Introduction

We are concerned with the critical threshold phenomena in the following Euler-Poisson system,
\[ p_t + (pu)_x = 0, \]
\[ u_t + uu_x = -k\phi_x - vu, \quad \phi_{xx} = \rho - c, \quad \phi |_{x=0} = \phi |_{x=L} = 0, \quad \phi(0) = \phi(L) = 0, \]

on \( \mathbb{R} \times (0, \infty) \) subject to \( C_1 \) initial conditions. Here \( c > 0 \) is the constant background, \( v > 0 \) is the damping coefficient, and \( k \) is the force direction coefficient. \( k > 0 \) signifies regressive force and vice-versa. Finite time breakdown vs global stability of the solution is an important aspect of such systems. This question has been addressed in a series of works.

Key Idea

Along the particle path, \( \Gamma = \{(x(t), x'(t)) = u(x(t), t), x(0) = x \in \mathbb{R} \} \), the dynamics of \( \rho \) and \( d := u_t \) are governed by,
\[ \frac{d^2}{dt^2} + vd = k(p - c). \]

We introduce a change of variables,
\[ r = \frac{d}{p}, \quad s = \frac{1}{p}, \]

Consequently, we obtain the following linear system,
\[ r' = -vr - k(1 - cs), \quad s' = r, \]

with appropriate initial conditions. We employed the method of characteristics to convert the PDE system (1) to ODE system (2). On observation, \( r'(t) > 0 \) \( \forall t > 0 \) globally exists, this linear ODE system can be uniquely solved to obtain the critical threshold. We further characterize the threshold curve using a uniquely defined curve.

Results

Theorem

For the given 1D Euler Poisson system (1), \( \exists (x^*, t^*) \) for which \( \lim_{t \to t^*} u(x(t), t) = -\infty \iff \exists x \in \mathbb{R} \) such that
1. \((v > 2k/c)\) Strong damping
\[ \max\{u_0(x), u_0'(x)\} + \lambda_1(c - p_0(x)) \leq 0, \quad \text{and} \quad \frac{c^2u_0(x) - u_0'(x) - \lambda_1(c - p_0(x))}{k_0(x)} \leq \frac{c^2u_0(x) - u_0'(x) - \lambda_1(c - p_0(x))}{k_0(x)} \]
where \( \lambda_1 = \sqrt{\frac{4v^2 - 4kc}{2c}} \) \( \lambda_2 = \sqrt{\frac{4v^2 - 4kc}{2c}} \).
2. \((v = 2k/c)\) Borderline damping
\[ \max\{u_0(x), u_0'(x)\} + \frac{v(c - p_0(x))}{2c} < 0, \quad \text{and} \quad \ln\left(\frac{2c\nu_x + v(c - p_0(x))}{2c
\[ 2c\nu_x + v(c - p_0(x))} \right) \geq \frac{2c\nu_x + v(c - p_0(x))}{2c\nu_x + v(c - p_0(x))} \]
where \( \nu = \sqrt{\frac{kc - v^2}{4k}} \).
3. \((v < 2k/c)\) Weak damping
\[ \left(\frac{u_0(x) + \sqrt{c(c - p_0(x))}}{2c}\right)^2 \leq \left(\frac{p_0(x) + 2c\nu_x + v(c - p_0(x))}{c}\right)^2 \]
\[ \beta = \frac{2c\nu_x + v(c - p_0(x))}{2c\nu_x + v(c - p_0(x))} \]

Application to an aggregation system [1]

Instead of electric force governed by the Poisson equation, here we have non-local interactions between particles. The system then reads,
\[ \rho_t + (\rho u)_x = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty), \]
\[ u_t + uu_x = -\nabla \cdot (\rho \phi) - \rho \beta, \quad \phi_{xx} = -\rho \beta. \]

subject to \( C_1 \) initial conditions with \( \int_{\mathbb{R}} \rho_0(x) \, dx = M_0 \).

Theorem

For the 1D pressureless damped Euler system of equations (3), there exists a unique global solution \( \rho, u \in C^1(\mathbb{R} \times [0, \infty)) \) \( \forall x \in \mathbb{R} \),
1. (Subcritical mass \( M_0 < 1/4 \))
\[ \|u_0(x), u_0'(x)\| < \|\rho_0(x) - \rho_0' (1/\rho)\|, \rho > 0, \]
where \( R_1(0, \infty) \to \mathbb{R}, \]
\[ \frac{dR_1}{ds} = 1 + \frac{1}{R_1^2} (2 - 2M_0), \quad R_1(0) = 0. \]
2. (Critical mass \( M_0 = 1/4 \))
\[ \|u_0(x), u_0'(x)\| < \|\rho_0(x) - \rho_0' (1/\rho)\|, \rho > 0, \]
where \( R_2(0, \infty) \to \mathbb{R}, \]
\[ \frac{dR_2}{ds} = 1 + \frac{1}{R_2^2} (2 - 2M_0), \quad R_2(0) = 0. \]
3. (Supercritical mass \( M_0 > 1/4 \))
\[ \|u_0(x), u_0'(x)\| < \|\rho_0(x) - \rho_0' (1/\rho)\|, \rho > 0, \]
where \( \gamma = \frac{2}{M_0} \left(\frac{\rho_0(x) - \rho_0' (1/\rho)}{\rho_0(x) - \rho_0' (1/\rho)}\right)^2, \]
and \( R_1(0, \infty) \to \mathbb{R}^+ \cup \{0\} \) is a continuous function satisfying
\[ \frac{dR_1}{ds} = 1 + \frac{1}{R_1} (2 - 2M_0), \quad R_1(0) = 0, \]
and \( R_2(0, \infty) \to \mathbb{R}^+ \cup \{0\} \) is another continuous function satisfying
\[ \frac{dR_2}{ds} = 1 + \frac{1}{R_2} (2M_0 - \gamma - 2), \quad R_2(0) = 0. \]

References