Asymptotic Decay to Relaxation Shock Fronts in Two Dimensions

Hailiang Liu *

Abstract

We prove nonlinear stability of planar shock fronts for certain relaxation system in two spatial dimensions. If the subcharacteristic condition is assumed and the initial perturbation is sufficiently small, the mass carried by perturbations is not necessarily finite, then the solution converges to a shifted planar shock front solution as time $t \uparrow \infty$. The asymptotic phase shift of shock fronts is in general non-zero and governed by a similarity solution to heat equation. The asymptotic decay rate to the shock front is proved to be $t^{-1/4}$ in $L^\infty(\mathbb{R}^2)$ without imposing extra decay rate in space for initial perturbations. The proofs are based on an elementary weighted energy analysis to the error equation.

Key words. relaxation, shock fronts, nonlinear stability, convergence rate
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1 Introduction

In this paper we prove the time-asymptotic stability of relaxation shock fronts under perturbations of infinite mass in two dimensions, where the front is associated with a non-convex flux without the forces transverse to the shock front. We also investigate the time decay rates to the shock fronts. For the specific system considered below we thus extend the result of [10] to the case of nonvanishing phase shift and the result of [12] to the case of two dimensions.

*Department of Mathematics, Henan Normal University, Xinxiang, 453002, P.R.China. Supported in part by the National Natural Science Foundation of China and a Humboldt Fellowship at the Otto-von-Guerick-University Magdeburg.

This paper concerns a simple example of Jin-Xin's relaxation systems, see [5]

$$\begin{aligned} &u_t + v_{1x} + v_{2y} = 0, \qquad (t,x,y) \in \mathrm{IR}^+ \times \mathrm{IR}^2, \\ &\quad v_{1t} + a_1 u_x = f(u) - v_1, \\ &\quad v_{2t} + a_2 u_y = -v_2. \end{aligned} \tag{1.1}$$

The unknows u, v_1, v_2 belong to IR, the function $f = f(u)$ is in C^2, and $a_i > 0,\ i = 1,2$ are fixed constants satisfying the well known subcharacteristic condition, see [8]

$$-\sqrt{a_1} < f'(u) < \sqrt{a_1}, \quad \text{for all } u \text{ under consideration}, \tag{1.2}$$

which plays an essential role in our stability analysis.

The initial data are asymptotically constants as $x \to \pm\infty$, i.e.,

$$(u, v_1, v_2)(0, x, y) = (u_0, v_{10}, v_{20})(x, y) \tag{1.3}$$

satisfy

$$\lim_{x \to \pm\infty} \|(u_0, v_{10}, v_{20})(x, \cdot) - (u_\pm, f(u_\pm), 0)\|_{L^\infty(\mathrm{IR})} = 0 \tag{1.4}$$

with $u_\pm$ being two given constants such that $u_- > u_+$.

A planar relaxation shock front is a solution of form $(u, v_1, v_2) = (U, V_1, V_2)(z)$, $z = x - st$, where $(U, V_1, V_2)(z)$ satisfy

$$\begin{aligned} -sU' + V_1' &= 0, \\ -sV_1' + a_1 U' &= f(U) - V_1, \\ -sV_2' &= -V_2, \end{aligned} \tag{1.5}$$

$$(U, V_1, V_2)(\pm\infty) = (u_\pm, f(u_\pm), 0). \tag{1.6}$$

It can be verified that such a solution exists if the flux function f satisfies the subcharacteristic condition (1.2) and the Oleinik entropy condition

$$Q(u) := f(u) - f(u_\pm) - s(u - u_\pm) < 0, \qquad \text{for } u_+ < u < u_-. \tag{1.7}$$

It is noted that when $s \neq f'(u_\pm)$ condition (1.7) implies the following Lax's shock condition

$$f'(u_+) < s < f'(u_-). \tag{1.8}$$

Here the constants $u_\pm$ and s (shock speed) must satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0. \tag{1.9}$$

We are concerned with the long-time behavior of solutions whose initial data is close to a planar relaxation shock front.

Our purpose in this paper is to study the stability of the shock front with perturbation carrying infinite mass, i.e.,

$$\int_{\mathbb{R}^2} (u_0(x,y) - U(x))dxdy = \infty,$$

in this case the asymptotic state is

$$(U, V_1, V_2)(x - st + d(t,y)),$$

and the effective phase shift $d(t,y)$ in the shock front may have different value at $y = \pm\infty$.

In fact, following [2] and [10], the location of the shock front may be determined by solving a wave equation of $d(t,y)$, which satisfies

$$d(t,y) = \frac{1}{u_+ - u_-}\int_{\mathbb{R}} [u(t,x,y) - U(x)]dx,$$

or equivalently

$$\int_{\mathbb{R}} [u(t,x,y) - U(x + d(t,y))]dx = 0$$

for all $y,\ t$. It will be shown that the effective phase shift $d(t,y)$ is time-asymptotically governed by a solution to the heat equation

$$d_t = a_2 d_{yy},$$

interpolating two different states $d_\pm$ at far fileds $y = \pm\infty$. This fact indicates that, as $t \to \infty$, the wave equation has a parabolic structure, see [16]. Such kind of phenomena was originally observed by Hsiao-Liu [4] for a system of hyperbolic conservations with damping.

The simplest case is $d \equiv const$ which is determined by

$$\frac{1}{u_+ - u_-}\int_{\mathbb{R}} [u(t,x,y) - U(x)]dx = d.$$

This is in consistent with the one-dimensional theory, cf. [8], [11].

The second aim of this paper is to investigate the asymptotic decay rates toward the relaxation shock front. Our result shows that the decay rate of perturbations could not be faster than t^{-1} even if a stronger localization of perturbation may be imposed. However the decay rate is always not slower than $t^{-1/4}$. This is in contrast to the one dimensional case, see [12], [14]. The rate of asymptotic convergence to the viscous shock fronts in two space dimentions was investigated in [18].

In two dimensional cases, Luo and Xin [10] have investigated the stability of weak shock front solution to (1.1) with transverse forces, i.e., system (1.1) with last equation replaced by

$$v_{2t} + a_2 u_y = -\frac{1}{\varepsilon}(v_2 - g(u)).$$

In order to control the effect from $g(u)$ the convexity of f in their case is essentially assumed. They showed that if the initial derivation from a given weak relaxation shock profile is sufficiently small, then the corresponding solution to the relaxation problem approaches the same planar shock front time asymptotically without a phase shift. Our result shows that the asymptotic phase shift is in general non-zero. A nonvanishing phase shift will be studied here for the perturbations with possibly infinite mass. Our proof is given by the elementary weighted energy method following [6], [10] and [12]. It should be mentioned here that in our considered case there is no assumption on the convexity of flux f, and we do not impose any restriction on the shock strength to the price that a more strictly subcharacteristic condition is assumed.

The analysis of the stability of various nonlinear waves for relaxation models, and in particular for Jin and Xin's approximation system, can be found in [1], [8], [9], [11] and [13]. Concerning the planar viscous shock front, see [2], [3] and [17].

The plan of this paper is as follows. Some preliminaries and main theorems will be given in Section 2. In Section 3 the dynamic shock location is analyzed. Energy analysis for the a priori estimates is given in Section 4. The decay rates shall be obtained in the final section.

2 Preliminaries and Main Results

We start by stating the properties of the relaxation shock fronts. Integrating the first equation in (1.5) over $(\pm\infty, z)$ and using the Rankine-Hugoniot condition (1.9), we have by (1.6)

$$(a_1 - s^2)U_z = Q(U), \quad V_1 = sU + f(u_\pm) - su_\pm \quad \text{and} \quad V_2 = 0. \tag{2.1}$$

Due to the subcharacteristic condition (1.2) and the Oleinik shock condition (1.7), the $u-$component U is a monotone function of z, i.e. $U_z < 0$. This fact will be used later in the stability analysis. In the present paper we restrict ourselves to the non-characteristic shock front in the sense that $s \neq f'(u_\pm)$ so that the profile U tends to $u_\pm$ exponentially as $z \to \pm\infty$.

Let us decompose the solution as

$$(u, v_1, v_2)(t, x, y) = (U, V_1, V_2)(z + d(t, y)) + (\phi_z, \psi_1, \psi_2)(t, z, y) \tag{2.2}$$

for $z = x - st$. Substituting (2.2) into (1.1), using the equation (2.1) of the shock front, we obtain

$$\begin{aligned} \phi_{zt} - s\phi_{zz} + U_z d_t + \psi_{1z} + \psi_{2y} = 0,& \\ \psi_{1t} - s\psi_{1z} + V_{1z} d_t + a_1 \phi_{zz} = f(U + \phi_z) - f(U) - \psi_1,& \\ \psi_{2t} - s\psi_{2z} + a_2 U_z d_y + a_2 \phi_{zy} = -\psi_2,& \end{aligned} \tag{2.3}$$

or the higher order equation for ϕ after eliminating ψ_1 and ψ_2 from the above system

$$\begin{aligned} &\Big[(\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - a_1\phi_{zz} - a_2\phi_{yy} + \phi_t - s\phi_z \\ &\quad + f(U + \phi_z) - f(U) - U_z F_1\Big]_z + U_z\Big[d_t + d_{tt} - a_2 d_{yy}\Big] = 0, \end{aligned} \tag{2.4}$$

where

$$F_1 := 2sd_t - d_t^2 + a_2 d_y^2.$$

Equation (2.4) will be integrated in z if the terms proportional to U_z cancel. So let $d(t, y)$ be governed by

$$\mathcal{L}_1(d) := d_t + d_{tt} - a_2 d_{yy} = 0, \tag{2.5}$$

then integrating (2.4) over $(-\infty, z)$ and linearizing the resultant equation around the wave profile U one obtains

$$\mathcal{L}_2(\phi) := (\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - a_1\phi_{zz} - a_2\phi_{yy} + \phi_t + Q'(U)\phi_z = F + F_1 U_z \tag{2.6}$$

where $Q'(U) = f'(U) - s$ and

$$F(U, \phi_z) = f(U + \phi_z) - f(U) - f'(U)\phi_z = O(1)(\phi_z^2)$$

is a higher order term.

From the system (1.1) the corresponding initial data for d and ϕ can be determined as follows

$$d(0, y) := d_0(y) = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} [u_0(x, y) - U(x)]dx, \tag{2.7}$$

$$d_t(0, y) := d_1(y) = \frac{-1}{u_+ - u_-} \int_{\mathbb{R}} \partial_y v_{20}(x, y)dx - s \tag{2.8}$$

and

$$\phi(0, z, y) := \phi_0(z, y) = \int_{-\infty}^{z} [u_0(x, y) - U(x + d_0(y))]\, dx, \tag{2.9}$$

$$\begin{aligned} \phi_t(0, z, y) := \phi_1(z, y) &= s[u_0(z, y) - U(z + d_0(y))] + V_1(x + d_0(y)) - v_{10}(z, y) \\ &- [U(z + d_0(y) - u_-]d_1 - \int_{-\infty}^{z} \partial_y v_{20}(x, y)dx. \end{aligned} \tag{2.10}$$

Therefore, our problem can be reformulated to the system (2.5) of d and (2.6) of ϕ with initial data

$$(d, d_t)(0, x, y) = (d_0, d_1)(x, y), \tag{2.11}$$

$$(\phi, \phi_t)(0, x, y) = (\phi_0, \phi_1)(x, y), \tag{2.12}$$

where (d_0, d_1) and (ϕ_0, ϕ_1) are given in (2.7)-(2.10).

The dynamic location of the planar shock front is determined by the equation (2.5) starting with (d_0, d_1). Assume $(d_0, d_1) \in L^\infty(\mathbb{R}) \times L^1(\mathbb{R})$ with $d_0(\pm\infty) = d_\pm$, let us consider two cases.

If $d_+ = d_-$, then the asymptotic profile for $\int_{\mathbb{R}}[d_0 - d_- + d_1]dy \neq 0$ is

$$D(t, y) = d_- + \int_{\mathbb{R}} [d_0 - d_- + d_1]dyG(t + 1, y) \quad \text{with} \quad G(t, y) = \frac{1}{\sqrt{4\pi a_2 t}} \exp[-\frac{y^2}{4a_2 t}].$$

The solution satisfy

$$(d - d_-, d_t) \in C^0([0, \infty); H^3) \times C^0([0, \infty); H^2)$$

and

$$||d(t) - d_-||_3^2 + ||d_t||_2^2 + \int_0^t ||(d_t, d_z)(\tau)||_2^2 d\tau \leq C\left[||d_0 - d_-||_3^2 + ||d_1||_2^2\right]$$

provided that $(d_0 - d_-, d_1) \in H^3 \times H^2 \cap L^1 \times L^1$.

Now we introduce some notations. $H^l (l \geq 0)$ denotes the usual Sobolev space with norm $\|\cdot\|_l$. Let

$$L^2_w = \Big\{ u | \quad \sqrt{w} u \in L^2 \Big\},$$

with the associated norm

$$|u|_w = \left(\int_{\mathbb{R}^2} w|u|^2 dxdy \right)^{1/2}$$

where $w > 0$ is a weight function. When $C^{-1} \leq w(x,y) \leq C$, the $|\cdot|_w$ reduce to the usual L^2-norm $\|\cdot\|$. When $w = (1+x^2)^{\alpha/2}$, the weighted space is denoted by L^2_α with norm $|\cdot|_w = |\cdot|_\alpha$.

In this case we have

Theorem 2.1. *($d_+ = d_-$) Suppose that the subcharacteristic condition (1.2) is satisfied with suitably large a_1. Let $(U, V_1, V_2)(x - st)$ be the planar relaxation shock profile associated with (u_+, u_-, s) as described above. Assume $(\phi_0, \phi_1) \in H^3(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ and $(d_0 - d_-, d_1) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Then there exists a positive constant δ_1 such that if*

$$\|\phi_0\|_3 + \|\phi_1\|_2 + \|d_0 - d_-\|_3 + \|d_1\|_2 \leq \delta_1,$$

then the problem (1.1)-(1.2) has a unique global solution (u, v_1, v_2) satisfying

$$\lim_{t\to\infty} \sup_{\mathbb{R}^2} \Big| (u, v_1, v_2)(t,x,y) - (U, V_1, V_2)(x - st + d_-) \Big| = 0.$$

Next we consider the case $d_+ \neq d_-$. In such a case the asymptotic profile is given by

$$D(t,y) = \rho\left(\frac{y + y_0}{\sqrt{t+1}}\right) - m_1 \theta(y) e^{-t}$$

with y_0 uniquely determined by the relation

$$\int_{\mathbb{R}} (d_0 + d_1 - \rho(y + y_0)) dy = 0.$$

Here ρ is a similarity solution to the heat equation $d_t = a_2 d_{yy}$ and can be written as

$$\rho(y) = d_- \int_{\frac{y}{\sqrt{4a_2\pi}}}^{+\infty} e^{-\pi z^2} dz + d_+ \int_{-\infty}^{\frac{y}{\sqrt{4a_2\pi}}} e^{-\pi z^2} dz \to d_\pm \quad \text{as} \quad y \to \pm\infty, \qquad (2.13)$$

$m_1 := \int_{\mathbb{R}} d_1 dy$ and $\theta(y)$ is a smooth function with compact support and integral 1.

Further we have the following estimate on, instead of d, the integrated perturbation

$$\eta(t,y)=\int_{-\infty}^{y}[d-D](t,y)dy,$$

$$||\eta(t)||_3^2+||\eta_t||_2^2+\int_0^t ||(\eta_t,\eta_y)(\tau)||_2^2 d\tau \le C\left[||\eta_0||_3^2+||\eta_1||_2^2\right]. \tag{2.14}$$

In this case we will prove the following

Theorem 2.2. *($d_+\neq d_-$) Suppose that the subcharacteristic condition (1.2) is satisfied with a_1 suitably large. Let $(U,V_1,V_2)(x-st)$ be the planar relaxation shock profile associated with (u_+,u_-,s) as described above. Assume $(\phi_0,\phi_1)\in H^3(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ and $(\eta_0,\eta_1)\in H^3(\mathbb{R})\times H^2(\mathbb{R})$. Then there exists a constant $\delta_2>0$ such that if*

$$\|\phi_0\|_3+\|\phi_1\|_2+\|\eta_0\|_3+\|\eta_1\|_2+|m_1|+|d_+-d_-|\le\delta_2,$$

then the problem (1.1)-(1.2) has a unique global solution (u,v_1,v_2) satisfying

$$\lim_{t\to\infty}\sup_{\mathbb{R}^2}\left|(u,v_1,v_2)(t,x,y)-(U,V_1,V_2)(x-st+\rho(\frac{y+y_0}{\sqrt{t+1}}))\right|=0.$$

Theorem 2.3. (Convergence rate) *Let (u,v_1,v_2) be a solution obtained in Theorem 2.2. Asuume that $(\phi_0,u_0-U,v_{10}-V_1,v_{20})\in L^2_\alpha$ for some $\alpha\ge0$ and a_1 is suitably large. Then*

$$\sup_{\mathbb{R}^2}\left|(u,v_1,v_2)(t,x,y)-(U,V_1,V_2)(x-st+\rho(\frac{y+y_0}{\sqrt{t+1}}))\right|\le C(1+t)^{-\min\{1,\,\alpha/2+1/4\}}.$$

with some $C>0$ which does not depend on t.

Corollary 2.4. *Let (u,v_1,v_2) be a solution obtained in Theorem 2.2. Then the following convergence rate estimate holds*

$$\sup_{\mathbb{R}^2}\left|(u,v_1,v_2)(t,x,y)-(U,V_1,V_2)(x-st+\rho(\frac{y+y_0}{\sqrt{t+1}}))\right|\le C(1+t)^{-1/4}.$$

These theorems are proved by combining the local existence with the a priori estimates. For semi-linear hyperbolic systems (2.6) the local existence can be proved in a standard way. The key is to establish a priori estimates by using the elementary energy method, which will be given in Sections 4-5.

3 Dynamic location of the shock front

We first analyze the behavior of the effective phase shift d of the shock front. Consider the general data $(d_0, d_1) \in L^\infty(\mathrm{IR})^2$ with

$$d_0 \to d_\pm, \qquad d_1 \in L^1(\mathrm{IR}) \tag{3.1}$$

by distinguishing the two cases

Case 1. $d_+ = d_-$

In this case the solution can be expressed as

$$d - d_- = \partial_t(S(t)(d_0 - d_-)) + S(t)((d_0 - d_-) + d_1) \tag{3.2}$$

where

$$S(t)(d) = \frac{1}{2\sqrt{a_2}} e^{-t/2} \int_{x-\sqrt{a_2}t}^{x+\sqrt{a_2}t} I_0\left(\frac{1}{2a_2}\sqrt{a_2 t^2 - |x-y|^2}\right) d(y)dy,$$

and $I_0(y)$ is the Bessel function of order zero with imaginary argument, which satisfies

$$I_0''(y) + \frac{1}{y} I_0'(y) - I_0(y) = 0$$

and

$$I_0(y) \sim \sqrt{\frac{1}{2\pi y}} e^y \quad \text{as} \quad y \to \infty.$$

Observe that the first term in (3.2) decays faster than the second, so $d - d_-$ behaves like

$$G(t,\cdot) * (d_0 - d_- + d_1)$$

with

$$G(t,y) = \frac{1}{\sqrt{4\pi a_2 t}} \exp\left(-\frac{y^2}{4a_2 t}\right)$$

and $*$ being convolution.

It is clear that the above time asymptotic profile satisfies the following heat equation

$$d_t = a_2 d_{yy}, \tag{3.3}$$

$$d(x,0) = d_0 + d_1 \to d_-, \quad \text{as} \quad y \to \pm\infty. \tag{3.4}$$

In fact in this case the simple calculations of $\partial_y^k(2.5) \times \partial_y^k[d + 2d_t]$ with $k = 0,1,2$ inductively give the following basic estimate for d.

Lemma 3.1. *Assume that* $(d_0 - d_-, d_1) \in H^3 \times H^2$*, then it holds that*

$$\|(d-d_-)(t)\|_3^2 + \|d_t(t)\|_2^2 + \int_0^t \left[\|d_y\|_2^2 + \|d_t\|_2^2\right] d\tau \le C\left[\|d_0 - d_-\|_3^2 + \|d_1\|_2^2\right]. \quad (3.5)$$

Case 2. $d_+ \neq d_-$

As in case 1, the time-asymptotic profile is expected to be a similarity solution of the heat equation with initial data

$$d(y,0) = d_0(y) + d_1(y) \to d_\pm \quad \text{as} \quad y \to \pm\infty.$$

In fact the expected asymptotic profile can be written explicitly as (2.13) satisfying $\rho \to d_\pm$ as $y \to \pm\infty$.

Assuming that $(d_0 + d_1 - \rho)$ is integrable, then there exists a unique y_0 such that

$$\int_{\mathrm{IR}} (d_0 + d_1 - \rho(y + y_0))dy = 0. \quad (3.6)$$

Following [4] we remind the approach to decompose d. Due to the fact that $\rho_t = a_2\rho_{yy}$ we have

$$(d-\rho)_t + (d-\rho)_{tt} = a_2(d - \rho - \rho_t)_{yy}.$$

The integration over IR yields

$$\frac{d}{dt}\int_{\mathrm{IR}}(d-\rho) = \frac{d}{dt}\int_{\mathrm{IR}}(d-\rho)|_{t=0}\cdot e^{-t} = \int_{\mathrm{IR}} d_1 dy \cdot e^{-t},$$

and hence

$$\frac{d}{dt}\int_{\mathrm{IR}}(d(t,y) - \rho(\frac{y+y_0}{\sqrt{t+1}}) + m_1\theta(y)e^{-t})dy = 0, \quad (3.7)$$

where $m_1 = \int_{\mathrm{IR}} d_1(y)dy$ and $\theta(y)$ is a smooth function with compact support satisfying

$$\int_{\mathrm{IR}} \theta(y)dy = 1.$$

Thus due to (3.6) and (3.7) it holds that

$$\int_{\mathrm{IR}} \left[d(t,y) - \rho(\frac{y+y_0}{\sqrt{t+1}}) + m_1\theta(y)e^{-t}\right]dy = 0, \quad t > 0.$$

Then we reach the setting

$$d(t,y) = \rho(\frac{y+y_0}{\sqrt{t+1}}) - m_1\theta(y)e^{-t} + \eta_y(t,y)$$

with η satisfying

$$\eta_t + \eta_{tt} - a_2\eta_{yy} = F_2 \tag{3.8}$$

where $F_2 = a_2[m_1\theta(y)e^{-t} - \rho_t]_y$.

For later use, let us state some estimates on ρ which can be worked out from the exact expression of ρ in (2.13).

Lemma 3.2. *For* $\xi = \frac{y}{\sqrt{t+1}}$ *we have*

$$\sum_{k=1}^{3} |\frac{d^k}{dy^k}\rho(\xi)| + |\rho - d_+|_{y>0} + |\rho - d_-|_{y<0} \leq O(1)|d_+ - d_-|e^{-c\xi^2}$$

and

$$\|\partial_y^k \partial_t^q \rho(\frac{y}{\sqrt{t+1}})\|_{L^p}^p \leq O(1)|d_+ - d_-|(1+t)^{\frac{1}{2}-\frac{p}{2}(1+k+2q)}, \quad \text{for} \quad k,\ q \geq 0.$$

Moreover,

$$\int_0^t \|\partial_y^k \partial_t^q \rho(\frac{y}{\sqrt{t+1}})\|_{L^p}^p d\tau \leq O(1)|d_+ - d_-|^p \quad \textit{for} \quad p > \frac{3}{1+k+2q}. \tag{3.9}$$

Let us assume $(\eta_0, \eta_1) \in H^3 \times H^2$, then we have a local solution η of (3.8) with initial data $(\eta, \eta_t)|_{t=0} = (\eta_0, \eta_1)$.

To establish the convergence rates, we prepare the following estimate.

Lemma 3.3. *Suppose that both* $\delta = |d_+ - d_-| + |m_1|$ *and* $\|\eta_0\|_3 + \|\eta_1\|_2$ *are sufficiently small. Then there exists a unique global solution* (η, η_t) *of (3.8) satisfying*

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^k\|\partial_y^k\eta(t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_y^k\partial_t\eta(t)\|^2 \\ &\quad + \int_0^t \left[\sum_{j=1}^{3}(1+\tau)^{j-1}\|\partial_y^j\eta(\tau)\|^2 + \sum_{j=0}^{2}(1+\tau)^{j+1}\|\partial_y^j\partial_t\eta(\tau)\|^2\right] d\tau \\ &\leq C\left[\|\eta_0\|_3^2 + \|\eta_1\|_2^2 + \delta\right].\end{aligned} \tag{3.10}$$

Proof. Define

$$N(T) = \sup_{0\leq t\leq T}\left\{\sum_{q=0}^{2}\sum_{k=0}^{3-q}(1+t)^{k+2q}\|\partial_y^k\partial_t^q\eta(t)\|^2 + (1+t)^5\|\partial_t^3\eta(t)\|^2\right\}^{1/2}.$$

Following [16], we multiply $\partial_y^k\partial_t^q(3.8)$ by $\bar\eta_t$ and $\bar\eta+2\bar\eta_t$ with $\bar\eta=\partial_y^k\partial_t^q\eta$ and integrate the resultant equation over IR, respectively, to obtain

$$\frac{d}{dt}\left[\|\bar\eta_t(t)\|^2+a_2\|\bar\eta_y(t)\|^2\right]+2\|\bar\eta_t\|^2=2\int_{\mathrm{IR}}\bar F_2\cdot\bar\eta_t dy, \tag{3.11}$$

$$\frac{d}{dt}\left[\|\bar\eta_t(t)\|^2+\langle\eta,\eta_t\rangle+\frac{1}{2}\|\eta(t)\|^2+a_2\|\bar\eta_y(t)\|^2\right]+\|\bar\eta_t(t)\|^2+a_2\|\bar\eta_y(t)\|^2$$
$$=\int_{\mathrm{IR}}\bar F_2\cdot(\bar\eta+2\bar\eta_t)dy \tag{3.12}$$

where $\bar F_2=\partial_y^k\partial_t^q F_2$. The right hand side of (3.11) is bounded by

$$\begin{aligned}\left|2\int_{\mathrm{IR}}\bar F_2\bar\eta dy\right| &\le C\|\bar\eta_t\|_1\int_{\mathrm{IR}}\left[|m_1|e^{-t}+|\partial_y^{k+1}\partial_t^{q+1}\rho|\right]dy\\ &\le CN(T)(1+t)^{-k/2-q-1}[|m_1|e^{-t}+|d_+-d_-|(1+t)^{-3/2-k/2-q}]\\ &\le CN(T)\delta(1+t)^{-5/2-k-2q},\end{aligned}$$

where we have use the definition of $N(T), F_2$ in (3.8) and the Lemma 3.2. Likewise, the right hand side of (3.12) is bounded by

$$CN(T)\delta(1+t)^{-3/2-k-2q}.$$

Multiplying (3.11) by $(1+t)^{1+k+2q}$ and integrating the resultant inequality over $[0,t]$ leads to

$$(1+t)^{1+k+2q}\|(\bar\eta_t,\bar\eta_y)(t)\|^2+\int_0^t(1+\tau)^{1+k+2q}\|\bar\eta_t(\tau)\|^2d\tau \tag{3.13}$$
$$\le C\left(\|(\bar\eta_t,\bar\eta_y)(0)\|^2+\int_0^t(1+\tau)^{k+2q}\|(\bar\eta_t,\bar\eta_y)(\tau)\|^2d\tau+\delta N(T)\int_0^t(1+\tau)^{-3/2}d\tau\right).$$

Similar calculations to $(1+t)^i(3.12)$ with $0\le i\le k+2q$ yields

$$\begin{aligned}&(1+t)^i\|(\bar\eta,\bar\eta_t,\bar\eta_y)(t)\|^2+\int_0^t(1+\tau)^i\|(\bar\eta_t,\bar\eta_y)(\tau)\|^2d\tau\\ &\le C\left(\|(\bar\eta,\bar\eta_t,\bar\eta_y)(0)\|^2+i\int_0^t(1+\tau)^{i-1}\|(\bar\eta,\bar\eta_t,\bar\eta_y)(\tau)\|^2d\tau\right.\\ &\quad\left.+\delta N(T)\int_0^t(1+\tau)^{-3/2-k-2q+i}d\tau\right).\end{aligned} \tag{3.14}$$

Using the induction to the above inequality in the way that for a fixed $q\in\{0,1,2\}$ we estimate (3.14) with $k=2-q$ respectively, then obtain a final inequality for $i=k+2q$

by using the estimates for $i = 0, \cdots, k+2q-1$

$$
\begin{aligned}
&(1+t)^{k+2q}\|\partial_y^k \partial_t^q (\eta, \eta_t, \eta_y)(t)\|^2 + \int_0^t (1+\tau)^{k+2q}\|\partial_y^k \partial_t^q (\eta_t, \eta_y)(\tau)\|^2 d\tau \\
&\quad \le C\left[\|\eta_0\|_3^2 + \|\eta_1\|_2^2 + \delta\right].
\end{aligned}
$$

Based on this one obtains from (3.13)

$$
\begin{aligned}
&(1+t)^{k+2q+1}\|\partial_y^k \partial_t^q (\eta_t, \eta_y)\|^2 + \int_0^t (1+\tau)^{k+2q+1}\|\partial_y^k \partial_t^q (\eta_t, \eta_y)(\tau)\|^2 d\tau \\
&\quad \le C\left[\|\eta_0\|_3^2 + \|\eta_1\|_2^2 + \delta\right].
\end{aligned}
\tag{3.15}
$$

Collecting all the estimates completes the proof of Lemma 3.3. □

4 Energy Estimates

In this section we devote ourselves to the estimates of the solution $\phi(t, z, y)$, $0 < t < T$, of

$$
\mathcal{L}_2(\phi) := (\phi_t - s\phi_z)_t - s(\phi_t - s\phi_z)_z - a_1\phi_{zz} - a_2\phi_{yy} + \phi_t + Q'(U)\phi_z = F + F_1 U_z \tag{4.1}
$$

in the space $X(0,T)$, which is defined as

$$
\begin{aligned}
X(0,T) = \Big\{ &(\phi, \phi_z) \in C^0([0,T]; H^3(\mathbb{R}^2)) \cap C^1([0,T]; H^2(\mathbb{R}^2)), \\
&(\phi_t, \nabla\phi) \in L^2([0,T]; H^2) \Big\}
\end{aligned}
$$

under some a priori assumptions $N_i(T) \le \epsilon$, $(0 < \epsilon << 1)$ for a given $T > 0$. To simplify the presentation we first analyze the case $d_+ = d_-$ and then extend to the general case $d_+ \ne d_-$.

4.1 A priori estimate for the case $d_+ = d_-$

One may assume $d_- = 0$ by shifting $d \to d + d_-$. Putting

$$
N_1(T) \equiv \sup_{0 \le t \le T} \left\{ \|\phi(t)\|_3^2 + \|\phi_z(t)\|_3^2 + \|\phi_t\|_2^2 + \|\phi_{tz}\|_2^2 + \|d_0 - d_-\|_3^2 + \|d_1\|_2^2 \right\}^{1/2}.
$$

We first establish the basic estimate:

Lemma 4.1. *Let ϕ be a solution to (4.1) in $X(0,T)$ for a positive constant T. Then for suitably large a_1 there exists a constant $\delta_3 > 0$ such that if $N_1(T) \le \delta_3$, then $\phi(t,z,y)$ satisfies*

$$\begin{aligned}
&\|(\phi,\phi_t,\nabla\phi)(t)\|^2 + \int_0^t \|(\phi_t,\nabla\phi)(\tau)\|^2 d\tau + \int_0^t \int_{\mathbb{R}^2} |U_z|\phi^2 dz dy d\tau \\
\le & C\left[\|(\phi,\phi_t,\nabla\phi)(0)\|^2 + \int_0^t [\|d_t\|_1^2 + \|d_y\|_1^2] d\tau\right]
\end{aligned} \tag{4.2}$$

for $0 \le t \le T$.

Proof. Let $H(u)$ be a strictly convex function satisfying $H(u_\pm) = 0$, $H'(u_\pm) \ne 0$ and $H(u) < 0$. The weight function

$$w(U) = \frac{H(U)}{Q(U)} \tag{4.3}$$

is taken so that we could treat the nonconvexity of the flux function f, see [15].

Using the fact $H'(u_\pm) \ne 0$ we have for suitable constant $C > 0$

$$C^{-1} \le w(U) \le C, \quad |w'(U)| \le C. \tag{4.4}$$

Multiplying (4.1) by $2w(U)\phi$, we have

$$\begin{aligned}
&\Big[w\phi^2 + 2w\phi(\phi_t - s\phi_z) - (w_t - sw_z)\phi^2\Big]_t - 2w(\phi_t - s\phi_z)^2 + 2a_1 w\phi_z^2 + 2a_2 w\phi_y^2 \\
&+ A\phi^2 + \{\cdots\}_z + \{\cdots\}_y = 2(F + F_1 U_z)w\phi.
\end{aligned} \tag{4.5}$$

Here the abbreviation A reads

$$\begin{aligned}
A &= -w_t - (Q'(U)w)_z + (w_t - sw_z)_t - s(w_t - sw_z)_z - a_1 w_{zz} - a_2 w_{yy} \\
&= \big[(s^2 - a_1)w_{zz} - (Q'(U)w)_z\big] + \big[w_{tt} - 2sw_{zt} - a_2 w_{yy} - w_t\big] \\
&= A_1 + A_2.
\end{aligned}$$

By virtue of the above choice of weight w in (4.3) one gets

$$A_1 = \big[(s^2 - a_1)w_{zz} - (Q'(U)w)_z\big] = -H'' U_z \ge \mu |U_z| > 0,$$

for U is monotone decreasing in z and H is a convex function satisfying $H'' \ge \mu > 0$. Furthermore, using the equation (2.5) of d, $w_t = w_z d_t$ and $w_z = w' U_z$, one obtains

$$A_2 = [w_{tt} - 2sw_{zt} - a_2 w_{yy} - w_t] = -F_1 w_{zz} - 2d_t w_z.$$

Next multiplying the equation (4.1) by $2w(\phi_t - s\phi_z)$ yields

$$\begin{aligned}
&\Big[w(\phi_t - s\phi_z)^2 + a_1 w\phi_z^2 + a_2 w\phi_y^2\Big]_t + \Big[2w - w_t + sw_s\Big](\phi_t - s\phi_z)^2 \\
&+ 2\Big[a_1 w_z + f'(U)w\Big]\phi_z(\phi_t - s\phi_z) + 2a_2 w_y\phi_y(\phi_t - s\phi_z) \\
&- a_1(w_t - sw_s)\phi_z^2 - a_2(w_t - sw_z)\phi_y^2 + \{\cdots\}_y + \{\cdots\}_z \\
=&2(F + F_1U_z)(\phi_t - s\phi_z)w. \qquad (4.6)
\end{aligned}$$

Integrating (4.5) + 2 × (4.6) over $\mathbb{R}^2 \times [0,t]$ yields

$$\begin{aligned}
\int_{\mathbb{R}^2}(B_1(t))dzdy + \int_0^t\int_{\mathbb{R}^2}(B_2)dzdyd\tau + \int_0^t\int_{\mathbb{R}^2}|U_z||\phi|^2dzdyd\tau \\
\leq \int_{\mathbb{R}^2}(B_1(0))dzdy + \int_0^t\int_{\mathbb{R}^2}(B_3) + (B_4)dzdyd\tau, \qquad (4.7)
\end{aligned}$$

where the individual terms B_i, $i = 1, \cdots, 4$, are

$$\begin{aligned}
(B_1) &:= w\left[(\phi_t - s\phi_z)^2 + \phi(\phi_t - s\phi_z) + \left(\frac{1}{2} - w^{-1}(w_t - sw_z)\right)\phi^2 + a_1\phi_z^2 + a_2\phi_y^2\right], \\
(B_2) &:= w\Big[(\phi_t - s\phi_z)^2 + 2f'(U)\phi_z(\phi_t - s\phi_z) + a_1\phi_z^2 + a_2\phi_y^2\Big], \\
(B_3) &:= (w_t - sw_z)\Big[(\phi_t - s\phi_z)^2 + a_1\phi_z^2 + a_2\phi_y^2\Big] - 2\big[a_1w_z\phi_z + a_2w_y\phi_y\big](\phi_t - s\phi_z), \\
(B_4) &:= \big[F_1w_{zz} + 2d_tw_z\big]\phi^2 + 2(F + F_1U_z)\big[\phi + 2(\phi_t - s\phi_z)\big]w.
\end{aligned}$$

Since $w(U) \sim C$, $|w'(U)| \sim C$ and $(a_1 - s^2)U_z = Q(U)$ together with the boundedness of d_t from Lemma 3.1, we obtain

$$|w_t - sw_z| = |(d_t - s)w'\frac{Q(U)}{a_1 - s^2}| \leq \frac{1}{12}w$$

for a_1 suitably large. In order to handle the "mixed terms" in B_1, let us use the Young inequality with $\epsilon = 3/4$ to obtain

$$\begin{aligned}
(B_1) &\geq w\left[(1-\epsilon)(\phi_t - s\phi_z)^2 + \frac{1}{2}(1 - \frac{1}{2\epsilon} - \frac{1}{6})\phi^2 + a_1\phi_z^2 + a_2\phi_y^2\right] \\
&\geq w\left[\frac{1}{4}(\phi_t - s\phi_z)^2 + \frac{1}{12}\phi^2 + a_1\phi_z^2 + a_2\phi_y^2\right]. \qquad (4.8)
\end{aligned}$$

Again using the Young inequality and the subcharacteristic condition $(f')^2 < a_1$ we have

$$(B_2) \geq w\left[\frac{1}{4}(\phi_t - s\phi_z)^2 + \frac{1}{12}\phi^2 + \frac{2}{3}a_1\phi_z^2 + a_2\phi_y^2\right]. \qquad (4.9)$$

Noting that $\left|\frac{w_z}{w}\right|^2 \leq C$, the estimate on d and

$$|w_z| = |w' \frac{Q(U)}{a_1 - s^2}| \leq C a_1^{-1}, \quad |w_y| \leq |w_z||d_y| \leq C a_1^{-1},$$

then the third term (B_3) may be estimated as

$$\begin{aligned}(B_3) \leq & |w_z| \Big| (d_t - s)\big[(\phi_t - s\phi_z)^2 + a_1\phi_z^2 + a_2\phi_y^2\big] - 2(a_1\phi_z + a_2 d_y \phi_y)(\phi_t - s\phi_z) \Big| \\ \leq & C a_1^{-1} \big[(\phi_t - s\phi_z)^2 + a_1\phi_z^2 + a_2\phi_y^2\big]. \end{aligned} \tag{4.10}$$

Next we turn to estimate every terms involved in $\int_{\mathbb{R}^2}(B_4)dzdy$.

$$\begin{aligned}\left|\int_{\mathbb{R}^2} F_1 w_{zz}\phi^2 dydz\right| = & \left|\int_{\mathbb{R}^2} 2F_1 w_z \phi\phi_z dydz\right| \\ \leq & \frac{a_1}{6}\int_{\mathbb{R}^2} w\phi_z^2 dydz + C\int_{\mathbb{R}^2} F_1^2 w_z^2 / w\phi^2 dydz. \end{aligned} \tag{4.11}$$

Noting that $\int_{\mathbb{R}^2} w\phi^2 dydz \leq CN_1(T)^2$ and $\sup_y F_1^2 \leq \int_R (F_1^2 + F_{1y}^2)dy$, the last term in (4.11) is bounded from above by $CN_1(T)^2\|F_1\|_1^2$. Thus

$$\left|\int_{\mathbb{R}^2} F_1 w_{zz}\phi^2 dydz\right| \leq \frac{a_1}{6}\int_{\mathbb{R}^2} w\phi_z^2 dydz + CN_1(T)^2\|F_1\|_1^2.$$

Similarly, the second term in $\int_{\mathbb{R}^2}(B_4)dzdy$ is majored by

$$\left|\int_{\mathbb{R}^2} 2d_t w_z \phi^2 dydz\right| \leq \frac{a_1}{6}\int_{\mathbb{R}^2} w\phi_z^2 dydz + CN_1(T)^2\|d_t\|_1^2.$$

Taking the nonlinear higher order term $F = O(1)\phi_z^2$ into consideration and using the following Sobolev inequality

$$\sup_{0\leq t\leq T}[|\phi(\cdot,t)|_{C^1} + |\phi_z(\cdot,t)|_{C^1}] \leq CN_1(T),$$

one gets

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} |F||\phi + 2(\phi_t - s\phi_z)|w dydz\right| \\ &\leq C\sup_{\mathbb{R}^2}|\phi + 2(\phi_t - s\phi_z)| \int_{\mathbb{R}^2} w\phi_z^2 dydz \leq CN_1(T)\int_{\mathbb{R}^2} w\phi_z^2 dydz.\end{aligned}$$

The last term in $\int_{\mathbb{R}^2}(B_4)dzdy$ is estimated as

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} |F_1 U_z||\phi + 2(\phi_t - s\phi_z)|w dydz\right| \\ &\leq \int_{\mathbb{R}} |F_1| \left(\int_{\mathbb{R}} w|U_z|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}} w(\phi, \phi_t, \nabla\phi)^2) dz\right)^{1/2} dy.\end{aligned} \tag{4.12}$$

Note that the decaying property of the shock front U, $U_z \sim exp(-c|z|)$ as $z \to \pm\infty$, leads to $\int_{\mathbb{R}} w(U)|U_z|^2 dz \le C$. Thus (4.12) is further majored by

$$
\begin{aligned}
C \int_{\mathbb{R}} |F_1| & \left(\int_{\mathbb{R}} w(\phi,\phi_t,\nabla\phi)^2 dz \right)^{1/2} dy \\
& \le C \left(\int_{\mathbb{R}} F_1^2 dy \right)^{1/2} \left(\int_{\mathbb{R}^2} w(\phi,\phi_t,\nabla\phi)^2 dy dz \right)^{1/2} \\
& \le C N_1(T) \|F_1\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Combining the above estimates we get

$$
\int_{\mathbb{R}^2} (B_4) dy dz \le \left[\frac{a_1}{3} + C N_1(t) \right] \int_{\mathbb{R}^2} w\phi_z^2 dy dz + C N_1(t)(\|F_1\|_1^2 + \|F_1\|).
$$

Thus

$$
\begin{aligned}
|(\phi,\phi_t,\nabla\phi)(t)|^2_{w(U)} &+ \int_0^t |(\phi_t,\nabla\phi)(\tau)|^2_{w(U)} d\tau + \int_0^t \|\sqrt{-U_z}\phi(\tau)\|^2 d\tau \\
& \le C \left\{ |(\phi,\phi_t,\nabla\phi)(0)|^2_{w(U)} + N_1(T) \int_0^t \|F_1\|_1^2 d\tau + N_1(T) \int_0^t \|F_1\| d\tau \right\}.
\end{aligned}
$$

Note that $F_1 = 2sd_t - d_t^2 + a_2 d_y^2$ and the fact that $\int_0^t \|d_t\| d\tau$ is not uniformly bounded with respect to t and nor is $\int_0^t \|F_1\| d\tau$, we have to treat the following term separately

$$
\begin{aligned}
\left| \int_{\mathbb{R}^2} |d_t U_z||\phi| w dy dz \right| &\le \epsilon_1 \int_{\mathbb{R}^2} |U_z|\phi^2 dy dz + C \int_{\mathbb{R}^2} |U_z| w^2 d_t^2 dz dy \\
&\le \epsilon_1 \int_{\mathbb{R}^2} |U_z|\phi^2 dz dy + C\|d_t\|^2.
\end{aligned}
$$

for suitably small $\epsilon_1 > 0$. Thus the terms in (4.12) involving F_1 are bounded from above by

$$
C \int_0^t \|(d_t, d_y)\|_1^2 d\tau.
$$

Plugging the above estimates into (4.7) and using $w \sim C$ yields the desired basic estimate (4.2). □

Next we sketch the proof of the higher order estimates with notation $D^l = \partial_z^{l_1}\partial_y^{l_2}$, $l = l_1 + l_2$.

Lemma 4.2. *It holds that*

$$
\begin{aligned}
\sum_{|l|=1} & \left[\|D^l(\phi,\phi_t,\nabla\phi)(t)\|^2 + \int_0^t \|D^l(\phi_t,\nabla\phi)(t)\|^2 \| d\tau \right] \\
& \le C \left\{ \|\phi,\phi_t,\nabla\phi)(0)\|_1^2 + \int_0^t \|(d_t,d_y)\|_1^2 d\tau \right\}. \qquad (4.13)
\end{aligned}
$$

Proof. Differentiating $\mathcal{L}_2(\phi) = F + F_1 U_z$ with respect y and z, respectively, leads to

$$\mathcal{L}_2(\phi_y) = -Q'(U)_z d_y \phi_z + F_y + (F_1 U_z)_y \tag{4.14}$$

and

$$\mathcal{L}_2(\phi_z) = -Q'(U)_z \phi_z + F_z + (F_1 U_z)_z. \tag{4.15}$$

Thus $\int_{\mathbb{R}^2}(4.14)[\phi + 2(\phi_t - s\phi_z)]_y dydz$ gives

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2}\left[\partial_y(\phi_t - s\phi_z)^2 + \partial_y\phi\partial_y(\phi_t - s\phi_z) + \frac{1}{2}(\partial_y\phi)^2 + a_1\phi_{zy}^2 + a_2\phi_{yy}^2\right] dzdy \\ &\quad + \int_{\mathbb{R}^2}\left[\partial_y(\phi_t - s\phi_z)^2 + 2f'(U)\partial_y\phi_z\partial_y(\phi_t - s\phi_z) + a_1\phi_{zy}^2 + a_2\phi_{yy}^2\right] dzdy \\ &= \int_{\mathbb{R}^2} Q'(U)_z\phi_y^2 dydz + 2\int_{\mathbb{R}^2}\left[F_y + (F_1U_z)_y - Q'(U)_z d_y\phi_z\right]\left[\phi + 2(\phi_t - s\phi_z)\right]_y dydz. \end{aligned} \tag{4.16}$$

Integrating (4.16) over $[0, t]$ and using the Young inequality, the right hand side is bounded by

$$(\epsilon + CN_1(T) + C\|d\|_2)\int_0^t \|\partial_y(\phi_t, \nabla\phi)(\tau)\|^2 d\tau + C\int_0^t \|\nabla\phi(\tau)\|^2 d\tau + C(\epsilon)\int_0^t \|(d_t, d_y)\|_1^2 d\tau.$$

Using the basic estimate in Lemma 4.1 and the smallness of $(\epsilon + CN_1(T) + C\|d\|_2)$, the resultant estimate is

$$\|\partial_y(\phi, \phi_t, \nabla\phi)(t)\|^2 + \int_0^t \|\partial_y(\phi_t, \nabla\phi)(\tau)\|^2 d\tau \le C\left[\|(\phi, \phi_t, \nabla\phi)(0)\|_1^2 + \int_0^t \|(d_t, d_y)\|_1^2 d\tau\right].$$

Similarly, $\int_{\mathbb{R}^2}(4.15)[\phi + 2(\phi_t - s\phi_z)]_z dydz$ yields

$$\|\partial_z(\phi, \phi_t, \nabla\phi)\|^2 + \int_0^t \|\partial_z(\phi_t, \nabla\phi)\|^2 d\tau \le C\left[\|(\phi, \phi_t, \nabla\phi)(0)\|_1^2 + \int_0^t \|(d_t, d_y)\|_1^2 d\tau\right].$$

Hence the proof is complete. □

Similarly the higher order estimates give us the following

Lemma 4.3. *It holds that*

$$\begin{aligned} &\sum_{|l|=2}\left[\|D^l(\phi, \phi_t, \nabla\phi)(t)\|^2 + \int_0^t \|D^l(\phi_t, \nabla\phi)\|^2 d\tau\right] \\ &\qquad \le C\left[\sum_{|l|\le 2}\|D^l(\phi, \phi_t, \nabla\phi)(0)\|^2 + \int_0^t \|(d_t, d_y)(\tau)\|_2^2 d\tau\right] \end{aligned} \tag{4.17}$$

for $0 \le t \le T$.

Finally in order to close the energy estimate a further estimate is necessary. After a few calculations of the indentity

$$\sum_{|l|=2}\int_0^t\int_{\mathbb{R}^2}[D^l\partial_z\mathcal{L}(\phi)-D^l\partial_z(F+F_1U_z)]D^l\partial_z[\phi+2(\phi_t-s\phi_z)]dydzd\tau=0,$$

one obtains

Lemma 4.4. *It holds that*

$$\sum_{|l|=2}\left[\|D^l\partial_z(\phi,\phi_t,\nabla\phi)(t)\|^2+\int_0^t\|D^l\partial_z(\phi_t,\nabla\phi)\|^2d\tau\right]$$
$$\leq C\left[\sum_{|l|\leq 2}\|D^l\partial_z(\phi,\phi_t,\nabla\phi)(0)\|^2+\int_0^t\|(d_t,d_y)(\tau)\|_2^2d\tau\right]$$

for $0\leq t\leq T$.

Combining the above lemmas 4.1-4.4 and Lemma 3.1, we arrive at the following a priori estimate.

Proposition 4.5. Let ϕ be a solution to (4.1) in $X(0,T)$ for a positive constant T. Then for suitably large a_1 there exists a constant $\delta_4>0$ such that if $N_1(T)\leq\delta_4$, then $\phi(t,z,y)$ satisfies

$$N_1^2(t)+\int_0^t\sum_{|l|\leq 2}\|D^l(1+\partial_z)(\phi_t,\nabla\phi)(\tau)\|^2d\tau\leq CN_1^2(0)\tag{4.18}$$

for $0\leq t\leq T$.

4.2 A priori estimate for the case $d_+\neq d_-$

Set

$$N_2(T)\equiv\sup_{0\leq t\leq T}\left\{\|\phi(t)\|_3^2+\|\phi_z(t)\|_3^2+\|\phi_t\|_2^2+\|\phi_{tz}\|_2^2+\|\eta_0\|_3^2+\|\eta_1\|_2^2+\delta\right\}^{1/2}.$$

The first basic estimate in this case is stated as

Lemma 4.5. *Let* ϕ *be a solution to (4.1) in* $X(0,T)$ *for a positive constant* T*. Then for a suitably large* a_1 *there exists a constant* $\delta_5>0$ *such that if* $N_2(T)\leq\delta_5$*, then*

$\phi(t,z,y)$ satisfies

$$\begin{aligned}&\|(\phi,\phi_t,\nabla\phi)(t)\|^2+\int_0^t\|(\phi_t,\nabla\phi)(\tau)\|^2d\tau+\int_0^t\int_{\mathbb{R}^2}|U_z|\phi^2dzdyd\tau\\ \le& C\Big[\|(\phi,\phi_t,\nabla\phi)(0)\|^2+\delta+\|\eta_0\|_3^2+\|\eta_1\|_2^2\Big]\end{aligned} \tag{4.19}$$

for $0\le t\le T$.

Proof. The same procedure as that in the proof of Lemma 4.1 leads to (4.7). The estimates of $(B_1)-(B_3)$ are similar to those in the case $d_+=d_-$. The estimate for (B_4) based on some previous estimates on η is given below. Making a similar analysis to the case $d_+=d_-$ we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^2}(B_4)dydzd\tau\le&\Big[\frac{a_1}{3}+CN_2(t)\Big]\int_0^t\int_{\mathbb{R}^2}w\phi_z^2dydz+CN_2(t)\int_0^t\|F_1\|_1^2\\ &+\left|\int_0^t\int_{\mathbb{R}^2}|F_1U_z(\phi+2(\phi_t-s\phi_z)\right|wdydz.\end{aligned} \tag{4.20}$$

Using the fact that $\int_{\mathbb{R}}w|U_z|dz\le C$ and $w(U)\sim C$, the last term is bounded from above by

$$\begin{aligned}&\int_{\mathbb{R}^2}|F_1U_z\phi|wdydz+2\int_{\mathbb{R}^2}|F_1U_z(\phi_t-s\phi_z)|wdydz\\ &\quad\le\epsilon_1\int_{\mathbb{R}^2}|U_z|\phi^2dydz+\epsilon_2\int_{\mathbb{R}^2}|(\phi_t,\nabla\phi)|^2wdydz+C(\epsilon_1,\epsilon_2)\int_{\mathbb{R}}|F_1|^2dy\end{aligned}$$

for suitably small constants $\epsilon_1,\ \epsilon_2$.

Note that $F_1=2sd_t-d_t^2+a_2d_y^2$ and $d=\rho+m_1\theta e^{-t}+\eta_y$, using Lemma 3.2-3.3 one obtains

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}|F_1|^2dyd\tau\le C\left\{\int_0^t\left(\|\rho_t\|^2+\|\rho_y\|_{L^4}^4\right)d\tau+|m_1|+\int_0^t\Big[\|\eta_{yt}\|^2+\|\eta_{yy}\|_{L^4}^4\Big]d\tau\right\}\\ &\quad\le C\Big[\delta+\|\eta_0\|_3^2+\|\eta_1\|_2^2\Big].\end{aligned}$$

Combinging the above estimates and previous ones proves the Lemma 4.5. □

Higher-order estimates are similar to the case of $d_+=d_-$. So we obtain the similar estimate to (4.18) for the general setting $d_+\ne d_-$ with N_2 replacing N_1. Theorem 2.2-2.3 are the consequences of these a priori estimates based on the local existence theorem. The details are omitted here.

5 Time Decay rates

Having the stability analysis in the previous sections, we proceed to measure the decay rates of the perturbation.

To establish the algebraic decay rates as claimed, we use the iteration introduced by Kawashima and Matsumura [6], and weighted energy estimates.

Since $U(z)$ is monotone and $U(0) \in]u_+, u_-[$, we may introduce a specific convex hull $H(u)$ of $Q(u)$ such that

$$H'(U(0)) = 0 \tag{5.1}$$

and define the weight $w(U) = H(U)/Q(U)$ as in [7].

To obtain the desired decay estimate, we use a composite weight

$$K(t, z, y) = (1+t)^\gamma \langle z + d(t, y)\rangle_M^\beta w(U) = (1+t)^\gamma \left(1 + (z+d)^2/M^2\right)^{\beta/2} w(U).$$

Here $M > 0$ is a large constant to be determined, γ and β are non-negative constants which are at our disposal.

Let us put

$$N_3(T) \equiv \sup_{0\le t\le T} \left\{ \sum_{0\le l+k\le 3} (1+t)^{\gamma+l} \|\partial_y^l \partial_z^k (\phi, \phi_t, \nabla\phi)(t)\|^2 + \|\eta_0\|_3^2 + \|\eta_1\|_2^2 + \delta \right\}^{1/2}$$

with $\delta = |m_1| + |d_+ - d_-|$. Multiplying (4.1) by $2K(t,z,y)\big[\phi + 2(\phi_t - s\phi_z)\big]$, suitably grouping of terms yields

$$\begin{aligned}
&\left[K(\phi_t - s\phi_z)^2 + K\phi(\phi_t - s\phi_z) + \frac{1}{2}(K + sK_z)\phi^2 + a_1 K\phi_z^2 + a_2 K\phi_y^2\right]_t \\
&\quad - \frac{\gamma}{1+t}\left[K(\phi_t - s\phi_z)^2 + K\phi(\phi_t - s\phi_z) + \frac{1}{2}(K + sK_z)\phi^2 + a_1 K\phi_z^2 + a_2 K\phi_y^2\right] \\
&\quad - K_z d_t\left[(\phi_t - s\phi_z)^2 + \phi(\phi_t - s\phi_z) + \frac{1}{2}\phi^2 + a_1\phi_z^2 + a_2\phi_y^2\right] - \frac{1}{2}sK_{zz}d_t\phi^2 \\
&\quad + (K + sK_z)(\phi_t - s\phi_z)^2 + 2(Kf'(U) + a_1K_z)\phi_z(\phi_t - s\phi_z) + 2a_2K_y\phi_y(\phi_t - s\phi_z) \\
&\quad + (K + sK_z)[a_1\phi_z^2 + a_2\phi_y^2] + \frac{1}{2}A\phi^2 + \{\cdots\}_z + \{\cdots\}_y \\
&= [F + F_1U_z]\Big[\phi + 2(\phi_t - s\phi_z)\Big]K
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
A &= -[(a_1 - s^2)K_z + Q'(U)K]_z - a_2K_{yy} \\
&= -\Big[(a - s^2)\overline{K}_z w(U)\Big]_z - a_2K_{yy} + A_\beta
\end{aligned} \tag{5.3}$$

with $A_\beta(z) = -\overline{K}_z H' - \overline{K} H'' U_z$. To make (5.2) useful we now regroup the term $A\phi^2$ as follows

$$\begin{aligned}\frac{1}{2}A\phi^2 = & -[\frac{1}{2}\phi^2(a_1 - s^2)\overline{K}_z w(U)]_z + (a_1 - s^2)\overline{K}_z w(U)\phi\phi_z \\ & -[\frac{1}{2}a_2\phi^2 K_y]_y + a_2 K_y \phi\phi_y + \frac{1}{2}A_\beta\phi^2.\end{aligned}$$

For A_β, we have the following estimate

Lemma 5.1. *Let $\alpha \geq 0$ be a given number. For $\beta \in [0, \alpha]$, there is a positive number c_0 independent of β such that*

$$A_\beta \geq c_0\beta(1+t)^\gamma \left[\langle z+d\rangle_M^{\beta-1} + \langle z+d\rangle_M^\beta |U_z|\right] \quad \text{for any } z \in \mathrm{IR}. \tag{5.4}$$

Proof. Using the choice of w and $\overline{K} = (1+t)^\gamma \langle z+d\rangle_M^\beta$, we have

$$\begin{aligned}A_\beta = & -\overline{K}_z H' - \frac{1}{2}\overline{K} H'' U_z - \frac{1}{2}\overline{K} H'' U_z \\ = & \,(1+t)^\gamma \langle z+d\rangle_M^{\beta-1}\left[-\langle z+d\rangle_M U_z - 2\beta\partial_z\langle z+d\rangle_M (H'(U) - H'(U(0))\right] \\ & + \frac{1}{2}H''(U)(1+t)^\gamma\langle z+d\rangle_M^\beta |U_z|.\end{aligned}$$

Recall that $H'(U(0)) = 0$ and H is convex, the first two terms in A_β are bounded from below by

$$c_0\beta(1+t)^\gamma\langle z+d\rangle_M^{\beta-1}$$

with $c_0 = \min\left\{\frac{|Q(U(0))|}{2(a_1-s^2)\alpha}, \frac{|u_+-u_-|}{2}\mu\right\}$. The last term is exactly the corresponding term in (5.4) with $c_0 = \frac{1}{2}\mu$ and $\mu = \min_{u\in[u_+,u_-]}\{H''(u)\}$. □

From (5.2), we have

$$E_{1t} - \frac{\gamma}{1+t}E_1 + E_2 + \frac{1}{2}A_\beta\phi^2 = \sum_{i=1}^{3} T_i \tag{5.5}$$

where

$$E_1 := K\left[(\phi_t - s\phi_z)^2 + \phi(\phi_t - s\phi_z) + \frac{1}{2}(1 + sK_z/K)\phi^2 + a_1\phi_z^2 + a_2\phi_y^2\right],$$

$$E_2 := K\Big[(1 + sK_z/K)(\phi_t - s\phi_z)^2 + 2(f' + a_1K_z/K)\phi_z(\phi_t - s\phi_z)$$
$$+ a_1(1 + sK_z/K)\phi_z^2 + 2a_2\frac{K_y}{K}\phi_y(\phi_t - s\phi_z) + a_2(1 + sK_z/K)\phi_y^2\Big],$$

$$T_1 := (1+t)^\gamma\left\{\Big[a_2\langle z+d\rangle_M^\beta w(U)\Big]_{yy} + \Big[(a_1 - s^2)\langle z+d\rangle_M^\beta w(U)\Big]_z\right.$$
$$\left. + s\Big[\langle z+d\rangle_M^\beta w(U)\Big]_{tz}\right\}\frac{\phi^2}{2},$$

$$T_2 := (1+t)^\gamma\Big[\langle z+d\rangle_M^\beta w(U)\Big]_t\left[(\phi_t - s\phi_z)^2 + \phi(\phi_t - s\phi_z) + \frac{1}{2}\phi^2 + a_1\phi_z^2 + a_2\phi_y^2\right],$$

$$T_3 := (1+t)^\gamma\langle z+d\rangle_M^\beta(F + F_1U_z)\Big[\phi + 2(\phi_t - s\phi_z)\Big].$$

Observe that for a_1 suitably large

$$\begin{aligned}\left|\frac{K_z}{K}\right| &\le \left|\frac{w'}{w}\frac{Q(U)}{a_1 - s^2}\right| + \left|\frac{\beta}{M}\frac{(z+d)/M}{\langle z+d\rangle_M^2}\right| \\ &\le Ca_1^{-1} + \alpha M^{-1},\end{aligned}$$

which is small by choosing suitably large M. This fact together with the subcharacteristic condition (1.2) implies that there exists a constant $c > 0$ such that

$$\begin{aligned}&E_1 \sim K\big[\phi^2 + (\phi_t - s\phi_z)^2 + a_1\phi_z^2 + a_2\phi_y^2\big],\\ &cK\big[(\phi_t - s\phi_z)^2 + a_1\phi_z^2 + a_2\phi_y^2\big] \le E_2.\end{aligned} \tag{5.6}$$

Thus integrating (5.5) over $[0,t]\times \mathrm{IR}^2$, and using the fact $C^{-1}\overline{K} \le K \le C\overline{K}$ ($K = \overline{K}w(U)$), we have

$$\begin{aligned}&(1+t)^\gamma|(\phi,\phi_t,\nabla\phi)(t)|_\beta^2 + \int_0^t (1+\tau)^\gamma\Big[\beta|\phi(\tau)|_{\beta-1}^2 + |(\phi_t,\nabla\phi)(\tau)|_\beta^2\Big]d\tau\\ &\quad + \int_0^t (1+\tau)^\gamma\int_{\mathrm{IR}^2}\langle z+d\rangle_M^\beta|U_z|\phi^2\,dz\,dy\,d\tau\\ &\le C\Big\{|(\phi,\phi_t,\nabla\phi)(0)|_\beta^2 + \gamma\int_0^t(1+\tau)^{\gamma-1}|(\phi,\phi_t,\nabla\phi)(\tau)|_\beta^2 d\tau\\ &\quad + \left|\int_0^t\int_{\mathrm{IR}^2}(T_1+T_2+T_3)dz\,dy\,d\tau\right|\Big\}.\end{aligned} \tag{5.7}$$

It remains to estimate the last three terms on the right hand of (5.7).

Estimate for $\int_0^t \int_{\mathbb{R}^2} T_1 dzdyd\tau$. Because of the exponential decay of $U(z)$ at $z = \pm\infty$, we always have

$$\int_{\mathbb{R}} |U_z| < z + d >_M^\beta dz \le C$$

for $\beta \in [0, \alpha]$, which will be used repeatedly.

$$\begin{aligned}
& C \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} [a_2 \langle z+d \rangle_M^\beta w(U)]_{yy} \phi^2 dzdyd\tau \\
& \quad \le C \int_0^t (1+\tau)^\gamma \left| \int_{\mathbb{R}^2} 2a_2 \Big[\langle z+d \rangle_M^\beta w(U) \Big]_z d_y \phi \phi_y dzdy \right| d\tau \\
& \quad \le C \left\{ \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d \rangle_M^\beta |U_z| |(\phi, \phi_y)|^{8/3} dzdyd\tau \right. \\
& \qquad + \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d \rangle_M^\beta |U_z| |d_y|^4 dzdyd\tau \\
& \qquad \left. + \beta \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d \rangle_M^\beta |d_y| |(\phi, \phi_z)|^2 dzdyd\tau \right\} \\
& \quad \le C \left\{ N_3(T)^{2/3} \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d \rangle_M^\beta |U_z| |(\phi, \phi_y)|^2 dzdyd\tau \right. \\
& \qquad \left. + \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}} |d_y|^4 dyd\tau + \beta N_3(T) \int_0^t (1+\tau)^\gamma \Big[|\phi(\tau)|_{\beta-1}^2 + |\phi_y(\tau)|_\beta^2 \Big] d\tau \right\}.
\end{aligned}$$

Similarly the third term in T_1 is bounded as

$$\begin{aligned}
& \left| sC \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \Big[\langle z+d \rangle_M^\beta w(U) \Big]_{tz} \phi^2 dzdyd\tau \right| \\
& \quad \le C \left\{ N_3(T)^{2/3} \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d \rangle_M^\beta |U_z| |(\phi, \phi_y)(\tau)|^2 dzdyd\tau \right. \\
& \qquad \left. + \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}} |d_t|^4 dyd\tau + \beta N_3(T) \int_0^t (1+\tau)^\gamma \Big[|\phi(\tau)|_{\beta-1}^2 + |\phi_y(\tau)|^\beta \Big] d\tau \right\}.
\end{aligned}$$

The remaining term in T_1 is treated as

$$
\begin{aligned}
&(a_1 - s^2)C \left| \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \left[\partial_z \langle z+d\rangle_M^\beta w(U)\right]_z \phi^2 dzdyd\tau \right| \\
&\quad \le C\beta \left| \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d\rangle_M^{\beta-1} \frac{\partial}{\partial z}\langle z+d\rangle_M w(U)\phi\phi_z dzdyd\tau \right| \\
&\quad \le C\beta \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d\rangle_M^{\beta-1} |\phi||\phi_z| dzdyd\tau \\
&\quad \le \frac{1}{4}\beta \int_0^t (1+\tau)^\gamma |\phi|_{\beta-1}^2 \\
&\qquad + C\beta \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}} \left(\int_{|z+d|\ge R} + \int_{|z+d|\le R} \right) \langle z+d\rangle_M^{\beta-1} \phi_z^2 dzdyd\tau \\
&\quad \le \frac{1}{4}\beta \int_0^t (1+\tau)^\gamma |\phi|_{\beta-1}^2 d\tau + \frac{1}{4}\int_0^t (1+\tau)^\gamma |\phi_z|_\beta^2 d\tau \\
&\qquad + C_R\beta \int_0^t (1+\tau)^\gamma |\phi_z|_0^2 d\tau
\end{aligned}
$$

where we have chosen R suitably large such that $R \ge 4CM\beta$ to bound the term $\int_{|z+d|\ge R} \langle z+d\rangle_M^{\beta-1}\phi_z^2 dzdy$ by $\frac{1}{4}|\phi_z|_{\beta-1}$.

Combining the above estimates together completes the estimate for $\int_0^t \int_{\mathbb{R}^2} T_1 dzdyd\tau$.

Estimate for $\int_0^t \int_{\mathbb{R}^2} T_2 dzdyd\tau$.

$$
\begin{aligned}
C\left| \int_0^t \int_{\mathbb{R}^2} T_2 dzdyd\tau \right| &\le C \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} |d_\tau| \Big[\langle z+d\rangle_M^\beta |U_z||w'(U)| \\
&\qquad + \beta \langle z+d\rangle_M^{\beta-1} w(U) \Big] \left[(\phi_t - s\phi_z)^2 + \phi^2 + \nabla\phi^2 \right] dydzd\tau \\
&\le C \Big\{ \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d\rangle_M^\beta |U_z||(\phi, \phi_t - s\phi_z, \nabla\phi)|^{8/3} dydzd\tau \\
&\qquad + \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d\rangle_M^\beta |U_z||d_\tau|^4 dydzd\tau \\
&\qquad + \beta \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} |d_\tau| \langle z+d\rangle_M^{\beta-1} |(\phi, \phi_t - s\phi_z, \nabla\phi)|^2 dydzd\tau \Big\} \\
&\le C \Big\{ N_3(T)^{2/3} \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}^2} \langle z+d\rangle_M^\beta |U_z||(\phi, \phi_t - s\phi_z, \nabla\phi)|^2 dzdyd\tau \\
&\qquad + \int_0^t (1+\tau)^\gamma \int_{\mathbb{R}} |d_t|^4 dyd\tau \\
&\qquad + \beta N_3(T) \int_0^t (1+\tau)^\gamma \left[|\phi|_{\beta-1}^2 + |(\phi_t, \nabla\phi)|_\beta^2 \right] d\tau \Big\}
\end{aligned}
$$

Estimate for $\int_0^t \int_{\rm I\!R^2} T_3 dydzd\tau$.

Due to the fact that $F = O(1)(\phi_z)^2$, one gets

$$\begin{aligned}&\int_0^t (1+\tau)^\gamma \int_{\rm I\!R^2} \langle z+d\rangle_M^\beta w(U) F\big[\phi + 2(\phi_t - s\phi_z)\big] dydzd\tau \\ &\quad \le CN_3(T)\int_0^t (1+\tau)^\gamma |\phi_z(\tau)|_\beta^2 d\tau.\end{aligned}$$

Using the fact that $w(U), |w'(U)| \sim C$ and $|U_z| \sim \exp(-|z|)$ as $|z| \to \infty$, we have

$$\begin{aligned}&\int_0^t (1+\tau)^\gamma \int_{\rm I\!R^2} \langle z-d\rangle_M^\beta F_1 U_z w(U)\big[\phi + 2(\phi_t - s\phi_z)\big] dydzd\tau \\ &\quad \le \frac{1}{4}\int_0^t (1+\tau)^\gamma \int_{\rm I\!R^2} \langle z+d\rangle_M^\beta |U_z|\phi^2 dzdyd\tau + \frac{1}{4}\int_0^t (1+\tau)^\gamma |\phi_t - s\phi_z|_\beta^2 \\ &\quad + C\int_0^t (1+\tau)^\gamma \int_{\rm I\!R} \Big[|d_t|^2 + |d_t|^4 + |d_y|^4\Big] dy.\end{aligned}$$

Plugging all above estimates into (5.7) and noting the smallness of $N_3(T)$, one obtains

$$\begin{aligned}&(1+t)^\gamma |(\phi, \phi_t, \nabla\phi)(t)|_\beta^2 + \int_0^t (1+\tau)^\gamma \Big[|(\phi_t, \nabla\phi)(\tau)|_\beta^2 + \beta|\phi(\tau)|_{\beta-1}^2\Big] d\tau \\ &\quad \le C\Bigg\{|(\phi, \phi_t, \nabla\phi)(0)|_\beta^2 + \gamma\int_0^t (1+\tau)^{\gamma-1} |(\phi, \phi_t, \nabla\phi)(\tau)|_\beta^2 d\tau \\ &\quad +\beta\int_0^t (1+\tau)^\gamma \|\phi_z(\tau)\|^2 d\tau + \int_0^t (1+\tau)^\gamma \Big[\|d_t\|^2 + \|(d_t, d_y)\|_{L^4}^4\Big] d\tau\Bigg\}. \qquad (5.8)\end{aligned}$$

To estimate the last term of (5.8), we establish the following lemma.

Lemma 5.2. *Suppose that $(\eta_0, \eta_1) \in H^3 \times H^2 \cap L^1 \times L^1$ and $\delta = |m_1| + |d_+ - d_-|$ are sufficiently small. Then the effective phase shift $d(t, y)$ satisfies*

$$\int_0^t \|\partial_y^k \partial_t^q d(t,y)\|_{L^p}^p d\tau \le C_p N_3(0) \qquad (5.9)$$

if $p > 3/(k+2q+1)$ for $0 \le k+2q \le 2$.

Proof. Note that $d = \rho\left(\frac{y+y_0}{\sqrt{t+1}}\right) - m_1\theta(y)e^{-t} + \eta_y$, we obtain

$$\|\partial_y^k \partial_t^q d\|_{L^p}^p \le C\left[\|\partial_y^k \partial_t^q \rho\|_{L^p}^p + |m_1|e^{-t}\|\partial_y^k \theta\|_{L^p}^p + \|\partial_y^{k+1}\partial_t^q \eta\|_{L^p}^p\right]. \qquad (5.10)$$

By Lemma 3.2, we have

$$\|\partial_y^k \partial_t^q \rho\|_{L^p}^p \le C|d_+ - d_-|(1+t)^{\frac{1}{2} - \frac{p}{2}(1+k+2q)}, \quad k,\ q \ge 0.$$

Using Lemma 3.3 and the basic inequality

$$\|g\|_{L^p}^p \le C\|g\|^{(p+2)/2}\|g_y\|^{(p-2)/2},$$

we estimate the last term in (5.10)

$$\begin{aligned}\|\partial_y^{k+1}\partial_t^q\eta\|_{L^p}^p &\le C\|\partial_y^{k+1}\partial_t^q\eta\|^{(p+2)/2}\|\partial_y^{k+2}\partial_t^q\eta\|^{(p-2)/2}\\ &\le C_pN_3(0)(1+t)^{1/2-3p/4-(k+2q)p/2}.\end{aligned}$$

Combining the above estimates leads to

$$\|\partial_y^k\partial_t^q d\|_{L^p}^p \le C_pN_3(0)(1+t)^{\frac{1-p}{2}-\frac{p}{2}(k+2q)}.$$

Thus (5.9) follows immediately. □

Applying (5.9) to (5.8), one obtains the following basic decay estimate.

Lemma 5.3. *There exists a constant $\delta_6 > 0$ such that if $N_3(T) \le \delta_6$, then it holds that for $t \in [0,T]$*

$$\begin{aligned}&(1+t)^\gamma|(\phi,\phi_t,\nabla\phi)(t)|_\beta^2 + \int_0^t(1+\tau)^\gamma\Big[(\phi_t,\nabla\phi)(\tau)|_\beta^2 + \beta|\phi(\tau)|_{\beta-1}^2\Big]d\tau\\ &\le C\Big\{|(\phi,\phi_t,\nabla\phi)(0)|_\beta^2 + \gamma\int_0^t(1+\tau)^{\gamma-1}|(\phi,\phi_t,\nabla\phi)(\tau)|_\beta^2 d\tau\\ &+\beta\int_0^t(1+\tau)^\gamma\|\phi_z(\tau)\|^2d\tau\Big\} + CN_3(0)\int_0^t(1+\tau)^{\gamma-5/2}d\tau. \qquad (5.11)\end{aligned}$$

Applying the induction to (5.11), we have

Lemma 5.4. *Let $\alpha \ge 0$ be any given number and $\bar\alpha = \min\{\alpha, 3/2\}$, then it holds that for $\gamma = 0, [\bar\alpha]$*

$$\begin{aligned}&(1+t)^\gamma\|(\phi,\phi_t,\nabla\phi)(t)\|^2 + \int_0^t(1+\tau)^\gamma\Big[|(\phi_t,\nabla\phi)(\tau)|_{\alpha-\gamma-1}^2 + (\alpha-\gamma)|\phi(\tau)|_{\alpha-\gamma-1}^2\Big]d\tau \qquad (5.12)\\ &\le C\Big[|\phi,\phi_t,\nabla\phi)(0)|_\alpha^2 + N_3(0)\Big].\end{aligned}$$

The proof is similar to that in Kawashima and Matsumura [6]. Based on this basic estimate we get the following sharper estimate for $\bar\alpha$ noninteger.

Lemma 5.5. *It holds that for any $\epsilon > 0$*

$$(1+t)^{\bar{\alpha}}\|(\phi,\phi_t,\nabla\phi)(t)\|^2 + (1+t)^{-\epsilon}\int_0^t (1+\tau)^{\bar{\alpha}+\epsilon}\|(\phi_t,\nabla\phi)(\tau)\|^2$$
$$\leq C\Big[|\phi,\phi_t,\nabla\phi)(0)|_\alpha^2 + N_3(0)\Big]. \tag{5.13}$$

Proof. Taking $\beta = 0$ and $\gamma = \bar{\alpha}+\epsilon$ in (5.11) leads to

$$(1+t)^{\bar{\alpha}+\epsilon}\|(\phi,\phi_t,\nabla\phi)(t)\|^2 + \int_0^t (1+\tau)^{\bar{\alpha}+\epsilon}\|(\phi_t,\nabla\phi)(\tau)\|^2 d\tau$$
$$\leq C\left\{\|\phi,\phi_t,\nabla\phi)(0)\|^2 + \gamma\int_0^t (1+\tau)^{\gamma-1}\|(\phi,\phi_t,\nabla\phi)(\tau)\|^2 d\tau\right.$$
$$\left.+N_3(0)\int_0^t (1+\tau)^{\gamma-5/2}d\tau\right\}. \tag{5.14}$$

The second term on the right hand side may be estimated by a similar procedure to that in [12], [14] and is bounded from above by

$$\gamma\int_0^t (1+\tau)^{\gamma-1}\|(\phi,\phi_t,\nabla\phi)(\tau)\|^2 d\tau \leq C\Big[N_3(0) + \|(\phi,\phi_t,\nabla\phi)(0)\|^2\Big](1+t)^\epsilon.$$

The third term is

$$CN_3(0)\int_0^t (1+\tau)^{\bar{\alpha}+\epsilon-5/2}d\tau \leq CN_3(0)(1+t)^{\bar{\alpha}-3/2+\epsilon} \leq CN_3(0)(1+t)^\epsilon.$$

Combining the above estimates, we are done. □

For higher derivatives we have the following lemma.

Lemma 5.6. *For any given $\alpha \geq 0$ and $\bar{\alpha} = \min\{\alpha, \frac{3}{2}\}$, there exists a constant $\delta_6 > 0$ such that if $N_3(T) \leq \delta_6$, then it holds that*

$$\sum_{0\leq l+k\leq 3}\left\{(1+t)^{\bar{\alpha}+l+\epsilon}\|\partial_y^l\partial_z^k(\phi,\phi_t,\nabla\phi)(t)\|^2 + \int_0 (1+\tau)^{\bar{\alpha}+l+\epsilon}\|\partial_y^l\partial_z^k(\phi_t,\nabla\phi)(\tau)\|^2 d\tau\right\}$$
$$\leq C(1+t)^\epsilon\Big[N_3(0) + (\phi,\phi_t,\nabla\phi)(0)|_\alpha^2\Big]$$

for $t \in [0,T]$.

Proof. Apply $\partial_y^l\partial_z^k$ to (4.1) and multiply it by $(1+t)^{\bar{\alpha}+l+\epsilon}\partial_y^l\partial_z^k\big[\phi+2(\phi_t - s\phi_z)\big]$, then integrate the obtained equation over $[0,t]\times \mathbb{R}^2$, one can obtain the desired estimate in the same way as in the previous lemmas; The details are omitted. Let us point out that the faster decay order of $\|\partial_y^l(\phi, \phi_t, \nabla\phi)\|$ than $\|\partial_z(\phi, \phi_t, \nabla\phi)\|$ can be seen from the simple case $l = k = 1$ in (4.14) and (4.15), the right hand of (4.14) is

$$d_y\Big[-Q'(U)_z\phi_z + F_1 U_{zz}\Big] + F_y + F_{1y}U_z,$$

but the right hand side of (4.15) is

$$-Q'(U)_z\phi_z + F_1 U_{zz} + F_z.$$

The presence of d_y in former case allows us to get faster decay as claimed. $\square$

Finally, we obtain the estimate

$$\sup_{\mathbb{R}^2}\big|u(t,x,y) - U(x - st + d(t,y))\big| = \sup_{\mathbb{R}^2}\big|\phi_z(t,z,y)\big|$$
$$\leq C\Big[\|\phi_z\|\|\phi_{zz}\|\|\phi_{zy}\|\|\phi_{zzy}\|\Big]^{1/4} \leq CN_3(0)(1+t)^{-\bar{\alpha}/2-1/4}$$

where we have used the estimate in Lemma 5.5-5.6.

Returning to the original coordinate $(x,t) = (z+st, t)$, we have

$$\psi_{1t} + \psi_1 = f(U+\phi_x) - f(U) - a_1\phi_{xx} - sU'd_t.$$

It follows that

$$\psi_1 = e^{-t}\psi_1(0,x,y) + \int_0^t e^{-(t-\tau)}\big[f(U+\phi_x) - f(U) - a_1\phi_{xx} - sU'd_t\big]d\tau.$$

Using the previous estimates on ϕ and $d(t,y)$, we obtain

$$\sup_{\mathbb{R}^2}|f(U+\phi_x) - f(U) - a_1\phi_{xx}| \leq C(1+t)^{-\bar{\alpha}/2-1/4}$$

and

$$\sup_{\mathbb{R}^2}|U'd_t| \leq C(1+t)^{-3/2}$$

where we have used the fact that

$$\|\partial_y^k\partial_t^q d\|_{L^\infty} \leq C(1+t)^{-1/2-(k+2q)/2}.$$

Using the following estimate

$$\int_0^t e^{-(t-s)}(1+s)^{-\gamma}ds \leq C(1+t)^{-\gamma}$$

for any $\gamma \geq 0$, we immediately get

$$\sup_{\mathbb{R}^2}|\psi_1| \leq C(1+t)^{-\bar{\alpha}/2-1/4}.$$

The estimate on ψ_2 is similar. Hence the proof of Theorem 2.3 is complete.

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