Measure-valued spline curves: An optimal transport viewpoint

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Sep 25, 2020
Introduction

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2. Splines in $\mathcal{P}_2(\mathbb{R}^d)$

3. Formulation in phase space

4. Fluid dynamical formulation
Smooth interpolation of distributions
Optimal Mass Transport

Move mass from distribution $\rho_0$ to $\rho_1$. 

\[ y = T(x) \]
Monge’s formulation:

$$\inf_{T \# \rho_0 = \rho_1} \int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) \, dx.$$ 

- $c(x, y)$ is the cost function. Generally, $c(x, y) = |x - y|^2$.
- $T \# \rho_0 := \rho_0 \circ T^{-1}$ is the pushforward of $\rho_0$ through the transport plan, $T$.
- Quite challenging because of the constraint.
Kantorovich’s formulation:

\[
\inf_{\gamma \in \Pi(\rho_0, \rho_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \gamma(dx\,dy).
\]

- \( \Pi(\rho_0, \rho_1) = \{ \gamma \in P(\mathbb{R}^d \times \mathbb{R}^d) : \gamma \# \pi_1 = \rho_0, \gamma \# \pi_2 = \rho_1 \} \).
- More generalized and easier to handle.
- The solution defines Wasserstein distance.
Interpolation

Figure: Interpolation between the optimal transport framework (left) and Euclidean space (right).

Wasserstein geodesic is essentially a ”linear” distribution between distributions.
Linear interpolation vs Spline
Splines in $\mathbb{R}^d$

Definition 1

Let \( \{t_i, x_i\}_{i=0}^N \subset [0, 1] \times \mathbb{R}^d \) be given time-space data:

- A function $S \in C^2([0, 1]; \mathbb{R}^d)$ is a **cubic spline** if $S$ is a cubic polynomial in each interval $[t_i, t_{i+1}]$, $i = 0, \ldots, n$.
- A cubic spline is an **interpolating cubic spline** if $S_{t_i} = x_i$ for all $i = 0, 1, \ldots, n$.
- An interpolating spline is called **natural** if $\partial_{tt}S_{t_0} = \partial_{tt}S_{t_1} = 0$.

WLOG, assume $0 = t_0 < t_1 < \cdots < t_N = 1$.

**Aim:** To use this concept to interpolate distributions smoothly.
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Variational Interpretation

Theorem 2 (Holladay '57)

Let $\{t_i, x_i\}_{i=0}^N \subset [0, 1] \times \mathbb{R}^d$ be given time-space data. The variational problem

$$\inf_X \int_0^1 |\partial_{tt} S_t|^2 dt,$$

$X \in H^2([0, 1]; \mathbb{R}^d)$,

$X_{t_i} = x_i$,

admits a unique solution, which is the natural interpolating spline.

$H^2([0, 1]; \mathbb{R}^d)$ is the space of twice continuously differentiable functions with square-integrable second order derivative. Natural cubic spline minimizes mean-squared acceleration!
Splines in $\mathcal{P}_2(\mathbb{R}^d)$

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Spline Interpolation of Distributions

The problem of transporting mass configuration $\rho_0$ into the mass configuration $\rho_i$ at time $t_i$ while minimizing the mean squared acceleration.

- A **transport plan** $P$ is a probability distribution on space $\Omega = C^0$.
- For $A \subset \Omega$, $P(A)$ represents the total mass that flows along the paths in $A$. 
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$X_t$ is the projection map: $\forall \omega \in \Omega$, $X_t(\omega) = \omega_t$.

**Definition 3**

Let $\{t_i, \rho_i\}_{i=0}^N \subset [0, 1] \times \mathcal{P}_2(\mathbb{R}^d)$ be given data. The marginal flow $\rho_t$ of an optimal solution of the problem

$$
\inf_P \int_0^1 \int_\Omega |\partial_{tt}X_t|^2 \, dP dt,
$$

$P \in \mathcal{P}(\Omega)$, $P(H^2) = 1$,

$$(X_{t_i})^\# P = \rho_i, \ i = 0, 1, \ldots, N,$$

is an **interpolating spline** for the given data.

We can guarantee existence but not uniqueness.
Consistency

**Proposition**

Let \( \{t_i, x_i\} \subset [0, 1] \times \mathbb{R}^d \) and set \( \rho_i := \delta_{x_i} \) for \( 0 \leq i \leq N \). Then the unique optimal solution is

\[
P^* = \delta_S,
\]

where \( S \) is the natural interpolating spline for \( \{t_i, x_i\}_{i=0}^N \).

All the particles are tied to each other and move together.
Existence

\[ X_T := (X_{t_0}, X_{t_1}, \ldots, X_{t_N}), \]
\[ \Pi(\rho_0, \rho_1, \ldots, \rho_N) = \{ \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d \ldots \times \mathbb{R}^d) : (Y_i)_{\#} \pi = \rho_i \}. \]

**Theorem 4**

Let \( \{\rho_i\}_{i=0}^N \subset \mathcal{P}_2(\mathbb{R}^d) \). Then there exists at least an optimal solution to the problem. Moreover, the following are equivalent

(a) \( \hat{P} \) is an optimal solution.

(b) \( \hat{P}(S_0^0) = 1 \) and \( \hat{\pi} := (X_T)_{\#} \hat{P} \) is an optimal solution for

\[ \inf_{\pi} \int C(x_0, x_1, \ldots, x_N) \, d\pi, \]
\[ \pi \in \Pi(\rho_0, \rho_1, \ldots, \rho_N). \]

\[ C(x_0, x_1, \ldots, x_N) = \int_0^1 |\partial_{tt} S_t(x_0, x_1, \ldots, x_N)|^2 \, dt. \]
An optimal solution is supported on natural splines of $\mathbb{R}^d$.

Its joint distribution at times $(t_0, t_1, \ldots, t_N)$ solves a multimarginal optimal transport problem whose cost function is $C$.

$C(x_0, x_1, \ldots, x_N)$ is the optimal value as in the original variational problem in $\mathbb{R}^d$ (Mean-squared acceleration of natural spline).

The spline thus obtained is analogous to the fact that the geodesics of $\mathcal{P}_2(\mathbb{R}^d)$ are constructed pushing forward the optimal coupling of the Monge-Kantorovich problem through geodesics of $\mathbb{R}^d$.

This approach is computationally burdensome.
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Enlarge the space to $H^1 \times H^1$.

\[
\inf_Q \int_0^1 \int_{\Omega \times \Omega} |\partial_t V_t|^2 \, dQ \, dt,
\]

\[Q \in \mathcal{P}(\Omega \times \Omega), \quad Q(\mathcal{H}^1 \times \mathcal{H}^1) = 1,
\]

\[Q(\partial_t X_t = V_t \forall t \in [0,1]) = 1,
\]

\[(X_{t_i})\# Q = \rho_i, \quad i = 0, 1, \ldots, N.
\]

This is equivalent to the previous problem.

Analogous $C$ has a closed form solution.
Fluid dynamical formulation
Recall: Fluid dynamical formulation of the MP problem is due to Benamou and Brenier (2000). The optimal value for

$$\inf_{\mu, \nu} \int_0^1 \int_{\mathbb{R}^d} |\nu_t|^2(x) \mu_t(x) \, dx \, dt,$$

$$\partial_t \mu_t(x) + \nabla \cdot (\nu_t \mu_t)(x) = 0,$$

$$\mu_0 = \rho_0, \ \mu_1 = \rho_1.$$

is the squared Wasserstein distance $W_2^2(\rho_0, \rho_1)$ and the optimal curve is the displacement interpolation.
Splines

Problem:

\[
\inf_{\mu, a} \int_0^1 \int_{\mathbb{R}^d} |a_t(x, \nu)|^2 \mu_t(x, \nu) \, dx \, d\nu,

\partial_t \mu_t(x, \nu) + \langle \nabla_x \mu_t(x, \nu), \nu \rangle + \nabla_{\nu} \cdot (a_t \mu_t)(x, \nu) = 0,

\int_{\mathbb{R}^d} \mu_{t_i}(x, \nu) \, d\nu = \rho_{t_i}, \quad i = 0, 1, \ldots, N.
\]

This problem is equivalent to the variational form.
- Y. Chen, G. Conforti, T. Georgiou.
  *Measure-valued spline curves: An optimal transport viewpoint.*

- J. Holladay.
  *A smoothest curve approximation.*

- J.-D. Benamou, Y. Brenier.
  *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem.*
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*A convexity principle for interacting gases.*
Thank You!

Questions?