Constraint Satisfaction Problems (CSPs)

Outline

I. Job shop scheduling

II. The diet problem and linear programming

III. Binary CSPs

IV. Constraint Propagation

* Figures/images are from the textbook site (or by the instructor). Otherwise, the source is cited unless such citation would make little sense due to the triviality of generating the image.

I. Example 2: Job-Shop Scheduling

Car assembly with 15 tasks:

- install axles (front and back): 2
- affix wheels (right and left, front and back): 4
- tighten nuts for each wheel: 4
- affix hubcaps: 4
- inspect the final assembly: 1

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$\mathcal{D}$: time space for each variable.

$Axle_F =$ starting time for installation of the front axle.

Precedence Constraints

$$T_1 + d_1 \leq T_2$$

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- The axles have to be in place before the wheels are put on (axle installation takes 10 minutes).

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- ♣ The axles have to be in place before the wheels are put on (axle installation takes 10 minutes).

$$Axle_F + 10 \leq Wheel_{RF} \qquad Axle_F + 10 \leq Wheel_{LF}$$

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- ♣ Affix each wheel (1 minutes), then tighten the nuts (2 minutes), and finally attach the hubcap (1 minute, but not represented)

$$Wheel_{RF} + 1 \leq Nuts_{RF} \qquad Nuts_{RF} + 2 \leq Cap_{RF}$$

$$Wheel_{LF} + 1 \leq Nuts_{LF} \qquad Nuts_{LF} + 2 \leq Cap_{LF}$$

$$Wheel_{RB} + 1 \leq Nuts_{RB} \qquad Nuts_{RB} + 2 \leq Cap_{RB}$$

$$Wheel_{LB} + 1 \leq Nuts_{LB} \qquad Nuts_{LB} + 2 \leq Cap_{LB}$$

More Constraints

- Inspection comes last and take 3 minutes

$X + d_X \leq Inspect$ for every $X \in \mathcal{X}$

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Limit the domain of all variables (discretization)

$$\mathcal{D} = \{1, 2, \ldots, 27\}$$

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- Discrete, finite domains:

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II. The Diet Problem*

How much money to spend in order to get what Polly needs every day?

- energy (2,000 kcal)
- protein (55 g)
- calcium (800 mg)

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Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oat meal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

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Daily Serving Limits

	Servings at most per day
Oatmeal	4
Chicken	3
Eggs	2
Milk	8
Cherry pie	2
Pork with beans	2

Task: Design the *most economical* menu.

Formulating the Problem

x_1: servings of oatmeal
x_2: servings of chicken
x_3: servings of eggs
x_4: servings of whole milk
x_5: servings of cherry pie
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subject to

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Servings-per-day limits

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Servings-per-day limits

and

energy $\quad 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000$

protein $\quad 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \ge 55$

calcium $\quad 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \ge 800$

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Linear Program!

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Constraints (linear)

Linear Programming (LP)

Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\, x_d$

subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \leq b_2$$
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$\boldsymbol{x} = (x_1, x_2, \dots, x_d)$

$\boldsymbol{c} = (c_1, c_2, \dots, c_d)$

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Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\, x_d \qquad = \boldsymbol{c}\boldsymbol{x}^{\boldsymbol{T}}$

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Solvable in time polynomial in d.

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Solvable in time polynomial in d.

Simplex method $O(2^d)$

(best performance in practice)

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Alldiff$(v_1, \dots, v_k)$: variables $v_1, \dots, v_k$ must have different values.

e.g., Sudoku (all variables in a row, column, or 3×3 box).

Cryptarithmetic Puzzle

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

Cryptarithmetic Puzzle

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\begin{array}{cccc}
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\hline
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\end{array}
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Addition constraints on the four columns:

$$O + O = R + 10 \cdot C_1$$
$$C_1 + W + W = U + 10 \cdot C_2$$
$$C_2 + T + T = O + 10 \cdot C_3$$
$$C_3 = F$$

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carry-overs into the tens, hundreds, and thousands, resp.

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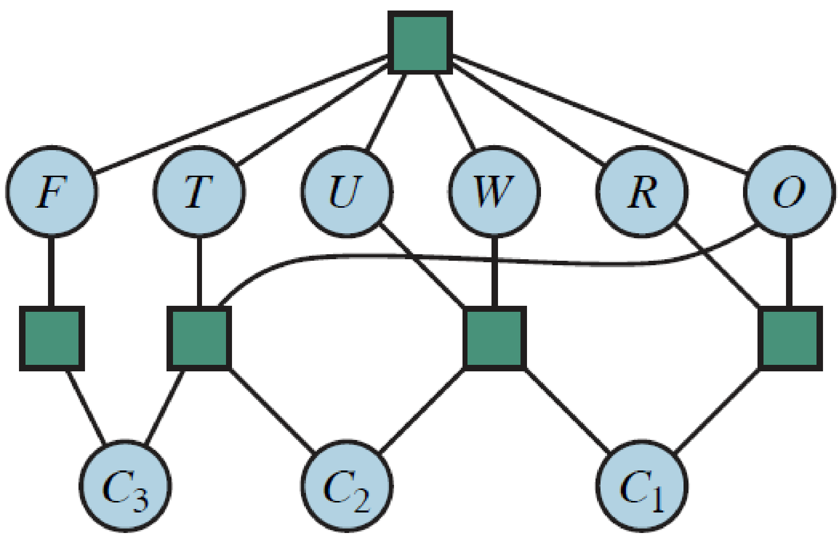

Constraint hypergraph

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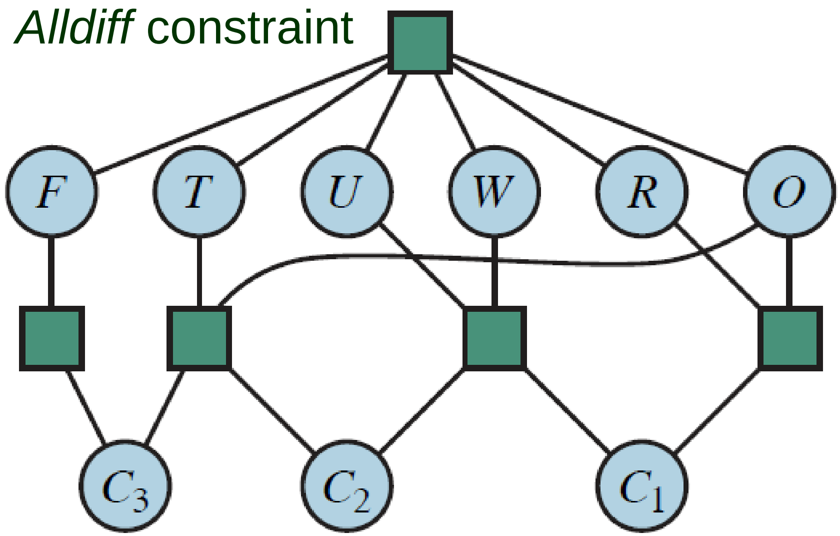
Alldiff constraint



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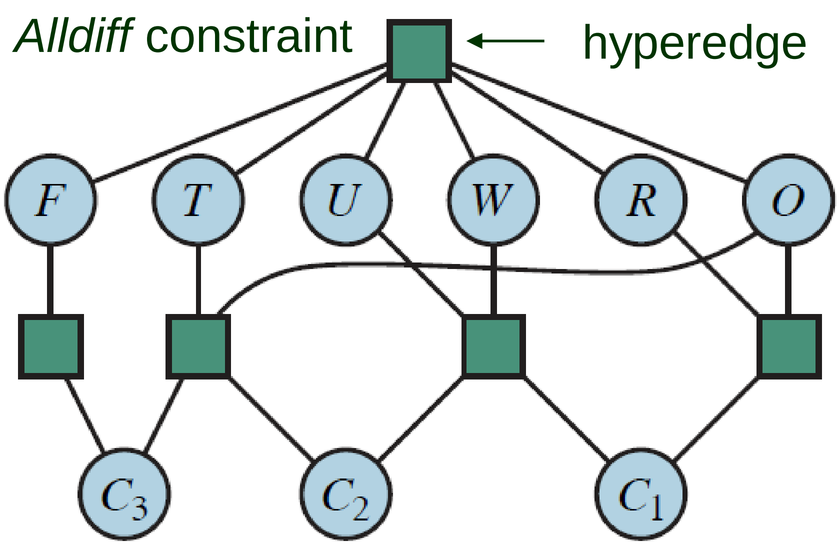

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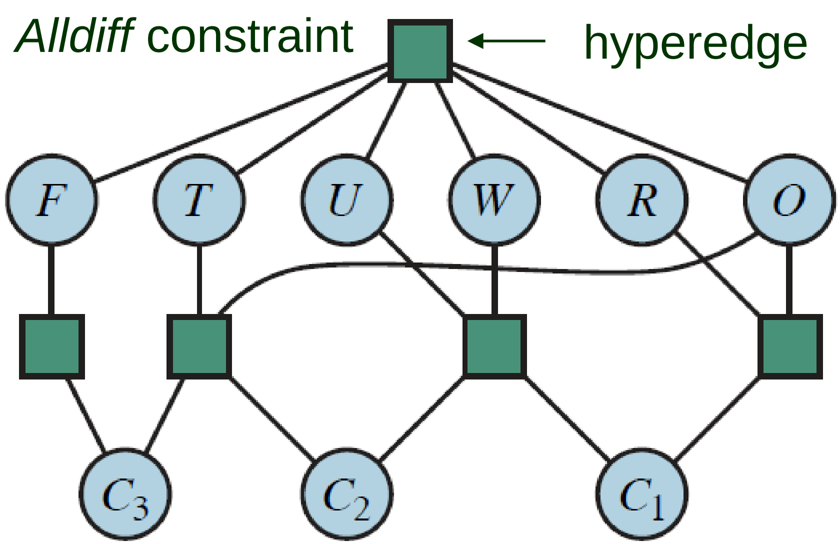

Constraint hypergraph

A *hypergraph* generalizes a graph in that an edge (called an *hyperedge*) can connect more than two vertices.

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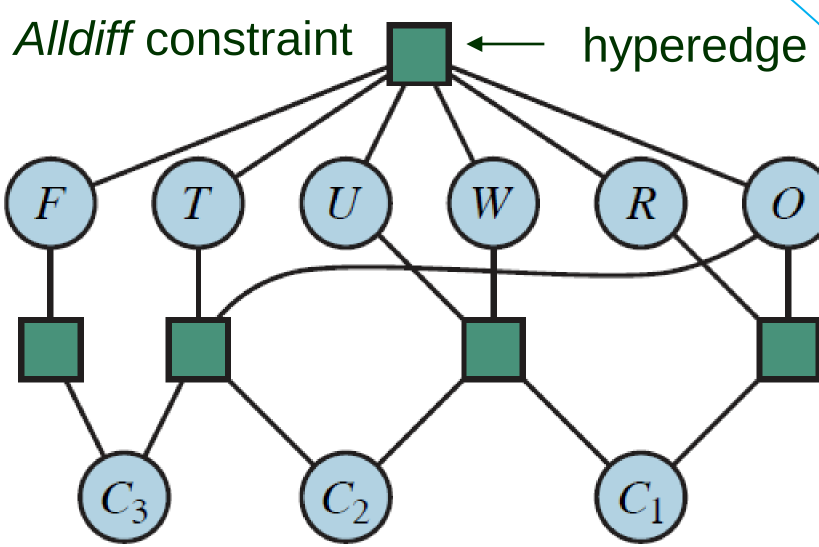

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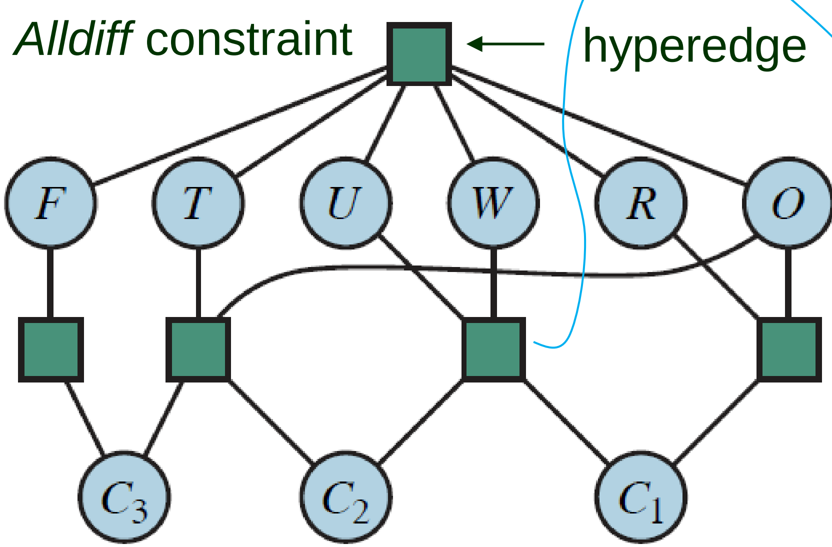

A *hypergraph* generalizes a graph in that an edge (called an *hyperedge*) can connect more than two vertices.

Constraint hypergraph

Transformation to Binary Constraints

Any n-ary constraint can be reduced to a set of binary constraints by introducing auxiliary variables.

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$\Downarrow$

Any CSP can be transformed into a binary CSP.

Advantages of a global constraint such as *Alldiff*:

- Easier and less error-prone to write.
- Allowing efficient special-purpose inference algorithms.

IV. Constraint Propagation

Use constraints to reduce the number of legal values for a variable, which in turn reduce those for another variable, and so on.

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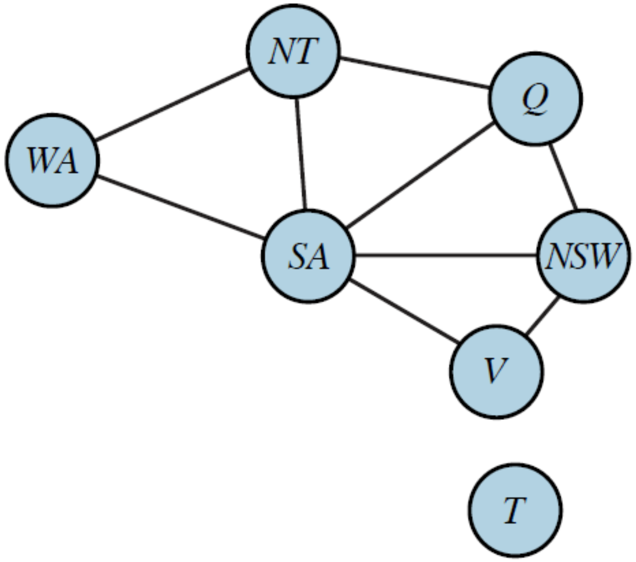

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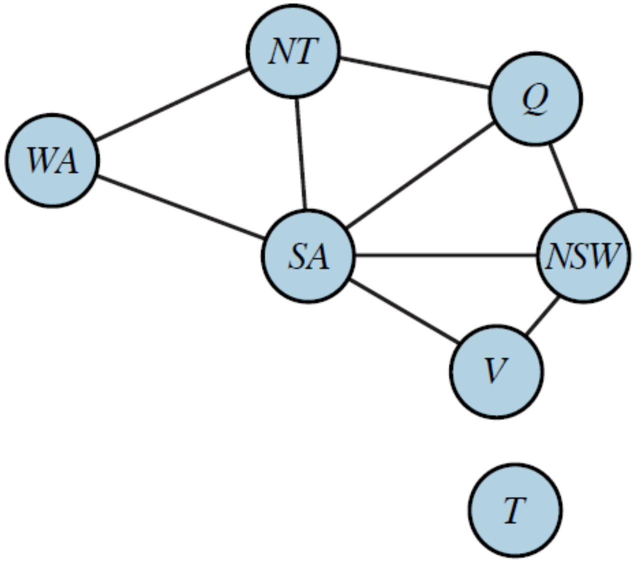

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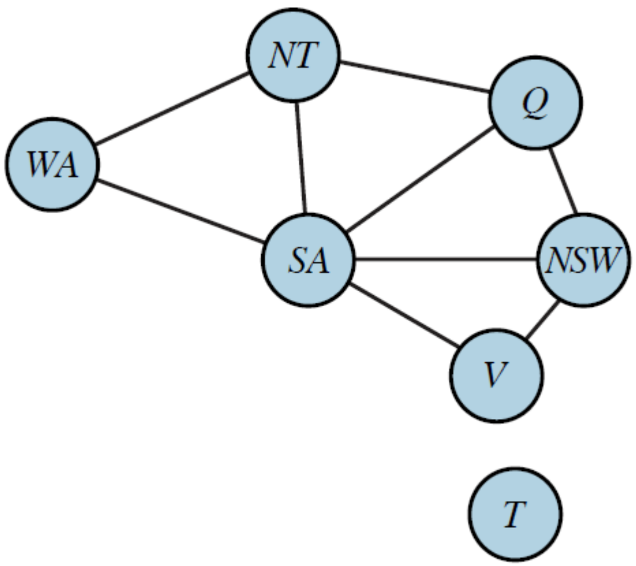

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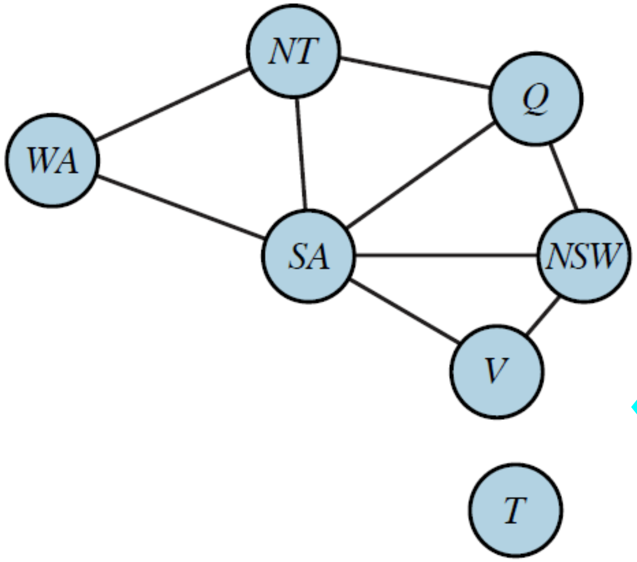

- Node consistency

Suppose South Australians dislike green.

⇩

Domain for *SA*: {red, blue}

♦ Eliminate all the unary constraints by reducing the domains of the involved variables at the start.

♦ A variable is *node-consistent* if all the values in its domain satisfy its unary constraints.

Arc Consistency

A variable X_i is *arc-consistent* with respect to (w.r.t.) another variable X_j if for every value in D_i there exists a value in D_j that satisfies the binary constraint on the arc (i.e., edge (X_i, X_j)).

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The constraint graph is *arc-consistent* if every variable is arc consistent with every other variable.

Application of Arc Consistency

Example 1

$$Y = X^2 \quad \text{where } X, Y \in \{0, 1, \ldots, 9\}$$

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Example 2 Australia map-coloring

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Example 2 Australia map-coloring

$SA \neq WA$

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Example 2 Australia map-coloring

SA≠WA

No effect on the domain {*red*, *green*, *blue*} of each variable.

Arc-Consistency Algorithm AC-3

```
function AC-3( csp) returns false if an inconsistency is found and true otherwise
  queue ← a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← POP(queue)
    if REVISE(csp, X_i, X_j) then
      if size of D_i = 0 then return false
      for each X_k in X_i.NEIGHBORS - {X_j} do   // propagate to other variables sharing a
        add (X_k, X_i) to queue                  // constraint with X_i since its domain has
  return true                                    // reduced.

function REVISE( csp, X_i, X_j) returns true iff we revise the domain of X_i
  revised ← false
  for each x in D_i do
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j then
      delete x from D_i
      revised ← true
  return revised
```

Arc-Consistency Algorithm AC-3

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  queue ← a queue of arcs, initially all the arcs in *csp*

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      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
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  return *true*

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Domain size $\leq d$, c binary constraints.

- Each arc inserted into the queue $\leq d$ times (X_i has $\leq d$ values to delete).
- Each checking takes $O(d^2)$ time.

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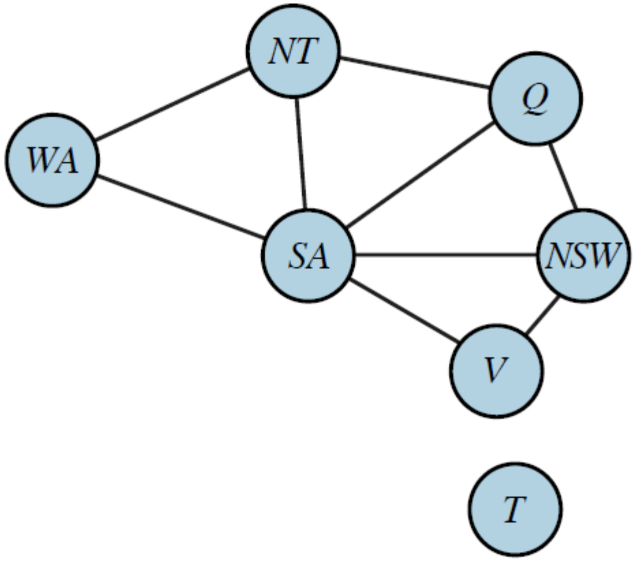
$$O(cd^3)$$

Path Consistency

Arc consistency reduces the domains.



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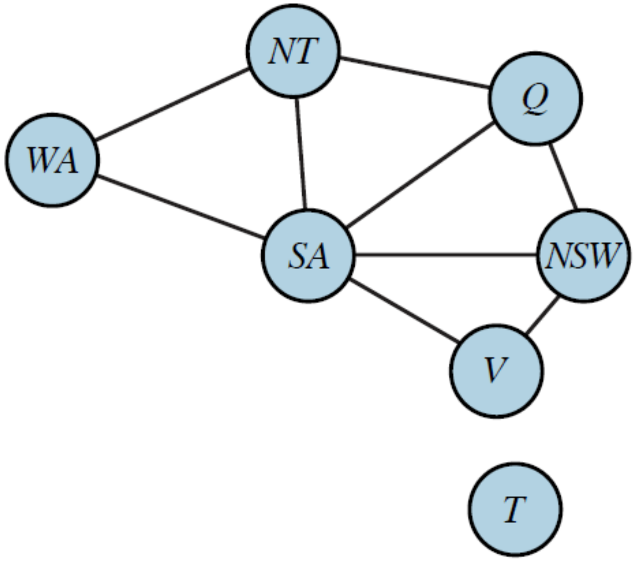

No solution if using two colors even though every constraint can be satisfied individually.

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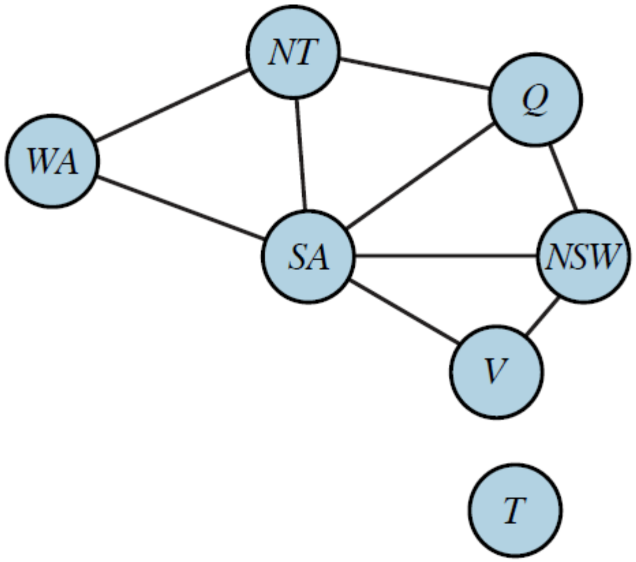

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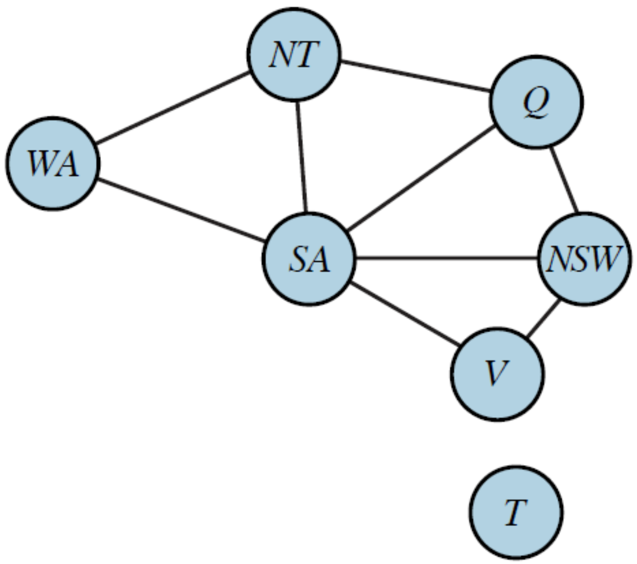

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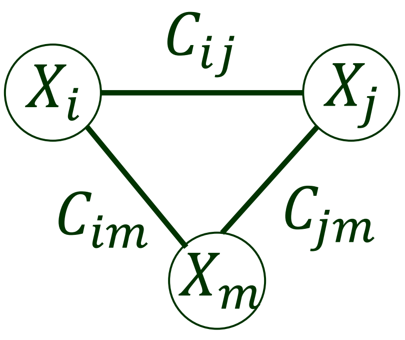
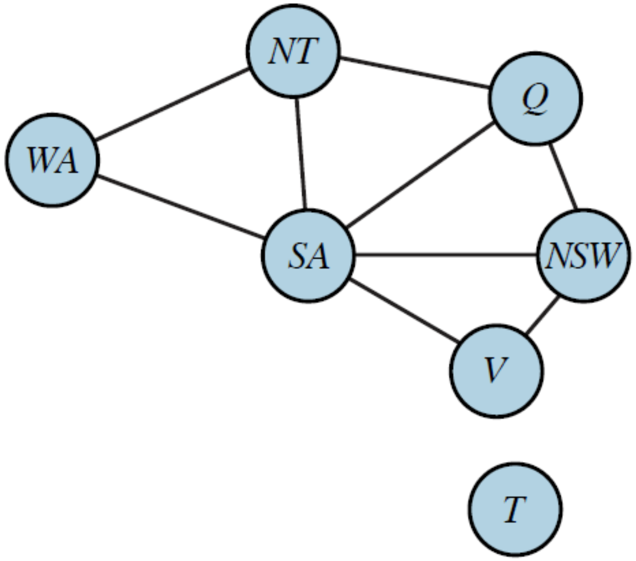
Arc consistency reduces the domains.

Path consistency checks implicit constraints inferable across triples of variables along a path.

$\{X_i, X_j\}$ is *path-consistent* w.r.t. X_m if for every assignment to X_i, X_j consistent with their constraint C_{ij} (if exists), there exists an assignment to X_m that satisfies the constraints C_{im} and C_{jm} between them and X_m.



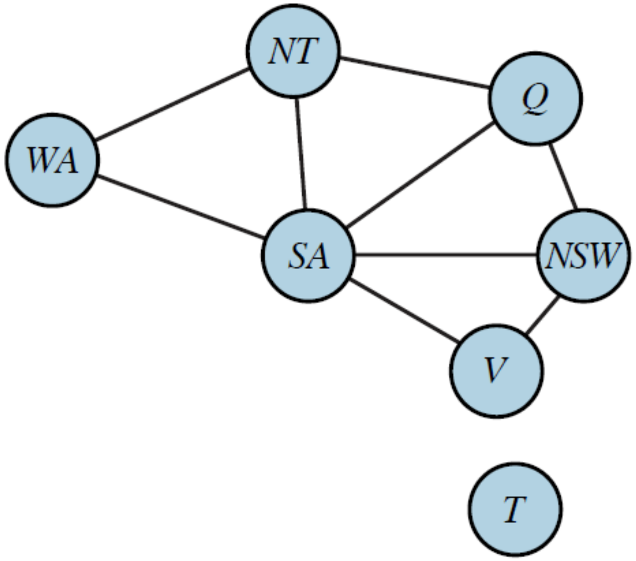
Application of Path Consistency



Make {*WA*, *SA*} path-consistent w.r.t. *NT*.

Assume two colors only: *red*, *blue*.

Application of Path Consistency



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Only two possible assignments:

{*WA* = *red*, *SA* = *blue*}

{*WA* = *blue*, *SA* = *red*}

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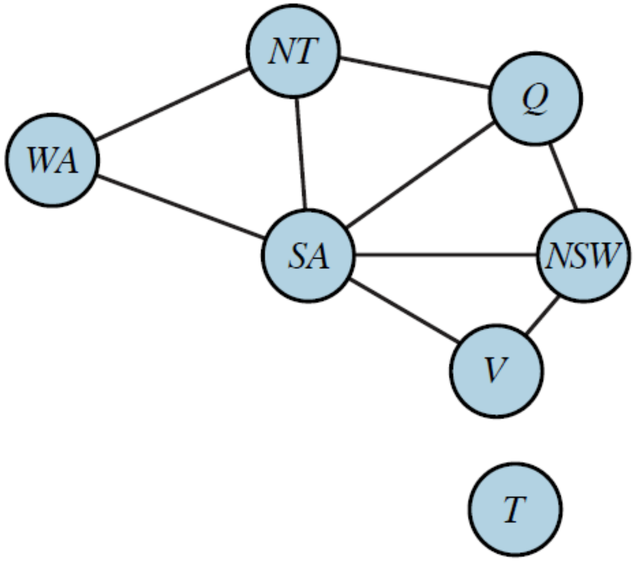

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⇓

No assignment for *NT* in either case!

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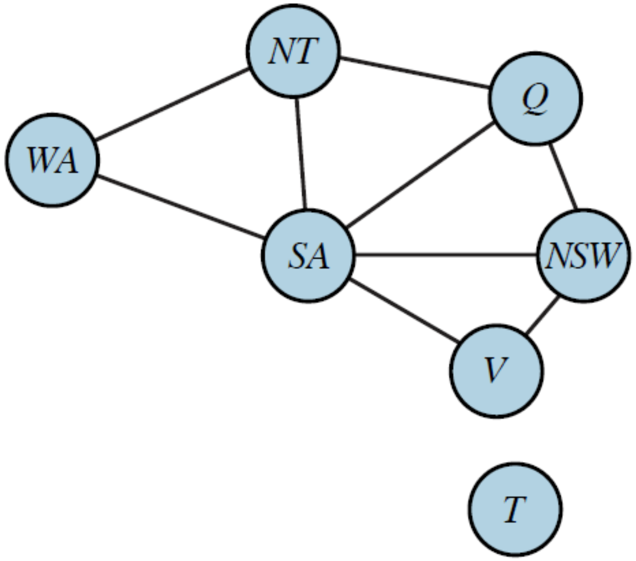

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Eliminate both assignments to *WA* and *SA*.

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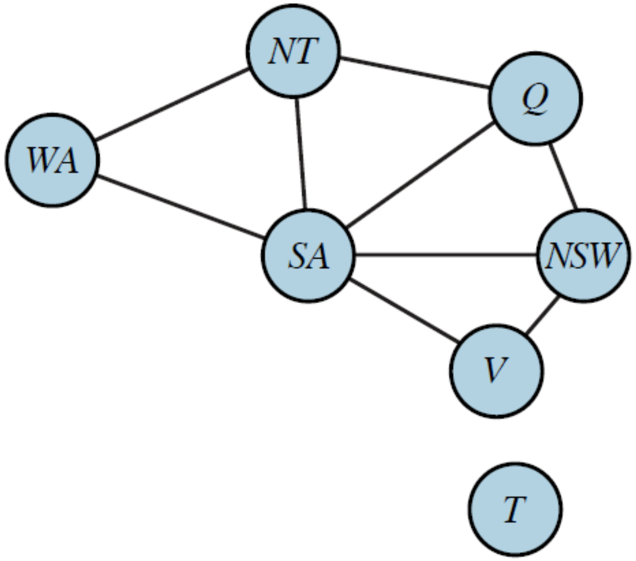

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⇩

No solution to the problem.

Bounds Propagation

Two flights F_1 and F_2 have domains:

$$D_1 = [0,165] \text{ and } D_2 = [0,385]$$

max capacity

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Bounds-consistent if for any two variables X, Y, whether X takes on its lower- or upper-bound values, there exists some value of Y that satisfies the constraint between X and Y.

* By the above definition, for some value of X between its lower and upper bounds, there may not exist a value of Y to meet the constraint (e.g., a nonlinear constraint).

Sudoku

Fill the digits 1 to 9 in a 9×9 grid such that no digit appears twice in any row, column, or 3×3 box.

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
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D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
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Variables: $A1, \ldots, A9, B1, \ldots I9$

Domain: $D = \{1,2, \ldots, 9\}$

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27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

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Sudoku as a CSP

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Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

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Alldiff$(A7, A8, A9, B7, B8, B9, C7, C8, C9)$

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⋮

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Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

⋮

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

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⋮

Alldiff$(A7, A8, A9, B7, B8, B9, C7, C8, C9)$

A CSP solver can handle thousands of puzzles per second!

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Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

⋮

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

⋮

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

⋮

Alldiff$(A7, A8, A9, B7, B8, B9, C7, C8, C9)$

A CSP solver can handle thousands of puzzles per second!
Only the simplest ones can be solved by AC-3.

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
D			8	1		2	9		
E	7					?			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Consider E6:

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
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E	7					?			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
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Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
D			8	1		2	9		
E	7					?			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

$$= \{3,4,5,6,9\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
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G			2	6		9	5		
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I			5		1		3		

Consider E6:

$D_{E6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the box

$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$

$= \{3,4,5,6,9\}$

$\Downarrow$ constraints in the column

$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$

$= \{4\}$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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E	7					4			8
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Consider I6:

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	1	2	3	4	5	6	7	8	9
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Consider I6: $D_{I6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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C			1	8		6	4		
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E	7					4			8
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G			2	6		9	5		
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider I6: $D_{I6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

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G			2	6		9	5		
H	8			2		3			9
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Consider E6:

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Consider A6:

$$D_{A6} \leftarrow \{1,2,\ldots,9\}$$

$\Downarrow$ constraints in the column

$$D_{A6} \leftarrow \{1\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2	1	6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7					4			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider E6:

$D_{E6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the box

$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$

$= \{3,4,5,6,9\}$

$\Downarrow$ constraints in the column

$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$

$= \{4\}$

Consider I6:

$D_{I6} \leftarrow \{1,2,\ldots,9\}$

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$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$

$\Downarrow$ constraints in the box

$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$

Consider A6:

$D_{A6} \leftarrow \{1,2,\ldots,9\}$

$\Downarrow$ constraints in the column

$D_{A6} \leftarrow \{1\}$