Construction of a Trapezoidal Map

Outline:

I. Randomized incremental construction

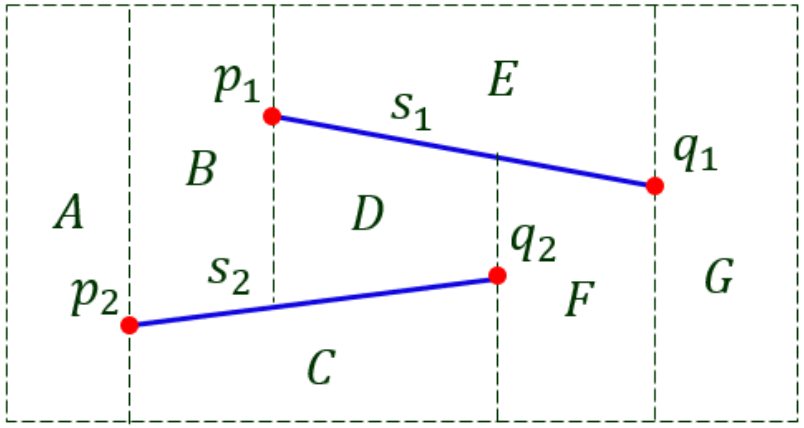
II. Trapezoidal map modification

III. Search structure update

IV. Analysis of expected query time

Search Structure

Trapezoidal map $T(S)$



Doubly-connected edge list (DCEL)

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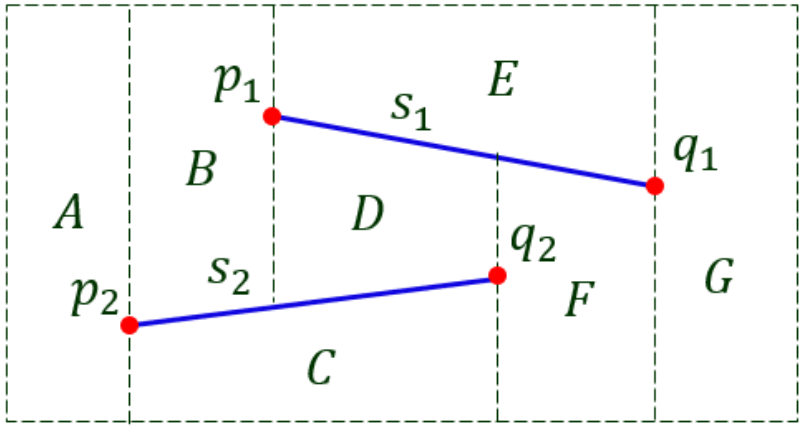

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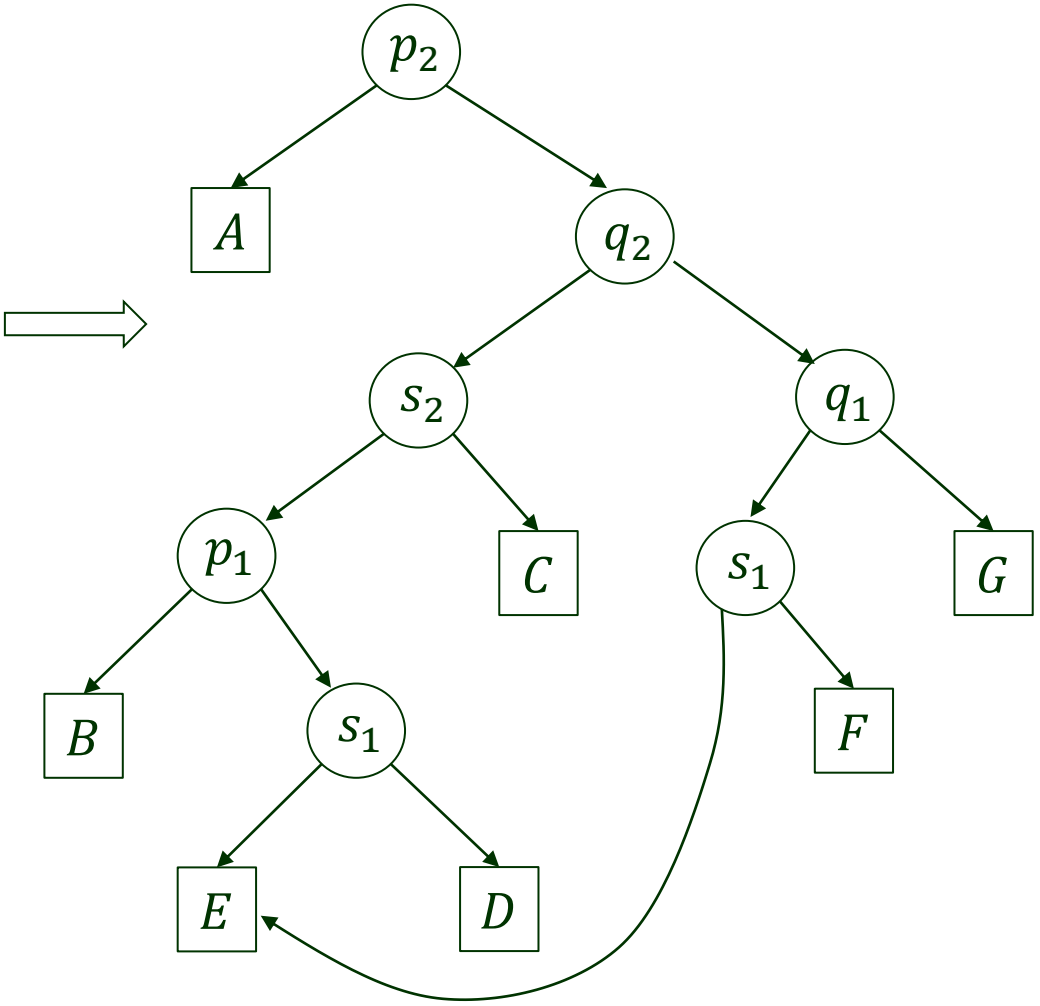
Search structure D



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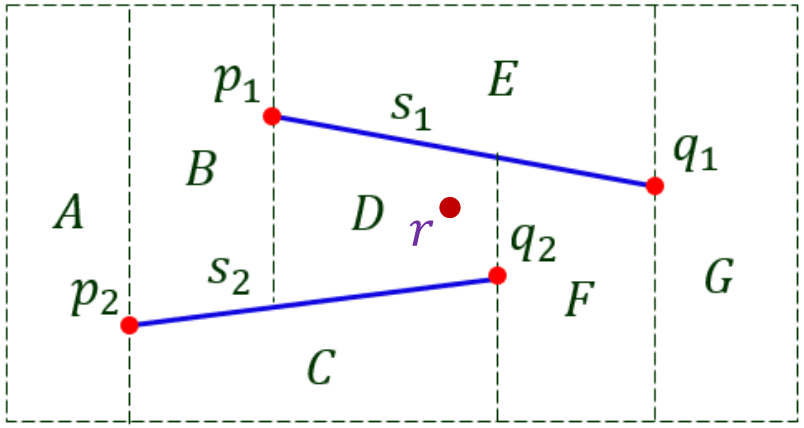

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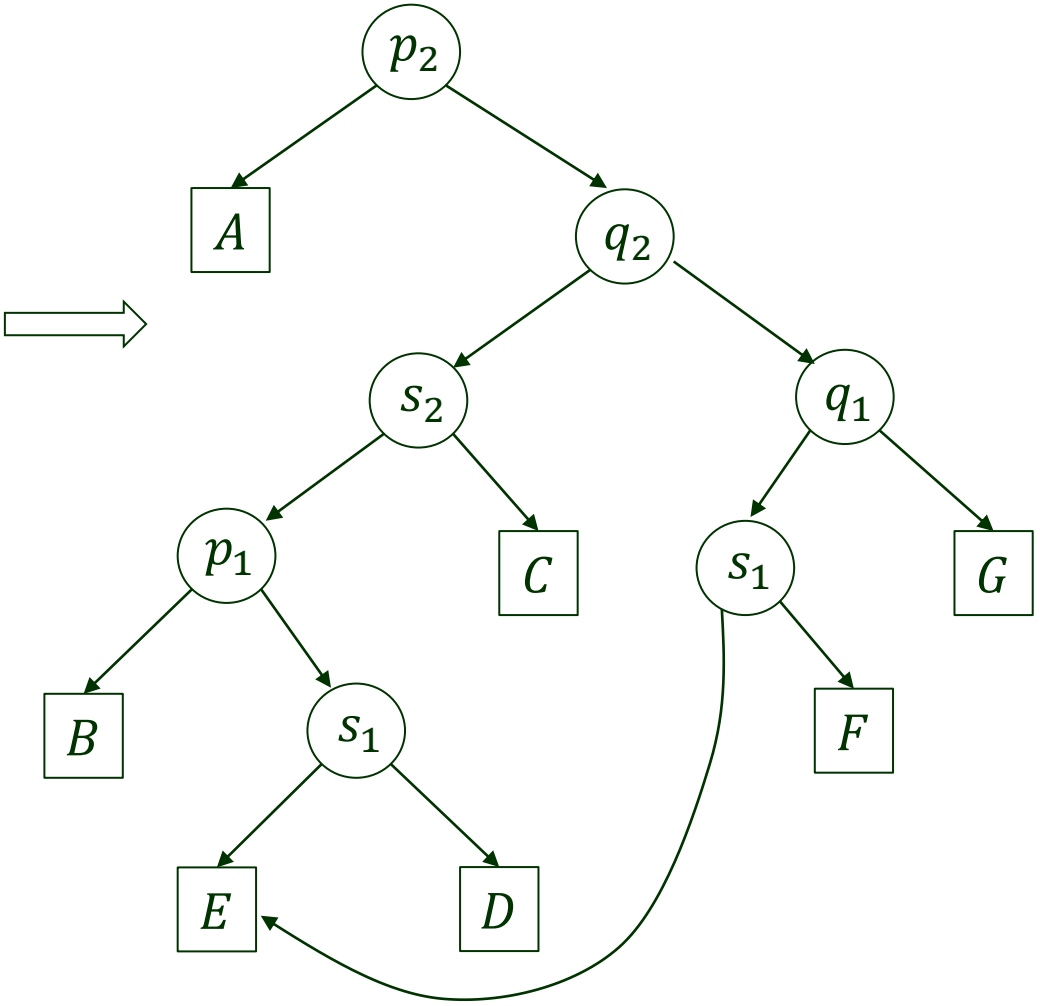

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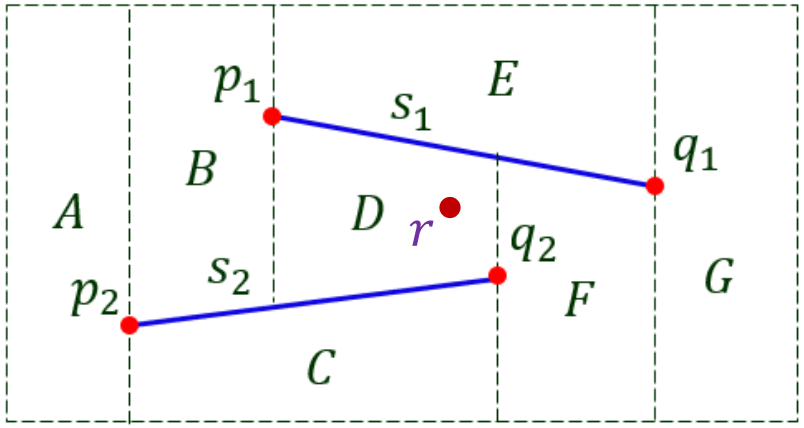

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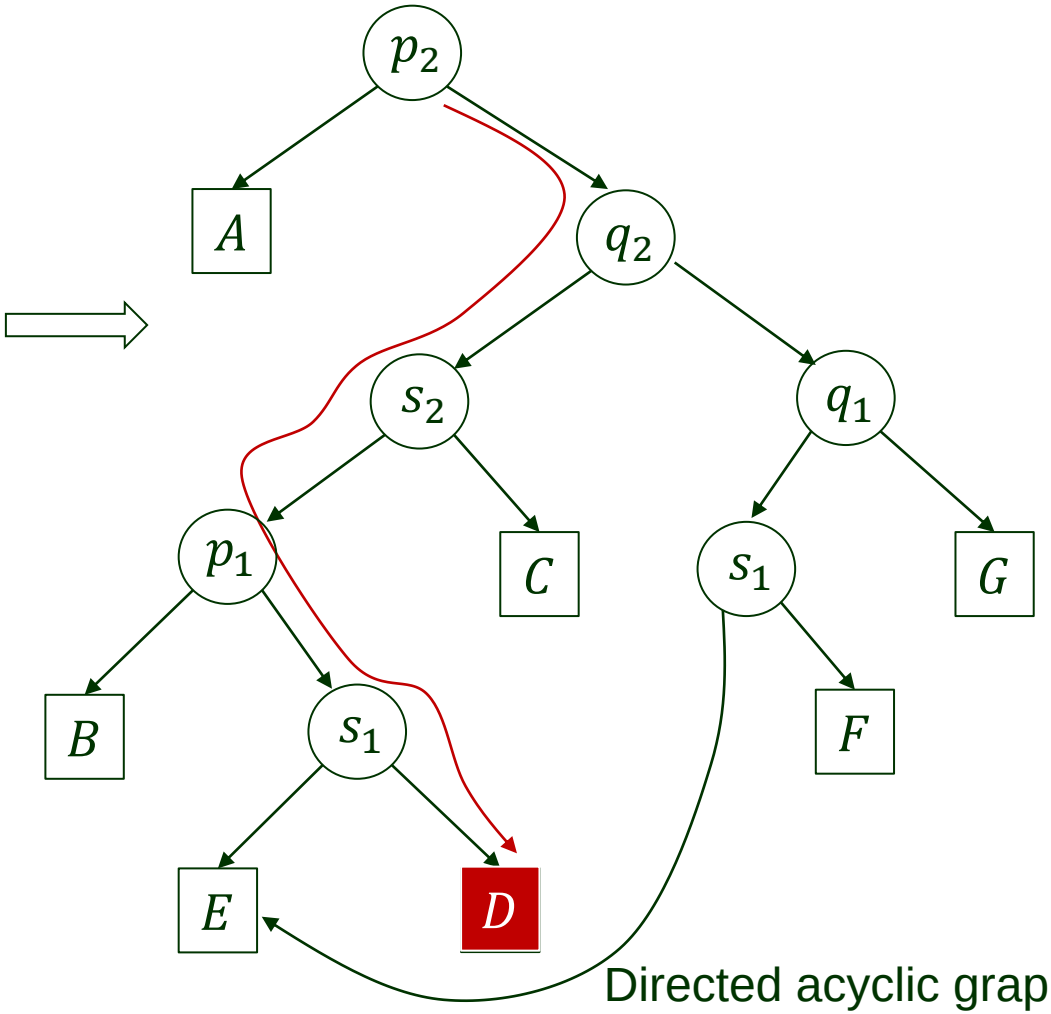
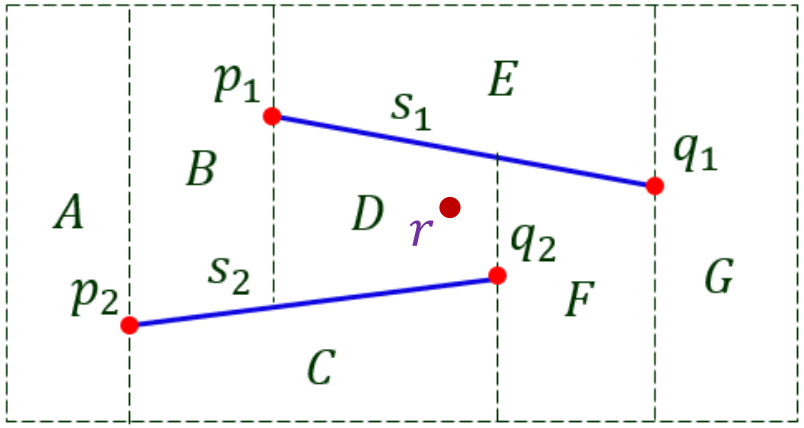


Directed acyclic graph (DAG)

Search Structure

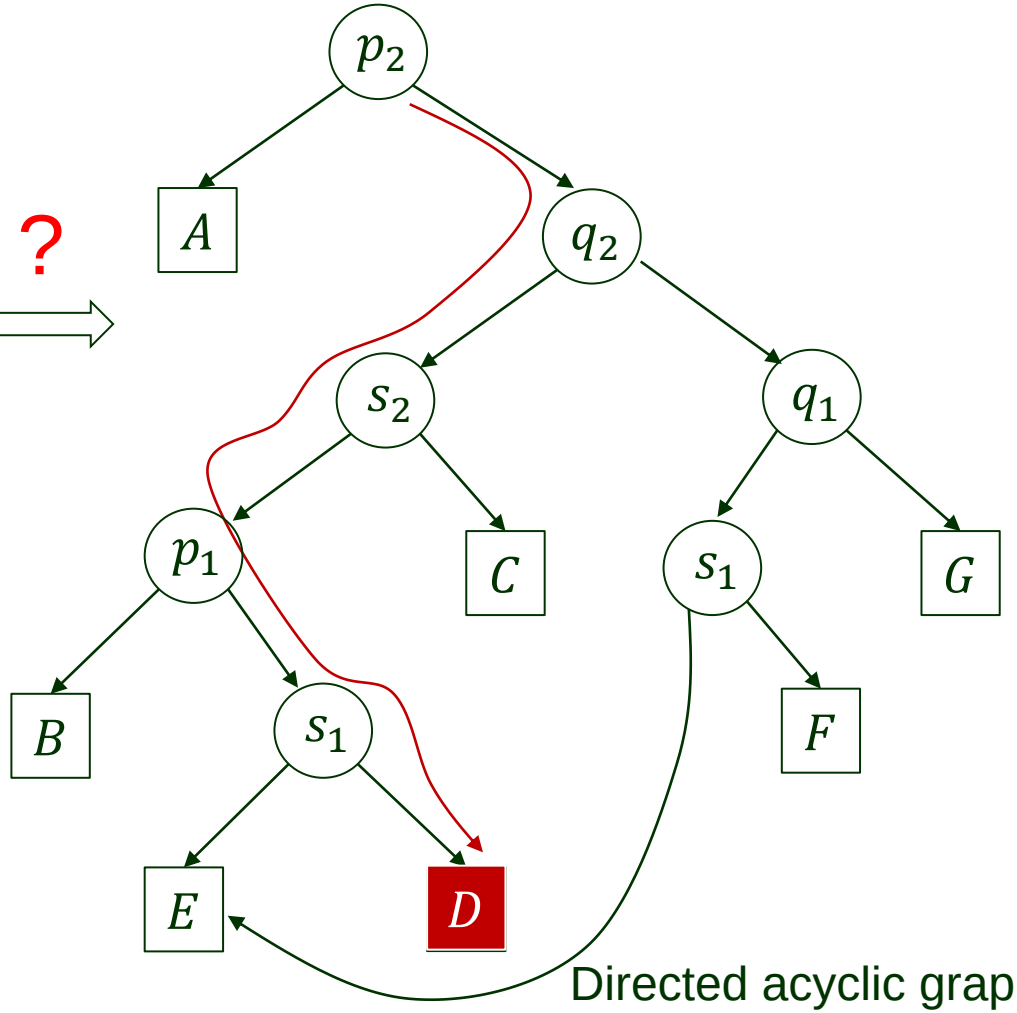
Trapezoidal map $T(S)$



Doubly-connected edge list (DCEL)

?

Search structure D



Directed acyclic graph (DAG)

I. Randomized Incremental Construction of D

- Randomly order segments: $s_1, s_2, \ldots, s_n$.
- Add one segment at a time.

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```
for i ← 1 to n
    T(S_i) → T(S_{i+1})
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for $i \leftarrow 1$ to n
  $T(S_i) \rightarrow T(S_{i+1})$

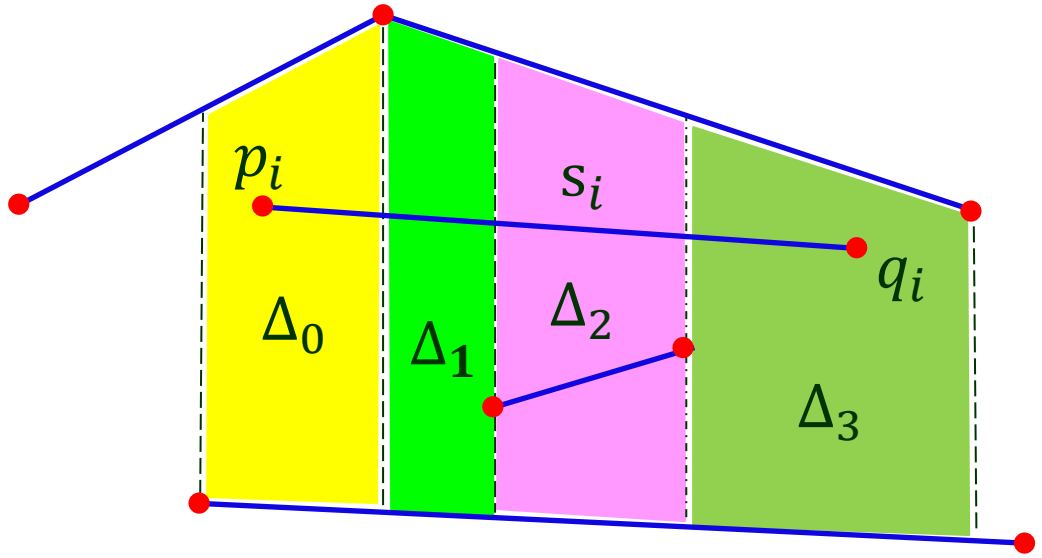
Loop invariants: a) T is the trapezoidal map for S_i;
b) D is a search structure for T.

Algorithm

1. Determine the bounding box R.
2. Initialize the trapezoidal map $T(S)$.
3. Initialize the search structure D.
4. Compute a random permutation $s_1, s_2, \ldots, s_n$ of segments.
5. for $i \leftarrow 1$ to n
6. do Find trapezoids $\Delta_0, \Delta_1, \ldots, \Delta_k$ properly intersected by s_i.
7. Replace with new trapezoids due to intersection by s_i.
8. Remove leaves for $\Delta_0, \Delta_1, \ldots, \Delta_k$ from D.
9. Create the leaves for the new trapezoids.

II. Trapezoidal Map Modification

How to add s_i?



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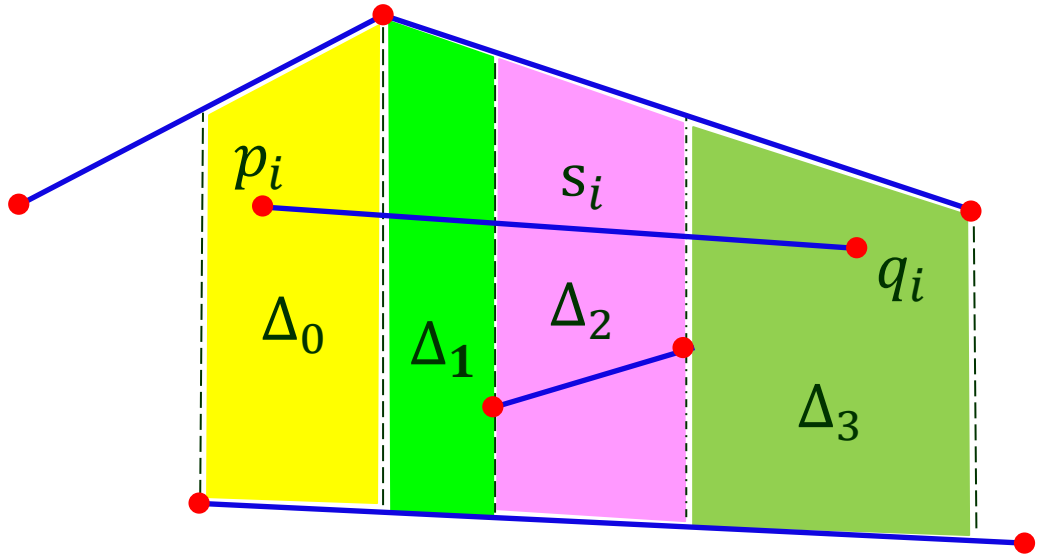


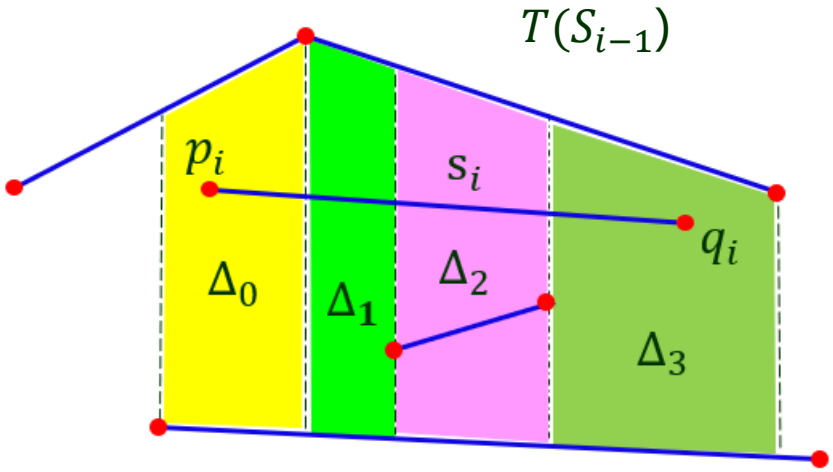
For every trapezoid $\Delta \in T(S_{i-1})$

$\Delta \notin T(S_i)$ iff Δ is intersected by s_i.

Intersected Trapezoids

Find trapezoids $\Delta_0, \Delta_1, \ldots, \Delta_k$ intersected by s_i

- Locate Δ_0.



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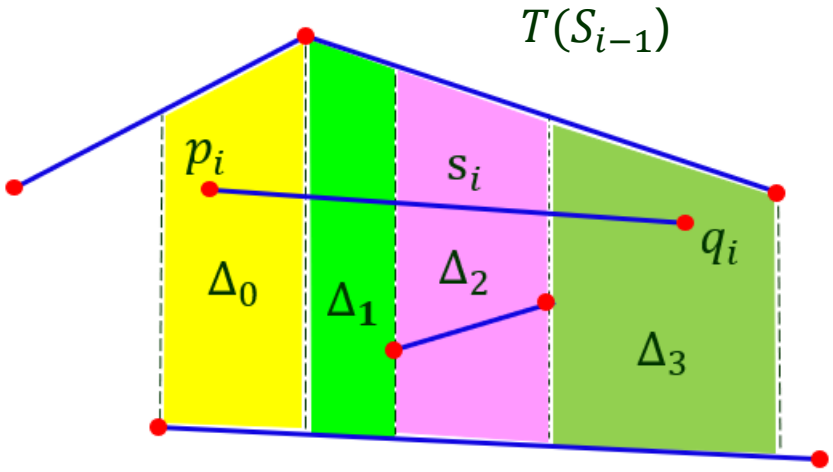

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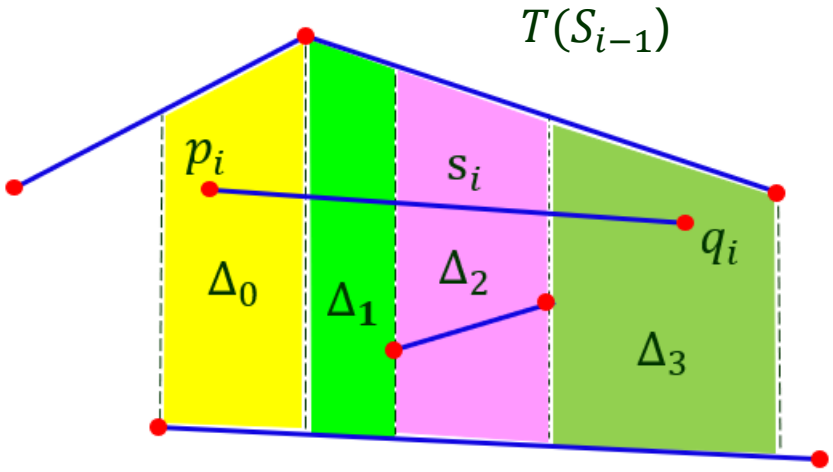
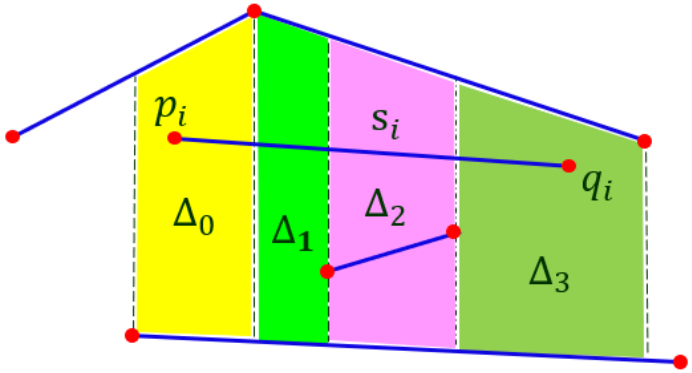
  Use the current search structure $D(S_{i-1})$ to perform query with p_i.



The Right Neighbor

- $\Delta_0 \Rightarrow$ locate $\Delta_1 \Rightarrow \cdots \Rightarrow$ locate Δ_k



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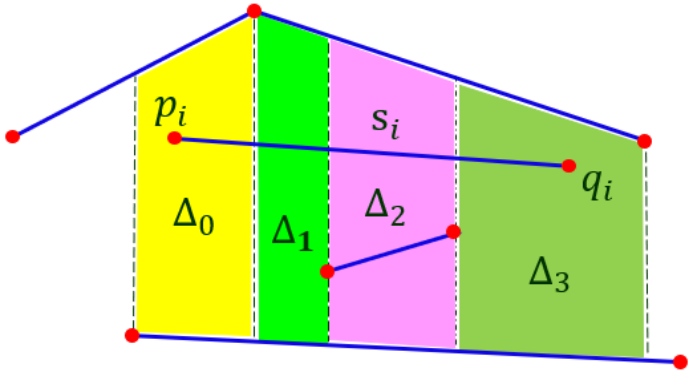

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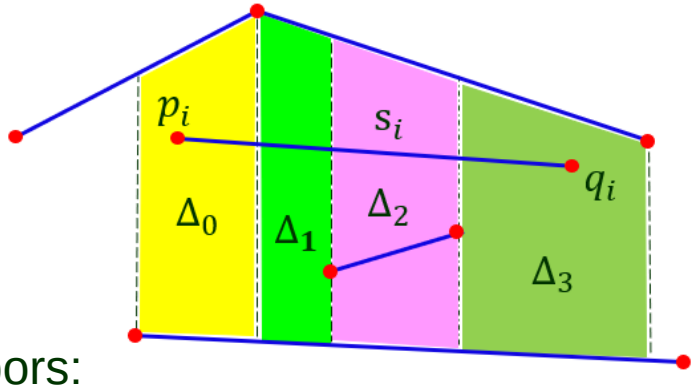

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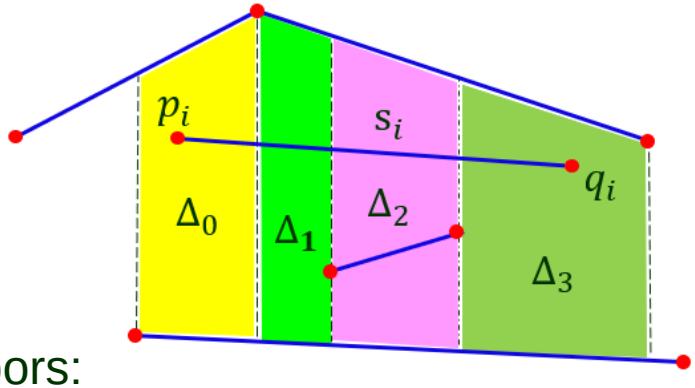

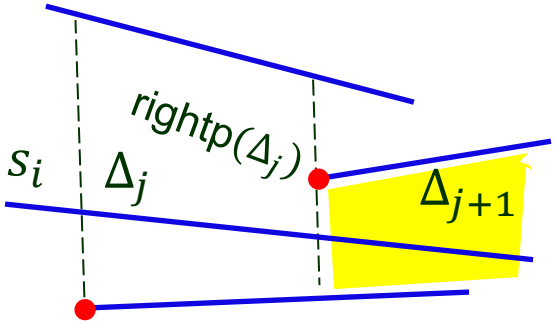

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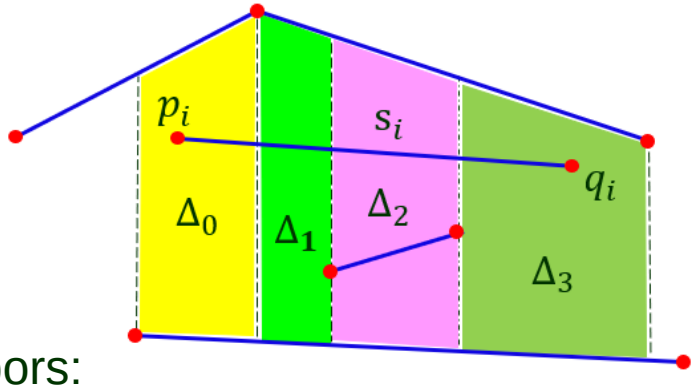

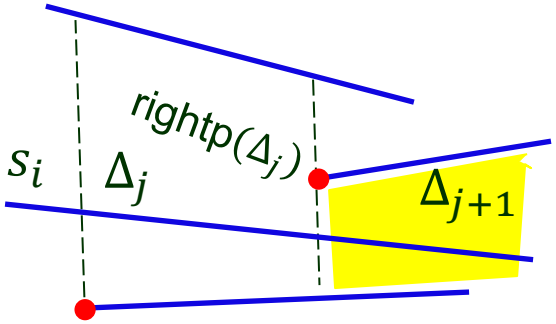

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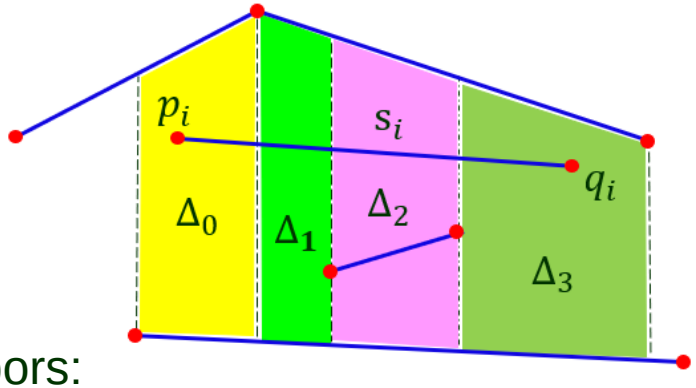

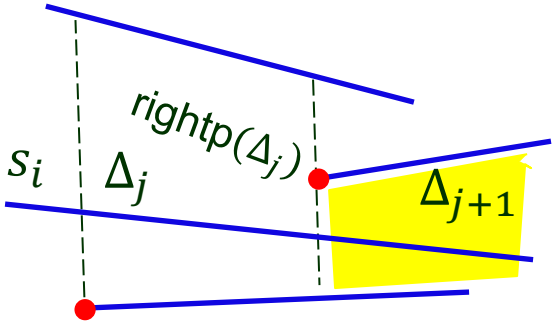

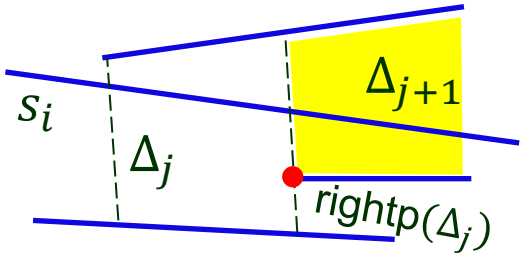

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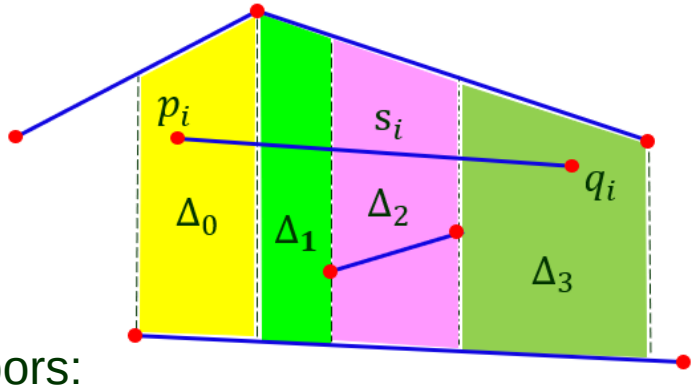
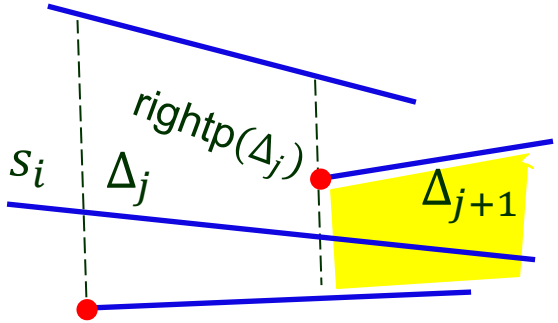
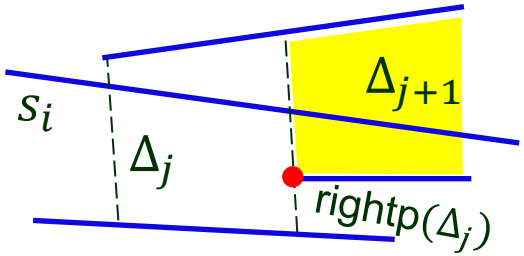
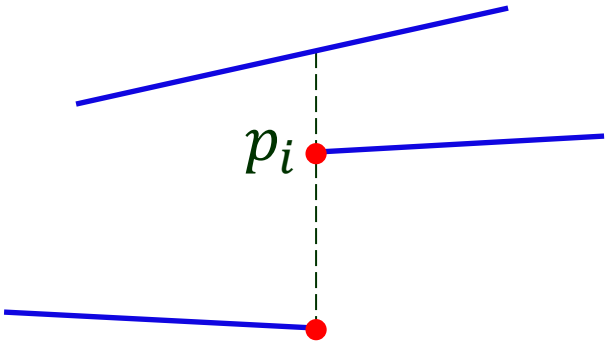




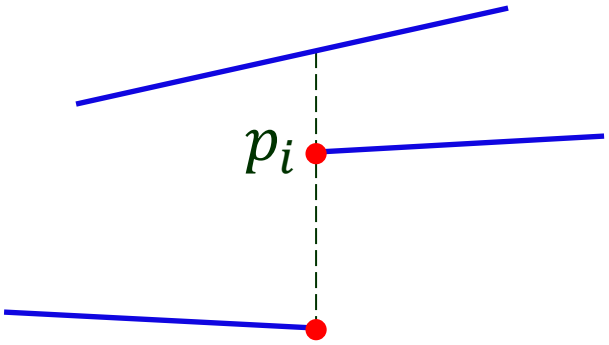


Two Degeneracies



- p_i lies on the vertical line of an x-node in $T(S_{i-1})$.

Two Degeneracies



- p_i lies on the vertical line of an x-node in $T(S_{i-1})$.

Treat it as lying "to the right" of the x-node.

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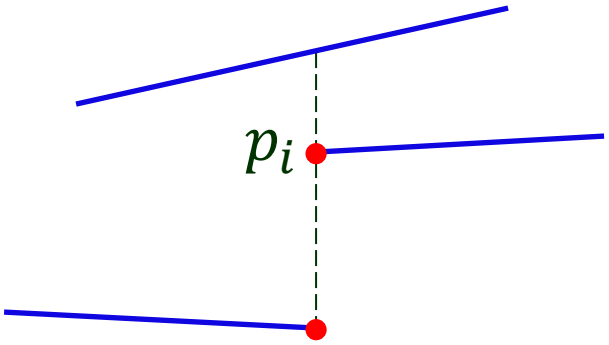

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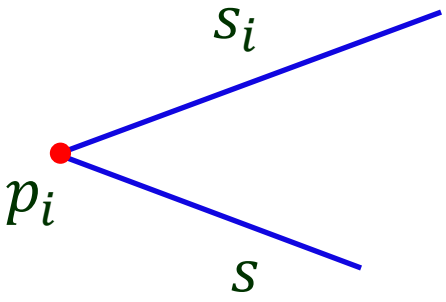

- p_i is the left endpoint of a segment s of a y-node.

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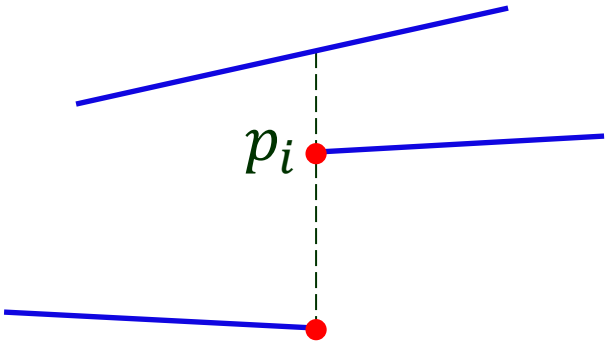

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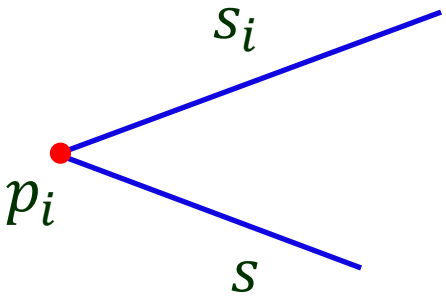


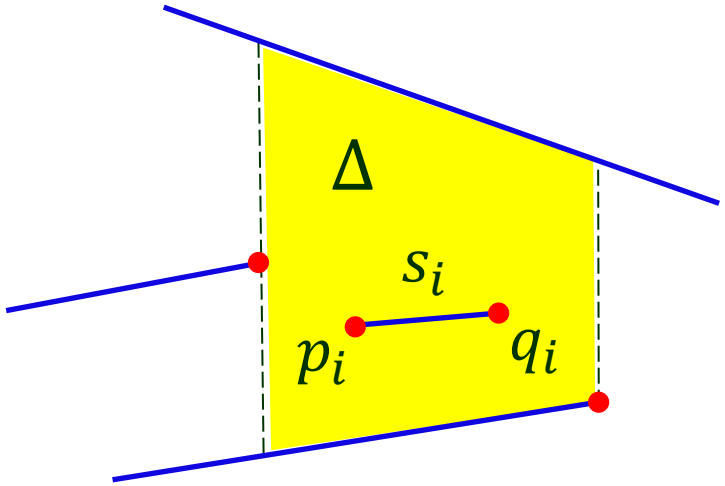
- p_i is the left endpoint of a segment s of a y-node.

Compare the slopes of s and s_i to determine "above" or "below".

III. Updating T and D (Simple Case)

s_i contained in Δ_0

- Non-degenerate situation: 4 new trapezoids.



$T(S_{i-1})$

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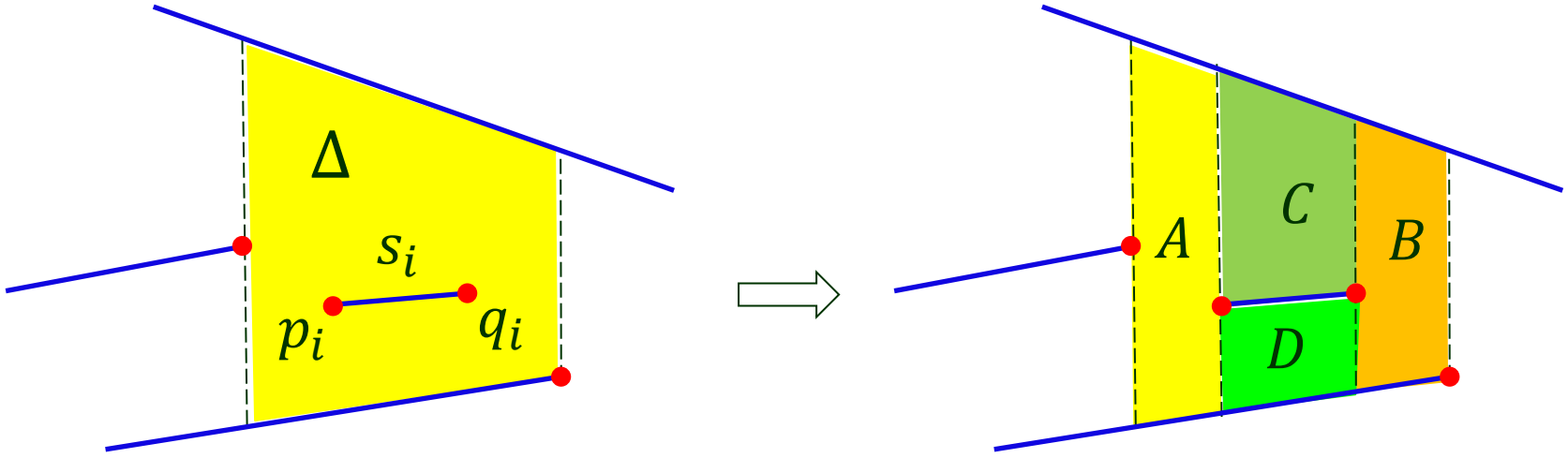
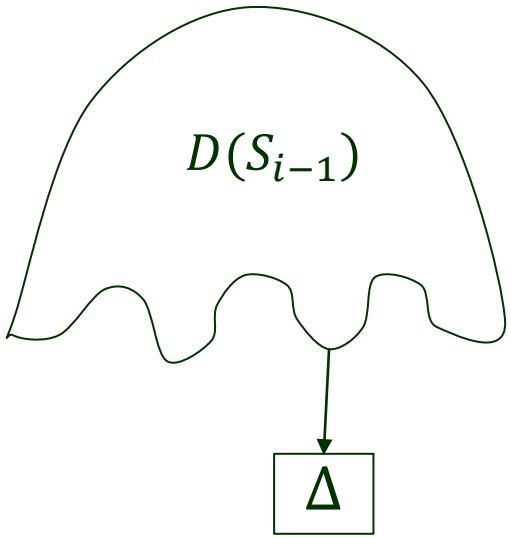
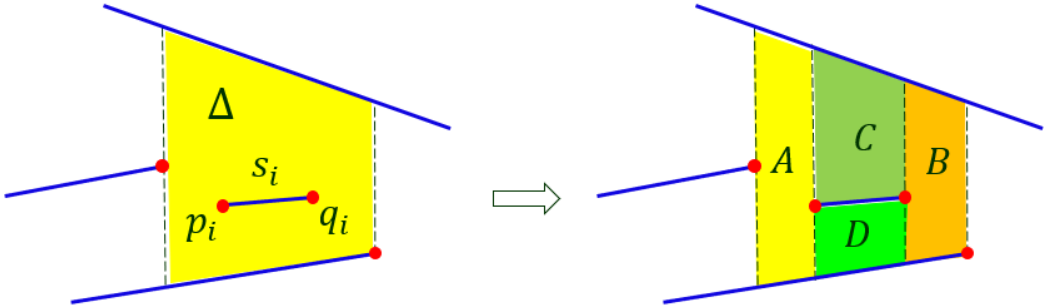


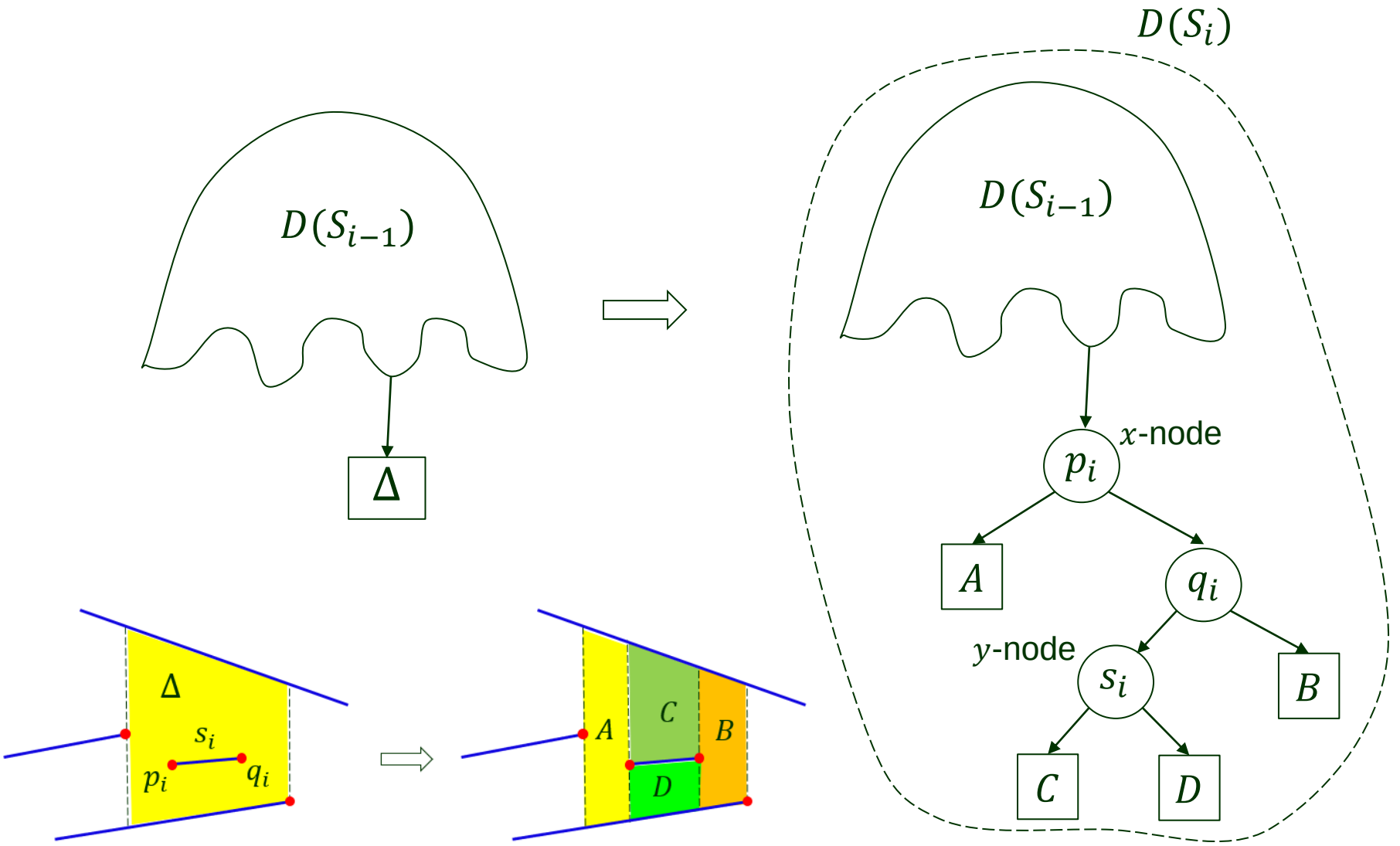
$T(S_{i-1})$ $T(S_i)$

Simple Update (cont'd)

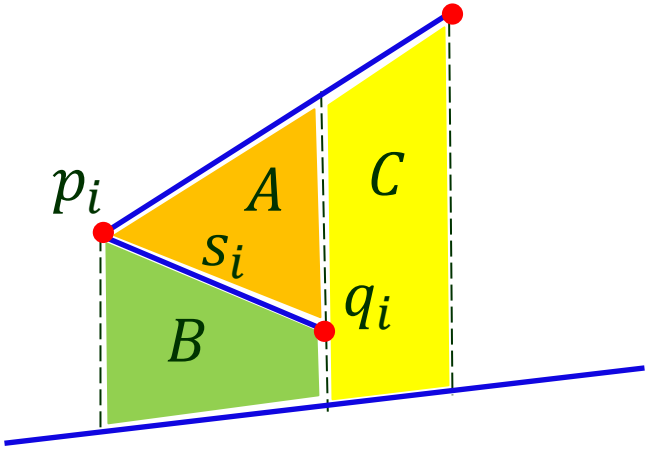




Simple Update (cont'd)



Degenerate Situations



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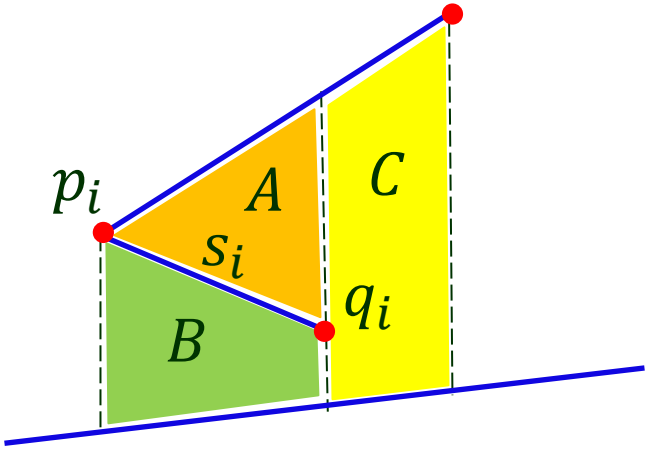

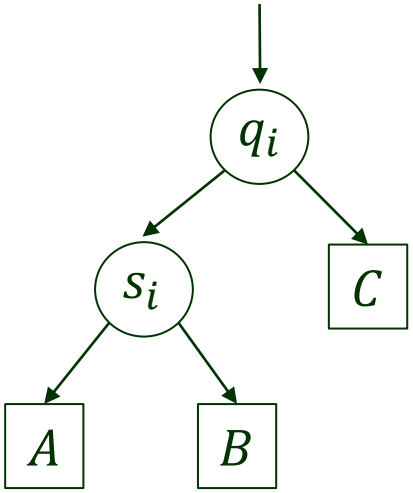

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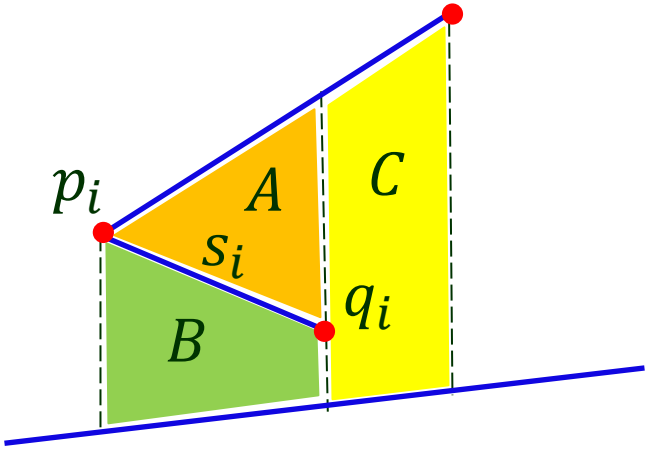

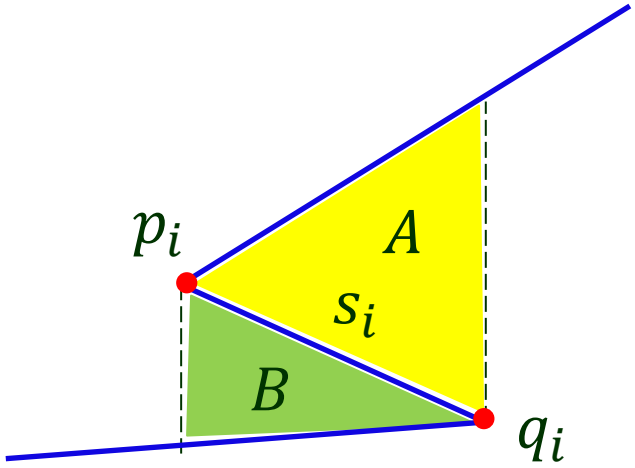

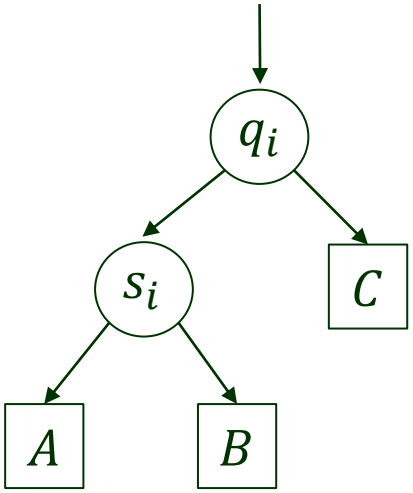

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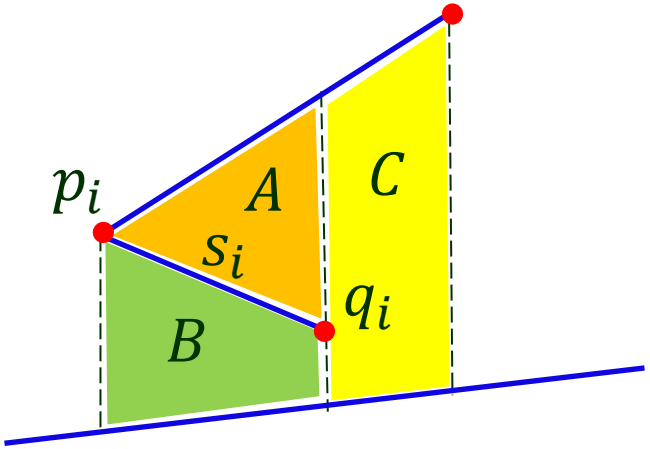
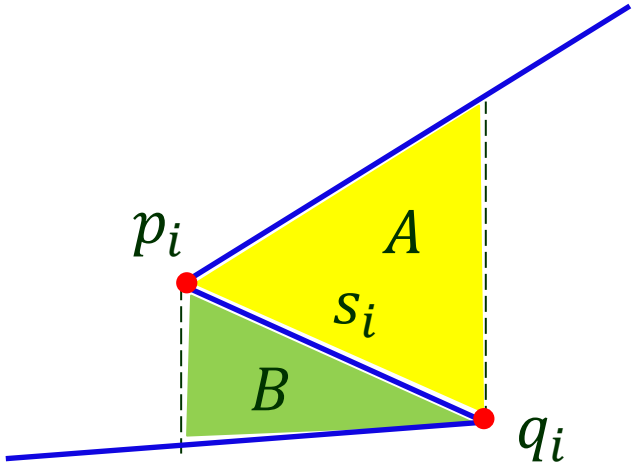
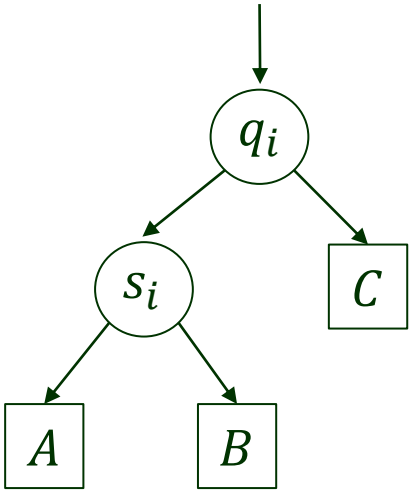
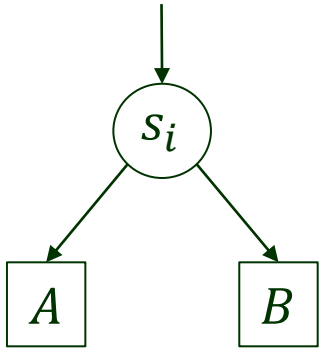


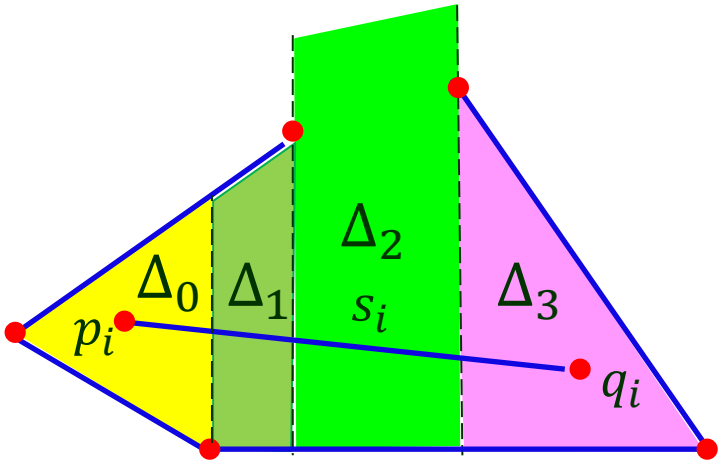






Updating T and D (General Case)

s_i intersects ≥ 2 trapezoids $\Delta_0, \Delta_1, \ldots, \Delta_k$.



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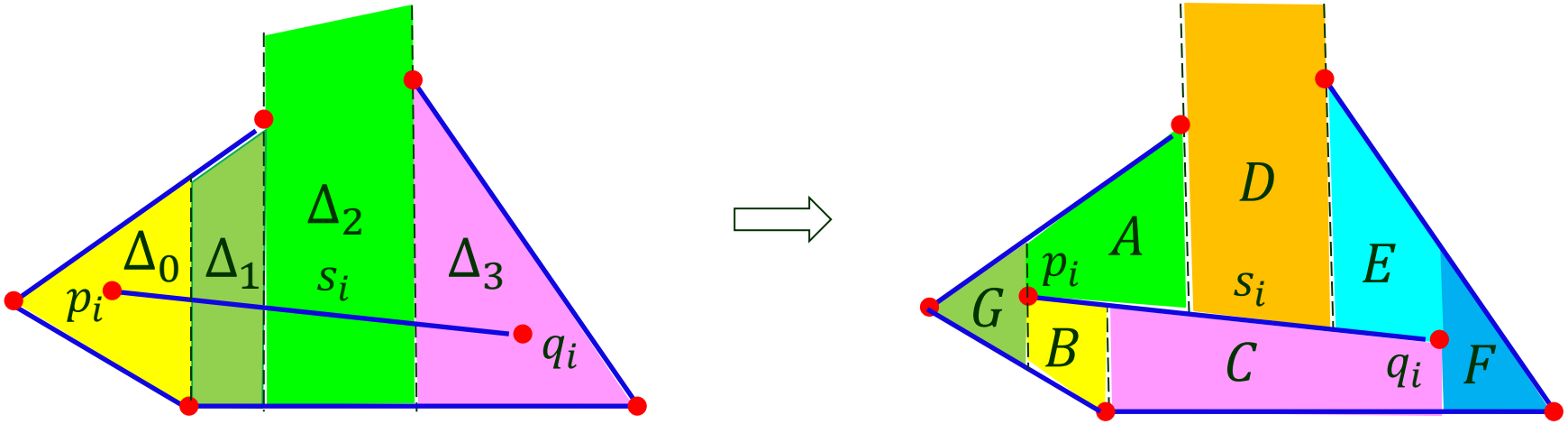
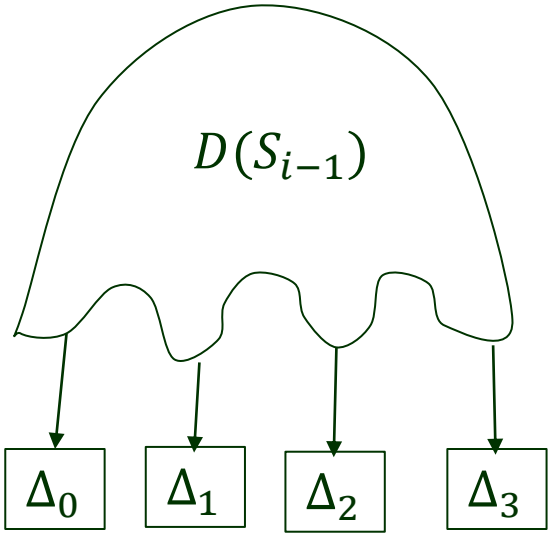
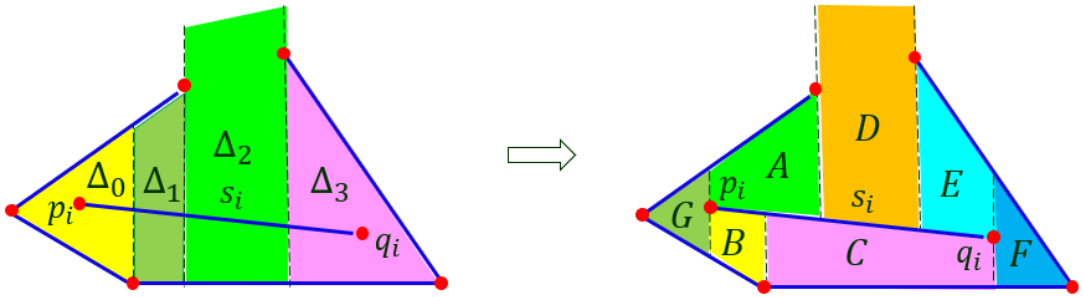


General Update (cont'd)





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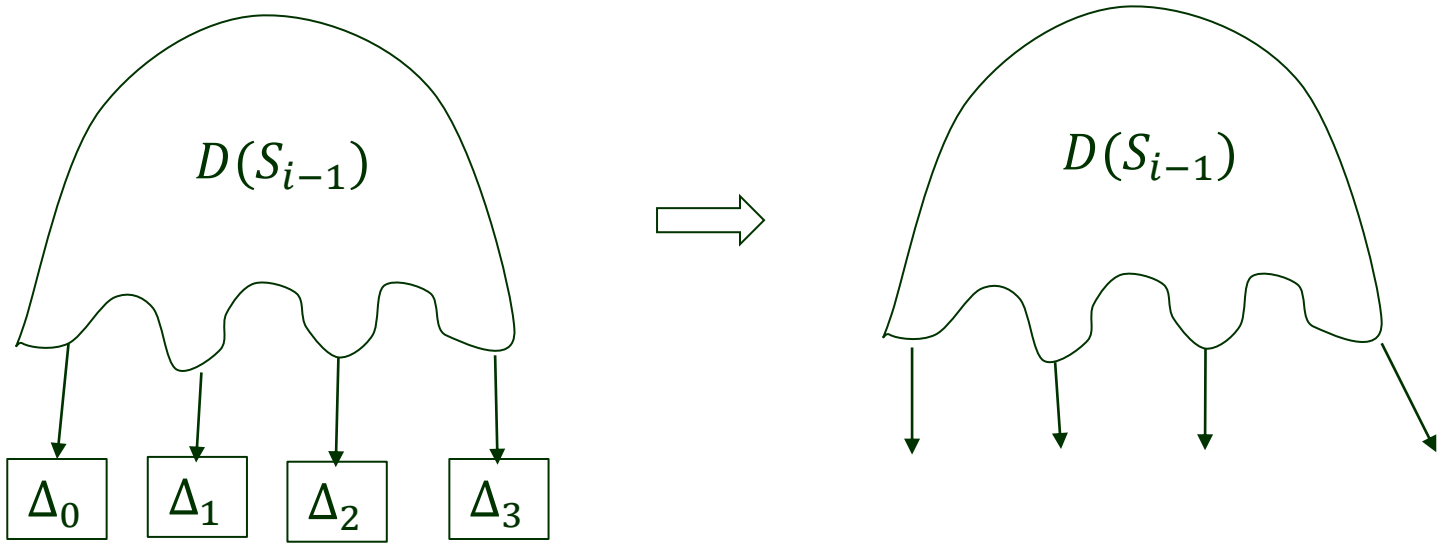

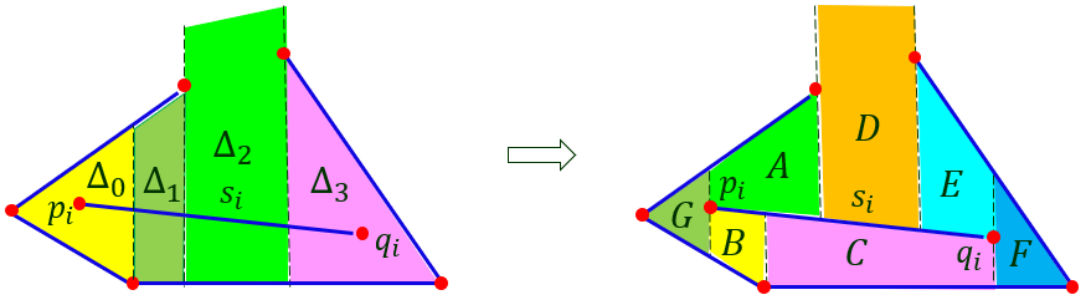

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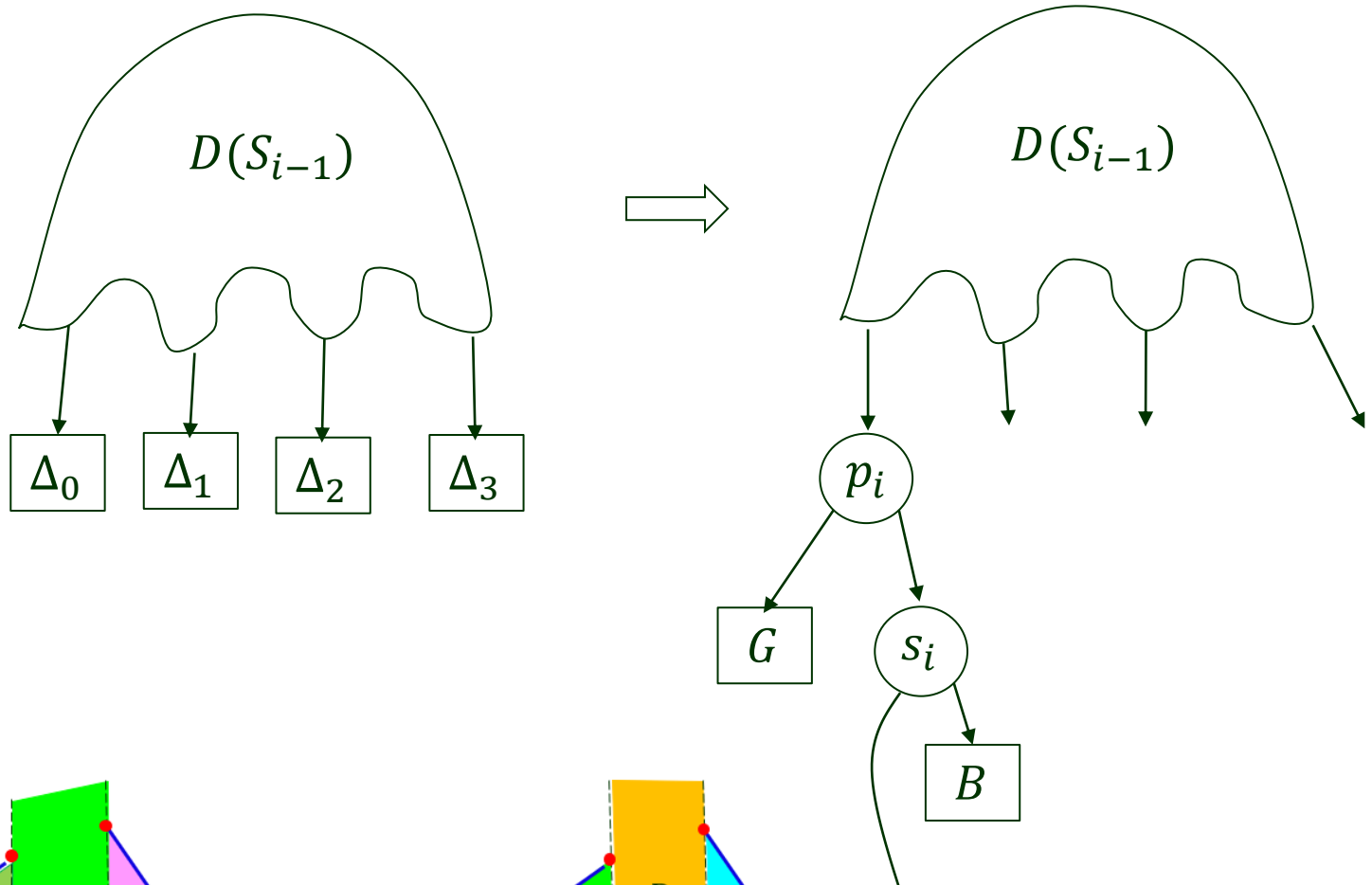

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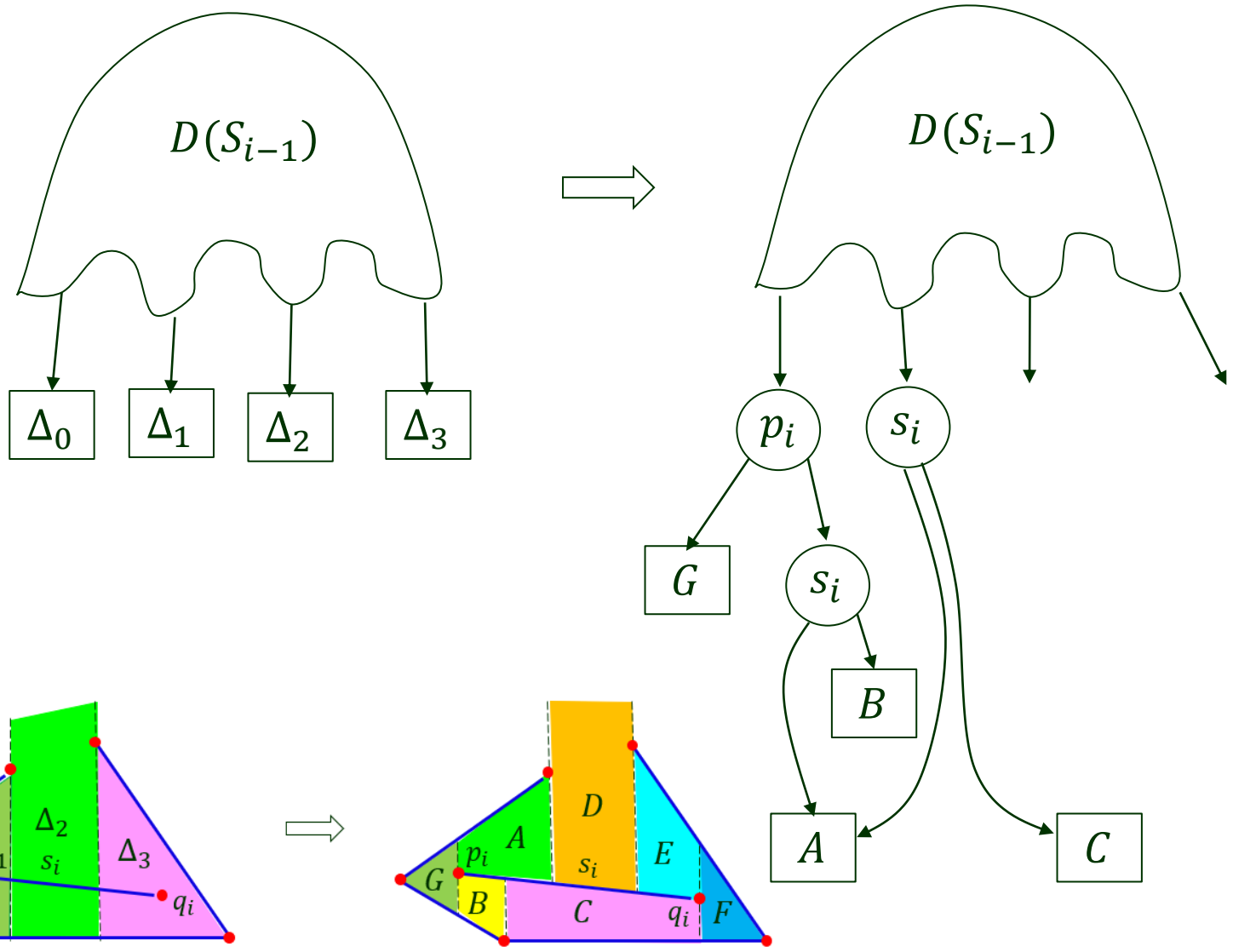

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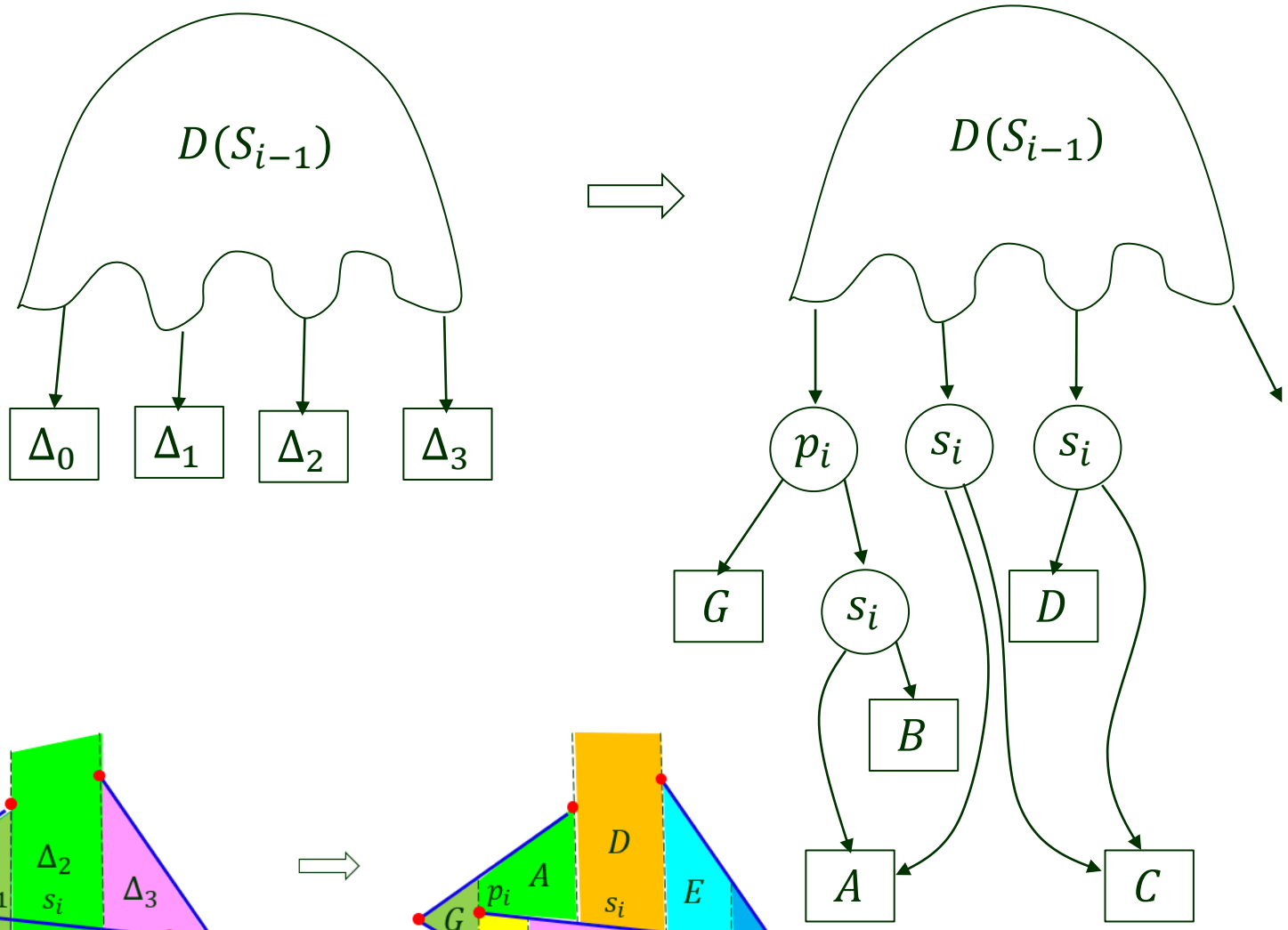

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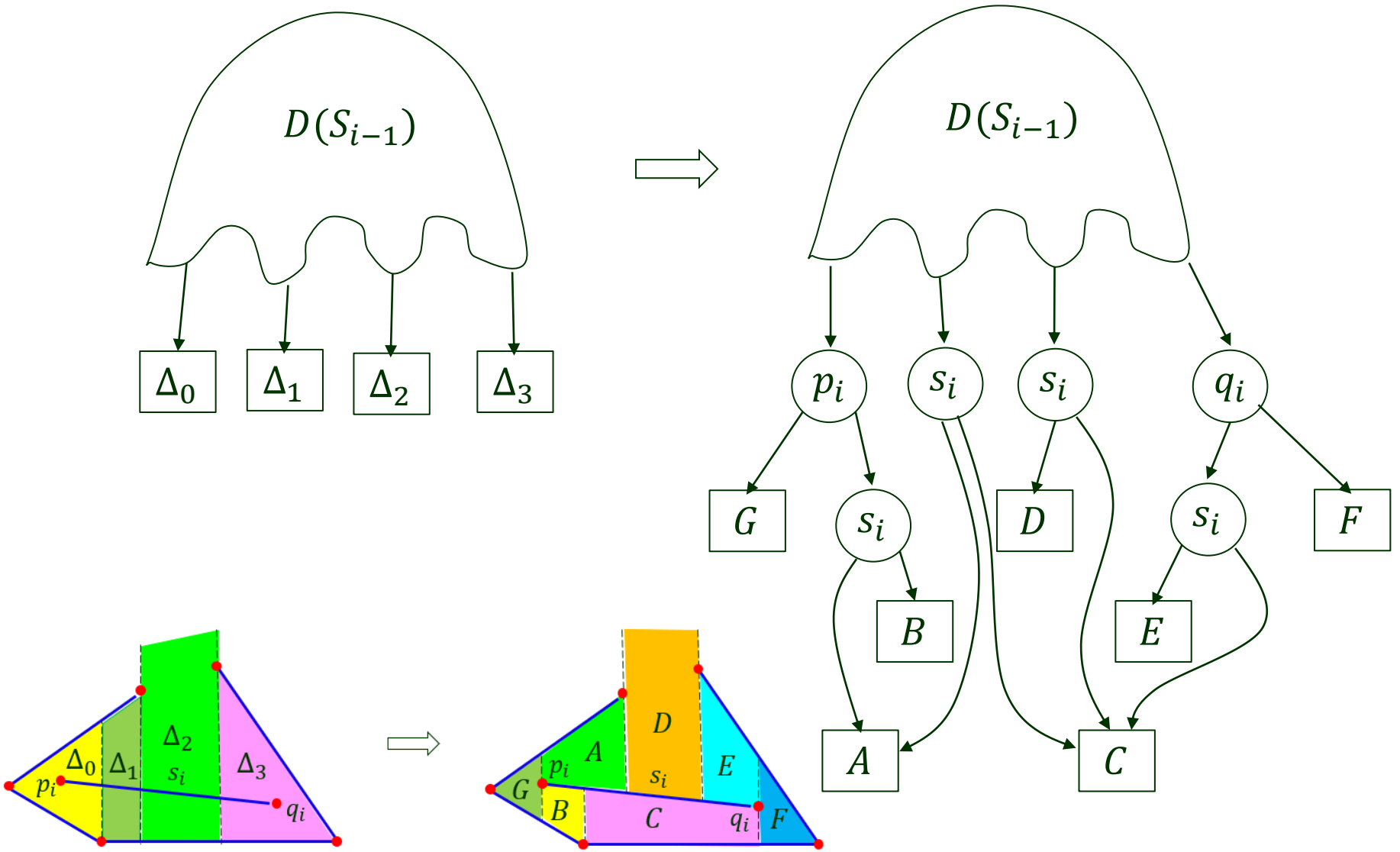
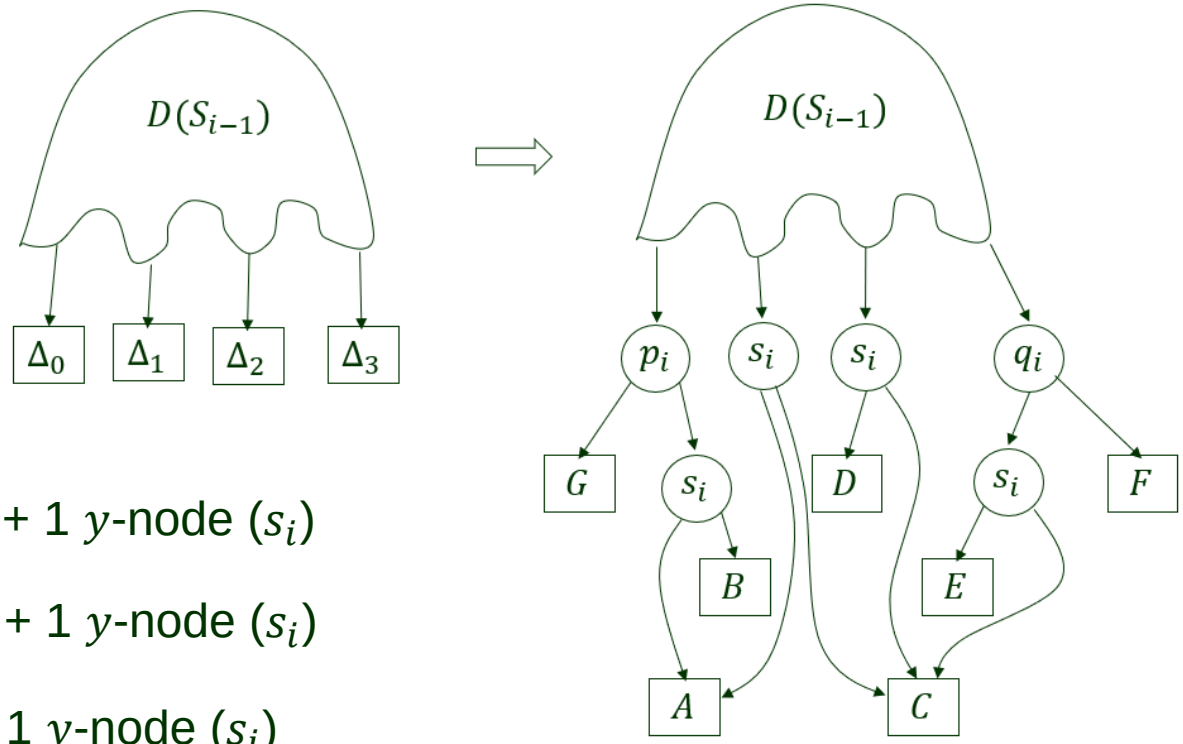


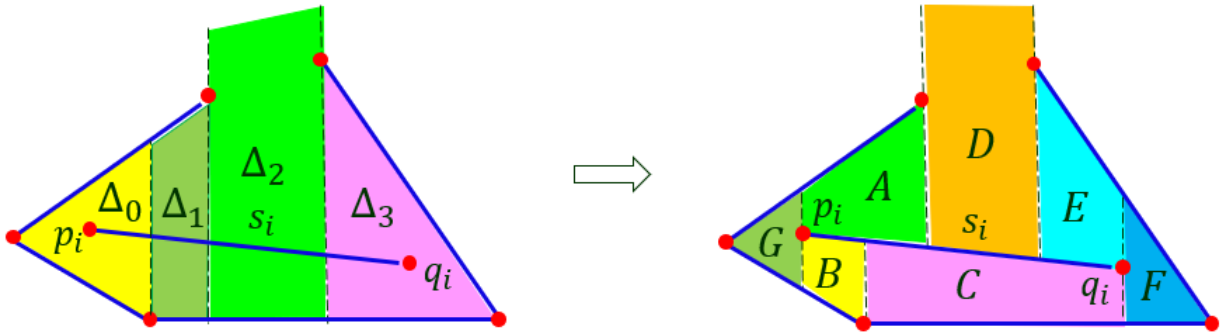
Search Structure Update



Δ_0: replaced with 1 x-node (p_i) + 1 y-node (s_i)

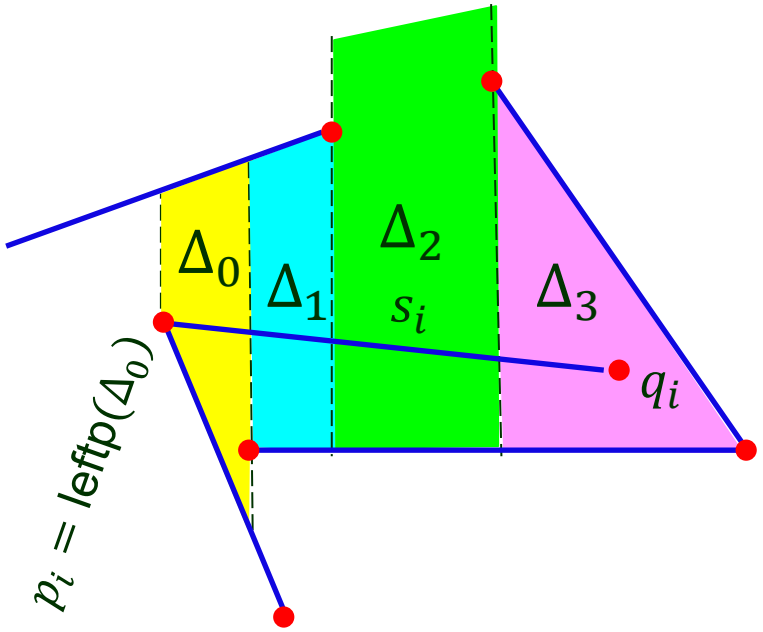
Δ_k: replaced with 1 x-node (q_i) + 1 y-node (s_i)

$\Delta_j, 1 \leq j \leq k-1$: replaced with 1 y-node (s_i)



Degeneracies

- p_i coincides with leftp(Δ_0).



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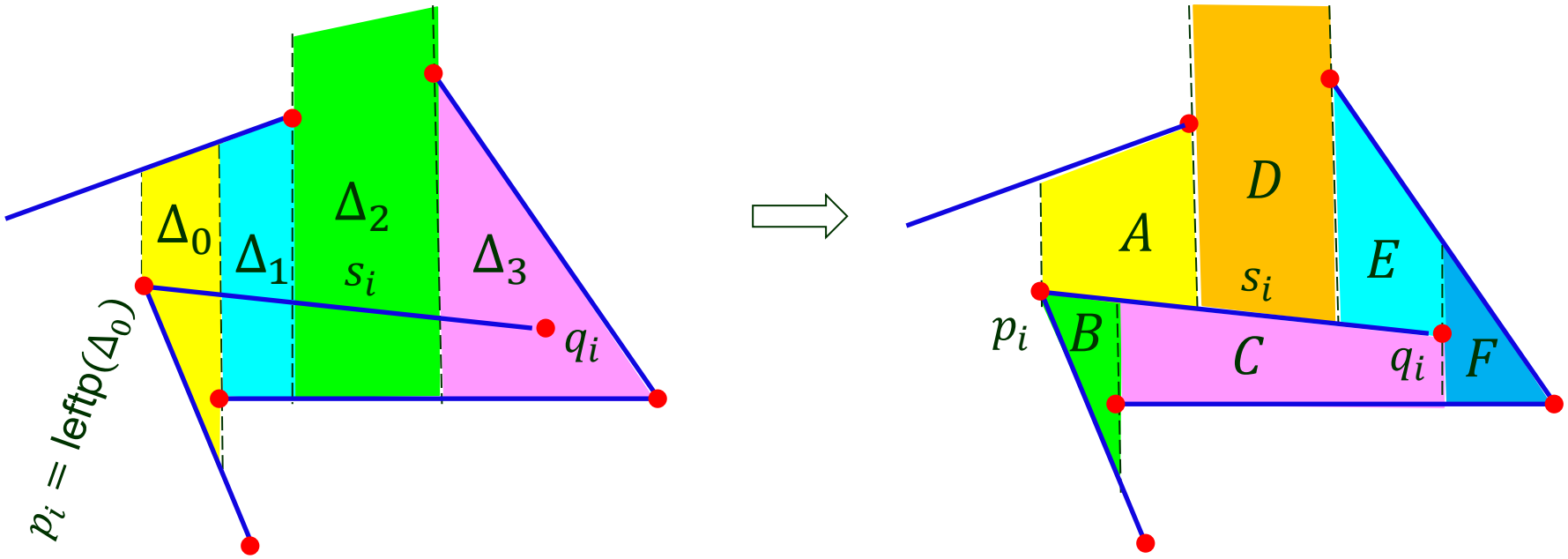

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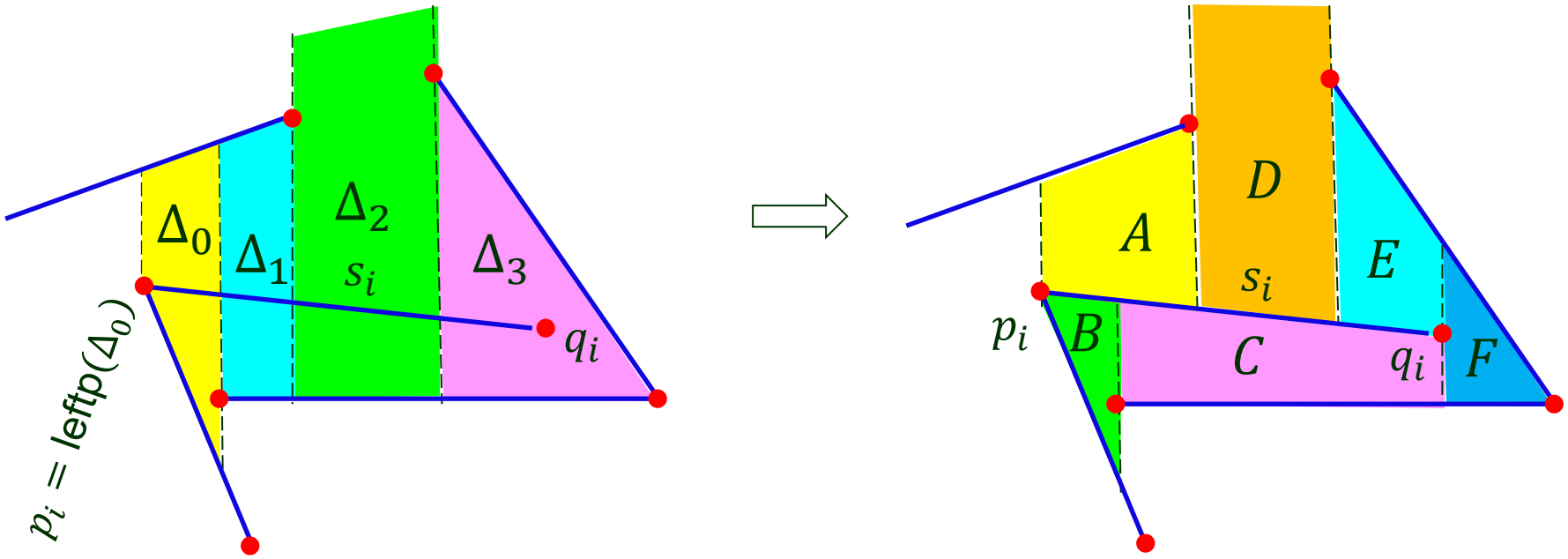

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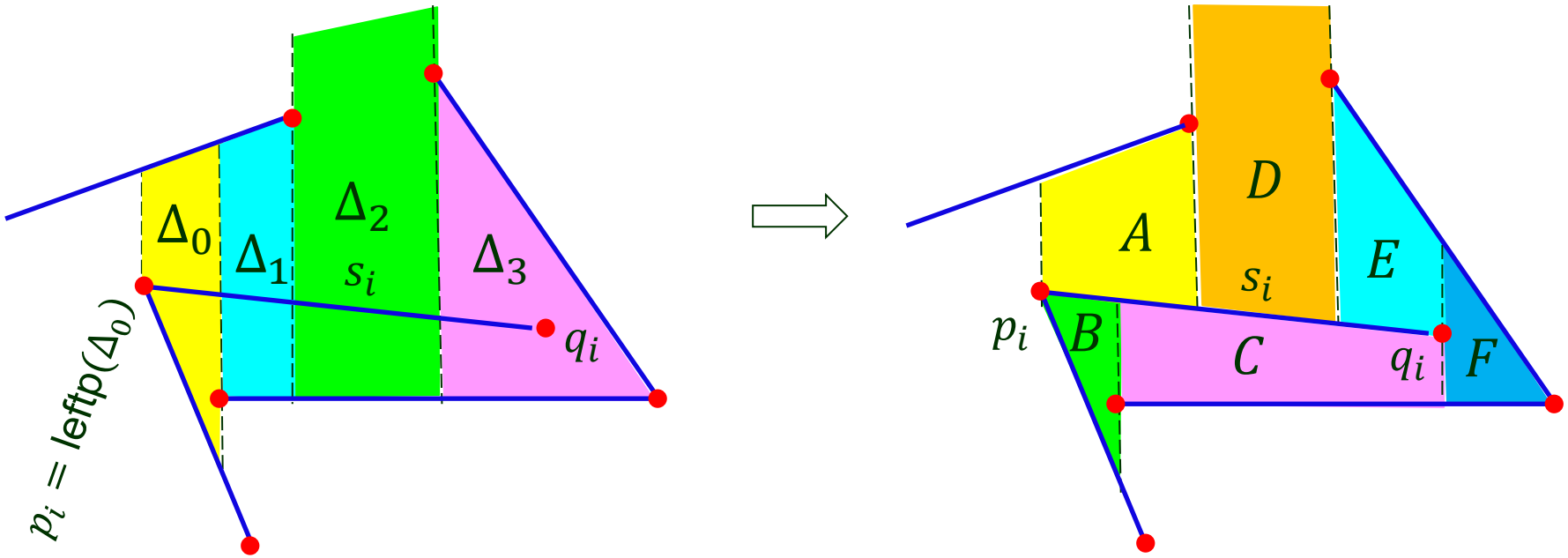

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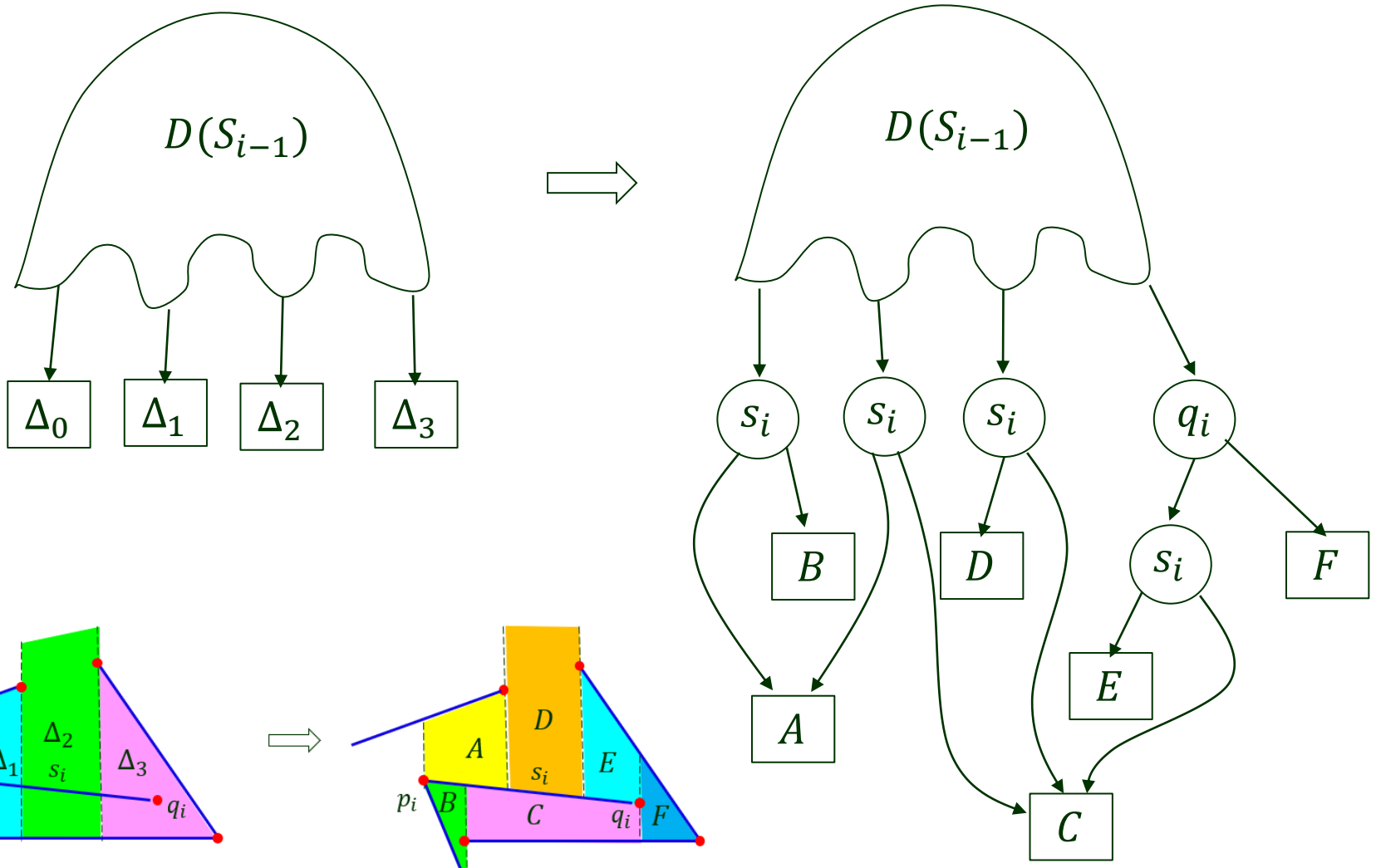
♣ q_i coincides with rightp(Δ_k) – similarly handled.

Degeneracies (cont'd)



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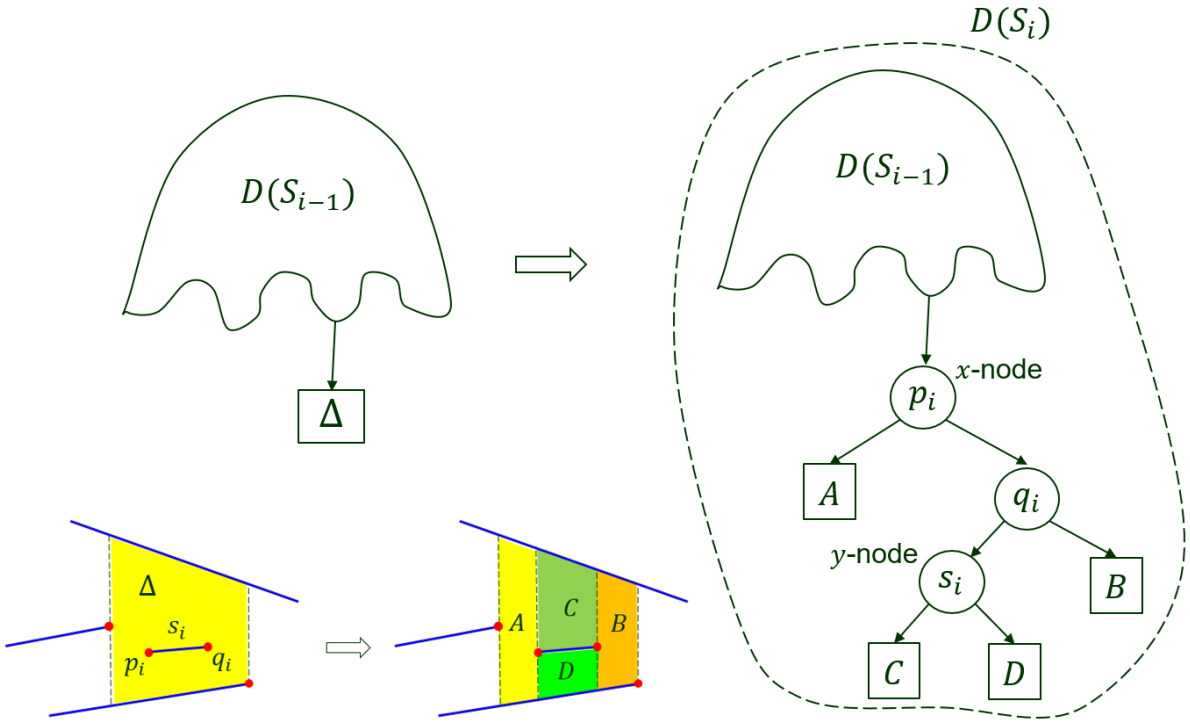
Expected path length: $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$

Nodes Added in One Iteration

Fact Any iteration adds ≤ 3 nodes to the search path γ.



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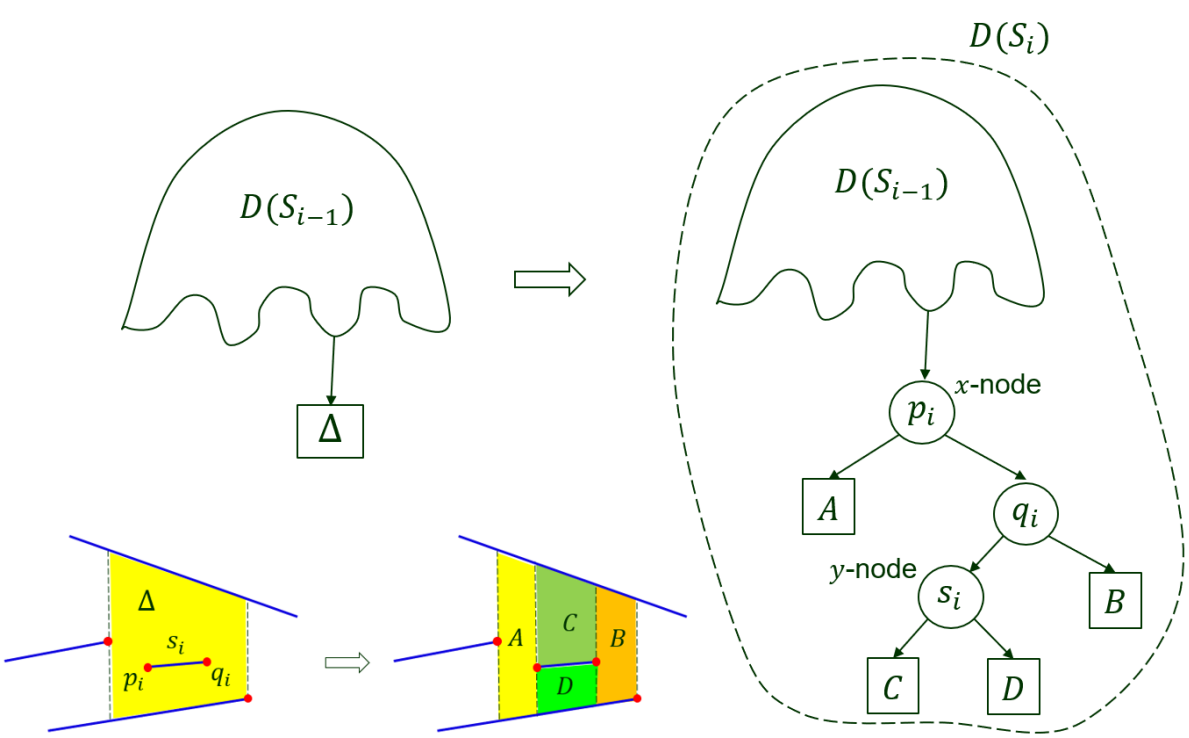
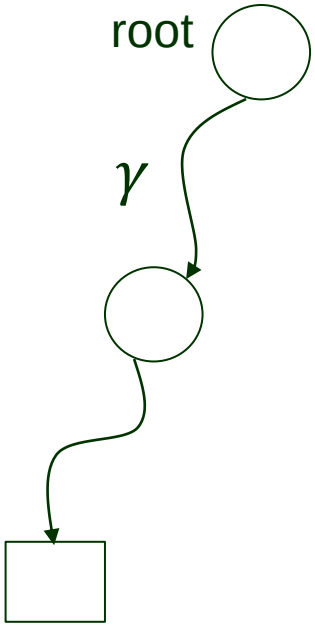


$$X_i \leq 3$$

The case where 3 nodes are added along a path.

Probability of Node Creation

P_i: probability that there is a node on the search path γ of the query point q which is created in iteration i.



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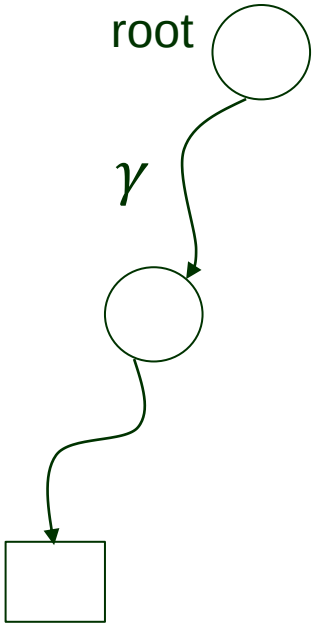

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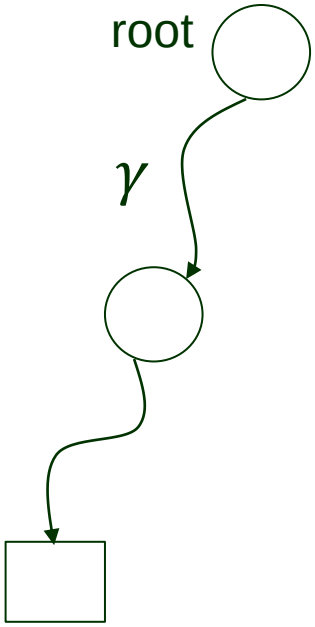

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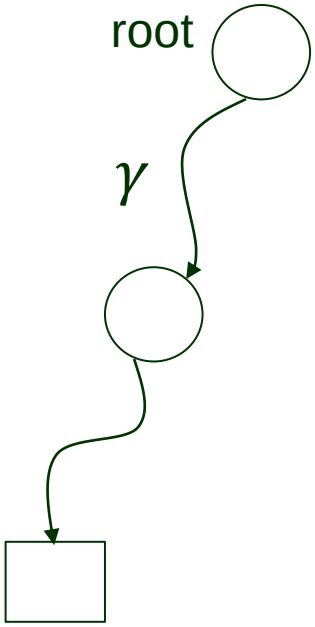

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$$P_i = \Pr(\Delta_q(S_i) \neq \Delta_q(S_{i-1}))$$

Cont'd

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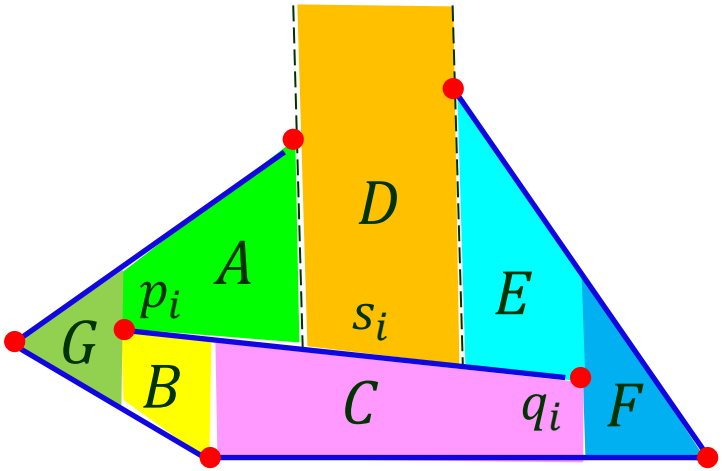

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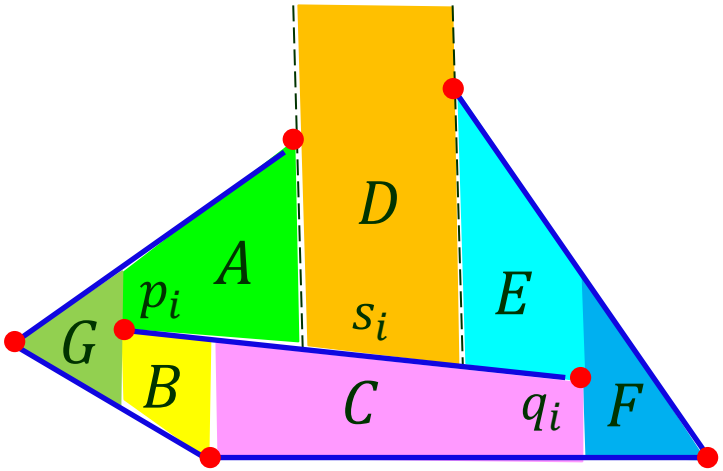

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$\Delta = \Delta_q(S_i)$

$\implies$ either top$(\Delta) = s_i$

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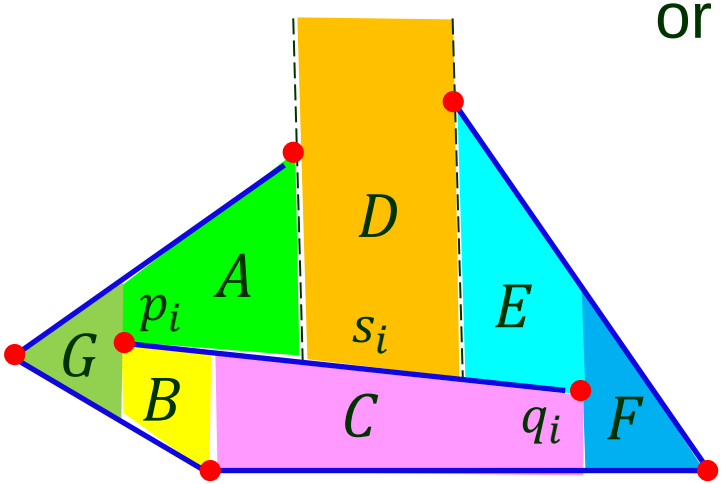

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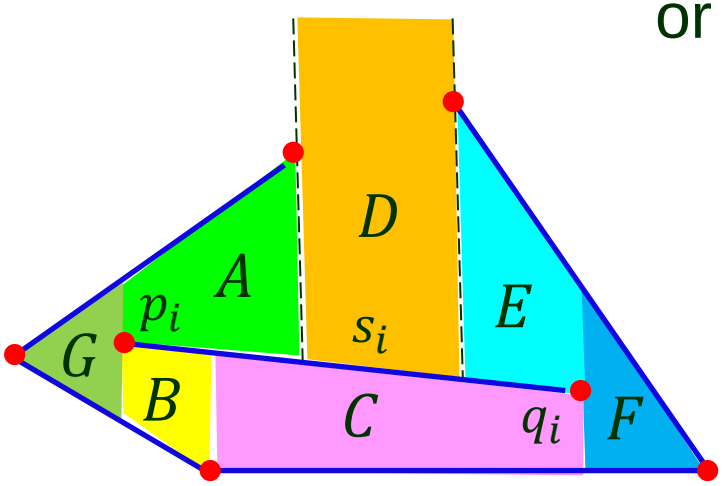

$\Delta_q(S_i)$ as a trapezoid in $T(S_i)$ does not depend on the order of segments in S_i.

Backward Analysis

Idea Look at the probability that $\Delta = \Delta_q(S_i)$ disappears when the segment s_i is removed.

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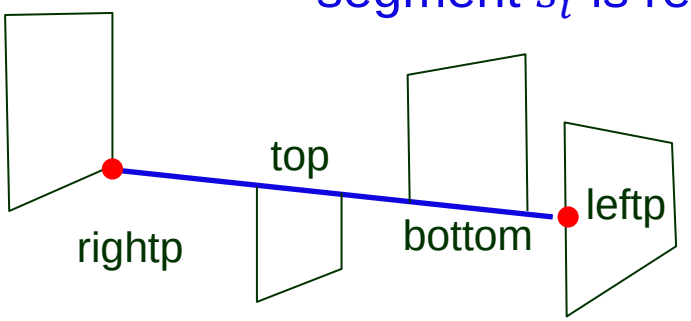

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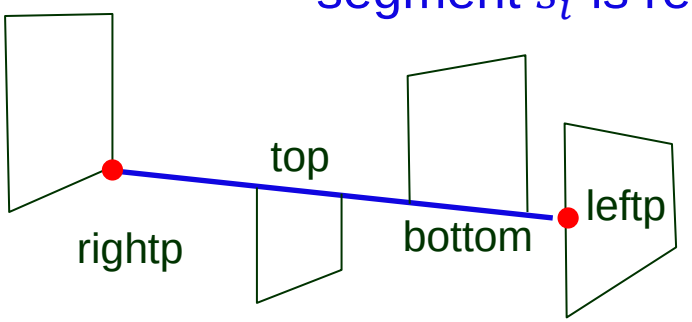

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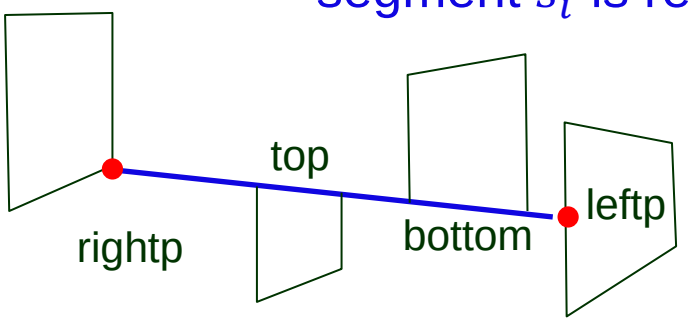

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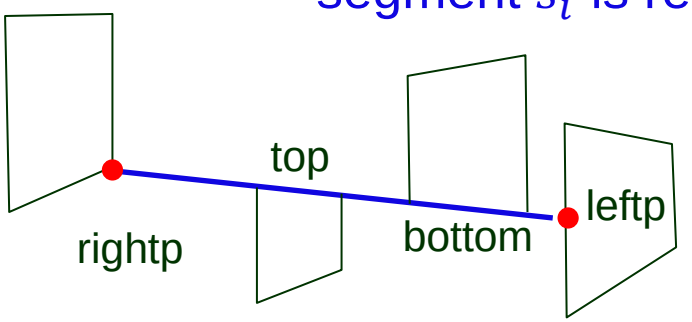

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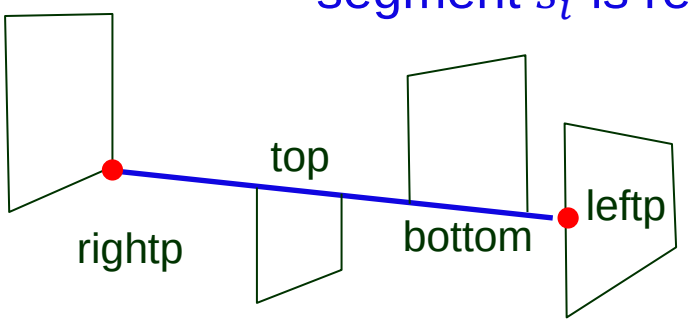

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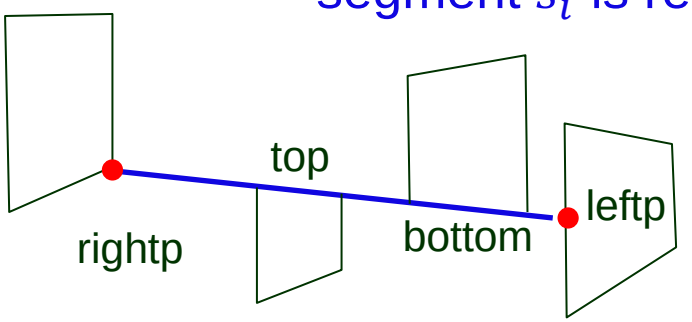

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Analysis (cont'd)

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$$P_i = \Pr\left(\Delta_q(S_i) \neq \Delta_q(S_{i-1})\right)$$
$$= \Pr\left(\Delta_q(S_i) \notin T(S_{i-1})\right)$$
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$$\ln n \leq H_n \leq \ln n + 1$$

Time and Space

Theorem a) The trapezoidal map $T(S)$ and search structure D are computed in $O(n \log n)$ expected time.
// not discussed

b) Expected size of D is $O(n)$. // not discussed

c) Expected query time is $O(\log n)$.