Long–Time Diffusive Behavior of Solutions to a Hyperbolic Relaxation System*

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Abstract

We study the large time behavior of the solutions to the Cauchy problem

$$\begin{aligned} u_t + v_x &= 0, \qquad (x,t) \in \mathrm{IR} \times \mathrm{IR}^+, \\ v_t + a^2 u_x &= f(u) - v, \\ (u,v) &= (u_0, v_0) \quad \text{in} \quad \mathrm{IR} \times \{0\} \end{aligned}$$

where $u_0 \in L^1(\mathrm{IR}) \cap L^\infty(\mathrm{IR})$, $f(u) = \alpha u^2/2$ and $|v_0| \leq a u_0$. Under the sub-characteristic condition we show that, as $t \to \infty$, nonnegative component u of solutions tends towards a diffusion wave of the convection-diffusion equation

$$u_t + f(u)_x = a^2 u_{xx}$$

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in the L^p norm, at a rate faster than $t^{-(p-1)/2p}$. This diffusion wave carries an invariant mass and has a self-similar structure.

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1 Introduction

In this paper we study the long-time behavior of solutions to a hyperbolic relaxation system of the form

$$\left.\begin{aligned} &u_t + v_x = 0, \\ &v_t + a^2 u_x = f(u) - v, \end{aligned}\right\} \tag{1.1}$$

with $f = \alpha\frac{u^2}{2}$. The variables u and v are the unknowns, $a > 0$ is a given constant.

The initial data are prescribed by

$$(u, v) = (u_0, v_0) \quad \text{in} \quad \mathbb{R} \times \{0\}, \tag{1.2}$$

where

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \;\; u_0 \geq a|v_0| \quad a.e. \;\; x \in \mathbb{R}. \tag{1.3}$$

Solutions of (1.1)-(1.2) with L^1 initial data satisfy the following property

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx = m, \quad \forall t \geq 0. \tag{1.4}$$

To ensure the global existence and the L^1 contraction of the solution we assume that a is large enough such that a subcharacteristic condition, [23], [36]

$$|f'(u)| < a, \tag{1.5}$$

holds for u under consideration. Recall that, from [29], for every $(u_0, v_0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})^2$, there exists a unique weak solution of (1.1), (1.2), with $(u, v) \in C([0,\infty[; L^1_{\rm loc}(\mathbb{R}))$ provided the stability condition (1.5) is satisfied. Define

$$\sup_{u\in I} \frac{|f'(u)|}{a} = \eta < 1 \tag{1.6}$$

where I is the (bounded) interval of range of u.

Let $\theta_m : \mathbb{R} \to \mathbb{R}$ be the wave solution to the diffusion equation

$$u_t + f(u)_x = a^2 u_{xx} \tag{1.7}$$

starting with a Dirac Mass $m\delta(x)$. The main goal of this paper is to establish the following result.

Theorem 1.1. *Assume the subcharacteristic condition (1.5) be satisfied. Let (u, v) be a solution of (1.1)-(1.3) with initial data $(u_0, v_0) \in L^1(\mathbb{R})^2$ carrying finite mass $m = \int_{\mathbb{R}} u_0 dx > 0$. Assume that there exists a constant $K > 0$ such that $|v_0| \le a u_0$ and $a u_{0x} \pm v_{0x} \le K$ for $x \in \mathbb{R}$. Then for every $p \in [1, \infty)$*

$$\lim_{t\to\infty} t^{\frac{p-1}{2p}} \|u(\cdot, t) - \theta_m(\cdot, t)\|_{L^p(\mathbb{R})} = 0. \tag{1.8}$$

The study on the behavior of solutions to hyperbolic relaxation systems has been caught much interest after the seminal work by T.P. Liu [23], consult [31] for a recent survey. Actually there are two different kinds of asymptotic limits. The first one is the relaxation limit, which is essentially driven by the hyperbolic equilibrium equation

$$u_t + f(u)_x = 0. \tag{1.9}$$

A different behavior is found if we look at the long–time effects of relaxation. In [23], T.P. Liu proved via the Chapman-Enskog expansion that for large times the effect of relaxation is close to a viscous wave when the solution is near a constant equilibrium state for a class of 2×2 hyperbolic relaxation system. Later, I. Chern [5] showed that the global smooth solution of relaxation system approach the diffusive wave $\theta(x,t)$ in the L^p–norm at a certain rate if the initial data are sufficiently small, in the sense that

$$\int u_0 dx + \| \int_{-\infty}^{x} (u_0 - \theta_0) dx \|_{H^3 \cap L^1} + \|v_0 - f(\theta_0)\|_{H^2 \cap L^1} << 1.$$

Respect to that result, the main achievement of Theorem 1.1 consists in removing the restriction on initial data by using a quite different approach. This seems remarkable for hyperbolic relaxation problems since the dissipation from the relaxation effect is known to be weaker than that in viscous approximation. The method is due to a parabolic scaling for a hyperbolic system. The scaling arguments of such kind have been widely used in studying the large time behavior of solutions for equations of parabolic type, see for instance [34, 16, 10] and the Lecture Notes [35], see also [9, 11, 7, 8] for a discussion of hyperbolic effects.

Let us also notice that this scaling is strongly connected with the so–called diffusive kinetic limits. See for example [6], Ch. 11, and references therein for a discussion of the diffusive limits of the Boltzmann equation. Here, we just give a very incomplete list of papers where diffusive limits have been investigated, see [4, 12, 13, 2, 25, 26].

For discrete velocities model, theoretical investigations start with the works [19, 27] for the diffusive limit of Carleman equations. The general theory of convergence to equilibrium for two velocities models can be found in [21]. For some recent numerical investigations we refer to [14, 28]. A general kinetic framework for approximating convection–diffusion equations has been recently proposed in [3].

The main theorem is proven by introducing the family of scaled functions

$$u^\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t),$$
$$v^\lambda(x,t) := \lambda^2 v(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

Upon substitution in (1.1)-(1.2) with $f = \alpha u^2/2$, $\alpha > 0$, one finds that for any $\lambda > 0$ the function (u^λ, v^λ) solves the problem

$$\left.\begin{array}{l}\partial_t u^\lambda + \partial_x v^\lambda = 0, \\ \lambda^{-2}\partial_t v^\lambda + a^2\partial_x u^\lambda = f(u^\lambda) - v^\lambda,\end{array}\right\} \tag{1.10}$$

$$(u^\lambda, v^\lambda)(x,0) = (\lambda u_0(\lambda x), \lambda^2 v_0(\lambda x)), \quad \text{in } \mathbb{R}.$$

As $\lambda \to \infty$, (1.10) formally reduces to

$$\partial_t u^\infty + \partial_x f(u^\infty) = a^2\partial_x^2 u^\infty \tag{1.11}$$

and

$$v^\infty = f(u^\infty) - a^2\partial_x u^\infty.$$

The investigation of the asymptotic behavior of the solution $\{u, v\}$ can be reduced by standard arguments to studying the convergence of the family $\{u^\lambda, v^\lambda\}$, as $\lambda \to +\infty$. The main difficulty is to obtain the appropriate a priori bounds on the family $\{u^\lambda\}$. Observe that if we want $\{u^\lambda\}$ to be uniformly bounded in $L^p(\mathbb{R})$ for some $p \geq 1$ and $t > 0$, we need to establish the following estimate

$$\|u(t)\|_{L^p(\mathbb{R})} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{1.12}$$

One of the main tools for doing this is the following entropy-type estimate.

Theorem 1.2. *Assume (u, v) is the solution of (1.1) with initial data (u_0, v_0) of mass $\int_{\mathbb{R}} u_0 dx = m$ such that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v_0 \in L^\infty(\mathbb{R})$. Under the stability condition (1.5). Then there exists a constant $K > 0$ such that if $u_{0x} \pm v_{0x}/a \leq K$, then*

$$\left[\frac{\partial u}{\partial x}\right]^+ \leq Ct^{-1}, \tag{1.13}$$

$$\|u\|_{L^\infty(\mathbb{R})} \leq C\frac{m}{\sqrt{t}}, \quad t > 0, \tag{1.14}$$

for a constant $C > 0$ depending only on K.

Let us notice that similar one-side inequalities were already obtained in various papers [18, 33, 22] to study the regularity of solutions and error rates, but without any estimate of the time decay. However this decay is a crucial tool in the following to obtain our results.

The following sections are organized as follows. In Section 2 we discuss the parabolic scaling limit for studying the large time behavior of solutions. In Section 3 we establish some compactness estimates needed in the following section, where we pass into the limit. Finally we briefly discuss the sharpness of our decay rate.

2 Scaling Limit and the Diffusion Wave

Recall that the mass in $u-$component in system (1.1) is conserved, i.e.

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx = m.$$

To determine the large wave pattern, it is useful to consider a parabolic–type scaling in all the variables.

Let (u,v) be the solution to system (1.1), and define for every $\lambda > 0$

$$u^\lambda(t,x) = \lambda^\gamma u(\lambda^\alpha x, \lambda^\beta t) = \lambda^\gamma u(y,\tau),$$

$$v^\lambda(t,x) = \lambda^\theta v(\lambda^\alpha x, \lambda^\beta t) = \lambda^\theta v(y,\tau)$$

with parameters $(\alpha,\beta,\gamma,\theta)$ to be determined. Substituting in (1.1) yields

$$\partial_t u^\lambda + \partial_x v^\lambda = \lambda^{\gamma+\beta}\partial_\tau u(y,\tau) + \lambda^{\alpha+\theta}\partial_y v(y,\tau)$$

and then, introducing two further parameters μ and ν, we have

$$\begin{aligned}\lambda^\mu \partial_t v^\lambda + a^2\partial_x u^\lambda - \Big(\lambda^\nu f(u^\lambda) - v^\lambda\Big) \\ = \lambda^{\mu+\theta+\beta}\partial_\tau v(y,\tau) + a^2\lambda^{\gamma+\alpha}\partial_y u(y,\tau) - \Big(\lambda^\nu f(\lambda^\gamma u(y,\tau)) - \lambda^\theta v\Big).\end{aligned}$$

Assume, more generally that the flux function f is homogeneous of degree q, i.e.: $f(\lambda^\gamma u) = f(u)\lambda^{\gamma q}$. Therefore we choose the parameters $(\alpha, \beta, \gamma, \theta)$ to respect the equations, which yields

$$\beta = 2\alpha, \quad \mu = -2\alpha, \quad \theta = \gamma + \alpha, \quad \nu = \gamma + \alpha - q\gamma.$$

To keep the mass unchanged in u component under scaling, it is necessary to assume $\gamma = \alpha$, for definiteness we may take $\alpha = 1$. Then we have

$$\beta = 2, \quad \mu = -2, \quad \theta = 2, \quad \nu = 2 - q.$$

Thus the scaled unknowns can be written as

$$\begin{aligned} u^\lambda(t, x) &= \lambda u(\lambda x, \lambda^2 t), \\ v^\lambda(t, x) &= \lambda^2 v(\lambda x, \lambda^2 t). \end{aligned}$$

These functions satisfy

$$\left.\begin{aligned} &\partial_t u^\lambda + \partial_x v^\lambda = 0 \\ &\lambda^{-2}\partial_t v^\lambda + a^2 \partial_x u^\lambda = \lambda^{2-q} f(u^\lambda) - v^\lambda \end{aligned}\right\} \tag{2.1}$$

with initial data

$$(u^\lambda, v^\lambda)(0, x) = (\lambda u_0(\lambda x), \lambda^2 v_0(\lambda x)), \quad \forall x \in \mathrm{IR}.$$

The idea of the proof of the main theorem is to pass into the limit in (2.1) as $\lambda \to \infty$, to say that in the limit we obtain a solution (U, V) of the reduced problem

$$\left.\begin{aligned} &\partial_t U + \partial_x V = 0 \\ &a^2 \partial_x U = f(U) - V \ \text{ for } \ q = 2 \ \text{ or } \ a^2 \partial_x U = -V \ \text{ for } \ q > 2. \end{aligned}\right\} \tag{2.2}$$

Since $u_0(x) \in L^1(\mathrm{IR})$ and $\int u_0 dx = m$, we have

$$\lambda u_0(\lambda x) \to U_0(x) = m\delta(x), \qquad \text{as } \ \lambda \to +\infty$$

denoting the concentration of the mass. Thus the limit $U(x,t)$ for $q=2$ can be formally identified with the solution of the following problem

$$\left.\begin{aligned} &\partial_t U + \partial_x f(U) = a^2 \partial_x^2 U, \quad (x,t) \in \mathrm{IR} \times \mathrm{IR}^+, \\ &U(x,0) = m\delta(x), \quad x \in \mathrm{IR}. \end{aligned}\right\} \tag{2.3}$$

Let us now recall some basic facts about the fundamental solution to the convection-diffusion equation. Consider the problem (2.3) with $f(u) = \alpha u^2/2$. By the Hopf-Cole transformation

$$U = -\frac{2a^2}{\alpha}(\ln W)_x,$$

the problem (2.3) is reduced to the standard heat equation

$$\left.\begin{aligned} &\partial_t W = a^2 \partial_x^2 W, \quad (x,t) \in \mathrm{IR} \times \mathrm{IR}^+, \\ &W(x,0) = \exp[\tfrac{-\alpha m}{4a^2}\mathrm{sign}(x)], \quad x \in \mathrm{IR}. \end{aligned}\right\} \tag{2.4}$$

More precisely, the solution can be explicitly written as

$$W(x,t) = e^{-\alpha m/(4a^2)} \int_{-\infty}^{x/2a\sqrt{\pi t}} e^{-\pi y^2}\,dy + e^{\alpha m/(4a^2)} \int_{x/2a\sqrt{\pi t}}^{\infty} e^{-\pi y^2}\,dy.$$

For given $m > 0$ this solution $U = \theta_m$ is unique and respects the following properties:

(i) The solution is self similar with respect to the scaling in u component, i.e.,

$$\theta_m(x,t) = \lambda \theta_m(\lambda x, \lambda^2 t).$$

(ii) θ_m is singular at $(0,0)$ and satisfies

$$\mathrm{esslim}_{t\to 0} \int_{\mathrm{IR}} \theta_m(x,t)\phi(x)dx = m\phi(0)$$

for any $\phi(x) \in C_0^2(\mathrm{IR})$. Moreover,

$$\theta_m \in L^\infty((0,\infty); L^1(\mathrm{IR})) \cap L^\infty((\tau,\infty), \mathrm{IR}), \;\; \forall \tau > 0.$$

The above facts will be used in passing to the limit of $\{u^\lambda\}$. In fact, note that for any fixed $t_0 > 0$

$$u^\lambda(x,t_0)-\theta_m(x,t_0) = \lambda u(\lambda x, \lambda^2 t_0)-\lambda\theta_m(\lambda x, \lambda^2 t_0) = \tau^{1/2}(u(y,\tau)-\theta_m(y,\tau))t_0^{-1/2},$$

where $y = \lambda x$ and $\tau = \lambda^2 t_0$. Therefore, if $u^\lambda(x,t_0)$ converges to $\theta_m(x,t_0)$ as $\lambda \to \infty$ in some norm, then we obtain the result in Theorem 1.1 in the same norm.

3 Regularity Estimates

In this section we establish some regularity estimates for the solution (u,v) of the problem

$$\begin{cases} u_t + v_x = 0, \\ v_t + a^2 u_x = f(u) - v, \end{cases} \tag{3.1}$$

with the initial data

$$(u,v) = (u_0, v_0) \quad \text{in } \mathbb{R} \times \{0\}. \tag{3.2}$$

Observe that the rescaled equation is of the form (3.1). Therefore investigating problem (3.1)-(3.2) enables us to prove estimates for the family $\{u^\lambda, v^\lambda\}$.

Rewriting (3.1) in terms of the Riemann invariants

$$R_{1,2} = \frac{1}{2}(u \pm v/a),$$

gives

$$R_t + \Lambda R_x = M(u) - R \tag{3.3}$$

where

$$M_{1,2}(u) = \frac{1}{2}(u \pm f(u)/a)$$

and $\Lambda = \mathrm{diag}(-a, a)$. Notice that by definition we have

$$u = R_1 + R_2 = M_1(u) + M_2(u).$$

Observe now that (1.6) implies that

$$\frac{1-\eta}{2} \le M_i'(u) \le \frac{1+\eta}{2}, \quad i=1,2, \quad u \in I. \tag{3.4}$$

We recall from [29, 30] that the solution $R(x,t)$ of (3.3) with initial data $R(x,0) \in L^\infty$ respects the following L^1 contraction property:

Lemma 3.1. *[29] If $R(x,0)$, $\tilde{R}(x,0) \in L^\infty$ and $R(x,0) - \tilde{R}(x,0) \in L^1(\mathrm{IR})$, then $R(x,t) - \tilde{R}(x,t) \in L^1(\mathrm{IR})$ for any $t \ge 0$ and $\|R(\cdot,t) - \tilde{R}(\cdot,t)\|_{L^1}$ is non-increasing in time t.*

Thanks to this lemma, we may restrict to the data $R(x,0) \in C_0^1(\mathrm{IR})$ since $C_0^1(\mathrm{IR})$ is dense in $L^1(\mathrm{IR})$. Set in the following

$$\beta = \frac{(1+\eta)}{\alpha(1-\eta)} \ .$$

Lemma 3.2. *Assume that $R_{0x} \le K$ for $K \le (1-\eta^2)/(4\alpha)$, R is the solution obtained with initial data $R(x,0)$ of total mass $\int u_0 dx = m > 0$. Then for every $t > 0$,*

$$R_{ix} \le \min\{K, \frac{\beta(1+\eta)}{2t}\}, \quad i = 1,2 \tag{3.5}$$

and

$$u_x \le \min\{2K, \frac{\beta}{t}\} \ . \tag{3.6}$$

Proof. Differentiating (3.3) with respect to x we obtain

$$(\partial_t + \Lambda \partial_x) R_x = H R_x \ ,$$

where $H := M'P - I$, $PR := R_1 + R_2$ and I is the unit matrix.

To estimate the bound of R_x from above, let us set

$$W = R_x - M'(u)B(t),$$

where $B(t) = \beta/(t+\delta)$. Then

$$(\partial_t + \Lambda\partial_x)W = HW + B(t)HM'(u) - (\partial_t + \Lambda\partial_x)[B(t)M'(u)]. \tag{3.7}$$

Applying the P operator to equation (3.3) yields

$$\partial_t(PR) + \partial_x(P\Lambda R) = 0.$$

Hence

$$(\partial_t + \Lambda\partial_x)(B(t)M') = B'(t)M'(u) + B(t)[\Lambda M''P - M''P\Lambda](W + B(t)M') .$$

Note that

$$HM'(u) = M'PM' - M' = 0.$$

Therefore substitution of the above equalities into (3.7) leads to

$$(\partial_t + \Lambda\partial_x)W = \tilde{H}W - Q, \tag{3.8}$$

where we define

$$\tilde{H} = H - B(t)[\Lambda M''P - M''P\Lambda],$$

and

$$Q = B'(t)M'(u) + B^2(t)[\Lambda M''P - M''P\Lambda]M''.$$

Let us check the terms $\tilde{H}$ and Q. Since $B'(t) = -\beta^{-1}B^2(t)$ and $P\Lambda M' = f'(u)$, we obtain

$$\begin{aligned} Q(t) &= B^2\beta^{-1}\{-M'(u) + \beta(\Lambda M'' - M''f'(u))\} \\ &= B^2\beta^{-1}\{-M' + \beta(\Lambda - f'(u)I)M''\} \\ &= B^2\beta^{-1}\{\beta f''(u)(M_2'(u), M_1'(u))^T - (M_1'(u), M_2'(u))^T\} \geq 0, \end{aligned}$$

since $\frac{1}{2}(1-\eta) \leq M_i' \leq \frac{1}{2}(1+\eta)$. A simple computation shows that

$$\tilde{H} = \begin{pmatrix} M_1' - 1 & M_1' - B(t)f''(u) \\ M_2' - B(t)f''(u) & M_2'(u) - 1 \end{pmatrix}$$

Taking $\delta = 2(1+\eta)/(1-\eta)^2$ one has

$$M_i'(u) - B(t)f''(u) \geq \frac{1}{2}(1-\eta) - \beta\delta^{-1}\alpha \geq 0, \qquad i = 1, 2.$$

To obtain the desired estimate, we use the following inequality on the initial data for equation (3.8)

$$W(x,0) = R_x(x,0) - \beta\delta^{-1}M'(u_0) \leq K - \frac{1}{2}\beta\delta^{-1}(1+\eta) \leq 0$$

for $K \leq (1-\eta^2)/(4\alpha)$. Thus using the maximum principle for the weakly coupled hyperbolic system (3.8), we obtain

$$W \leq 0.$$

Then we have

$$R_x \leq B(t)M'(u) < \frac{\beta(1+\eta)}{2t},$$

which leads to

$$u_x = PR_x \leq B(t) < \beta/t$$

due to the fact that $PM'(u) = 1$. □

From this estimate and the conservation of mass estimate one gets a bound for u.

Lemma 3.3. *For the initial data (u_0, v_0) satisfying $R_{0x} \leq K$ and $R_0(x) \geq 0$, it holds*

$$0 \leq u(x,t) \leq \left(\frac{2m\beta}{t}\right)^{1/2}. \qquad (3.9)$$

Proof. First we show that $u \geq 0$. By the definition of $M(u)$ and $f(0) = 0$ one has $M(0) = 0$. Thus using the Mean Value Theorem we obtain from (3.3), that

$$R_t + \Lambda R_x = (M'(\xi)P - I)R\ ,$$

for some value ξ between 0 and u. By using the subcharacteristic condition (1.5) and the comparison principle, see [29], one has

$$R(x,t) \geq 0$$

which gives us $u(x,t) = PR(x,t) \geq 0$.

Next we estimate the bound for u from above. Let $x_0 \in \mathrm{IR}$, $t > 0$ be fixed. Due to the entropy-type inequality in Lemma 3.2 we have

$$u(x_0,t) - u(x,t) \leq \frac{\beta}{t}(x_0 - x) \qquad \text{for any } \; x \leq x_0.$$

Hence setting $\gamma = \frac{t}{\beta} u(x_0,t)$ one has

$$\begin{aligned} m &= \int u(x,0)dx = \int u(x,t)dx \geq \int_{x_0-\gamma}^{x_0} u(x,t)dx \\ &= \gamma u(x_0,t) - \frac{\beta}{2t}\gamma^2 = \frac{t}{2\beta}u^2(x_0,t). \end{aligned}$$

This proves the result. □

From inequality (3.9) and the conservation of the mass, i.e. $\int u dx = m$, we obtain by interpolation the following estimate

$$\|u\|_p \leq C t^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{3.10}$$

The following lemma is another immediate consequence of Lemma 3.2.

Lemma 3.4. *Under the assumptions made in Lemma 3.3,*

$$f(u)_x \leq \frac{\alpha\beta u}{t}, \;\; t > 0 \;\; x \in \mathrm{IR}, \tag{3.11}$$

while for every $t > 0$,

$$\int_{\mathrm{IR}} |f(u)_x| dx \leq \frac{2\alpha\beta m}{t}. \tag{3.12}$$

Proof. Inequality (3.11) easily follows by Lemma 3.2. As for (3.12), we use (3.11) and the equations $\int u dx = m$ and $\int f(u)_x dx = 0$, i.e.

$$\begin{aligned} \int |f(u)_x| dx &= -\int f(u)_x dx + 2\int (f(u)_x)^+ dx \\ &\leq \frac{2\alpha\beta}{t}\int u dx = \frac{2\alpha\beta m}{t}. \end{aligned}$$

□

4 Asymptotic Behavior

This section is devoted to the proof of the main Theorem 1.1. To this end we are going to perform the following steps:

- establishing uniform bounds on the sequence $\{u^\lambda\}$;
- establishing the compactness of the sequence $\{u^\lambda\}$ in $L^1(\mathbb{R})$;
- Passing to the limit towards the function (θ_m).

Before going into the details of the proof, let us justify the decay rate to the desired limit. Assume the convergence result

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}}|u^\lambda(\cdot,t)-\theta_m(\cdot,t)|dx=0. \tag{4.1}$$

Since the function θ_m is invariant under the scaling, i.e.: $\lambda\theta_m(\lambda x,\lambda^2 t)=\theta_m(x,t)$, we choose $t=1$, and $\tau=\lambda^2$ in (4.1) to obtain

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}}|u(\cdot,\tau)-\theta_m(\cdot,\tau)|dx=0$$

which is the desired result for $p=1$. For $p>1$ one has

$$\int_{\mathbb{R}}|u(\cdot,\tau)-\theta_m(\cdot,\tau)|^p dx\le|u(\cdot,\tau)-\theta_m(\cdot,\tau)|_{L^\infty}^{p-1}\int_{\mathbb{R}}|u(\cdot,\tau)-\theta_m(\cdot,\tau)|dx.$$

By the estimate in Lemma 3.3 and the formula for θ_m one has

$$\|u(\tau)-\theta_m(\tau)\|_{L^\infty}\le C/\sqrt{\tau},$$

where C depends on m. Then the above estimates gives

$$\lim_{t\to\infty}t^{(p-1)/(2p)}\|u(\cdot,t)-\theta_m(\cdot,t)\|_{L^p}=0$$

which concludes the result asserted in Theorem 1.1.

4.1 Compactness Estimates for u^λ

By the estimates in Section 3 and the given scaling, we have

$$\int_{\mathbb{R}} f(u^\lambda)dx = \frac{\alpha\lambda^2}{2}\int_{\mathbb{R}} u^2(y,\lambda^2 t)dy \le C/\sqrt{t} \tag{4.2}$$

for $C = \frac{\alpha}{2}\sqrt{2m\beta}$. Moreover

$$\int_{\mathbb{R}} |f(u^\lambda(x,t))_x|dx = \lambda^2\int_{\mathbb{R}} |f(u^\lambda(y,\lambda^2 t))_y|dy \le (2\alpha\beta m)/t \tag{4.3}$$

and

$$\int_{\mathbb{R}} u^\lambda(x,t)dx = \int u(y,0)dy = m.$$

For every fixed $t_0 > 0$, the family $f(u^\lambda)(x,t_0)$ is uniformly bounded in $W^{1,1}(\mathbb{R})$. This allows us to pass into the limit along a sub-sequence, say $f(u^{\lambda_j})$, and obtain a limit $\bar{w}(\cdot,t_0)$ locally in $L^p(\mathbb{R})$ for $p < \infty$ and almost everywhere. It follows that for the same sub-sequence, u^{λ_j} converges to $\bar{u}(\cdot,t_0) = (2\alpha^{-1}\bar{w}(\cdot,t_0))^{1/2}$ in the same sense.

We have to show now that the limit $\bar{u}(x,t)$ is equal to θ_m. To this end we need to estimate the term $\lambda^{-2}v_t^\lambda$ in the equation (1.10).

Lemma 4.1. *There exists an increasing modulus of continuity ω_1 such that, for every $0 < \tau < t$ and every test function $\phi \in C_0^2(\mathbb{R})$, there holds*

$$\Big|\int\int_\tau^t \lambda^{-2}v_t^\lambda\phi(x)dxds\Big| \le e^{-\lambda^2(t-\tau)\theta}\lambda\omega_1(t-\tau) \tag{4.4}$$

for some $0 < \theta < 1$.

Proof. First we have that

$$\Big|\int\int_\tau^t \lambda^{-2}v_t^\lambda\phi(x)dxds\Big| = \Big|\int \lambda^{-2}(v(x,t)-v(x,\tau))\phi(x)dx\Big|.$$

From the second equation of (1.10) we can write

$$v^\lambda(x,t) = e^{-\lambda^2 t}\Big[v^\lambda(x,0) + \lambda^2 \int_0^t B(x,s)e^{\lambda^2 s}ds\Big] \tag{4.5}$$

with $B(x,t) = f(u^\lambda) - a^2 u^\lambda_x$. Then we find that

$$\begin{aligned}\Big|\int \lambda^{-2}(v(x,t) - v(x,\tau))\phi(x)dx\Big| &= \Big|\int \lambda^{-2}(e^{-\lambda^2 t} - e^{-\lambda^2\tau})v^\lambda(x,0) \\ &+ (e^{-\lambda^2 t} - e^{-\lambda^2 \tau})\int_0^\tau B(x,s)e^{\lambda^2 s}ds + e^{-\lambda^2 t}\int_\tau^t B(x,s)e^{\lambda^2 s}ds\Big| \\ &\le \sum_{i=1}^3 J_i.\end{aligned}$$

Let us now estimate the different terms on the right-hand side. First

$$\begin{aligned}J_1 &= \lambda^{-2}|e^{-\lambda^2 t} - e^{-\lambda^2\tau}|\Big|\int v^\lambda(x,0)\phi(x)dx\Big| \\ &\le \lambda^{-2}\lambda^2(t-\tau)e^{-\lambda^2(\tau+\theta(t-\tau))}\lambda\|\phi\|_\infty \int |v_0(x)|dx \le C\lambda(t-\tau)e^{-\lambda^2\theta(t-\tau)}\end{aligned}$$

for $\theta \in]0,1[$. Now we use inequality (3.9) to give

$$\begin{aligned}&\Big|\int\int_{t_1}^{t_2} B(x,s)e^{\lambda^2 s}\phi(x)dxds\Big| \\ &\le \Big|\int\int_{t_1}^{t_2} f(u^\lambda)e^{\lambda^2 s}\phi(x)dxds\Big| + \Big|\int\int_{t_1}^{t_2} e^{\lambda^2 s}a^2u^\lambda\phi_x(x)dxds\Big| \\ &\le \|\phi\|_\infty\frac{\alpha}{2}\int u^\lambda dx\int_{t_1}^{t_2}\frac{e^{\lambda^2 s}}{\sqrt{s}}\sqrt{2\beta m}ds + a^2\|\phi_x\|_\infty\int u^\lambda dx\int_{t_1}^{t_2} e^{\lambda^2 s}ds.\end{aligned}$$

Noting that $\int u^\lambda dx = m$ and

$$\int_{t_1}^{t_2}\frac{e^{\lambda^2 s}}{\sqrt{s}}ds \le 2e^{\lambda^2(t_1+\theta(t_2-t_1)}(\sqrt{t_2} - \sqrt{t_1})$$

for some $\theta \in]0,1[$, we have

$$\begin{aligned}&\Big|\int\int_{t_1}^{t_2} B(x,s)e^{\lambda^2 s}\phi(x)dxds\Big| \\ \le& Cme^{\lambda^2(t_1+\theta(t_2-t_1))}\Big\{(\sqrt{t_2} - \sqrt{t_1}) + (t_2 - t_1)\Big\}\end{aligned}$$

for some $0 < \theta < 1$, $C = \sup\{\alpha\|\phi\|_\infty\sqrt{2\beta m}, a^2\|\phi_x\|_\infty\}$.

This immediately yields

$$\begin{aligned} J_2 &= |e^{-\lambda^2 t} - e^{-\lambda^2 \tau}|\Big|\int\int_0^\tau B(x,s)e^{\lambda^2 s}\phi(x)dxds\Big| \\ &\le \lambda^2(t-\tau)e^{-\lambda^2(\tau+\theta_1(t-\tau))}Cme^{\lambda^2\theta\tau}(\sqrt{\tau}+\tau) \\ &\le Cm\lambda\tau e^{-\lambda^2(1-\theta)\tau}\lambda(t-\tau)e^{-\lambda^2\theta_1(t-\tau)} \end{aligned}$$

and

$$\begin{aligned} J_3 &= e^{-\lambda^2 t}\Big|\int\int_0^\tau B(x,s)e^{\lambda^2 s}\phi(x)dxds\Big| \\ &\le e^{-\lambda^2 t}Cme^{\lambda^2(\tau+\theta(t-\tau))}(\sqrt{t}-\sqrt{\tau}+t-\tau) \\ &\le Cm(\sqrt{t}-\sqrt{\tau}+t-\tau)e^{-\lambda^2(1-\theta)(t-\tau)}\ . \end{aligned}$$

The proof is complete. □

Lemma 4.2. *There exists an increasing modulus of continuity ω_2 such that for any $L > 0$ it holds*

$$\int_{-L}^{+L}|u^\lambda(x,t) - u^\lambda(x,\tau)|dx \le \omega_2(t-\tau), \tag{4.6}$$

for any $t > \tau > 0$.

Proof. From a slight modification of the time-continuity Kružkov Lemma (see [17], Lemma 5), it is enough to prove that there exists another increasing modulus of continuity ω_3 such that for any $\phi \in C_0^2(\mathrm{IR})$, it holds

$$\Big|\int_{\mathrm{IR}}(u^\lambda(x,t) - u^\lambda(x,\tau))\phi(x)dx\Big| \le \Big(Cm + e^{-\lambda^2(t-\tau)\theta}\lambda\Big)\,\omega_3(t-\tau), \tag{4.7}$$

for any $t > \tau > 0$. Now we have that, for $t > \tau > 0$ and every test function $\phi \in C_0^2(\mathrm{IR})$, we have

$$\Big|\int(u^\lambda(x,t) - u^\lambda(x,\tau))\phi(x)dx\Big| = \Big|\int\int_\tau^t\Big[B(x,s) - \lambda^{-2}v_t^\lambda\Big]\phi_x(x)dx\Big|.$$

As shown above we have also

$$\Big|\int\int_\tau^t B(x,s)\phi_x(x)dx\Big| \le \|\phi_x\|_\infty \alpha m\sqrt{2\beta m}(\sqrt{t}-\sqrt{\tau})$$
$$+ a^2\|\phi_{xx}\|_\infty m(t-\tau) \le Cm\omega_1(t-\tau)$$

which combined with the estimate in Lemma 4.1 leads to (4.7). Therefore the conclusion follows. □

The above time estimate ensures that the subsequence $\{u^{\lambda_j}\}$ converge to a function $\bar{u}(x,t)$ not merely for each fixed $t>0$ but almost everywhere in $\mathrm{IR}\times\mathrm{IR}^+$. In fact, the estimate (3.9) gives

$$|u^\lambda(x,t)| = |\lambda u(\lambda x, \lambda^2 t)| \le (\frac{2m\beta}{t})^{1/2} = h(t).$$

We then obtain for any $L>0$

$$\int_{-L}^{L} |f(u^\lambda(x,t) - f(u^\lambda(x,\tau)|dx \le \alpha h(t)\int_{-L}^{L} |u^\lambda(x,t)-u^\lambda(x,\tau)|dx$$
$$\le \alpha h(t)\omega_2(t-\tau).$$

This estimate when combined with (4.2) and (4.3) implies that $f(u^\lambda)$ is precompact also in $L^1_{loc}(\mathrm{IR}^+\times\mathrm{IR})$, so is u^λ.

As a consequence of the estimates in Lemma 4.1 and Lemma 4.2 we have that $\bar{u}(x,t)$ is a weak solution of (2.3) in the sense of distributions. In fact using the strong convergence of $f(u^\lambda)$ and the equality

$$\int u^\lambda(x,t)\phi(x)dx - \int u^\lambda(x,\tau)\phi(x)dx = \int_{\mathrm{IR}}\int_\tau^t f(u^\lambda)\phi_x(x)dxds$$
$$+ a^2\int_{\mathrm{IR}}\int_\tau^t u^\lambda\phi_{xx}(x)dxds - \lambda^{-2}\int_{\mathrm{IR}}\int_\tau^t v_t^\lambda\phi_x(x)dxds,$$

we obtain into the limit that

$$\int \bar{u}(x,t)\phi(x)dx - \int \bar{u}(x,\tau)\phi(x)dx$$
$$= \int\int_\tau^t f(\bar{u})\phi_x(x)dxds + a^2\int\int_\tau^t \bar{u}\phi_{xx}(x)dxds.$$

Moreover we have that $\lim_{t\to 0} \bar{u}(x,t) = m\delta(x)$ in the weak sense of measures in $\mathbb{R}$. In fact, let $\phi \in C_0^2(\mathbb{R})$ be a test function, then the above estimates yield

$$\left| \int u^\lambda(x,t)\phi(x)dx - \int u^\lambda(x,0)\phi(x)dx \right| \le Cm\omega_2(t) + C\lambda t e^{-\lambda^2\theta_1 t}.$$

Since $\int u^\lambda(x,0)\phi(x)dx \to m\phi(0)$ as $\lambda \to \infty$ (passing to a subsequence if necessary), the conclusion follows easily. Finally let us notice that, by standard arguments, the convergence of u^λ towards $\bar{u}$ takes place not only in L^1_{loc} but actually in L^1, since

$$\lim_{k\to\infty} \int_{|x|\ge k} u^\lambda(x,t)dx = 0 \ , \tag{4.8}$$

uniformly with respect to λ.

Therefore, from the estimates obtained above we observe that

$$\bar{u}(x,t) \in L^\infty((0,\infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (\tau,\infty)),$$

is a solution to equation (2.3) in the sense of distributions and $\lim_{t\to 0} \bar{u}(x,t) = m\delta(x)$ in the sense of bounded measures. Due to the uniqueness we have that $\bar{u}(x,t) = \theta_m$ and has been uniquely identified. We can replace now the sequence $\{u^{\lambda_j}\}$ by the whole family $\{u^\lambda\}$ in the limit process and we can conclude that u^λ converges to θ_m as $\lambda \to \infty$.

5 Remarks on the decay rate

Let us discuss the sharpness of our decay rate. Set

$$M_p(t) = t^{\frac{p-1}{2p}} \|u(\cdot,t) - \theta_m(\cdot,t)\|_{L^p(\mathbb{R})},$$

our main result reads

$$\lim_{t\to\infty} M_p(t) \to 0,$$

which means that $t^{\frac{p-1}{2p}}\theta$ is the L^p attractor of $t^{\frac{p-1}{2p}}u$. Different arguments by Chern [5] give the precise algebraic decay rate for a class of restricted initial data. His approach is to explore Fourier Transform combined with the weighted energy method. We would like to point out that usually one can only get the decay rate $t^{-\frac{p-1}{2p}}$ for the general initial data as we considered in this paper. The extra decay rate $t^{-1/2}$ is possible only by further specifying the initial data. Following Chern, we shall use Fourier transform to clarify the source of the decay rate for $p = 2$. Setting

$$z := u - \theta, \quad w := v - f(\theta) - a^2\theta_x,$$

then one gets

$$\begin{aligned} z_t + w_x &= 0, \\ w_t + a^2 z_x &= \lambda^* z - w + H \end{aligned}$$

with $\lambda^* = f'(0)$ and $H = O(1)(|\theta\theta_t||\theta_{xt}| + |z\theta| + |z|^2)$. By taking the Fourier transform of the linear part one gets the solution operator associated with the Green's matrix, we have

$$\hat{G}(\xi, t) = e^{tR(i\xi)}, \quad R(i\xi) = \begin{pmatrix} 0 & -i\xi \\ \lambda^* - i\xi a^2 & -1 \end{pmatrix}$$

By perturbations around $\xi = 0$ and $|\xi| = \infty$, one gets the following decay rate, see [5]

$$|\hat{G}(\xi, t)| \leq \begin{cases} O(1)e^{-\beta t}, & |\xi| \geq \xi_0 \\ O(1)[e^{-a^2\xi^2 t} + |\xi|e^{-\alpha\xi^2 t} + e^{-\beta t}], & |\xi| \leq \xi_0 \end{cases}$$

for some positive constants $\alpha > 0$, $\beta > 0$. The above estimates give us the following

$$\|z(t)\|_{L^2}^2 \leq C_1 t^{-1/2}[t^k\|D_x^k z_0\|_{L^1}^2 + t^{-1}\|(z_0, w_0)\|_{L^1}] + C_2 e^{-2\beta t}\|(z_0, w_0)\|_{L^2}^2. \tag{5.1}$$

Estimates for solutions of the full system then follow by estimating the solution formula obtained by the Duhamel's principle combined with the energy method. In (5.1) there are three decay factors: $t^{-1/2+k}$, $t^{-3/2}$ and $e^{-\beta t}$. For the integral data we have $k = 0$, then

$$M_2(t) = O(1),$$

that is even worse than our decay rate. Such rate may be really improved only by choosing $k = -1$, i.e. $z_0 \in W^{-1,1}$, then

$$M_2(t) = O(1)t^{-1/2}.$$

Let us recall that for the quasiliner hyperbolic equation with linear damping, the decay rates have been intensively studied. One way, see e.g. K. Nishihara [32], is to recover the extra decay rate in two steps by using weighted energy method combined with the pointwise analysis. First construct an approximate diffusion wave $\tilde{\theta}$ starting with the initial data $u_0 - v_{0x}$, then the extra rate is obtained because the leading term for u and $\tilde{\theta}$ is canceled in pointwise analysis for the chosen initial data. The second step is to measure the distance between θ and $\tilde{\theta}$ by further assuming $\theta_0 - \tilde{\theta}_0 \in W^{-1,1}$. Clearly for general data with $z_0 \in L^1(\mathrm{I\!R})$ one could only recover the rate $t^{-\frac{p-1}{2p}}$, which is optimal for u as well as for θ. For their difference z we show the fast decay in the sense of

$$\lim_{t\to\infty} M_p(t) = 0, \quad 1 \le p < \infty,$$

but probably at no uniform decay rate. However, it does not preclude an extra decay estimate of the form

$$M_p(t) \le O(1)t^{-1/2}\|V^0\|_{L^1\cap H^s} \tag{5.2}$$

with $V^0 = (\int_{-\infty}^x z_0(y)dy, w_0)$. Chern [5] obtained exactly this sort of estimate like (5.2). Such an extra decay, roughly speaking, comes from the initial extra decay rate in space.

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