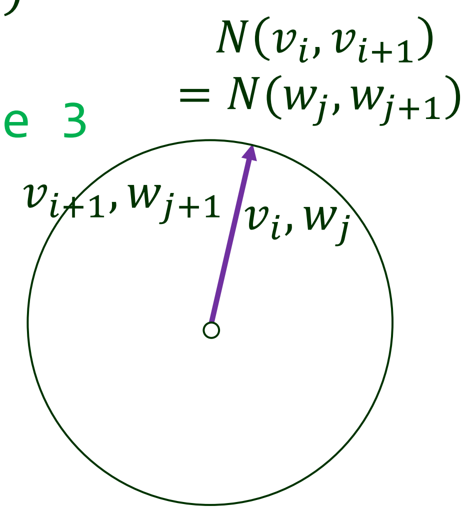
10. **until** $i = n + 1$ and $j = m + 1$



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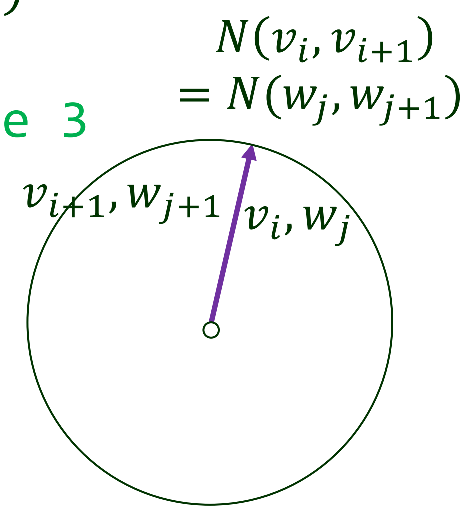

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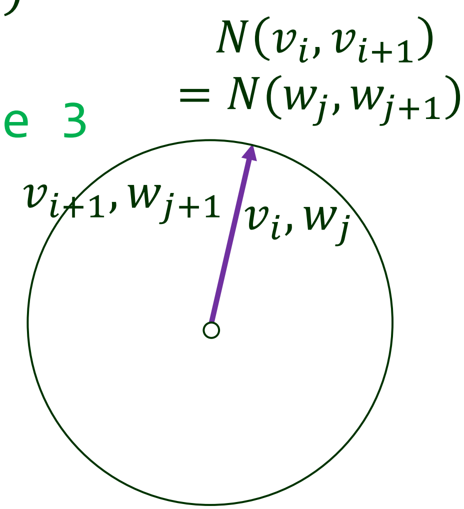

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Running time $O(n + m)$

IV. Nonconvex Robot or Obstacle

Triangulate whichever is nonconvex.

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Make use of the following equality for three sets S_1, S_2 and S_3:

$$S_1 \oplus (S_2 \cup S_3) = (S_1 \oplus S_2) \cup (S_1 \oplus S_3)$$

Complexity of $P \oplus R$

- Both P and R are nonconvex.

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We are given

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$$C_{forb} = \bigcup_{i=1}^{n} CP_i = \bigcup_{i=1}^{n} T_i \oplus (-R(0,0))$$

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Complexity $O(n)$

Computing the Forbidden C-Space

Divide-and-conquer

1. $C_{forb}^1 \leftarrow \bigcup_{i=1}^{\frac{n}{2}} CP_i$

2. $C_{forb}^2 \leftarrow \bigcup_{i=\frac{n}{2}+1}^{n} CP_i$

3. Compute $C_{forb} = C_{forb}^1 \cup C_{forb}^2$

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$T(n)$: time of computation

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n)$$

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$T(n)$: time of computation

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n) \implies T(n) = O(n \log^2 n)$$

C_{free} is the complement of C_{forb}.

Time Complexity for Path Planning

Theorem 3 $\mathcal{C}_{free}$ can be computed in time $O(n \log^2 n)$.

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Total preprocessing time:

$$O(n \log^2 n + n \log n) = O(n \log^2 n)$$

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Theorem 4 Translational motion planning for a convex robot of $O(1)$ complexity among polygonal obstacles of $O(n)$ total complexity can be solved in $O(n)$ time with preprocessing in $O(n\log^2 n)$ expected time.