Lecture Notes for *Introduction to Computational Geometry* (Com S 418/518)
Yan-Bin Jia, Iowa State University

# Range Trees

## Outline:

I. Construction of a 2D range Tree

II. Query with a 2D range Tree

III. High-dimensional range trees

Textbook: *Computational Geometry: Algorithms and Applications (*3rd ed.) by M. de Berg et al., Springer-Verlag, 2008.

# Improvement on a Kd-Tree

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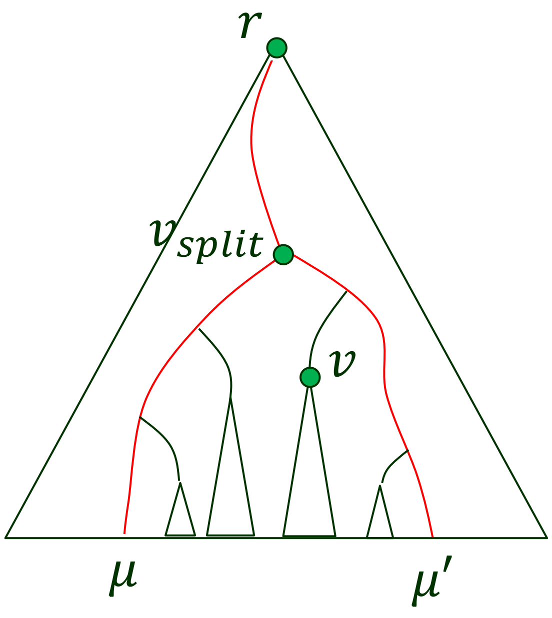
How?

# I. 2D Query

Query range: $[x, x'] \times [y, y']$

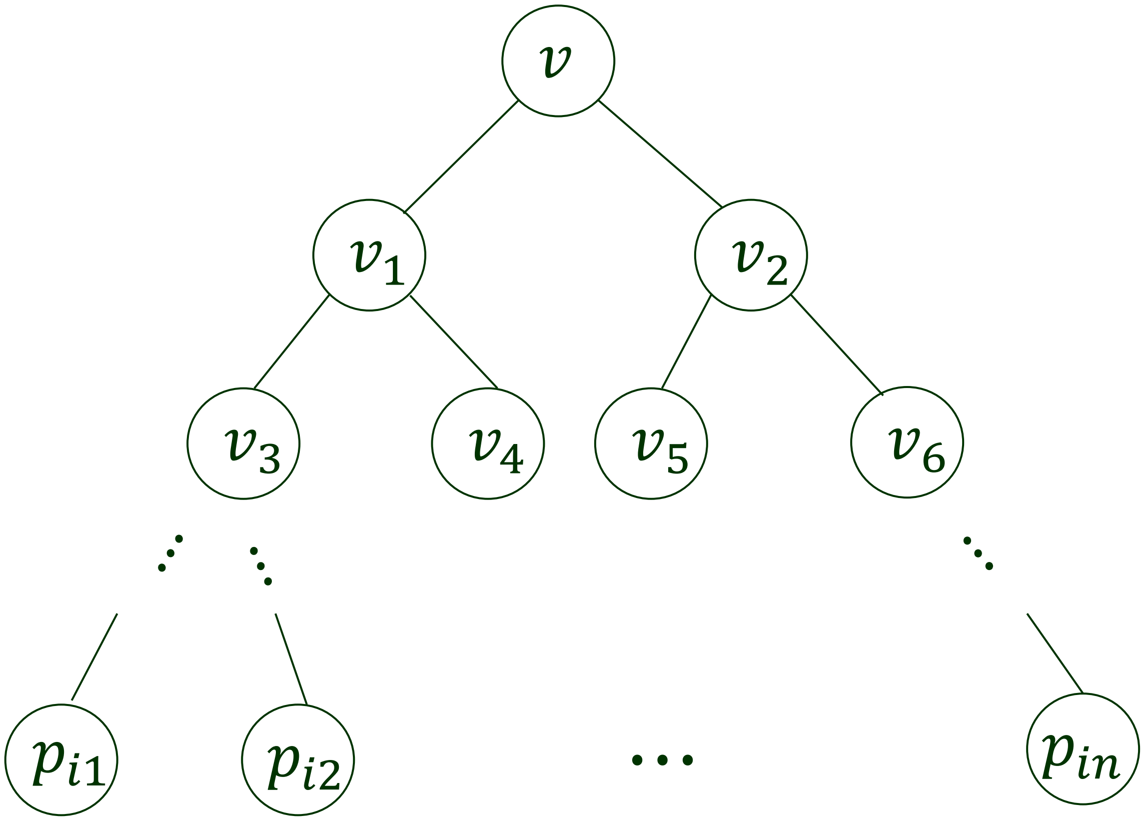
1. Find points with $x$-coordinate $\in [x, x']$




- Searches with $x$ and $x'$ in BST end at leaves $\mu$ and $\mu'$.
- Select a collection of subtrees which together contain exactly the points with $x$-coordinates $\in [x, x']$.

# Canonical Subsets

$P(v)$: leaves (points) of the subtree rooted at $v$



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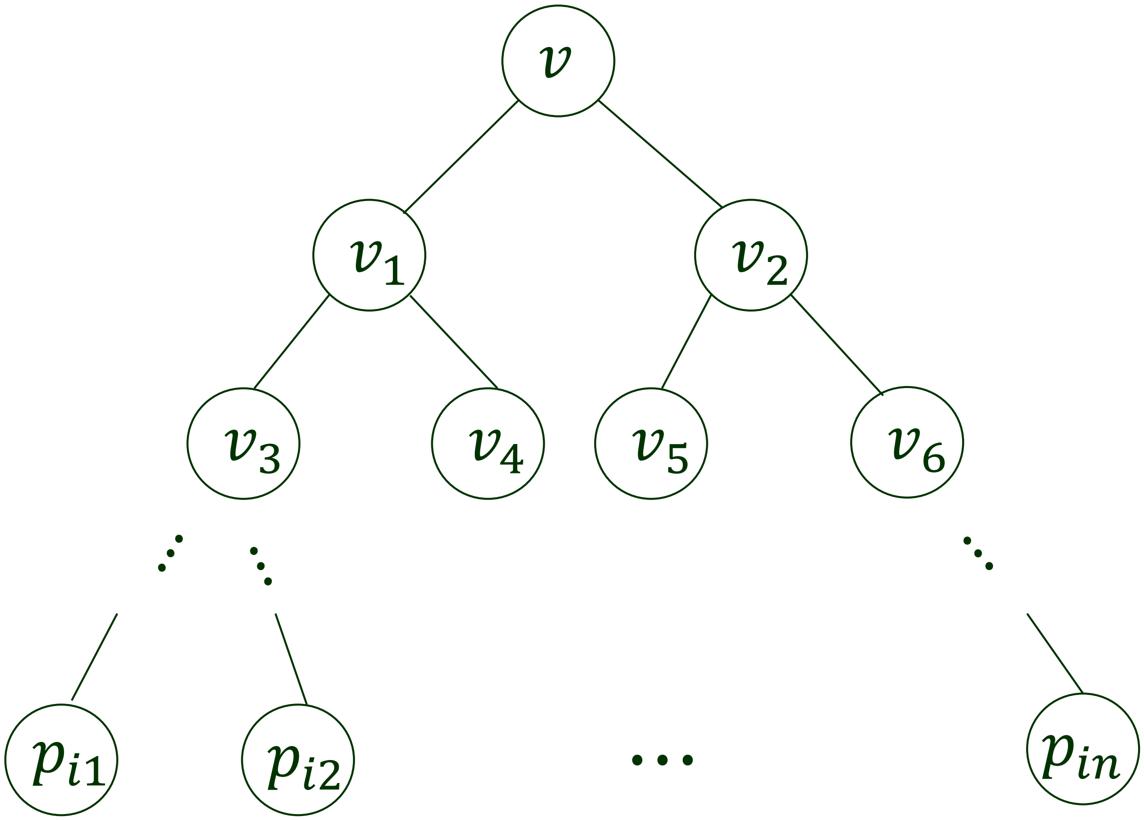


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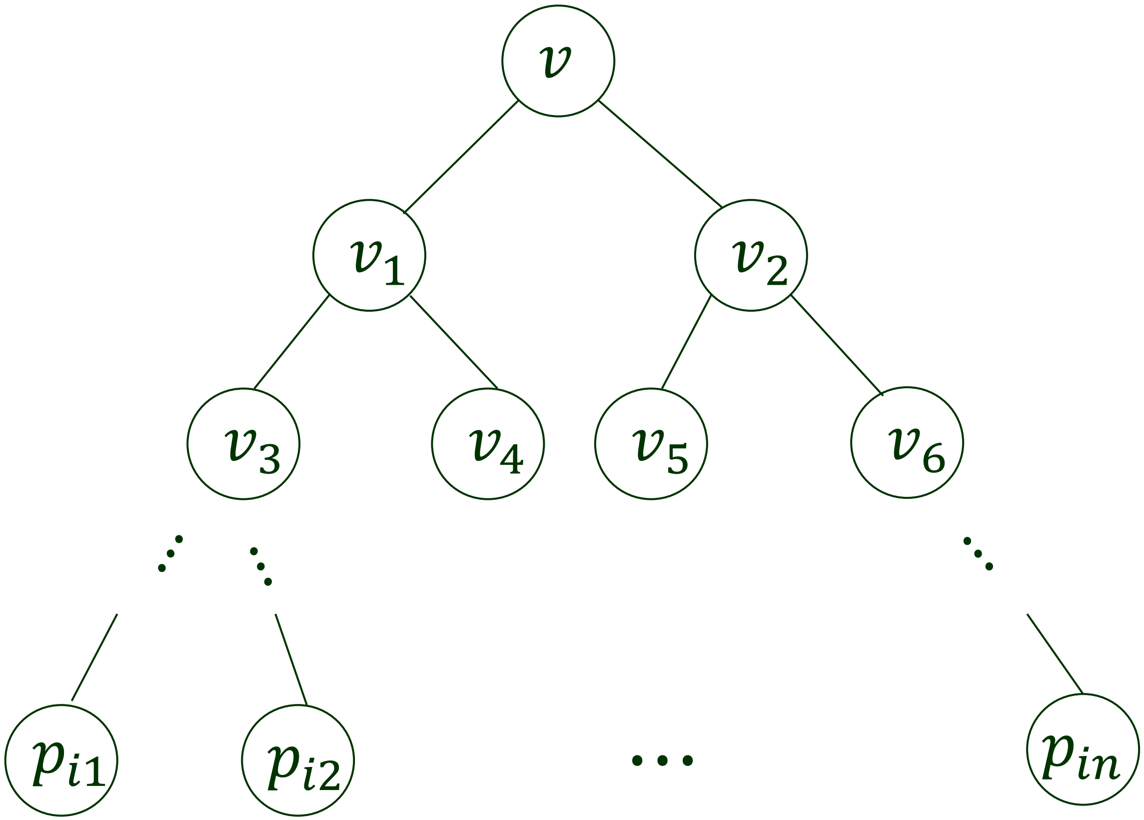


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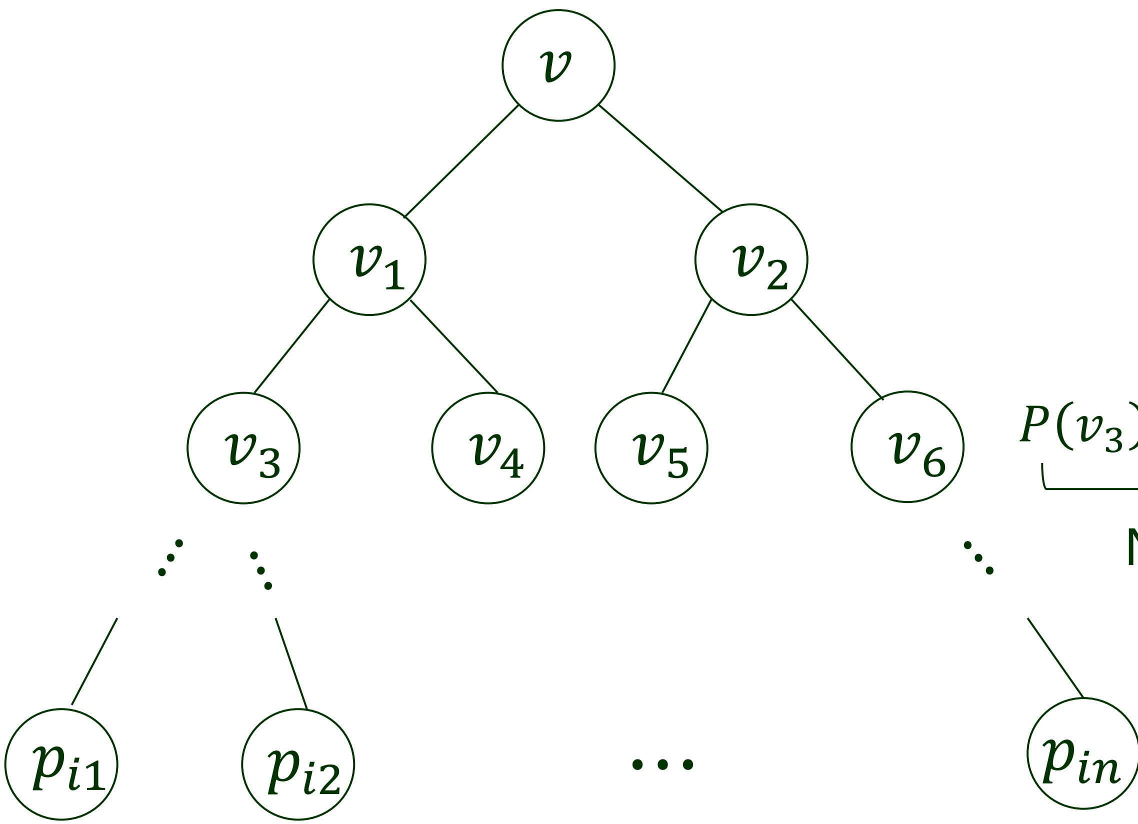


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$$\underbrace{P(v_3) \cup P(v_4) \cup P(v_5) \cup P(v_6)}_{\text{Non-intersecting subsets}} = P$$

# Back to 2D Query

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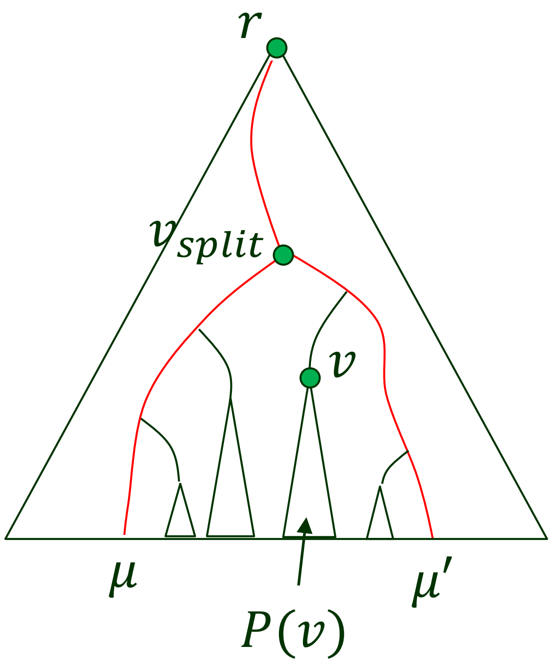

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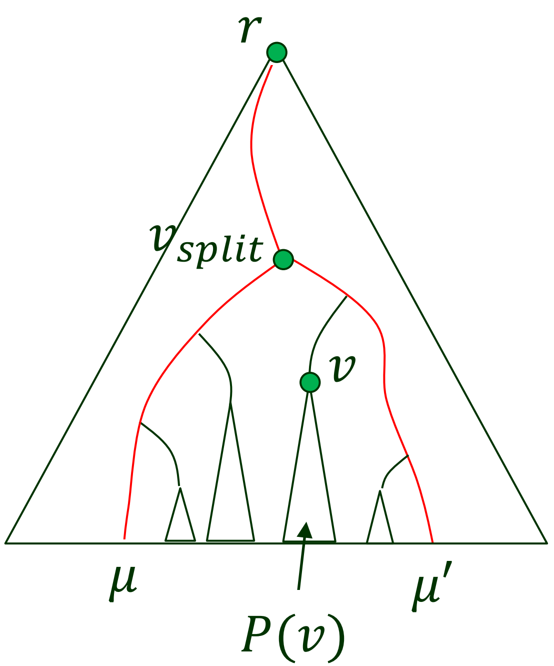


- Query $[x, x']$ yields $O(\log n)$ *disjoint* subsets:

$$\bigcup_{v} P(v)$$

where $v$ is

- right child of a node on the path $r \rightsquigarrow \mu$ when it makes a left turn, or
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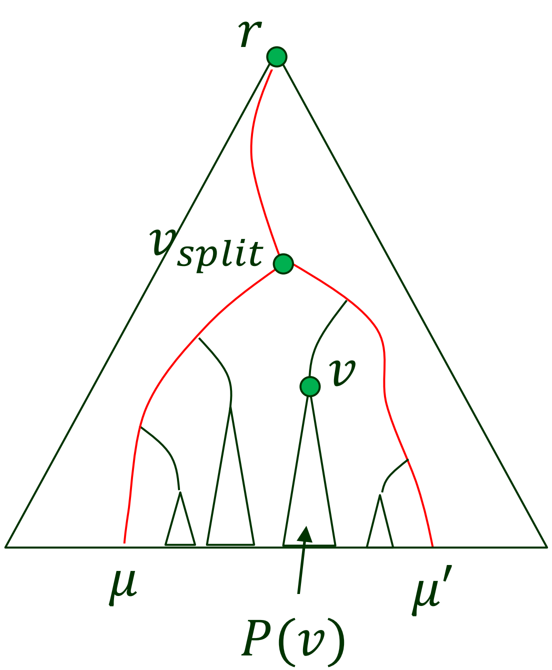


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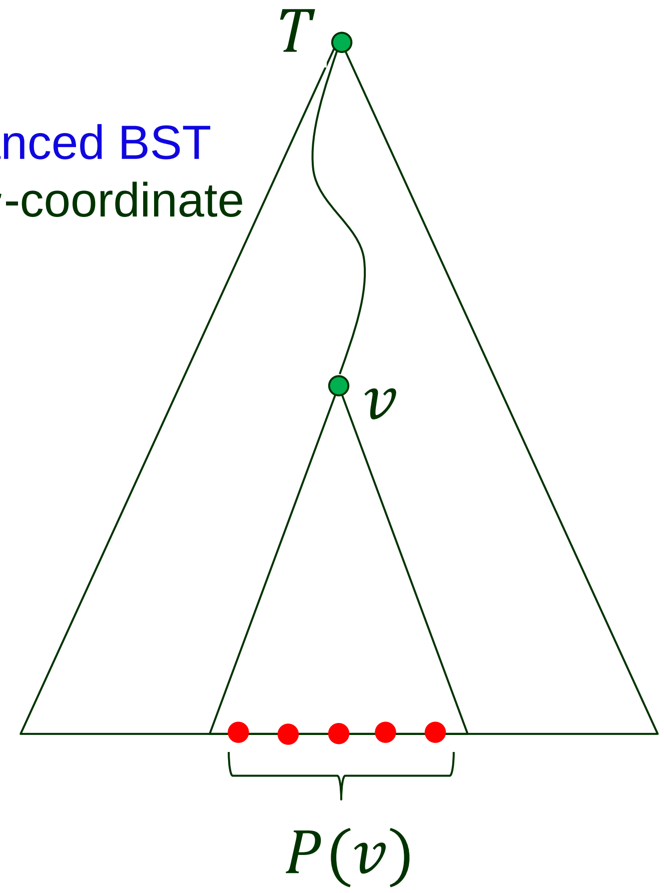
2. For each such $v$, query $[y, y']$ on $T(v)$.

How to make this query fast?

# 2D Range Tree

Main tree

balanced BST
on $x$-coordinate



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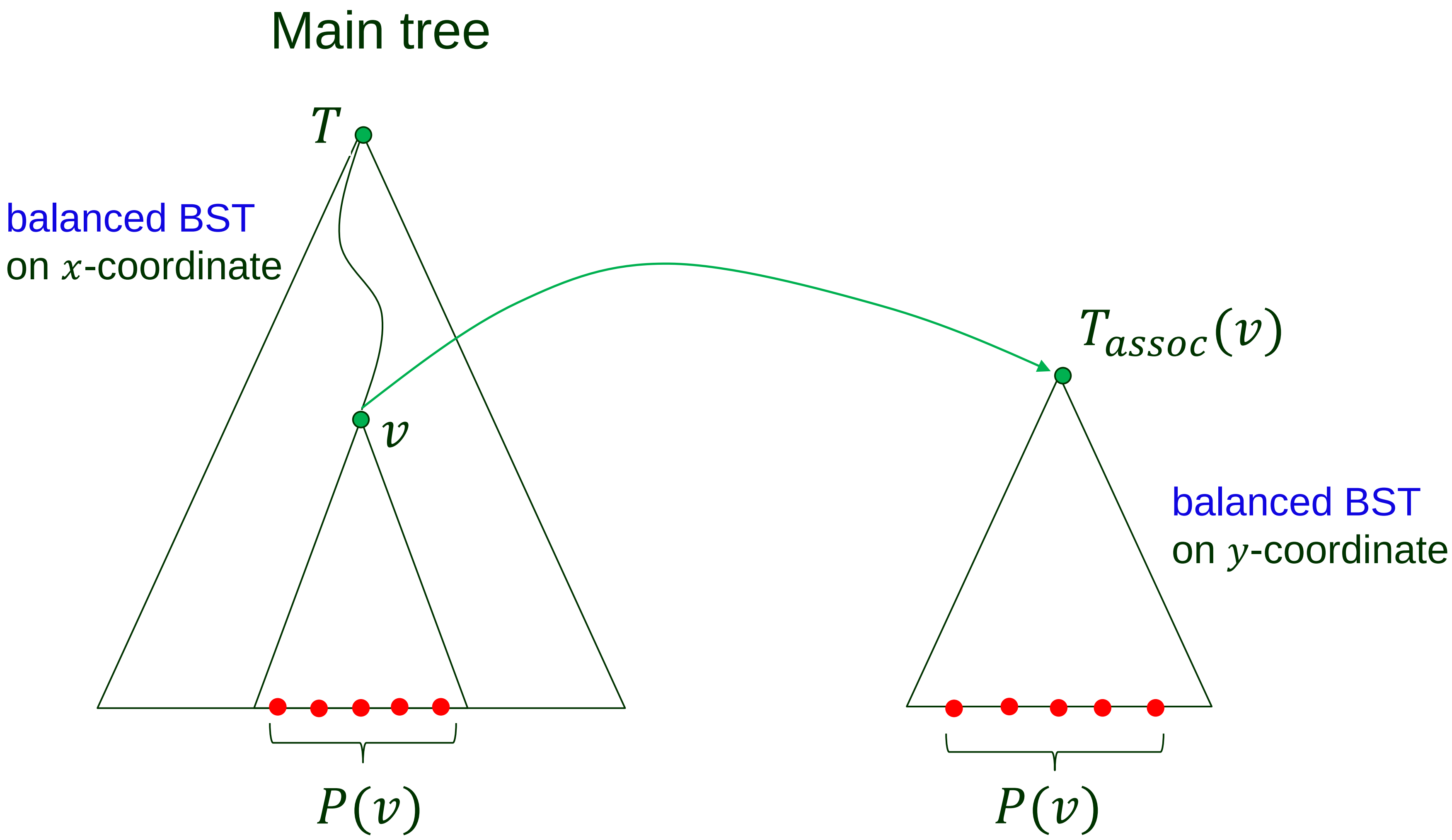

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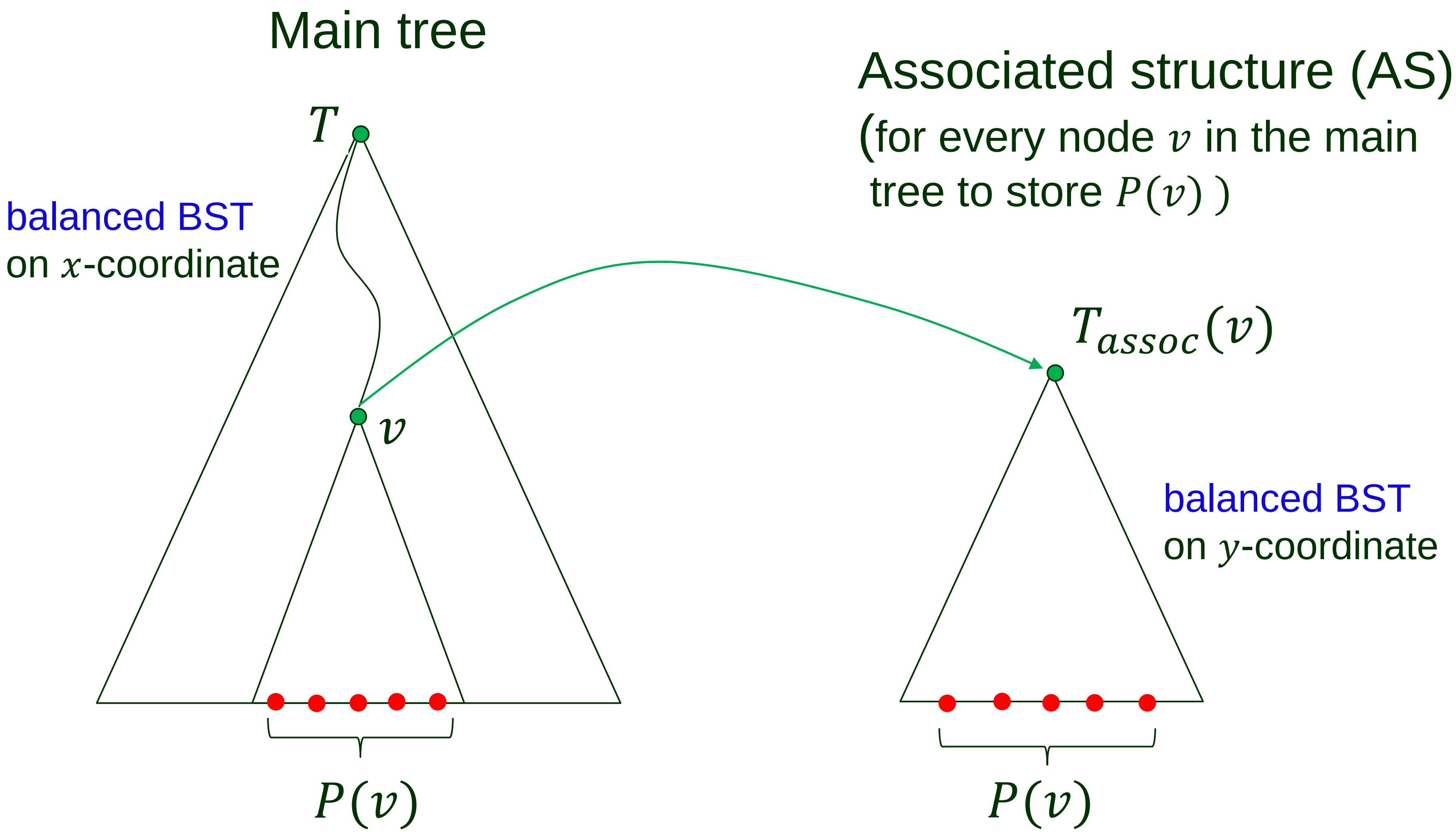

# Recursive Construction

$P = \{p_1, \cdots, p_n\}$ sorted on $x$-coordinate

Build2DRangeTree($P$)

1. build $T_{assoc}$ on $P_y = \{y_i \mid (x_i, y_i) \in P\}$ // leaves storing points
2. if $n = 1$
3. then
4. else
5.
6.

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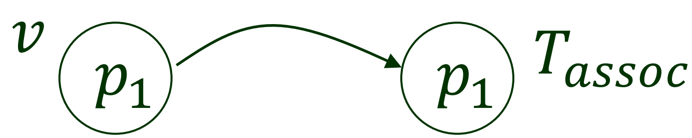


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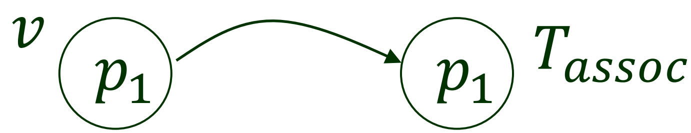


4. else split $P$ at the medium $x$-coordinate ($x_{mid}$)
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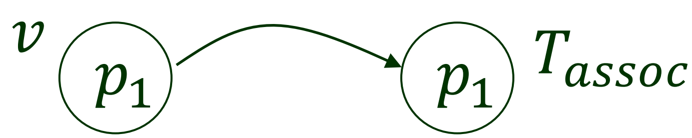



4. else split $P$ at the medium $x$-coordinate ($x_{mid}$)
   into $P_{left}$ and $P_{right}$

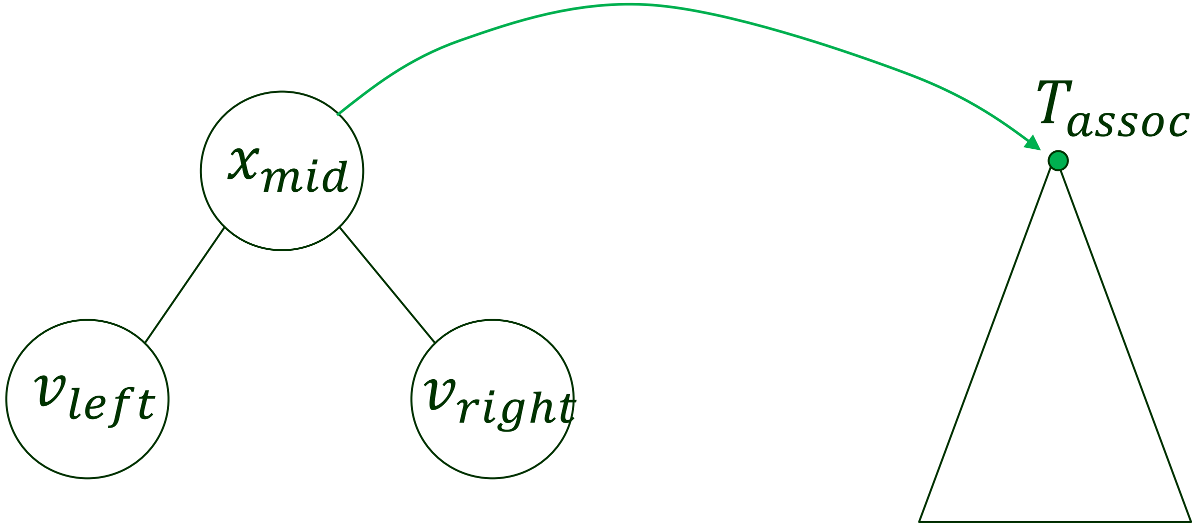
5. $v_{left} \leftarrow$ Build2DRangeTree($P_{left}$)
6. $v_{right} \leftarrow$ Build2DRangeTree($P_{right}$)

# Combining Range Trees

⋮

5. $v_{left} \leftarrow$ Build2DRangeTree($P_{left}$)
6. $v_{right} \leftarrow$ Build2DRangeTree($P_{right}$)

7.




8. return $v$

# Storage Analysis

Total size of the main tree &
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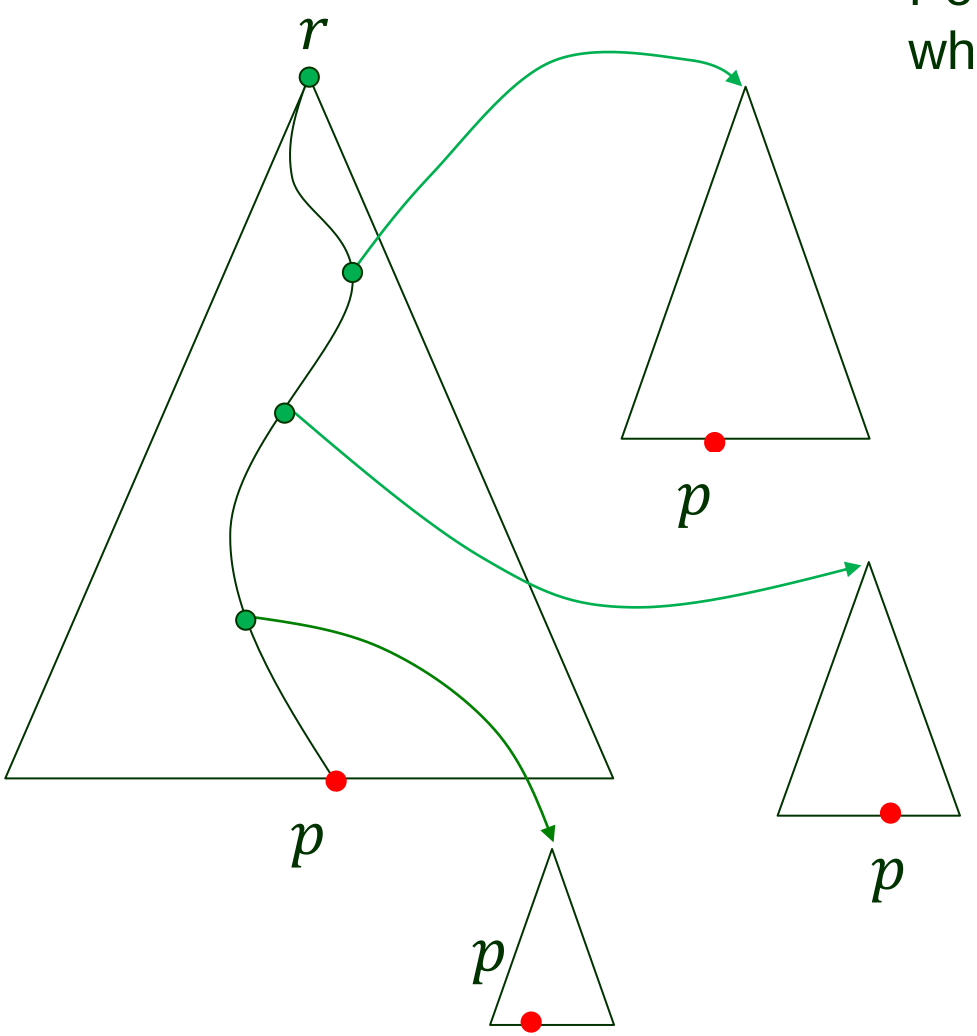
\# *times* the points are stored.

# How Often is a Point Stored?




Point $p$ is stored only in $T_{assoc}(v)$ where $v$ is on the path $r \rightsquigarrow p$.

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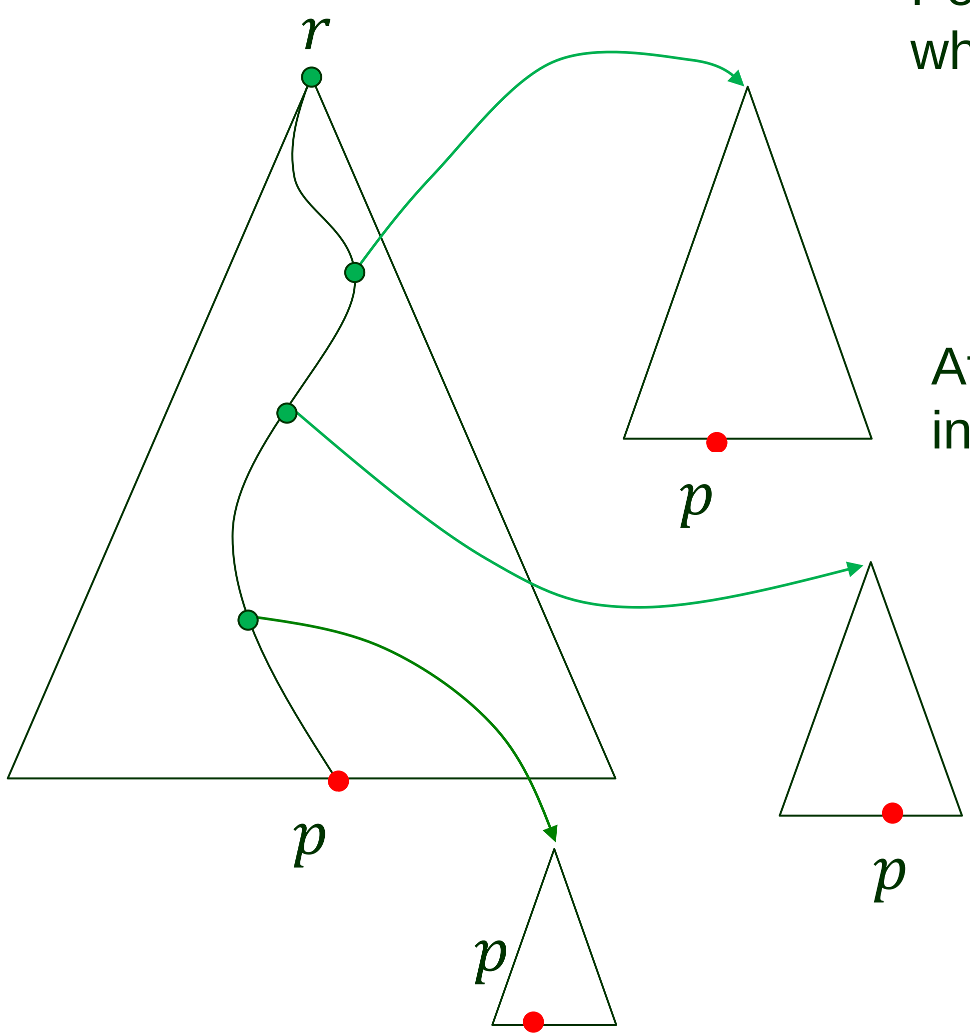


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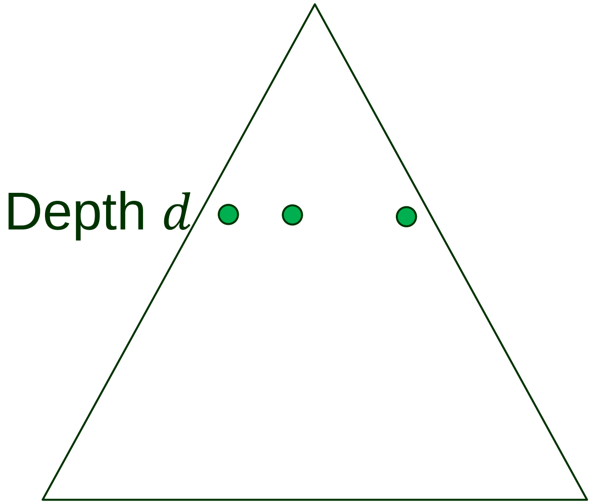
$\Downarrow$

At every depth $d$, $p$ is stored in exactly one AS.

# Depth $d$

At depth $d$ every point is stored at a leaf of *exactly one* AS.



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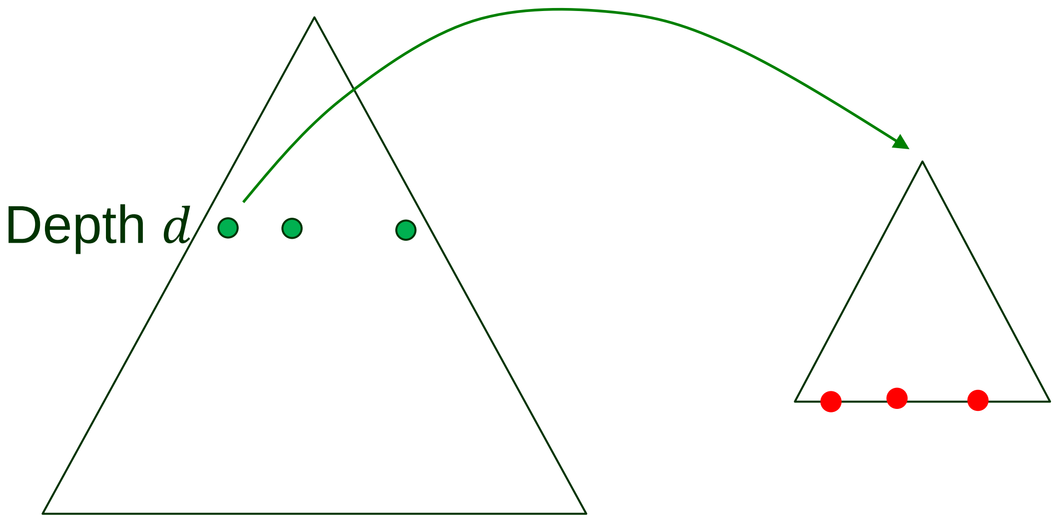

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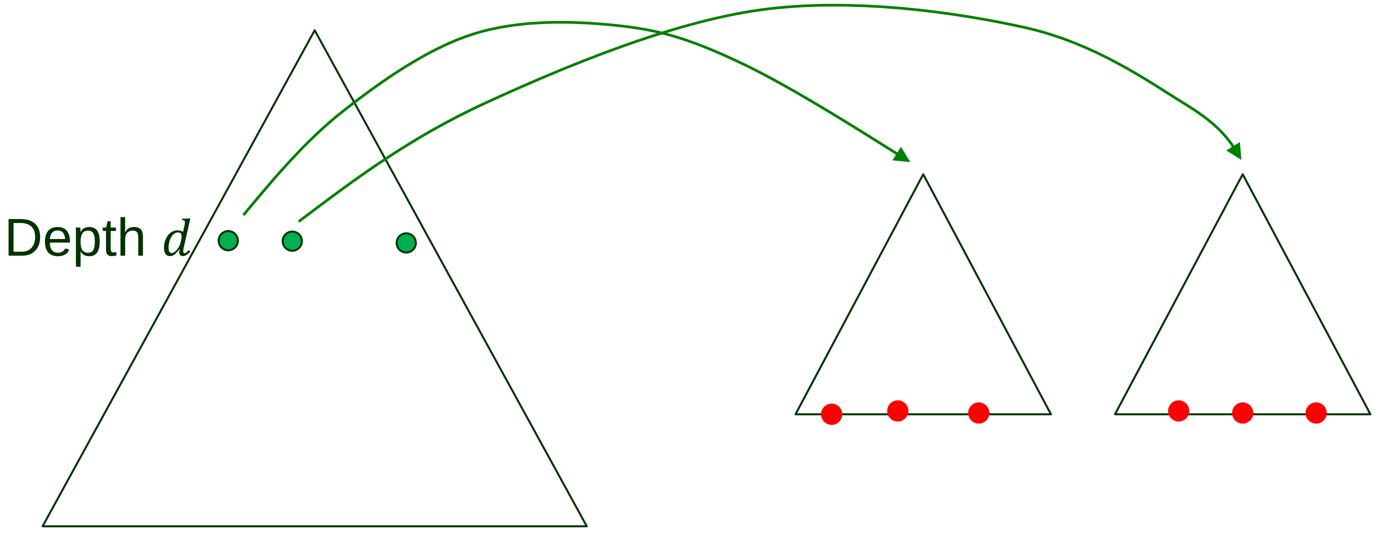

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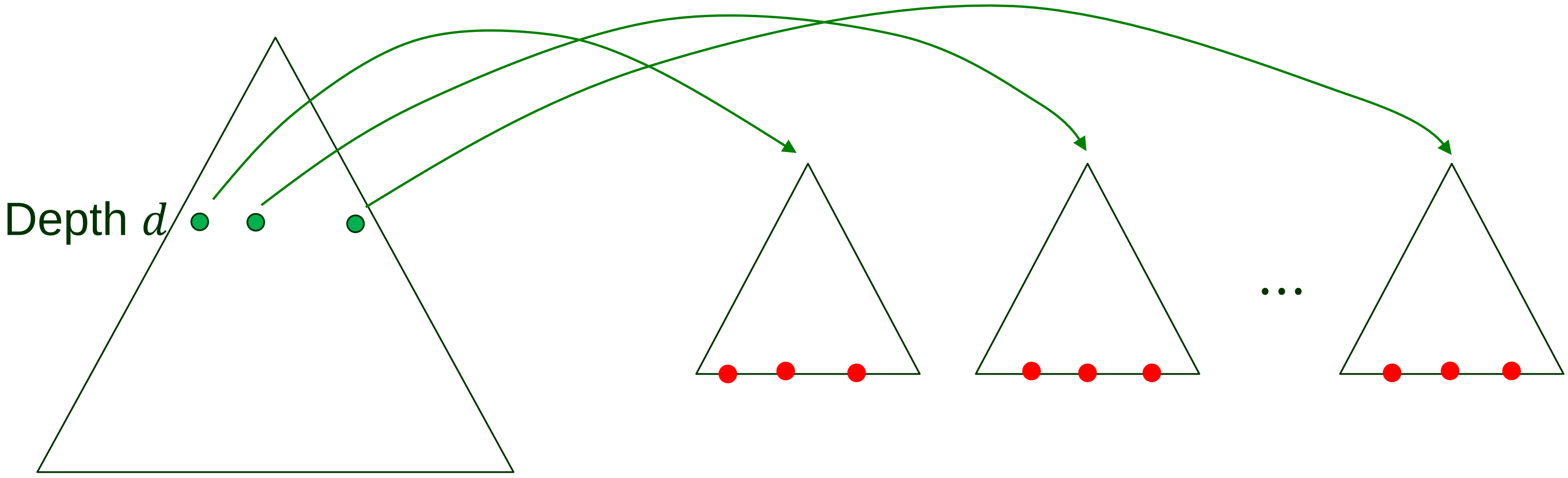

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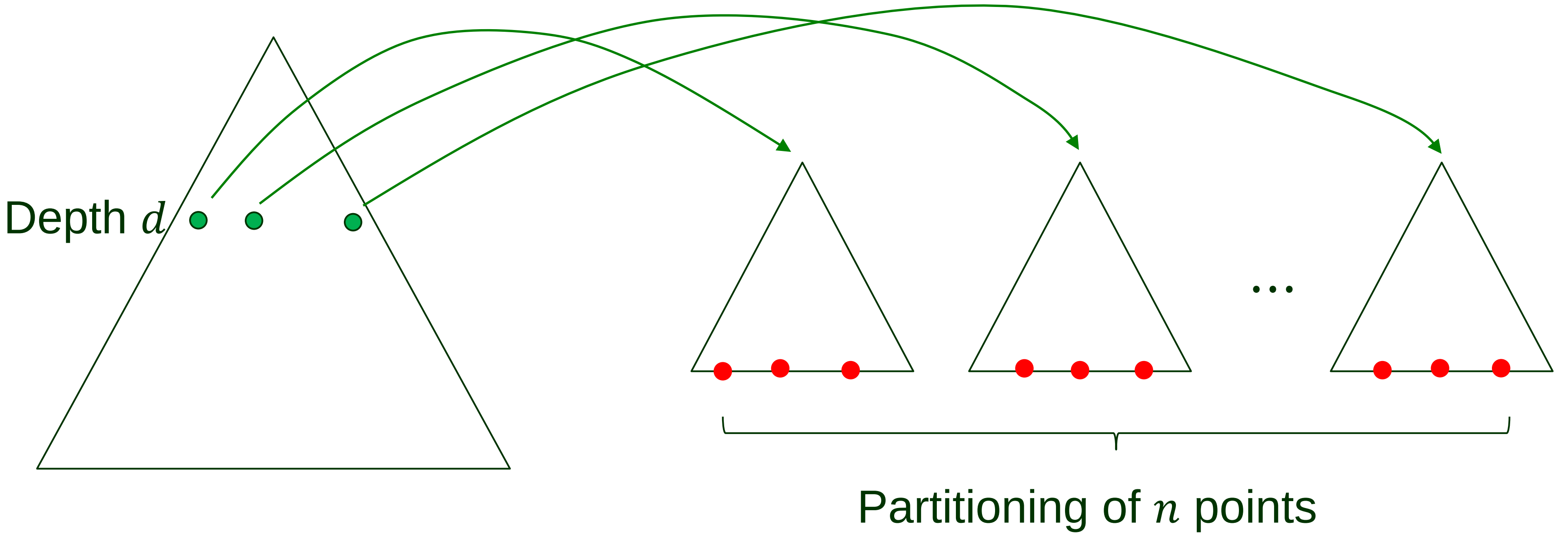

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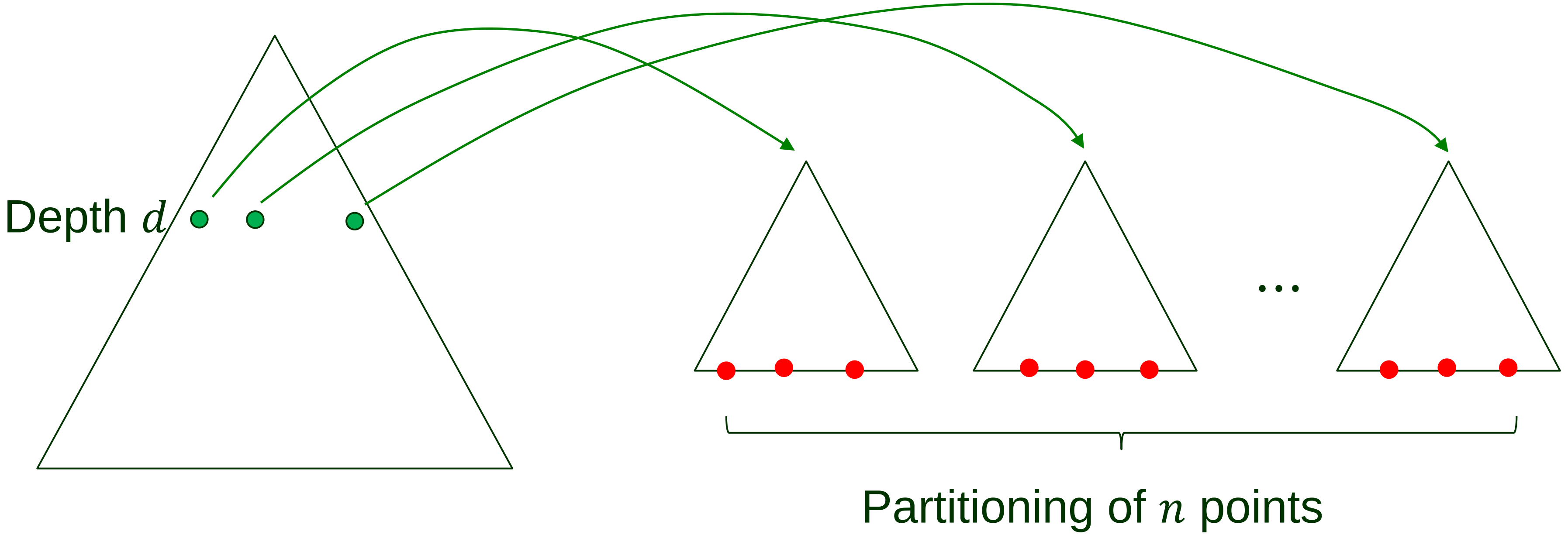


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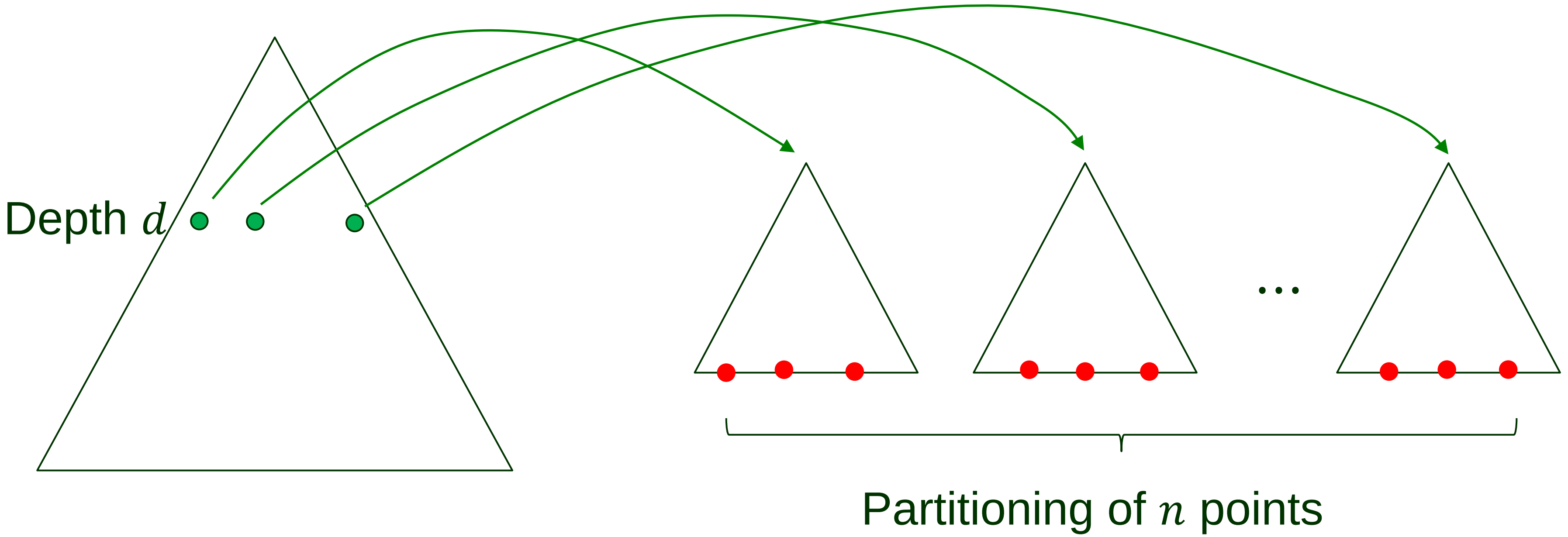


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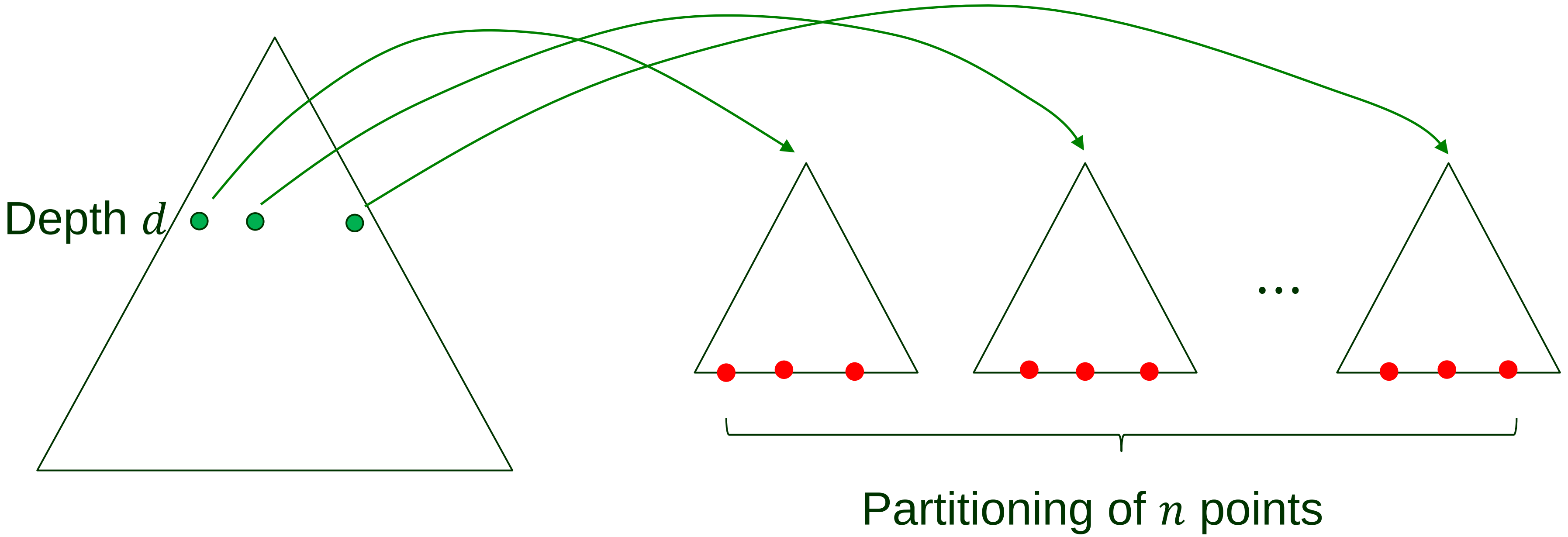


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$\Downarrow$

All the ASes at depth $d$ use $O(n)$ storage.

# Wrapping It Up

Each depth of $T$ requires $O(n)$ of total storage for ASes.

$T$ has height $O(\log n)$.

$T$ needs storage $O(n)$.

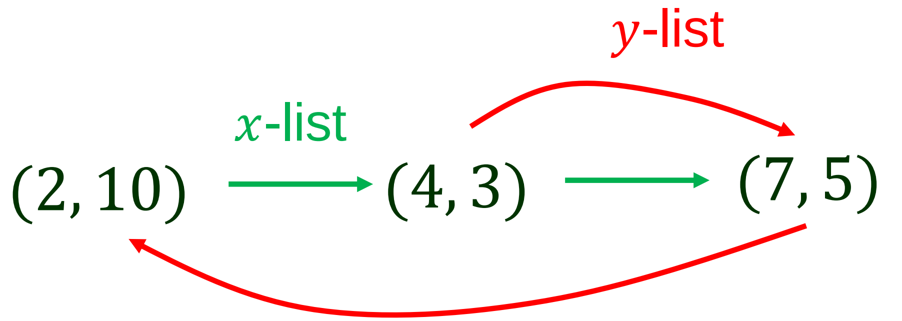
$O(n \log n)$ total storage

# Time of Construction

Maintain $n$ points in two lists:

- sorted on $x$-coordinate

- sorted also on $y$-coordinate



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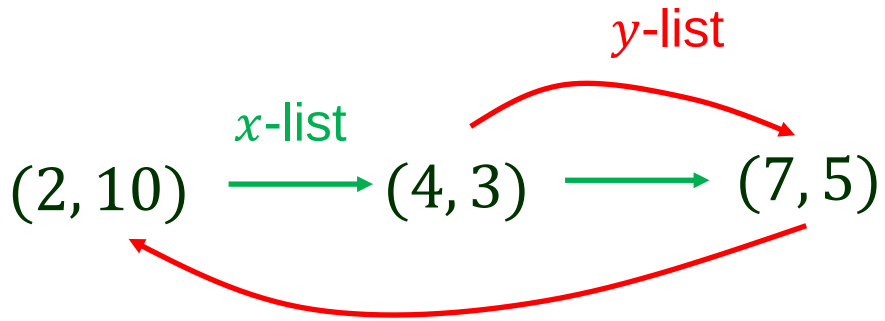


Construction time of each BST is linear to its size.

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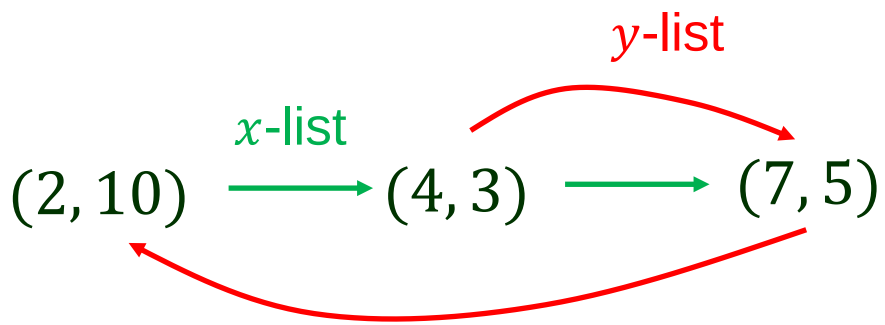


Construction time of each BST is linear to its size.

Total time: $O(n \log n)$

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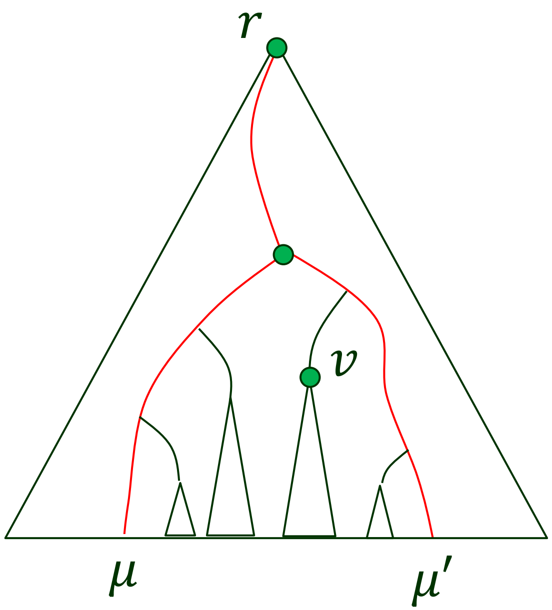
- Selects $O(\log n)$ canonical subsets which together contain points with $x$-coordinate in $[x, x']$.



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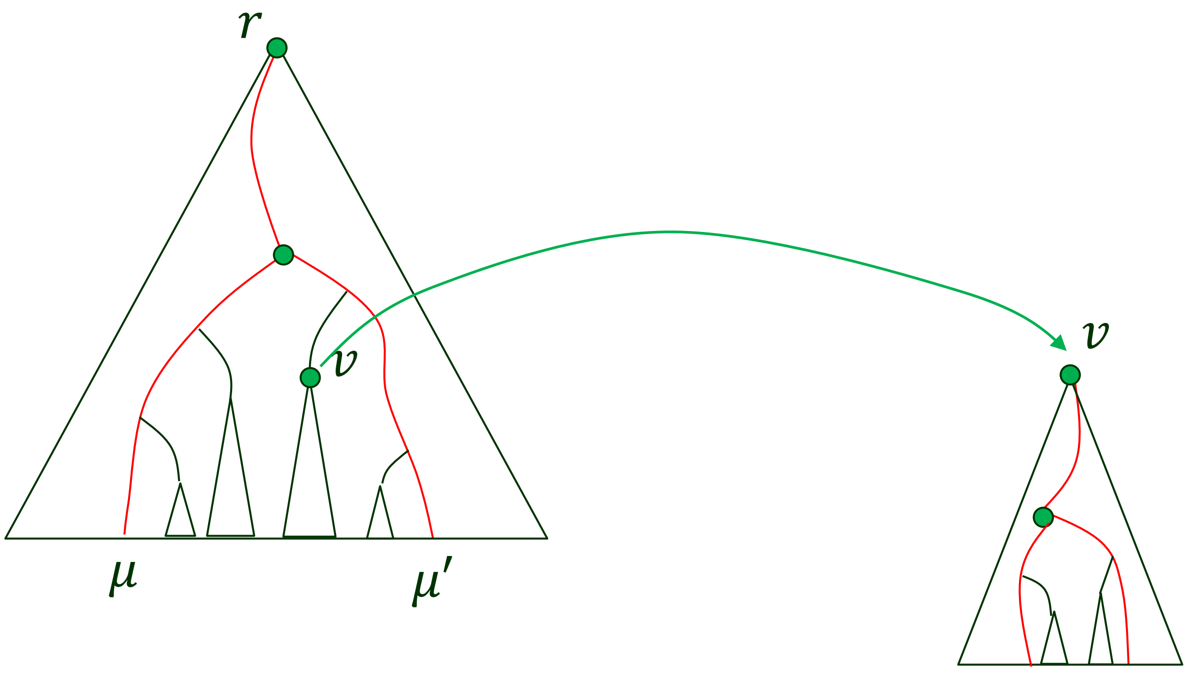
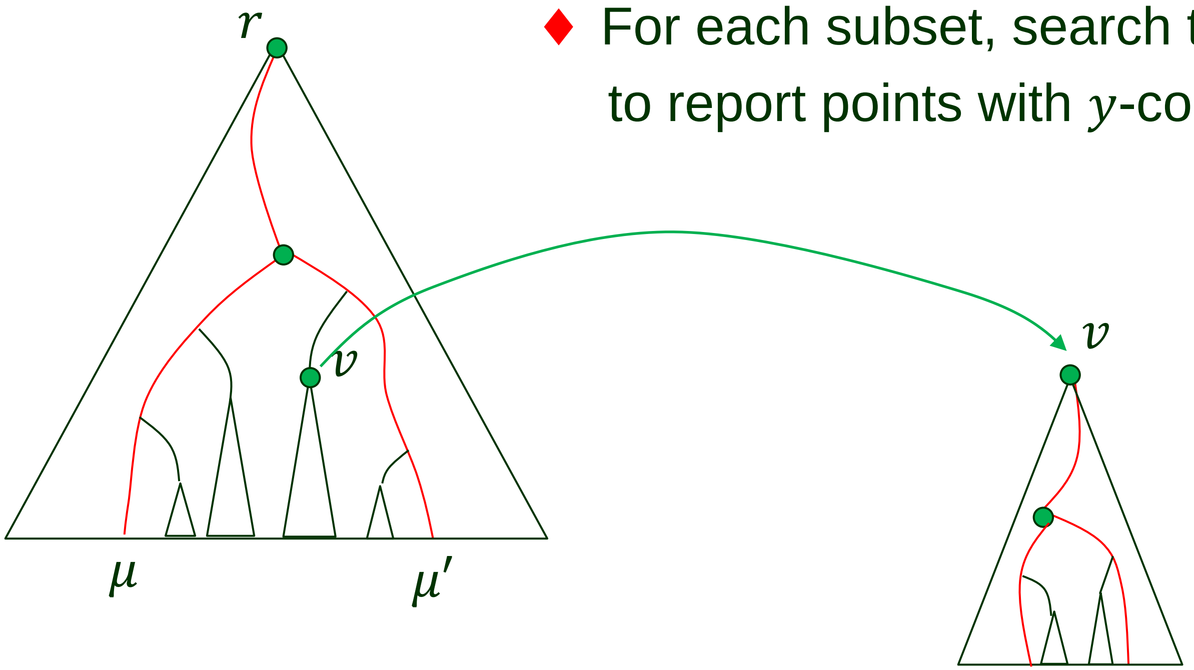


# II. Query

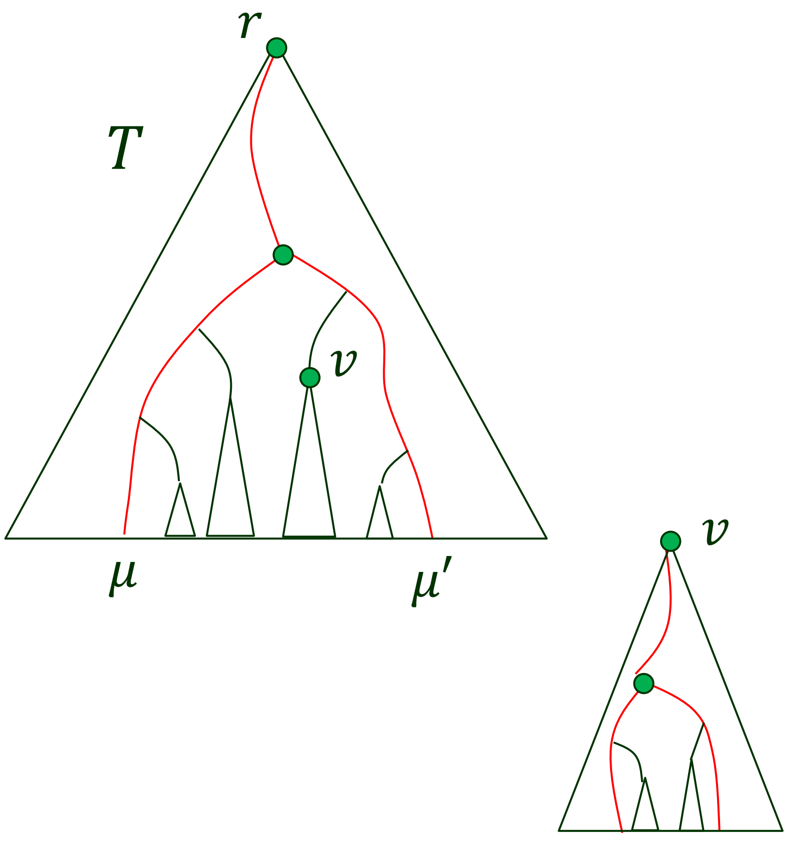
- Selects $O(\log n)$ canonical subsets which together contain points with $x$-coordinate in $[x, x']$.

- For each subset, search the associate structure to report points with $y$-coordinate in $[y, y']$.



# Time of a Recursive Call

A recursive call happens at a node $v$ in the main tree $T$ *if $v$ is a child of some node on one of the two search paths.*



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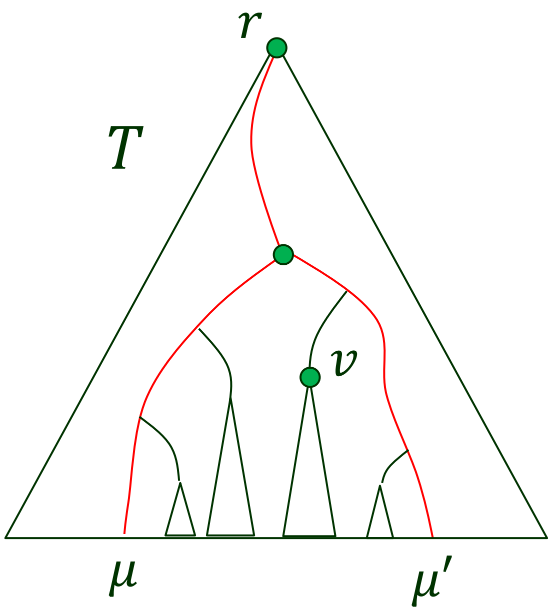
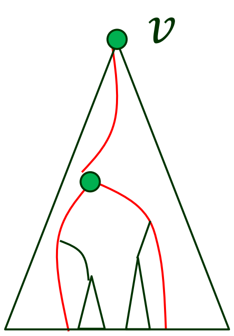



Time of this call:

$$O(\log n + k_v)$$



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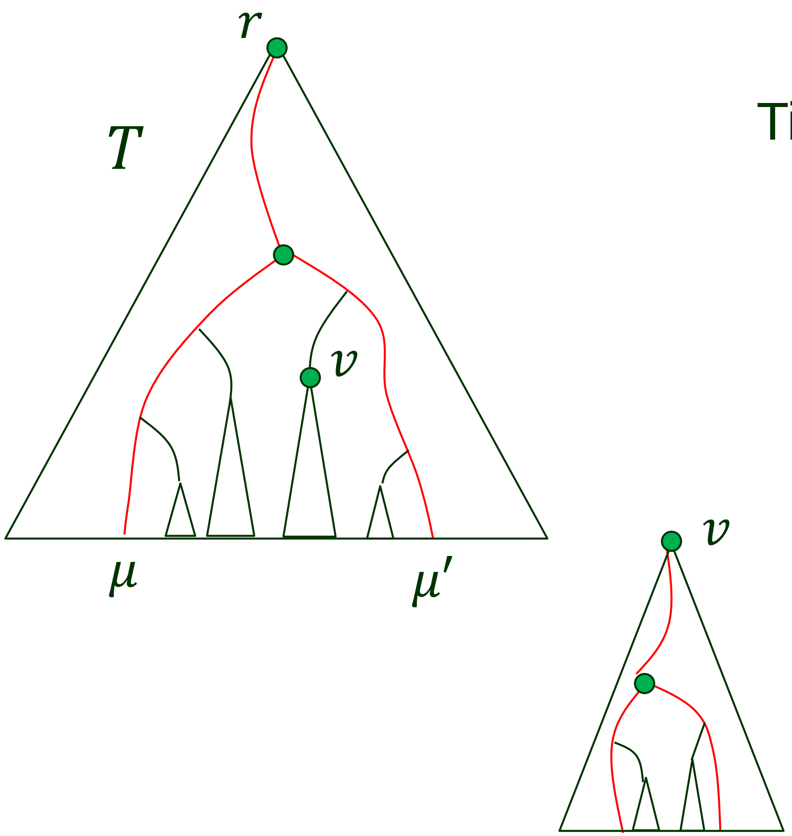


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# points reported in this call

# Query Time

$$\sum_{v} O(\log n + k_v) = \sum_{v} O(\log n) + \sum_{v} k_v$$

child of some node on a
search path in the main tree

$$= \sum_{v} O(\log n) + k$$

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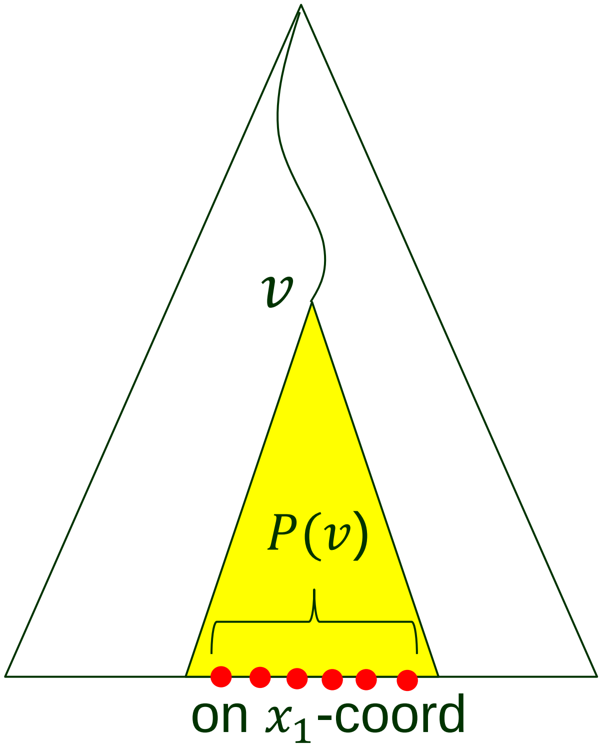
# III. High-Dimensional Range Trees

- Balanced BST (BBST) on $x_1$-coordinate (main tree).



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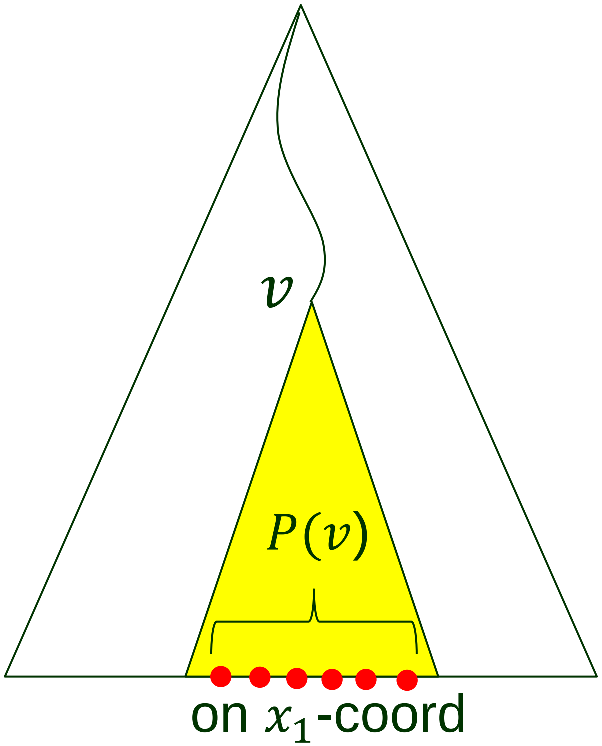

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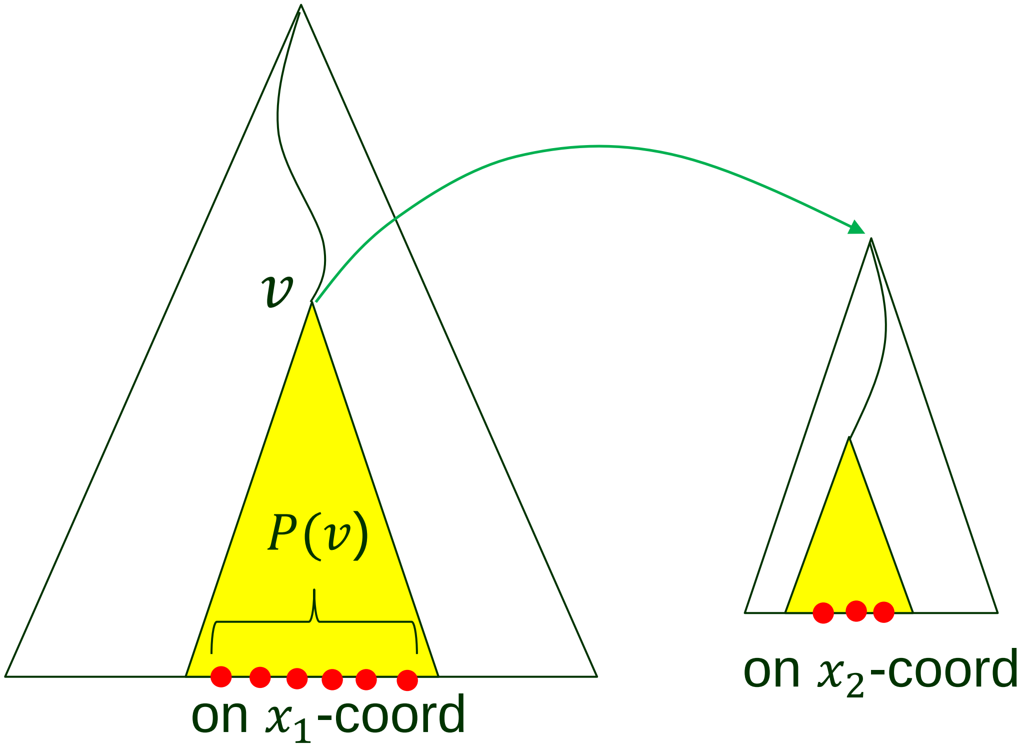

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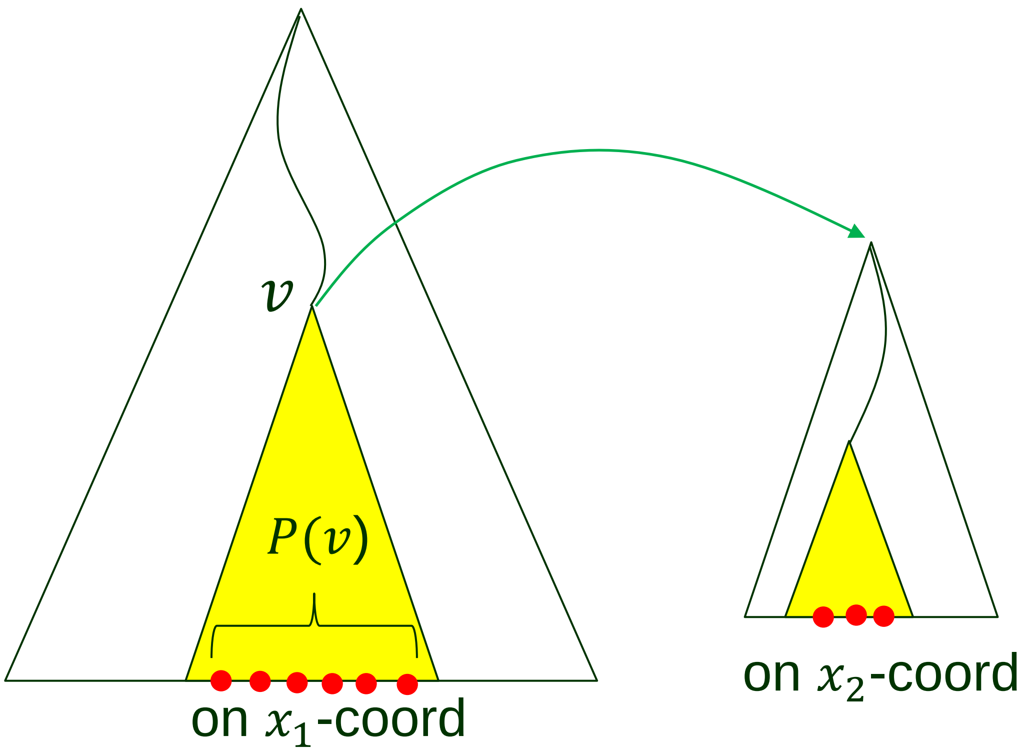

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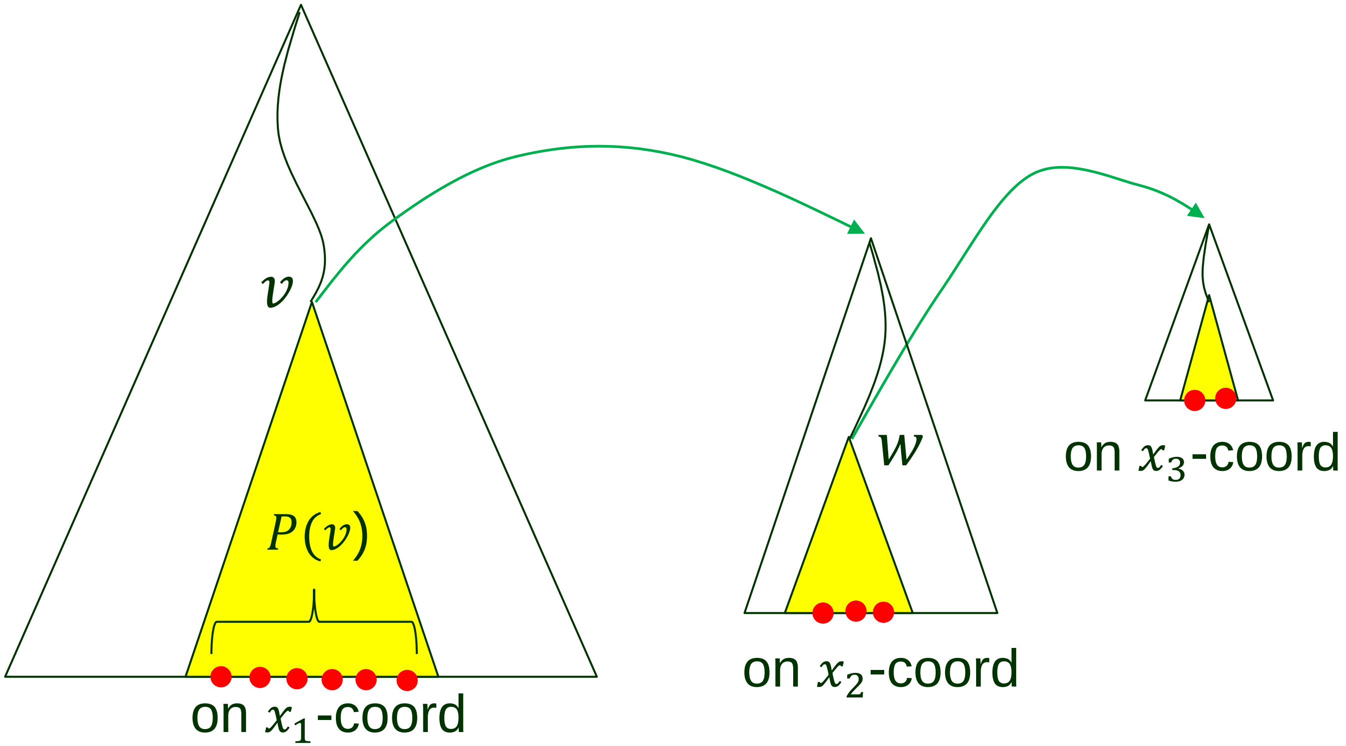

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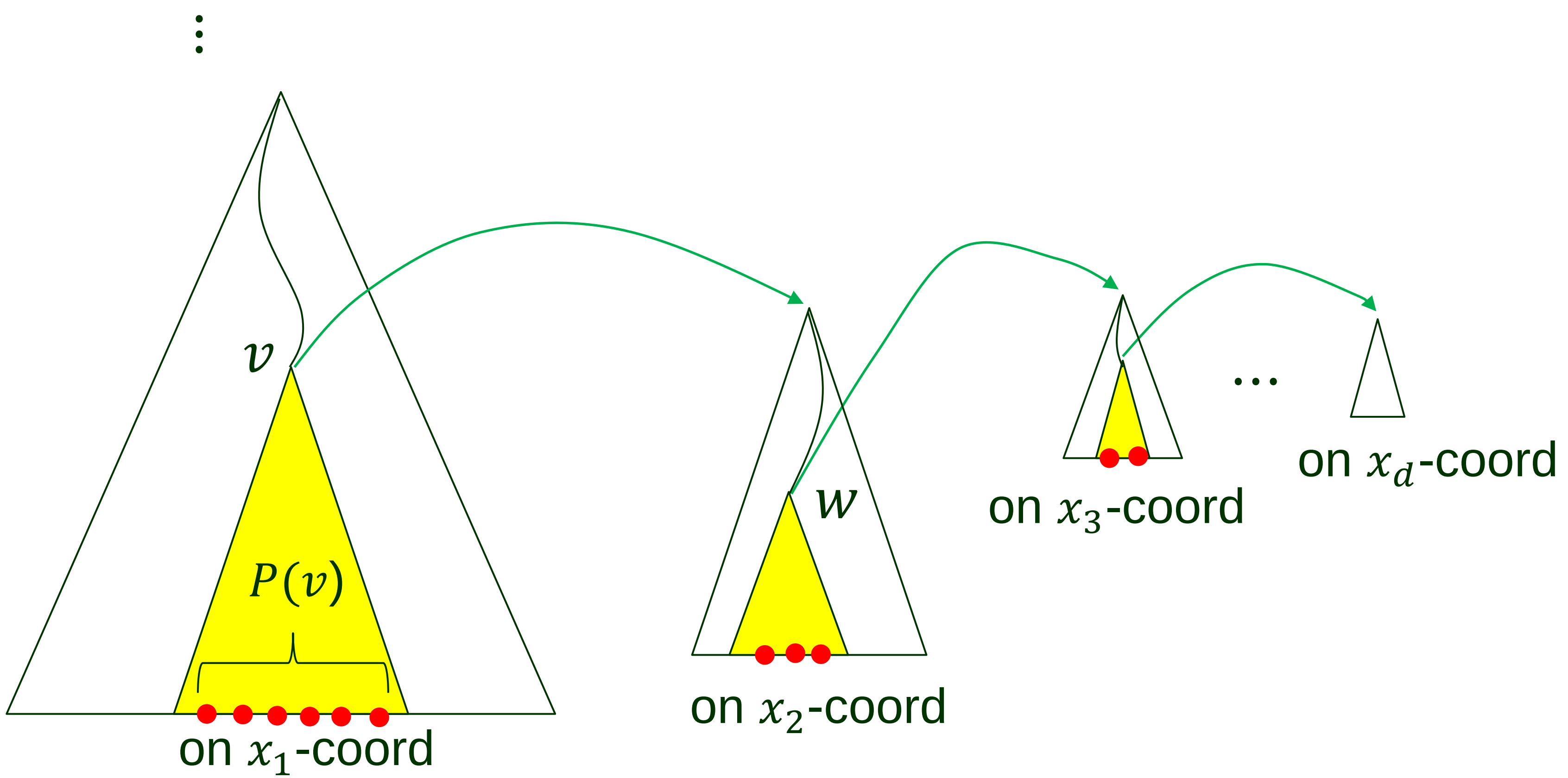

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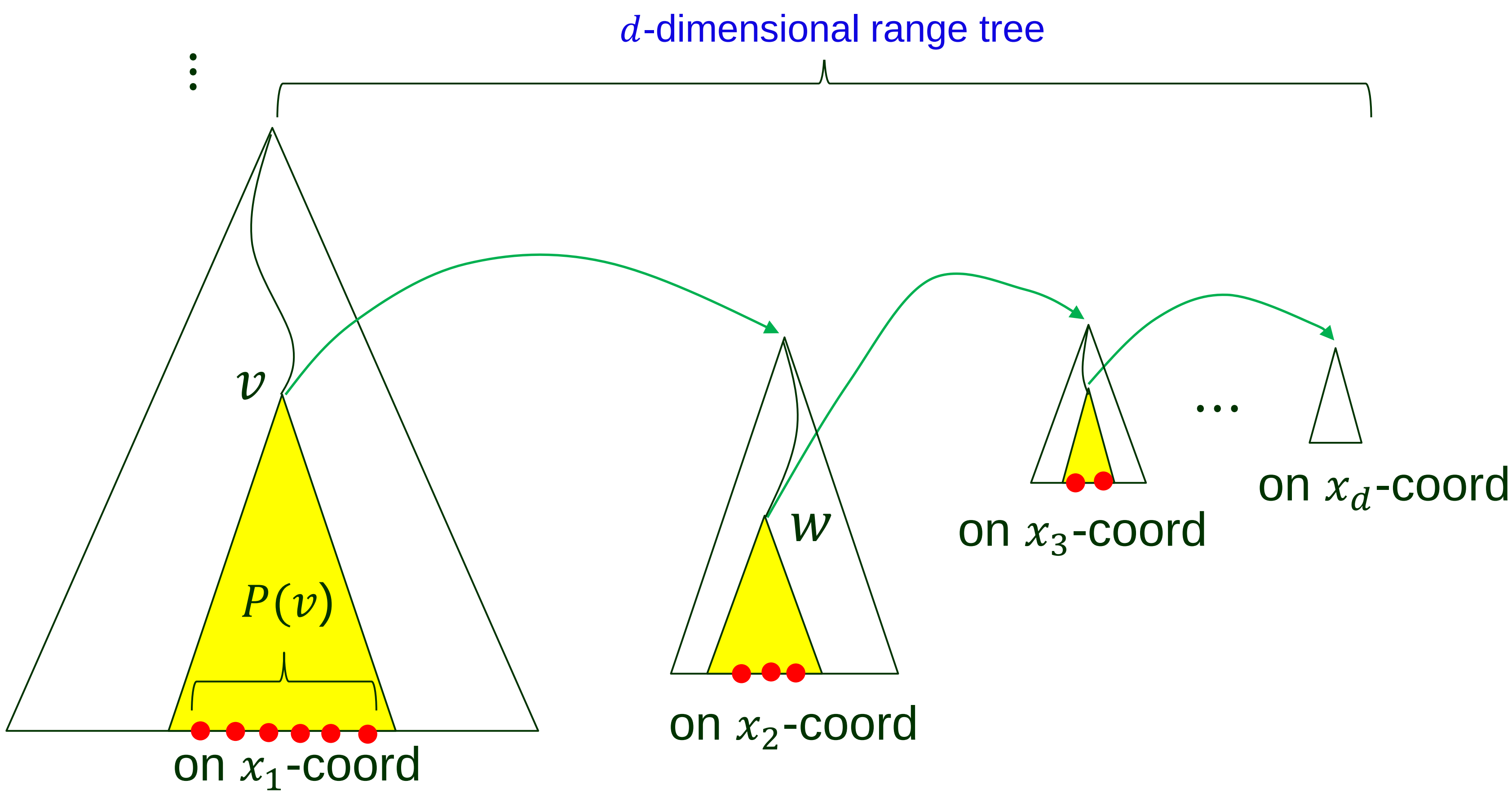

# III. High-Dimensional Range Trees

- Balanced BST (BBST) on $x_1$-coordinate (main tree).
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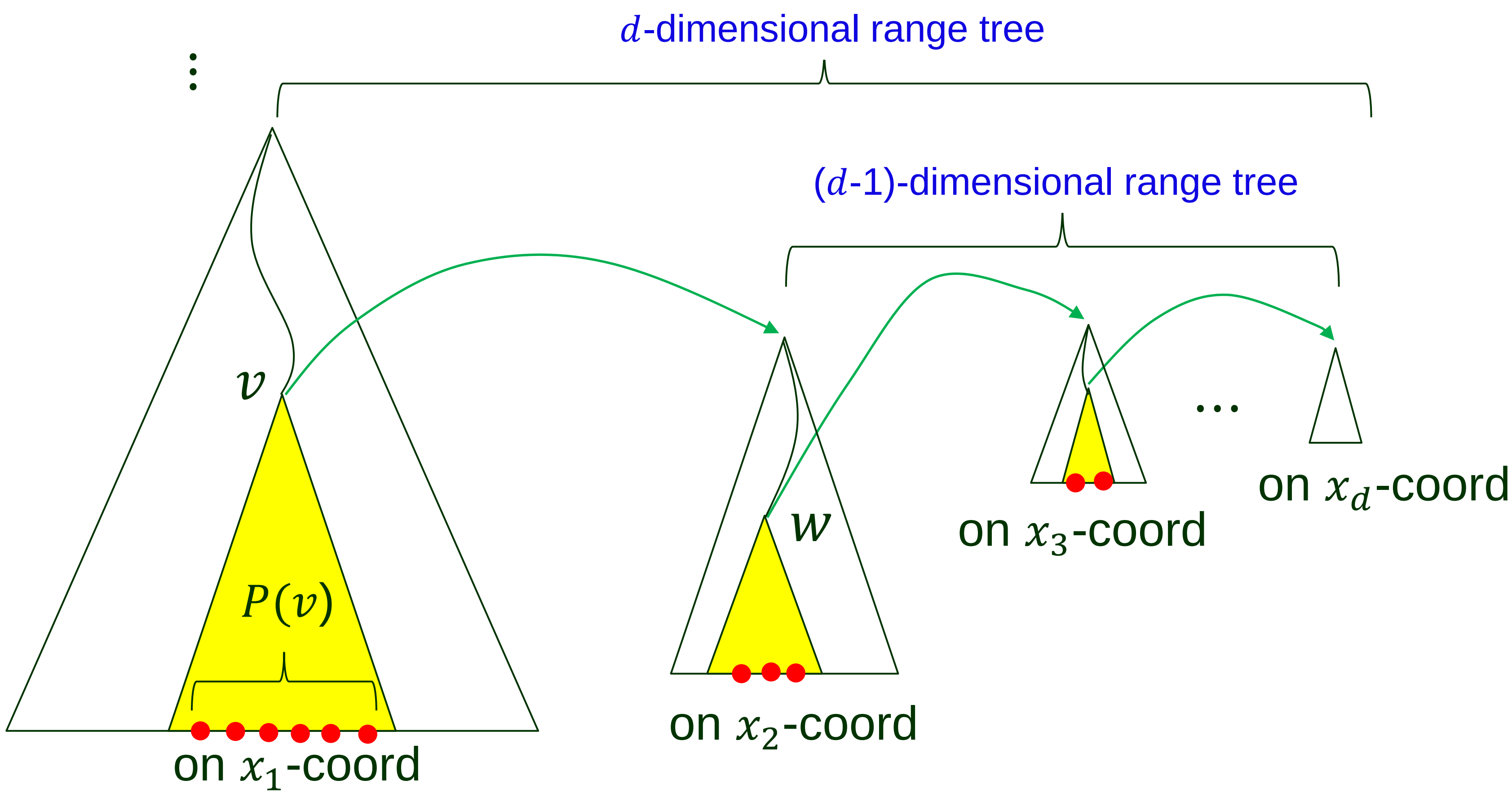

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