ALTERNATING EVOLUTION SCHEMES FOR HYPERBOLIC
CONSERVATION LAWS

HASEENA SARAN† AND HAILIANG LIU†

Abstract. The alternating evolution (AE) system of Liu [J. Hyperbolic Differ. Equ., 5 (2008), pp. 421–447], \(\partial_t u + \partial_x f(v) = \frac{1}{\epsilon}(v - u), \partial_t v + \partial_x f(u) = \frac{1}{\epsilon}(u - v)\), serves as a refined description of systems of hyperbolic conservation laws \(\partial_t \phi + \partial_x f(\phi) = 0\), \(\phi(x, 0) = \phi_0(x)\). The solution of conservation laws is precisely captured when two components take the same initial value as \(\phi_0\), or is approached by two components exponentially fast when \(\epsilon \downarrow 0\) if two initial states are sufficiently close. This nice property enables us to construct novel shock capturing schemes by sampling the AE system on alternating grids. In this paper we develop a class of local AE schemes by taking advantage of the AE system. Our approach is based on an average of the AE system over a hypercube centered at \(x\) with vertices at \(x \pm \Delta x\). The numerical scheme is then constructed by sampling the averaged system over alternating grids. Higher order accuracy is achieved by a combination of high order nonoscillatory polynomial reconstruction from the obtained averages and a matching Runge–Kutta solver in time discretization. Local AE schemes are made possible by letting the scale parameter \(\epsilon\) reflect the local distribution of nonlinear waves. The AE schemes have the advantage of easier formulation and implementation, and efficient computation of the solution. The formulation procedure of AE schemes in multiple dimensions is given, followed by both the first and second order AE schemes for two-dimensional conservation laws. Numerical experiments for both scalar conservation laws and compressible Euler equations are presented to demonstrate the high order accuracy and capacity of these AE schemes.

Key words. conservation laws, alternating evolution schemes, shock capturing methods, TVD stability

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1. Introduction. A multidimensional hyperbolic conservation law has the form

\[
\phi_t + \nabla_x \cdot f(\phi) = 0, \quad x \in \mathbb{R}^d, \quad t > 0,
\]

where \(\phi \in \mathbb{R}^m\) denotes a vector of conserved quantities, and \(f: \mathbb{R}^m \to \mathbb{R}^d\) is a nonlinear convection flux. The compressible Euler equation in gas dynamics is a canonical example. These equations are of great practical importance since they model a variety of physical phenomena that appear in fluid mechanics, astrophysics, groundwater flow, traffic flow, semiconductor device simulation, and magneto-hydrodynamics, among many others.

The need for devising accurate and efficient numerical methods for nonlinear hyperbolic conservation laws and related models has prompted and sustained the abundant research in this area; see, for example, [25, 39, 38, 6]. The notorious difficulty encountered for the satisfactory approximation of the exact solutions of these systems lies in the presence of discontinuities in the solution. A well-known recipe

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†Mathematics Department, Iowa State University, Ames, IA 50011 (hsaran@eqecat.com, hliu@iastate.edu). The research of the second author was partially supported by the National Science Foundation under grant DMS09-07963.
to achieve both high order accuracy and convergence to the entropy solution is the so-called high-resolution schemes. Their success is due to two factors: the local enforcement of nonlinear conservation laws and the nonoscillatory piecewise polynomial reconstruction from evolved local moments (cell averages).

In this work we present new alternating evolution (AE) schemes to (1.1), together with stability analysis and performance tests through numerical experiments. These schemes can be viewed as modifications of the global AE scheme proposed in [27]. In the formulation of these schemes we borrow techniques of both the local enforcement and the high order polynomial reconstruction from the literature, but apply them to a novel approximation system,

\begin{align}
 u_t + \nabla_x \cdot f(v) &= \frac{1}{\epsilon} (v - u), \\
 v_t + \nabla_x \cdot f(u) &= \frac{1}{\epsilon} (u - v).
\end{align}

Here $\epsilon > 0$ is a scale parameter of the user’s choice. One distinguishing feature of this system is the following inequality:

$$
\|u(\cdot, t) - v(\cdot, t)\|_{L^1} \leq \|u_0(\cdot) - v_0(\cdot)\|_{L^1} e^{-2t/\epsilon}.
$$

This implies that the exact solution for nonlinear conservation laws is captured by this system if $u(x, 0) = v(x, 0) = \phi_0(x)$. When $\|u_0(\cdot) - v_0(\cdot)\|_{L^1}$ is sufficiently small, both components approach each other exponentially fast as $\epsilon \downarrow 0$; see [27]. Such a feature allows a sampling of the system over alternating grids when performing spatial discretization. On the other hand if the system is rewritten as

\begin{align}
 u + \epsilon u_t &= v - \epsilon \nabla_x \cdot f(v), \\
 v + \epsilon v_t &= u - \epsilon \nabla_x \cdot f(u),
\end{align}

one can see for $\epsilon = O(\Delta t)$ that the system is able to transfer spatial changes of one component into temporal changes of another component. The communication between two components is thus realized by the choice of parameter $\epsilon$. A word of caution: The AE system may well lose hyperbolicity in the general setting, which by itself is an interesting issue. But this can be remedied by recasting the original conservation laws on a fast moving reference. Numerically such a remedy is unnecessary since the dissipation produced from the averaging operation has provided an ideal fix.

The numerical schemes, presented in [27] and this work, take advantage of these remarkable features of the AE system. There are three steps involved in the discretization procedure: (i) the AE system is locally enforced by taking the spatial averaging over a hypercube centered at $x$ with vertices being its neighboring grids; (ii) the averaged system is then sampled over alternating grids for its two components; (iii) the piecewise polynomial reconstruction from averages of two components is carried out, respectively, followed by an ODE solver with matching accuracy for time discretization of the resulting semidiscrete system. The strong stability preserving method [11] is particularly suitable. Extension of this procedure to multidimensional problems is straightforward. The scheme obtained in this manner is thus called the alternating evolution (AE) scheme. The result is a simple and high-resolution finite difference scheme, whose derivation involves no numerical fluxes and avoids dimensional splitting.

To put our study in the proper perspective, we recall a few of the references from the considerable amount of literature available on numerical methods for hyperbolic
conservation laws. The finite difference/volume solution techniques for nonlinear conservation laws fall under two main categories according to their way of sampling [33]: upwind and central schemes. The forerunners for these two large classes of high-resolution schemes for nonlinear conservation laws are the first order Godunov [10] and Lax–Friedrichs schemes [9, 24], respectively.

In [10] Godunov proposed evolving a piecewise cell average representation of the solution and evaluating the fluxes at cell interfaces. Godunov-type upwind schemes are then obtained by using an exact or approximate Riemann solver to distribute the nonlinear waves between two neighboring computational cells. Various higher order extensions of the Godunov scheme have been rapidly developing since the 1970s, employing higher order reconstruction of piecewise polynomials from the cell averages, including MUSCL, TVD, PPM, ENO, and WENO schemes [40, 41, 12, 7, 13, 35, 36, 28].

In contrast, Godunov-type central schemes are more diffusive, yet easy to formulate and implement since no Riemann solvers are required. In the one-dimensional case, examples of such schemes for conservation laws are the second order Nessyahu–Tadmor scheme [33] and the higher order schemes in [29, 1, 26]. Second order multidimensional central schemes were introduced in [16], and their higher order extensions were developed in [21]. For nonstaggered central schemes consult [15].

The two categories of schemes are somehow interlaced during their independent developments; the upwind scheme becomes Riemann solver-free when a local numerical flux is used to replace the exact Riemann solver (see Shu and Osher [35, 36]), and the central scheme becomes less diffusive when variable control volumes are used in deriving the scheme (see Kurganov and Tadmor [22]). The upwind feature can be further enforced [14, 20] in central-upwind schemes; see [19] for a recent derivation of such a scheme. The relaxation scheme of Jin and Xin [17] provides another approach for solving nonlinear conservation laws.

More closely related is the recent work of Liu [30], who generalizes the Nessyahu–Tadmor scheme [33] by evolving two pieces of information over redundant overlapping cells, therefore allowing easy formulation of semidiscrete schemes. The numerical solution is represented as a convex combination of overlapping cell averages evolved by the Nessyahu–Tadmor scheme. The technique has recently been applied in the development of a central-type discontinuous Galerkin method [31].

The scheme that we design here is different in that it is based on alternatively sampling the AE system [27] with spatial accuracy enhanced by interlaced local reconstructions. The procedure discussed in this paper opens a new way to derive robust symmetric finite difference schemes for hyperbolic conservation laws. Also the semidiscrete scheme thus obtained offers an economic approach for achieving a matching accuracy in time. Another attractive feature of the AE scheme is the amount of leverage in the choice of $\epsilon$. Indeed different choices of the scale parameter in such a procedure yield different AE schemes such as local AE schemes presented in this paper.

The AE schemes like other symmetric finite difference schemes are very easy to implement and efficient, but in some cases they cannot match more compact high order methods such as the discontinuous Galerkin method [34]. We also note that some schemes in [30] when rescaling the mesh size $\Delta x$ to $2\Delta x$ are the same as the global AE schemes presented in [27]. Hence, in terms of accuracy, the global AE scheme with $N$ grid points is comparable only with overlapping cell schemes [30] with $N/2$ grid points. But the AE scheme is more simple to implement. Actually such simplicity in implementation is even more pronounced in the two-dimensional case.
The AE system also distinguishes itself from the relaxation approximation [17, 18, 32] to nonlinear conservation laws. In relaxation systems the solution will deviate from the entropy solution with a distance of order $O(\sqrt{\epsilon})$ (here $\epsilon$ is the relaxation parameter; see, e.g., [18]); even initial data are at equilibrium. One thus has to resolve such a parameter with a proper choice of the mesh size. In contrast, the solution of the AE system when taking the same initial data for two components becomes the exact solution of conservation laws (independent of $\epsilon$); the choice of $\epsilon$ thus becomes a gift which can be wisely selected to stabilize the time discretization.

We now conclude this section by outlining the rest of the paper. In section 2 we give the first and second order formulations of the numerical method for both global and local schemes. A general algorithm is presented for constructing AE schemes of any desired order. Under some CFL-type conditions, both the total variation diminishing (TVD) property and the maximum principle are proved in section 3 for first and second order local AE schemes, respectively. Extension to multiple dimensions is discussed in section 4, where both first and second order schemes are explicitly given. Numerical tests for both scalar conservation laws and compressible Euler equations for up to third order schemes are presented in section 5. Our numerical results show that the AE schemes can capture shock waves as well as other existing high resolution central schemes.

2. Scheme formulation. The detailed formulation and stability analysis of the first and second order global AE schemes is given in [27]. We present the idea here again and give a general algorithm for constructing higher order AE schemes.

2.1. Global AE schemes. Let $\bar{u}(x, t)$ denote the sliding average of $u(\cdot, t)$ over interval $I_x = \{\xi | x - \Delta x \leq \xi \leq x + \Delta x\}$:

$$
\bar{u}(x, t) = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} u(\xi, t) d\xi.
$$

Integration of the one-dimensional AE system (1.2)–(1.3) over $I_x$ gives

\begin{align}
\frac{d}{dt} \bar{u}(x, t) &= -\frac{\bar{u}(x, t)}{\epsilon} + \frac{1}{\epsilon} L[v](x, t), \\
\frac{d}{dt} \bar{v}(x, t) &= -\frac{\bar{v}(x, t)}{\epsilon} + \frac{1}{\epsilon} L[u](x, t),
\end{align}

where $x$ serves as a moving parameter and

$$
L[\Phi](x, t) = \dot{\Phi}(x, t) - \frac{\epsilon}{2\Delta x} (f(\Phi(x + \Delta x, t)) - f(\Phi(x - \Delta x, t))).
$$

For simplicity we take uniform distributed grids $x_j, j \in \mathbb{Z}$, in a one-dimensional setting. The numerical scheme is constructed by sampling both formulations (2.1) and (2.2) over alternating grids for $u$ and $v$, respectively, with proper approximation of $L[\Phi]$ defined in (2.3). For example, if we sample (2.1) at $x_{2j}$ and (2.2) at $x_{2j+1}$, respectively, then we have

\begin{align}
\frac{d}{dt} u_{2j}(t) &= \frac{1}{\epsilon} [-u_{2j}(t) + L_{2j}[v](t)], \\
\frac{d}{dt} v_{2j+1}(t) &= \frac{1}{\epsilon} [-v_{2j+1}(t) + L_{2j+1}[u](t)].
\end{align}
Here we define $L_k[\Phi](t) := L[\Phi](x_k, t)$. In general, the numerical solution at time $t = t_n$ can be represented as

$$\Phi^n_k = \begin{cases} u^n_k, & k = 2j, \\ v^n_k, & k = 2j + 1, \end{cases}$$

so that the semidiscrete scheme takes a compact form:

$$\frac{d}{dt} \Phi_k = \frac{1}{\epsilon} [-\Phi_k + L_k[\Phi]],$$

where $L_k[\Phi]$ is an approximation of $L[\Phi]$ using nonoscillatory reconstructions based on averages $\Phi$. Let $p[\Phi]$ be a piecewise polynomial reconstruction on $I_k = [x_{k-1}, x_{k+1}]$, being conservative,

$$\Phi_k = \frac{1}{2\Delta x} \int_{I_k} p[\Phi](x) \, dx,$$

and accurate of a desired order. With such a reconstructed $p[\Phi]$, we proceed to evaluate $L[\Phi]$ as follows: at any node $x_k$, using piecewise polynomials $p[\Phi]$ constructed on $I_{k+1}$ and $I_{k-1}$, instead of the one on $I_k$, we obtain

$$L_k[\Phi] = \frac{1}{2\Delta x} \left[ \int_{I_k \cap I_{k+1}} + \int_{I_k \cap I_{k-1}} \right] p[\Phi] \, dx - \frac{\epsilon}{2\Delta x} \left[ f(p[\Phi]_{k+1}) - f(p[\Phi]_{k-1}) \right].$$

This leads to a general algorithm:

- **Initialization**: For $I_k := [x_{k-1}, x_{k+1}]$, we define $\Phi^0 = \{u^0, v^0\}$ via

$$\Phi^0_k = \frac{1}{2\Delta x} \int_{I_k} \phi_0(\xi) \, d\xi, \quad k \in \mathbb{Z}.$$

- **Reconstruction**: Given $\Phi_k(t)$, we reconstruct piecewise nonoscillatory polynomials $p[\Phi(t)](x)$ over intervals $I_k$ for $k \in \mathbb{Z}$, such that (2.8) holds for all $k \in \mathbb{Z}$.

- **Evolution**: Given $\Phi^n = \{u^n, v^n\}$, we obtain $\Phi^{n+1}$ by applying a stable Runge–Kutta method to the ODE system (2.7) over $[t_n, t_{n+1}]$.

Here the spatial accuracy of the scheme is ensured by the order of the reconstruction $p[\Phi]$; then one may select a proper ODE solver with matching accuracy in time to solve the semidiscrete scheme (2.7) and (2.9).

Here we recall both the first and second order schemes derived in [27]. In global AE schemes, $\epsilon$ is chosen such that the stability condition

$$\Delta t < \epsilon \leq Q \frac{\Delta x}{\max |f'|}$$

is satisfied. In system case max $|f'|$ is replaced by the largest eigenvalue of the Jacobian $\partial_u f$. The choice of $Q$ depends on the order of the scheme.

If the reconstructed polynomial is piecewise constant, then the first order scheme becomes

$$\Phi_k^{n+1} = (1 - \kappa) \Phi_k^n + \kappa \left[ \frac{\Phi_{k+1}^n + \Phi_{k-1}^n}{2} - \frac{\epsilon}{2\Delta x} \left( f(\Phi_{k+1}^n) - f(\Phi_{k-1}^n) \right) \right],$$

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where $\kappa = \frac{\Delta t}{\epsilon}$. When $Q = 1$, this yields a class of monotone schemes, with the celebrated Lax–Friedrichs scheme being a special case of $\kappa = 1$.

Using a linear polynomial reconstruction of the form
\begin{equation}
 p(\Phi^n)(x) = \sum_k (\Phi^n_k + s^n_k(x-x_k)) \chi_k(x),
\end{equation}
where $\chi_k$ is the characteristic function which takes value one on interval $I_k$, we arrive at the second order AE scheme (2.7) with
\begin{equation}
 L_k[\Phi] := \frac{\Phi_{k+1} + \Phi_{k-1}}{2} + \frac{\Delta x}{4} (s_{k-1} - s_{k+1}) - \frac{\epsilon}{2\Delta x} (f(\Phi_{k+1}) - f(\Phi_{k-1})).
\end{equation}
The nonoscillatory property requires that $s^n_k$ be chosen with certain limiters. In our numerical tests we use the basic minmod limiter (see, e.g., [33, 29]):
\begin{equation}
 s_k = \text{minmod} \left\{ \frac{\Phi_{k+1} - \Phi_k}{\Delta x}, \frac{\Phi_k - \Phi_{k-1}}{\Delta x} \right\},
\end{equation}
where
\begin{equation}
 \text{minmod} \{a, b\} = \frac{1}{2} \left( \text{sgn}(a) + \text{sgn}(b) \right) \min\{|a|, |b|\}.
\end{equation}

For the third order AE scheme used in our numerical tests we follow the ENO reconstruction for achieving the third order spatial accuracy and the third order TVD Runge–Kutta method for time discretization; see [35, 36]. Considering the third order strong stability preserving Runge–Kutta solver has a better stability property, so one may use it even for the second order (in space) scheme in the numerical simulation.

For the third order AE scheme used in our numerical tests we follow the ENO reconstruction for achieving the third order spatial accuracy and the third order TVD Runge–Kutta method for time discretization; see [35, 36]. Considering the third order strong stability preserving Runge–Kutta solver has a better stability property, so one may use it even for the second order (in space) scheme in the numerical simulation.

Finally we note that $\epsilon$ plays an important role as a dissipation factor. Figures 1 and 2 show plots for the solution of Burgers’ equation when the solution is discontinuous. From the graphs we can see that as $\epsilon$ decreases, the dissipation increases and the solution is more smeared for a fixed time step $\Delta t$. In order to reduce the dissipation we introduce local AE schemes, which are presented in the next section.

**2.2. Local AE schemes.** In local schemes, we embed more local information into $\epsilon$ than in the global AE schemes. Instead of a single global parameter $\epsilon$, we also use local parameters to increase the numerical accuracy of the scheme.
the one-dimensional AE system (1.2)–(1.3) over

\[ \epsilon (2.19) \]

\[ (2.16) \]

Depending on the way \( \epsilon \) scheme to show the effect of change in \( \epsilon \).

2.2.1. First order local AE schemes. Using local parameters and integrating
the one-dimensional AE system (1.2)–(1.3) over \( I_x \) gives

\[
\frac{d}{dt} u(x, t) = -\frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \partial_x f(v) \, dx - \frac{1}{2\Delta x} \left[ \frac{1}{\epsilon_x-\Delta x/2} \int_{x-\Delta x}^{x} + \frac{1}{\epsilon_x+\Delta x/2} \int_{x}^{x+\Delta x} \right] (u - v) \, dx,
\]

(2.16)

\[
\frac{d}{dt} v(x, t) = -\frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \partial_x f(u) \, dx - \frac{1}{2\Delta x} \left[ \frac{1}{\epsilon_{x-\Delta x/2}} \int_{x-\Delta x}^{x} + \frac{1}{\epsilon_{x+\Delta x/2}} \int_{x}^{x+\Delta x} \right] (v - u) \, dx.
\]

(2.17)

Note that \( \epsilon_{x-\Delta x/2} \) uses the information to the left of \( x \), and \( \epsilon_{x+\Delta x/2} \) uses information to the right of \( x \). Sampling (2.16) at \( x_{2j} \) and (2.17) at \( x_{2j+1} \), respectively, and using piecewise constant polynomial reconstruction, we obtain

\[
\Phi_{k+1}^n = \Phi_k^n + \frac{\lambda}{2} (f(\Phi_{k+1}^n) - f(\Phi_{k-1}^n)) - \kappa_{k-1/2}^n \frac{1}{2} (\Phi_k^n - \Phi_{k-1}^n) - \kappa_{k+1/2}^n (\Phi_k^n - \Phi_{k+1}^n),
\]

(2.18)

where

\[
\kappa_{k+1/2}^n = \frac{\Delta t}{\epsilon_{k+1/2}}.
\]

Use \( M_{k+1/2} \) to denote the range \([\min(\Phi_k, \Phi_{k+1}), \max(\Phi_k, \Phi_{k+1})]\). For the system case it denotes a curve in phase space connecting \( \Phi_k \) and \( \Phi_{k+1} \) via the Riemann fan. Depending on the way \( \epsilon_{k+1/2} \) is defined, we can formulate the following local AE scheme.

Local AE1 scheme. We choose \( \epsilon_k \) such that

\[
\epsilon_{k+1/2} \leq Q \frac{\Delta x}{\max_{s \in M_{k+1/2}} |f'(s)|}, \quad \Delta t \leq \min_k \epsilon_{k+1/2}.
\]

(2.19)

\( Q \) is a factor that is dependent on the order of the scheme, and the stability conditions presented in section 3 provide the range of values \( Q \) can take.

Fig. 2. Plot for Burgers’ equation at discontinuity on \([0, 2]\), \( T = 0.7, \Delta x = \frac{1}{200} \), third order
AE scheme to show the effect of change in \( \epsilon \).
Remark 2.1. For the scalar equation, we may adopt a more accurate version of the local scheme, called the local AE2 scheme. We choose \( \epsilon \) such that

\[
\epsilon_{k+1/2} \leq \frac{Q}{1} \frac{\Delta x}{|M_{k+1/2}|} \int_{M_{k+1/2}} |f'(s)| \, ds, \quad \Delta t \leq \min_k \epsilon_{k+1/2}.
\]

This choice is motivated by the Engquist–Osher flux \([8, 39]\). This scheme suggests that even in the AE discretization, there are still different ways to incorporate local wave propagation into the scheme.

In section 3 we will derive conditions on \( \epsilon_{k+1/2} \) in order that the local AE scheme (2.18)–(2.20) is numerically stable.

### 2.2.2. Second order local AE schemes

If we sample (2.16) at \( x_{2j} \) and use piecewise linear polynomial reconstruction on \([x_{2j-1}, x_{2j+1}]\), forward Euler in time with time step \( \Delta t \), we obtain a prediction for \( u \) as \( u_{2j}^* \) which can be given as

\[
u_{2j}^* = u_{2j}^* = u_{2j} - \frac{\Delta t}{2\Delta x} (f(v_{2j+1}^n) - f(v_{2j-1}^n))
- \frac{\Delta t}{2\Delta x} \int_{x_{2j-1}}^{x_{2j+1}} [(u^n_{2j} + s^n_{2j}(x - x_{2j})) - (v^n_{2j-1} + s^n_{2j-1}(x - x_{2j-1}))] \, dx
- \frac{\Delta t}{2\Delta x} \int_{x_{2j}}^{x_{2j+1}} [(v^n_{2j+1} + s^n_{2j+1}(x - x_{2j})) - (v^n_{2j+1} + s^n_{2j+1}(x - x_{2j-1}))] \, dx
= u_{2j} - \frac{\Delta t}{2\Delta x} (f(v_{2j+1}^n) - f(v_{2j-1}^n)) - \frac{\Delta t}{2\epsilon_{2j-1/2}} \left( u^n_{2j} - v^n_{2j-1} - \frac{\Delta x}{2} (s^n_{2j} + s^n_{2j-1}) \right)
- \frac{\Delta t}{2\epsilon_{2j-1/2}} \left( u^n_{2j} - v^n_{2j+1} + \frac{\Delta x}{2} (s^n_{2j} + s^n_{2j+1}) \right)
= u_{2j}^* - \Delta t F_{2j} [\Phi^n],
\]

where

\[
F_k[\Phi] = \frac{1}{2\Delta x} (f(\Phi_{k+1}) - f(\Phi_{k-1}))
+ \frac{1}{2\epsilon_{k-1/2}} \left( \Phi_k - \Phi_{k-1} - \frac{\Delta x}{2} (s_k + s_{k-1}) \right)
+ \frac{1}{2\epsilon_{k+1/2}} \left( \Phi_k - \Phi_{k+1} + \frac{\Delta x}{2} (s_k + s_{k+1}) \right).
\]

Similarly we can sample (2.17) at \( x_{2j+1} \) and use piecewise linear polynomial reconstruction to obtain a prediction for \( v_{2j+1}^n \). Thus the second order semidiscrete AE scheme is

\[
\frac{d}{dt} \Phi_k = F_k[\Phi].
\]

The second order local AE1 scheme is designed as in the case of the local first order AE1 scheme.

We shall prove stability of the second order scheme,

\[
\Phi_k^* = \Phi_k - \Delta t F_k[\Phi^n],
\]

\[
\Phi_k^{n+1} = \frac{1}{2} \Phi_k^n + \frac{1}{2} \Phi_k^* - \frac{\Delta t}{2} F_k[\Phi^*],
\]

where \( F_k[\Phi] \) is defined in (2.21).
Again this procedure can be easily implemented to obtain third order local AE schemes following the algorithm outlined for global AE schemes in section 2.1.

Next we make some remarks regarding the properties of the scheme.

Remark 2.2 (conservative form). Both global and local AE schemes are conservative, i.e., they satisfy

$$\sum_k \Phi_k^{n+1} = \sum_k \Phi_k^n.$$  

This is ensured by the conservative form of $F_k[\Phi]$:

$$\sum_k F_k[\Phi] = 0.$$  

According to the celebrated Lax–Wendroff theorem [23], the conservative form is necessary for underlying schemes to correctly approximate discontinuous weak solutions to the conservation law. It remains to show convergence of the underlying schemes, i.e., compactness of the numerical solution. This is measured in terms of the TVD property [12], to be analyzed in the next section.

Remark 2.3 (numerical viscosity). The local AE2 scheme has the smallest numerical viscosity since it uses the average of the local speed over a cell, followed by the local AE1 scheme, which uses the maximum of the local speed over a cell, followed by the global AE scheme, which takes the maximum of the speed over all cells.

3. Stability analysis. We show that both first and second order AE schemes are stable in the sense of satisfying the maximum principle, as well as having bounded total variation. Let $\Phi^n := \{u^n, v^n\}$ be a computed solution. We use the following notation:

$$|\Phi^n|_{\infty} = \max_k |\Phi^n_k|, \quad TV[\Phi^n] = \sum_k |\Phi_{k+1}^n - \Phi_k^n|.$$  

3.1. Stability of first order AE schemes.

**Theorem 3.1.** Let $\Phi := \{u^n, v^n\}$ be computed from the first order local AE scheme (2.18) for scalar hyperbolic conservation laws. If,

(a) for local AE1,

$$\epsilon_{k+1/2} \leq \frac{\Delta x}{\max_{s \in M_{k+1/2}} |f'(s)|}, \quad \Delta t \leq \frac{1}{2} \min_k \epsilon_{k+1/2};$$  

(b) or, for local AE2,

$$\epsilon_{k+1/2} \leq \frac{\Delta x}{1/M_{k+1/2}} \int_{M_{k+1/2}} |f'(s)| \, ds, \quad \Delta t \leq \frac{1}{2} \min_k \epsilon_{k+1/2}$$

holds, then

$$|\Phi^{n+1}|_{\infty} \leq |\Phi^n|_{\infty}, \quad n \in \mathbb{N}.$$  

Moreover, we have

$$TV[\Phi^{n+1}] \leq TV[\Phi^n], \quad n \in \mathbb{N}.$$
Proof. We can rewrite (2.18) as

\[
\Phi_{k+1}^n = \Phi_k^n - \frac{\lambda}{2} f_k' (\Phi_{k+1}^n - \Phi_k^n) - \frac{\kappa_{k-1/2}}{2} (\Phi_k^n - \Phi_{k-1}^n) - \frac{\kappa_{k+1/2}}{2} (\Phi_k^n - \Phi_{k+1}^n),
\]

where \( \lambda = \frac{\Delta t}{\Delta x} \) and \( f_k' = \int_0^1 f'(\Phi_k + \eta(\Phi_{k+1}^n - \Phi_k^n)) \, d\eta \). Rearranging, we obtain

\[
\Phi_{k+1}^n = \left(1 - \frac{\kappa_{k-1/2}}{2} - \frac{\kappa_{k+1/2}}{2}\right) \Phi_k^n + \left(-\frac{\lambda}{2} f_{k+1/2} + \frac{\kappa_{k+1/2}}{2}\right) \Phi_{k+1}^n
\]

(3.6)

\[
+ \left(\frac{\lambda}{2} f_{k-1/2} + \frac{\kappa_{k-1/2}}{2}\right) \Phi_{k-1}^n.
\]

When (3.2) or (3.3) holds, the coefficients in the expression for \( \Phi_{k+1}^n \) in (3.6) are nonnegative, and we have a convex combination of the grid point values. We can write

\[
|\Phi_{k+1}^n| \leq \left(1 - \frac{\kappa_{k-1/2}}{2} - \frac{\kappa_{k+1/2}}{2}\right) \max_k |\Phi_k^n| + \left(-\frac{\lambda}{2} f_{k+1/2} + \frac{\kappa_{k+1/2}}{2}\right) \max_k |\Phi_{k+1}^n|
\]

\[
+ \left(\frac{\lambda}{2} f_{k-1/2} + \frac{\kappa_{k-1/2}}{2}\right) \max_k |\Phi_{k-1}^n|
\]

\[
\leq |\Phi^n|_{\infty}.
\]

Thus the maximum principle (3.4) follows.

The TVD property follows from [37, Lemma 2.1] when the scheme is recast into a standard viscosity form. The proof of the asserted properties is thus complete.

3.2. Stability of second order AE schemes. For second order AE schemes, the slope limiter is typically chosen to ensure that the total variation does not increase under the operation of reconstruction:

\[
TV[p[u^n](x)] \leq TV[u^n], \quad TV[p[v^n](x)] \leq TV[v^n].
\]

This property is satisfied by the minmod limiters defined in (2.14). With such a reconstruction we prove the following.

THEOREM 3.2. Let \( \Phi^n \) be computed from the second order AE scheme (2.22)–(2.23) for scalar hyperbolic conservation laws. Let the slope \( s_k \) be defined in (2.14). If,

(a) for local AE1,

\[
\epsilon_{k+1/2} \leq \frac{1}{4} \frac{\Delta x}{\max_{s \in M_{k+1/2}} |f'(s)|}, \quad \frac{1}{3} \leq \kappa_{k+1/2} \leq \frac{1}{2},
\]

(b) or, for local AE2,

\[
\epsilon_{k+1/2} \leq \frac{1}{4} \frac{1}{|M_{k+1/2}|} \int_{M_{k+1/2}} \frac{\Delta x}{|f'(s)|} \, ds, \quad \frac{1}{3} \leq \kappa_{k+1/2} \leq \frac{1}{2}
\]

holds, then

\[
|\Phi_{n+1}^n|_\infty \leq |\Phi^n|_\infty, \quad n \in \mathbb{N}.
\]
Also, we have that

\[(3.10)\quad TV[\Phi^{n+1}] \leq TV[\Phi^n], \quad n \in \mathbb{N}.\]

**Remark 3.3.** The specific choice of bounds given in (3.7) and (3.8) is for convenience only. The proof as given below remains valid if we take \(\theta_1 \leq \kappa_{k+1/2} \leq \theta_2\) with \(Q = 1 - \frac{\theta_2}{2\theta_1}\) for any positive numbers \(\theta_i\) satisfying

\[\theta_2 \leq \frac{1}{2} \quad \text{and} \quad \frac{\theta_2}{2} < \theta_1 < \theta_2.\]

**Proof.** We first want to show that the maximum principle (3.9) holds. We will show that

\[(3.11)\quad |\Phi^*|_\infty = |\Phi^n - \Delta t F[\Phi^n]|_\infty \leq |\Phi^n|_\infty.

Then, from (2.23), we have

\[|\Phi^{n+1}|_\infty \leq \frac{1}{2}|\Phi^n|_\infty + \frac{1}{2}|\Phi^* - \Delta t F[\Phi^*]|_\infty.\]

Using (3.11) twice,

\[|\Phi^{n+1}|_\infty \leq \frac{1}{2}|\Phi^n|_\infty + \frac{1}{2}|\Phi^*|_\infty \leq \frac{1}{2}|\Phi^n|_\infty + \frac{1}{2}|\Phi^n|_\infty = |\Phi^n|_\infty,

which is the desired property (3.9).

We now prove (3.11) as follows:

\[\Phi^*_k = \Phi^n_k - \frac{\lambda}{2} (f_{k+1} - f_{k-1}) - \frac{\kappa_{k-1/2}}{2} (\Phi^n_k - \Phi^n_{k-1} - \frac{\Delta x}{2} (s_k + s_{k-1}))

\[ - \frac{\kappa_{k+1/2}}{2} (\Phi^n_k - \Phi^n_{k+1} + \frac{\Delta x}{2} (s_k + s_{k+1})),\]

where \(f_k\) is used to denote \(f(\Phi^n_k)\). Define the modified flux as

\[\tilde{f}^+_k := f_k + \frac{\Delta x}{2\lambda} \kappa_{k+1/2} s_k, \quad \tilde{f}^-_k := f_k + \frac{\Delta x}{2\lambda} \kappa_{k-1/2} s_k,

and rewrite \((f_{k+1} - f_{k-1})\) as \((f_{k+1} - f_k + f_k - f_{k-1})\). We can now rewrite (3.12) in terms of the modified flux as

\[\Phi^*_k = \Phi^n_k - \frac{\lambda}{2} (\tilde{f}^-_{k+1} - \tilde{f}^-_k + \tilde{f}^+_k - \tilde{f}^+_{k-1}) - \frac{\kappa_{k-1/2}}{2} (\Phi^n_k - \Phi^n_{k-1}) - \frac{\kappa_{k+1/2}}{2} (\Phi^n_k - \Phi^n_{k+1}).\]

Also set

\[(3.13)\quad \beta_{k+1/2}^\pm := \begin{cases} \frac{\tilde{f}^\pm_{k+1} - \tilde{f}^\pm_k}{\Phi^n_{k+1} - \Phi^n_k} & \text{if} \quad \Phi^n_{k+1} \neq \Phi^n_k, \\
0 & \text{if} \quad \Phi^n_{k+1} = \Phi^n_k.
\end{cases}\]

Using this definition, we have

\[\Phi^*_k = \Phi^n_k - \frac{\lambda}{2} \beta^-_{k+1/2} (\Phi^n_{k+1} - \Phi^n_k) - \frac{\lambda}{2} \beta^+_{k-1/2} (\Phi^n_k - \Phi^n_{k-1})

\[ - \frac{\kappa_{k-1/2}}{2} (\Phi^n_k - \Phi^n_{k-1}) - \frac{\kappa_{k+1/2}}{2} (\Phi^n_k - \Phi^n_{k+1}).\]
Rearranging,
\[
\Phi_k^* = \left(1 - \frac{\kappa_{k-1/2}}{2} - \frac{\kappa_{k+1/2}}{2} - \frac{\lambda}{2} \beta_{k-1/2}^+ + \frac{\lambda}{2} \beta_{k+1/2}^-ight) \Phi_k^\circ + \left(\frac{\kappa_{k-1/2}}{2} + \frac{\lambda}{2} \beta_{k-1/2}^+\right) \Phi_{k-1}
\]  \tag{3.15}

(3.15) \quad + \left(\frac{\kappa_{k+1/2}}{2} - \frac{\lambda}{2} \beta_{k+1/2}^-\right) \Phi_{k+1}^*.

When (3.7) or (3.8) holds, all the coefficients in the expression for \(\Phi_k^*\) are nonnegative. We will show this by looking at the coefficient for \(\Phi_k^\circ\), and the coefficients for the other terms follow.

\[
\frac{\lambda}{\kappa_{k+1/2}} \beta_{k+1/2}^- \leq \frac{\lambda}{\kappa_{k+1/2}} \left|f_{k+1} - f_k\right| + \frac{\Delta x}{2\kappa_{k+1/2}} \left|\kappa_{k+1/2} s_{k+1} - \kappa_{k-1/2} s_k\right| \leq \frac{\lambda}{\kappa_{k+1/2}} \left|f'\right| + \frac{1}{2} \max\left\{1, \frac{\kappa_{k-1/2}}{\kappa_{k+1/2}}\right\}.
\]

From (2.14) it follows that the maximum value that \(\frac{\Delta x|s_{k+1}|}{|\Phi_{k+1}^\circ - \Phi_k^\circ|}\) or \(\frac{\Delta x|s_k|}{|\Phi_{k+1}^\circ - \Phi_k^\circ|}\) can take is 1. Note that the slopes \(s_{k+1}\) and \(s_k\) have the same signs since minmod limiters are used. Now if (3.7) holds, we have \(\frac{\kappa_{k-1/2}}{\kappa_{k+1/2}} \leq 3/2\), and

\[
\frac{\lambda}{\kappa_{k+1/2}} \beta_{k+1/2}^- \leq \frac{1}{4} + \frac{3}{4} = 1.
\]

This implies that

\[
\left(\frac{\kappa_{k+1/2}}{2} - \frac{\lambda}{2} \beta_{k+1/2}^-\right) \geq 0.
\]

Along similar lines and with \(\kappa_{k+1/2} \leq 1/2\) again we can show that

\[
\left(1 - \frac{\kappa_{k-1/2}}{2} - \frac{\kappa_{k+1/2}}{2} - \frac{\lambda}{2} \beta_{k-1/2}^+ + \frac{\lambda}{2} \beta_{k+1/2}^-\right) \geq 0 \quad \text{and} \quad \left(\frac{\kappa_{k-1/2}}{2} + \frac{\lambda}{2} \beta_{k-1/2}^+\right) \geq 0.
\]

We can take the maximum norm on (3.15) to obtain

\[
|\Phi_k^*| \leq \left(1 - \frac{\kappa_{k-1/2}}{2} - \frac{\kappa_{k+1/2}}{2} - \frac{\lambda}{2} \beta_{k-1/2}^+ + \frac{\lambda}{2} \beta_{k+1/2}^-\right) \max_k \left|\Phi_k^\circ\right| + \left(\frac{\kappa_{k-1/2}}{2} + \frac{\lambda}{2} \beta_{k-1/2}^-\right) \max_k \left|\Phi_{k-1}\right| + \left(\frac{\kappa_{k-1/2}}{2} - \frac{\lambda}{2} \beta_{k+1/2}^+\right) \max_k \left|\Phi_{k+1}\right|,
\]

which is a convex combination, and hence we have

\[
|\Phi^*_k| \leq |\Phi^*_k|^\infty \leq |\Phi^*_k|^\infty.
\]

We next want to show that the scheme satisfies the TVD property. It suffices to show that

\[
TV[\Phi^*_k] \leq TV[\Phi^*_k].
\]

Using (3.14), we can write an expression for \(\Phi_k^* - \Phi_{k-1}^*\) as

\[
\Phi_k^* - \Phi_{k-1}^* = (1 - \kappa_{k-1/2}) (\Phi_k^\circ - \Phi_{k-1}^\circ) + \frac{1}{2} \kappa_{k+1/2} (\Phi_{k+1}^n - \Phi_k^\circ) + \frac{1}{2} \kappa_{k-3/2} (\Phi_{k-1}^n - \Phi_{k-2}^n) - \frac{\lambda}{2} \beta_{k+1/2}^- (\Phi_{k+1}^n - \Phi_k^\circ) - \frac{\lambda}{2} \beta_{k-1/2}^+ (\Phi_{k-1}^n - \Phi_{k-2}^n) - \frac{\lambda}{2} \beta_{k+1/2}^- (\Phi_{k+1}^n - \Phi_k^\circ) - \frac{\lambda}{2} \beta_{k-1/2}^+ (\Phi_{k-1}^n - \Phi_{k-2}^n).
\]
Rearranging,
\[
\Phi_k^* - \Phi_{k-1} = \left(1 - \kappa_{k-1/2} - \frac{\lambda_2}{2} \beta_{k-1/2} + \frac{\lambda_2}{2} \beta_{k-1/2}^{-} \right) (\Phi_k^n - \Phi_{k-1}^n)
+ \left(\frac{\kappa_{k+1/2}}{2} - \frac{\lambda_2}{2} \beta_{k+1/2}^{-} \right) (\Phi_k^n - \Phi_k^n) + \left(\frac{\kappa_{k+3/2}}{2} + \frac{\lambda_2}{2} \beta_{k+3/2}^{+} \right) (\Phi_{k-1}^n - \Phi_{k-2}^n).
\]

When (3.7) or (3.8) holds, all the coefficients in the expression for \(\Phi_k^* - \Phi_{k-1}^*\) are nonnegative. We can now take the absolute value and sum over \(k \in \mathbb{Z}\) to obtain
\[
\sum_k |\Phi_k^* - \Phi_{k-1}^*| \leq \sum_k \left(1 - \kappa_{k-1/2} - \frac{\lambda_2}{2} \beta_{k-1/2}^{-} + \frac{\lambda_2}{2} \beta_{k-1/2}^{+} \right) |\Phi_k^n - \Phi_{k-1}^n|
+ \sum_k \left(\kappa_{k+1/2} - \frac{\lambda_2}{2} \beta_{k+1/2}^{+} \right) |\Phi_k^n - \Phi_{k-1}^n|.
\]

Hence
\[
TV[\Phi^*] \leq \sum_k |\Phi_k^n - \Phi_{k-1}^n| = TV[\Phi^n].
\]

The proof is thus complete. \(\Box\)

4. Extension to multiple dimensions. Consider the system of conservation laws
\[
(4.1) \quad u_t + \nabla_x \cdot f(u) = 0, \quad (x, t) \in \mathbb{R}^d \times (0, T),
\]
where \(u = (u_1, \ldots, u_m)^T\). For simplicity, we take uniform distributed grids at \(x_a\) with multi-index \(\alpha = (\alpha_1, \ldots, \alpha_d)\). Let \(I_\alpha\) be a rectangle with vertices at \(\{x_{\alpha+\beta}, \ |\beta| = 1\}\), labeled as \(x_{\alpha \pm 1}\), the number of which amounts to \(2^d\). We take the average of the AE equation,
\[
u_t + \nabla_x \cdot f(v) = \frac{1}{\epsilon} (v - u),
\]
over \(I_\alpha\) to obtain
\[
\frac{d}{dt} \bar{u}_\alpha + \frac{1}{\epsilon} \bar{u}_\alpha = \frac{1}{\epsilon} L_\alpha[v](t),
\]
where \(\bar{u}_\alpha = \frac{1}{|I_\alpha|} \int_{I_\alpha} u(x, t) \, dx\) and
\[
L_\alpha[v] = \frac{1}{|I_\alpha|} \int_{I_\alpha} v \, dx - \frac{\epsilon}{|I_\alpha|} \int_{\partial I_\alpha} f(v) \cdot dS.
\]

Here, \(|I_\alpha|\) denotes the volume of \(I_\alpha\), and \(\partial I_\alpha\) indicates the boundary of \(I_\alpha\). Let \(\Phi \sim \bar{u}_\alpha\) denote the numerical solution, and reconstruct nonoscillatory polynomials \(p[\Phi]\) over \(I_\alpha\) from available averages \(\Phi_\alpha\); then we obtain a semidiscrete scheme
\[
(4.2) \quad \frac{d}{dt} \Phi_\alpha + \frac{1}{\epsilon} \Phi_\alpha = \frac{1}{\epsilon} L_\alpha[\Phi],
\]
where \(p[\Phi]\)'s constructed over \(I_{\alpha \pm 1}\) are to be used to evaluate
\[
(4.3) \quad L_\alpha[\Phi] = \sum_{|\alpha|} \frac{1}{|I_\alpha|} \int_{I_\alpha \cap I_{\alpha \pm 1}} p(\Phi)[x] \, dx - \sum_{|\alpha|} \frac{1}{|I_\alpha|} \int_{\partial I_\alpha \cap I_{\alpha \pm 1}} f(p[\Phi])(x) \cdot dS.
\]
Finally, to obtain the same order accuracy in time, the semidiscrete (4.2)–(4.3) is to be complemented with an ODE solver with matching accuracy for time discretization. The evolution parameter is chosen such that

\[
\epsilon \sum_{j=1}^{d} \max_{\lambda_j} \frac{|f_j'(\lambda)|}{\Delta x_j} \leq Q, \quad \Delta t < \epsilon,
\]

where \(Q \leq 1\) depends on the order of the scheme; see (4.8) or (4.16). For the system case, \(f'_j\) needs to be replaced by the dominant eigenvalues over a Riemann curve.

For the two-dimensional setting we present the corresponding global AE scheme for

\[
\phi_t + f(\phi)_x + g(\phi)_y = 0.
\]

### 4.1. First order scheme.

If the reconstructed polynomial is piecewise constant, then we obtain the first order scheme as

\[
\Phi_{k,l}^{n+1} = (1 - \kappa) \Phi_{k,l}^n + \kappa L_{k,l}[\Phi^n],
\]

where \(L_{k,l}[\Phi^n]\) can be evaluated as

\[
L_{k,l}[\Phi^n] = \frac{1}{4} \left( \Phi_{k-1,l-1}^n + \Phi_{k+1,l-1}^n + \Phi_{k+1,l+1}^n + \Phi_{k-1,l+1}^n \right) - \frac{\epsilon}{4\Delta x} \left( f(\Phi_{k+1,l-1}^n) - f(\Phi_{k-1,l-1}^n) + f(\Phi_{k+1,l+1}^n) - f(\Phi_{k-1,l+1}^n) \right) - \frac{\epsilon}{4\Delta y} \left( g(\Phi_{k-1,l+1}^n) - g(\Phi_{k-1,l-1}^n) + g(\Phi_{k+1,l+1}^n) - g(\Phi_{k+1,l-1}^n) \right),
\]

with \(\kappa = \frac{\Delta t}{\epsilon}\). The parameter \(\epsilon\) is chosen so that

\[
\epsilon \left( \max_{\lambda_j} \frac{|f'|}{\Delta x} + \max_{\lambda_j} \frac{|g'|}{\Delta y} \right) \leq 1 \quad \text{and} \quad \Delta t < \epsilon,
\]

which ensures the scalar maximum principle

\[
|\Phi^{n+1}|_{\infty} \leq |\Phi^n|_{\infty}, \quad n \in \mathbb{N}.
\]

Again, for the system case, both \(|f'|\) and \(|g'|\) in the stability requirement (4.8) need to be replaced by dominant eigenvalues of the corresponding Jacobian matrices.

### 4.2. Second order scheme.

The second order scheme requires linear polynomial reconstruction for its formulation, which on the rectangle \(I_{kl} = [x_k - \Delta x, x_k + \Delta x] \times [y_k - \Delta y, y_k + \Delta y]\) has the form

\[
p_{k,l}[\Phi^n](x,y) = \sum_{k,l} \left( \Phi_{k,l}^n + s_{k,l}'(x-x_k) + s_{k,l}'(y-y_l) \right) \chi_{k,l}(x,y),
\]

where \(\chi_{k,l}\) is the characteristic function which takes value one on the rectangle \(I_{k,l}\), and \(s'\) and \(s'\) are the numerical derivatives corresponding to \(\Phi_x\) and \(\Phi_y\). Now, using the midpoint quadrature rule in evaluation of (4.3), we obtain the second order AE scheme

\[
\frac{d}{dt} \Phi_{k,l} = L_{k,l}[\Phi]
\]
with
\[
L_{k,l}[\Phi] = \frac{1}{4} (\Phi_{k-1,l-1} + \Phi_{k+1,l-1} + \Phi_{k+1,l+1} + \Phi_{k-1,l+1}) \\
+ \frac{\Delta x}{8} (s_{k-1,l-1} + s_{k+1,l-1} - s_{k+1,l+1} + s_{k-1,l+1}) \\
+ \frac{\Delta y}{8} (s_{k-1,l-1} + s_{k+1,l-1} - s_{k+1,l+1} - s_{k-1,l+1}) \\
- \frac{\epsilon}{4\Delta x} \left[ f \left( \Phi_{k+1,l-1} + \frac{\Delta y}{2} s_{k+1,l-1} \right) - f \left( \Phi_{k-1,l-1} + \frac{\Delta y}{2} s_{k-1,l-1} \right) \right] \\
+ f \left( \Phi_{k+1,l+1} - \frac{\Delta y}{2} s_{k+1,l+1} \right) - f \left( \Phi_{k-1,l+1} - \frac{\Delta y}{2} s_{k-1,l+1} \right) \right] \\
- \frac{\epsilon}{4\Delta y} \left[ g \left( \Phi_{k-1,l-1} + \frac{\Delta x}{2} s_{k-1,l-1} \right) - g \left( \Phi_{k-1,l+1} + \frac{\Delta x}{2} s_{k-1,l+1} \right) \right] \\
+ g \left( \Phi_{k+1,l+1} - \frac{\Delta x}{2} s_{k+1,l+1} \right) - g \left( \Phi_{k+1,l-1} - \frac{\Delta x}{2} s_{k+1,l-1} \right) \right].
\]

The nonoscillatory property requires that we choose \(s_{k,l}'\) and \(s_{k,l}\) with certain limiters as in the one-dimensional polynomial reconstruction. In our proofs we use the basic minmod limiter defined as below:
\begin{align}
\text{minmod} & \quad \bigg\{ \frac{\Phi^n_{k+2,l} - \Phi^n_{k,l}}{2\Delta x} , \frac{\Phi^n_{k+1,l} - \Phi^n_{k-1,l}}{2\Delta x} \bigg\}, \\
\text{minmod} & \quad \bigg\{ \frac{\Phi^n_{k,l+2} - \Phi^n_{k,l}}{2\Delta y} , \frac{\Phi^n_{k+1,l} - \Phi^n_{k-1,l}}{2\Delta y} \bigg\},
\end{align}

with
\[
\text{minmod} \{ a_1, a_2, \ldots \} = \begin{cases} 
\min_i \{a_i\} & \text{if } a_i > 0 \text{ for all } i, \\
\max_i \{a_i\} & \text{if } a_i < 0 \text{ for all } i, \\
0 & \text{otherwise}.
\end{cases}
\]

When the second order Runge–Kutta discretization in time is used, the scheme becomes
\begin{align}
\Phi_{k,l}^* = (1 - \kappa)\Phi_{k,l}^n + \kappa L_{k,l}[\Phi^n], \\
\Phi_{k,l}^{n+1} = \frac{1}{2}\Phi_{k,l}^n + \frac{1}{2}(1 - \kappa)\Phi_{k,l}^* + \frac{\kappa}{2} L_{k,l}[\Phi^*].
\end{align}

Indeed a careful verification shows that such a choice again yields the scalar maximum principle (4.9), provided
\[
\epsilon \left( \frac{\max |f'|}{\Delta x} + \frac{\max |g'|}{\Delta y} \right) \leq \frac{1}{4} \quad \text{and} \quad \Delta t < \epsilon.
\]

5. Numerical tests.

5.1. Scalar conservation laws. In this section we use some model problems to numerically test the first, second, and third order global and local AE schemes. If \(\phi\) is the exact solution and \(\Phi\) is the computed solution, then the numerical errors are calculated as
\[
L^1 \text{ error } = \sum_k |\phi_k - \Phi_k| \Delta x, \quad L^\infty \text{ error } = \max_k |\phi_k - \Phi_k|.
\]

We call \(Q\) in (2.19) and (2.20) the CFL number in our numerical tests.
Table 1

The $L^1$ and $L^\infty$ errors for Burgers’ equation (Example 5.1) using $N$ equally spaced cells for different first order numerical schemes at $T = \frac{0.1}{\pi}$, when the solution is continuous.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>Error ratio</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
<th>Error ratio</th>
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<td>6.149400E-02</td>
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<td>1.632366E-03</td>
<td>0.204639</td>
</tr>
<tr>
<td>80</td>
<td>LxF</td>
<td>9.240998E-03</td>
<td>1.02</td>
<td>7.976805E-03</td>
<td>1.01</td>
<td>7.976805E-03</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>5.630459E-03</td>
<td>1.01</td>
<td>4.981333E-03</td>
<td>1.00</td>
<td>4.981333E-03</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.834574E-03</td>
<td>1.02</td>
<td>1.632366E-03</td>
<td>0.99</td>
<td>1.632366E-03</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>4.630626E-03</td>
<td>1.01</td>
<td>3.998649E-03</td>
<td>1.01</td>
<td>3.998649E-03</td>
<td>1.01</td>
</tr>
<tr>
<td>160</td>
<td>LxF</td>
<td>2.317531E-03</td>
<td>1.00</td>
<td>2.001063E-03</td>
<td>1.00</td>
<td>2.001063E-03</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>2.317531E-03</td>
<td>1.00</td>
<td>1.250070E-03</td>
<td>1.00</td>
<td>1.250070E-03</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>4.600955E-04</td>
<td>1.01</td>
<td>1.250070E-03</td>
<td>1.00</td>
<td>1.250070E-03</td>
<td>1.00</td>
</tr>
<tr>
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<td>AE2</td>
<td>4.600955E-04</td>
<td>1.01</td>
<td>1.250070E-03</td>
<td>1.00</td>
<td>1.250070E-03</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Example 5.1. The one-dimensional Burgers equation has the form

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad x \in [0, 2],$$

$$u(x, 0) = 1 + \sin \pi x,$$

$$u(0, t) = u(2, t).$$

We use the Burgers equation with convex flux to make a comparison of the global and local AE schemes with some standard central schemes such as the first order Lax–Friedrich (LxF), second order KT (Kurganov–Tadmor [22]) scheme, which improves upon the original NT (Nessyahu–Tadmor [33]) scheme. In order to check the numerical accuracy, we compute the solution when it is continuous. We take the final time $T = \frac{0.1}{\pi}$ and use CFL number $0.8$ and $\Delta t = 0.8\epsilon$ for the first order scheme, and CFL number $0.6$ and $\Delta t = 0.6\epsilon$ for the second order scheme. We can see that both the global and local schemes give the desired order of accuracy from Tables 1 and 2. Note that we do not use nonlinear limiters while checking the numerical accuracy of the second order schemes when the solution is smooth.

For first order schemes, the numerical errors, the orders of accuracy for the numerical solution, and ratios of the numerical errors of the AE and AE1 schemes in comparison with the LxF scheme are shown in Table 1. All the schemes give the required numerical order of accuracy. We can see that the local AE1 scheme gives the smallest numerical error. From Table 2, we can see that for second order schemes, the local AE1 scheme gives the smallest numerical error compared to the AE and KT schemes. Our global AE scheme is comparable to the KT scheme: when $n = 20$ and


Table 2
The $L^1$ and $L^\infty$ errors for Burgers’ equation (Example 5.1) using $N$ equally spaced cells for different second order numerical schemes at $T = \frac{\pi}{4}$, when the solution is continuous.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>Error ratio</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
<th>Error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
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<td>8.066034E-03</td>
<td>1.000000</td>
<td>8.127727E-03</td>
<td>1.000000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>5.198982E-03</td>
<td>0.645032</td>
<td>6.680986E-03</td>
<td>0.828903</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>4.520158E-03</td>
<td>0.560811</td>
<td>5.540215E-03</td>
<td>0.687369</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>KT</td>
<td>1.267812E-03</td>
<td>2.86</td>
<td>1.000000</td>
<td>1.429797E-03</td>
<td>2.69</td>
<td>1.000000</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>1.167134E-03</td>
<td>2.23</td>
<td>0.920589</td>
<td>1.478923E-03</td>
<td>2.25</td>
<td>1.034539</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.083947E-03</td>
<td>2.13</td>
<td>0.854975</td>
<td>1.305326E-03</td>
<td>2.16</td>
<td>0.912945</td>
</tr>
<tr>
<td>40</td>
<td>KT</td>
<td>2.732028E-04</td>
<td>2.29</td>
<td>1.000000</td>
<td>3.289424E-04</td>
<td>2.20</td>
<td>1.000000</td>
</tr>
<tr>
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<td>AE</td>
<td>2.793864E-04</td>
<td>2.13</td>
<td>1.002693</td>
<td>3.350735E-04</td>
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</tr>
<tr>
<td></td>
<td>AE1</td>
<td>2.641463E-04</td>
<td>2.07</td>
<td>0.966851</td>
<td>3.124174E-04</td>
<td>2.10</td>
<td>0.949763</td>
</tr>
<tr>
<td>80</td>
<td>KT</td>
<td>6.602794E-05</td>
<td>2.09</td>
<td>1.000000</td>
<td>8.180660E-05</td>
<td>2.04</td>
<td>1.000000</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>6.663412E-05</td>
<td>2.06</td>
<td>1.009181</td>
<td>7.969743E-05</td>
<td>2.09</td>
<td>0.977518</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>6.557099E-05</td>
<td>2.03</td>
<td>0.993079</td>
<td>7.703799E-05</td>
<td>2.04</td>
<td>0.941709</td>
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<tr>
<td>160</td>
<td>KT</td>
<td>1.645922E-05</td>
<td>2.02</td>
<td>1.000000</td>
<td>2.042386E-05</td>
<td>2.02</td>
<td>1.000000</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>1.646475E-05</td>
<td>2.03</td>
<td>1.000336</td>
<td>1.949339E-05</td>
<td>2.05</td>
<td>0.954442</td>
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<tr>
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<td>0.992104</td>
<td>1.912202E-05</td>
<td>2.02</td>
<td>0.936259</td>
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<tr>
<td>320</td>
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<td>4.139366E-06</td>
<td>2.00</td>
<td>1.000000</td>
<td>5.107226E-06</td>
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</tr>
<tr>
<td></td>
<td>AE</td>
<td>4.093983E-06</td>
<td>2.01</td>
<td>0.989115</td>
<td>4.810993E-06</td>
<td>2.02</td>
<td>0.941821</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>4.077607E-06</td>
<td>2.01</td>
<td>0.985159</td>
<td>4.763512E-06</td>
<td>2.01</td>
<td>0.932700</td>
</tr>
</tbody>
</table>

$n = 40$, the $L^\infty$ error of AE is higher than that of KT, and for $n = 40, 80, 160$, the $L^1$ error of AE is higher than that of KT. In contrast, as the number of cells is increased, the AE, AE1, and KT schemes give nearly the same errors.

We also do some CPU run-time comparisons in Table 3. The comparisons are made on an HP desktop with Intel Core Duo CPU E8400 @ 3.00GHz, 2.99GHz with 1.95GB RAM. As expected, the run-times for the local AE schemes are larger than that for the global AE scheme. Also note that the run-time for the second order global AE scheme is slightly smaller than the run-time for the second order KT scheme.

We also plot the solution to the Burgers equation when it is discontinuous, which is at time $T = 0.7$. We use CFL number $0.2$ and $\Delta t = 0.8\epsilon$ for both the second and third order schemes. From Figures 3 and 4, we see that there is a significant increase in the resolution of discontinuities from second to third order schemes. We can also see that the local AE schemes perform much better than the global AE scheme.

Example 5.2. The one-dimensional nonlinear Buckley–Leverett problem has the form

$$u_t + \left(\frac{4u^2}{4u^2 + (1 - u)^2}\right)_x = 0, \quad x \in [-1, 1].$$

The initial condition is given by

$$u(x, 0) = \begin{cases} 1, & x \in [-\frac{1}{2}, 0], \\ 0, & \text{otherwise.} \end{cases}$$

The flux is nonlinear, and we use this example to test the scheme when the initial data is discontinuous. In order to check the numerical accuracy, we take the final
Table 3

Run-time comparisons for Burgers’ equation (Example 5.1) using $N$ equally spaced cells for different order numerical schemes at $T = 0.1$, when the solution is continuous.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Scheme</th>
<th>First order CPU time</th>
<th>Second order CPU time</th>
<th>Third order CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>Lax</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>KT</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>- 0.03</td>
<td>- 0.03</td>
<td>- 0.03</td>
</tr>
<tr>
<td>40</td>
<td>Lax</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>KT</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>- 0.03</td>
<td>- 0.03</td>
<td>- 0.03</td>
</tr>
<tr>
<td></td>
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<td>0.03</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>80</td>
<td>Lax</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>NT</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>KT</td>
<td>- 0.03</td>
<td>-</td>
<td>- 0.05</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>- 0.06</td>
<td>- 0.06</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>0.06</td>
<td>0.09</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>2.50</td>
<td>7.67</td>
<td>7.94</td>
</tr>
<tr>
<td>160</td>
<td>Lax</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>KT</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>- 0.06</td>
<td>-</td>
<td>- 0.13</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>0.11</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>320</td>
<td>Lax</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>KT</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>0.05</td>
<td>0.20</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>0.36</td>
<td>0.86</td>
<td>0.98</td>
</tr>
<tr>
<td>640</td>
<td>Lax</td>
<td>0.02</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>KT</td>
<td>- 0.75</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>AE</td>
<td>0.13</td>
<td>0.63</td>
<td>1.41</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.22</td>
<td>3.47</td>
<td>3.98</td>
</tr>
</tbody>
</table>

time $T = 0.4$ and $N = 80$. We choose CFL number 0.2 and $\Delta t = 0.8\epsilon$ for both the second and third order schemes. The numerical results for second and third order schemes are given in Figure 5. As in the case of Burgers’ equation, we can see that local AE schemes perform better than the global AE scheme. Also, the resolution of discontinuities is, as expected, higher for third order schemes than for second order schemes.

We now test the numerical accuracy using a two-dimensional scalar equation with continuous initial data.

Example 5.3. Linear two-dimensional advection:

$$\phi_t + \phi_x + \phi_y = 0, \quad -1 \leq x, y \leq 1,$$

with initial data,

$$\phi(x, y, 0) = \sin \pi (x + y).$$

This equation causes the transport of the initial data. Periodic boundary conditions are used and the errors are calculated at final time $T = 1$ using exact solution $\phi = \sin(\pi (x+y-2))$. We do not use any limiters while checking the accuracy of the second order scheme since the solutions are smooth for all times and the numerical derivative
for the linear reconstruction is taken to be the central difference $\frac{\Phi_{i+1,j} - \Phi_{i-1,j}}{2\Delta x}$ and $\frac{\Phi_{i,j+1} - \Phi_{i,j-1}}{2\Delta y}$ in the $x$- and $y$-directions, respectively. The results, given in Tables 4 and 5, indicate that the scheme gives the desired accuracy both in the $L^1$ and $L^\infty$ norms.

Next we test the global and local AE schemes for the Euler system of equations.

**5.2. Euler equations of gas dynamics.** In this section we apply our second and third order global AE and local AE1 schemes to the Euler equations of polytropic gas.
Example 5.4. The Euler system of equations has the form

\[
\begin{pmatrix}
\rho \\
\rho v \\
E
\end{pmatrix}_t + \begin{pmatrix}
\frac{\rho v}{E} \\
\frac{\rho v^2 + p}{E + p}
\end{pmatrix}_x = 0,
\]

where \( p = (\gamma - 1)(E - \frac{1}{2} \rho v^2); \gamma = 1.4 \). We test the Euler equation with different initial data that are commonly used [25, 39].

1. Lax initial data: Here, \( x \in [0, 1] \) and

\[
(\rho, \rho v, E)(0) = \begin{cases}
(0.445, 0.311, 8.928), & x \in [0, 0.5], \\
(0.5, 0, 1.4275), & x \in [0.5, 1].
\end{cases}
\]

The final time \( T = 0.16 \).

2. Sod initial data: Here, \( x \in [0, 1] \) and

\[
(\rho, \rho v, E)(0) = \begin{cases}
(1, 0, 2.5), & x \in [0, 0.5], \\
(0.125, 0, 0.25), & x \in [0.5, 1].
\end{cases}
\]

The final time \( T = 0.1644 \).

3. Osher–Shu problem: Here, \( x \in [-5, 5] \) and

\[
(\rho, \rho v, E)(0) = \begin{cases}
(3.857143, 2.629369, 10.333333), & x \in [-5, -4], \\
(1 + 2 \sin(5x), 0, 1), & x \in [-4, 5].
\end{cases}
\]
The final time $T = 1.8$.

4. Woodward–Colella problem: Here, $x \in [0, 1]$ and

$$
(\rho, \rho v, E) (0) = \begin{cases} 
(1, 0, 2500), & x \in [0, 0.1], \\
(1, 0, 0.025), & x \in [0.1, 0.9], \\
(1, 250), & x \in [0.9, 1].
\end{cases}
$$

The final time $T = 0.01, 0.03, 0.038$. 

Fig. 5. Comparison of plots for the Buckley–Leverett problem, Example 5.2, on $[-1, 1]$, $T = 0.4$, $\Delta x = \frac{1}{40}$, $\Delta t = 0.8\epsilon$, second (left) and third order (right) schemes.
Table 4
The $L^1$ and $L^\infty$ errors for linear advection, Example 5.3, using $N$ equally spaced cells for global first order AE scheme at $T = 1$, when the solution is continuous.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.32E+00</td>
<td>-</td>
<td>4.67E-01</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>6.84E-01</td>
<td>0.9824</td>
<td>2.55E-01</td>
<td>0.9039</td>
</tr>
<tr>
<td>80</td>
<td>3.54E-01</td>
<td>0.9679</td>
<td>1.35E-01</td>
<td>0.9288</td>
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<tr>
<td>160</td>
<td>1.80E-01</td>
<td>0.9798</td>
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<td>320</td>
<td>9.12E-02</td>
<td>0.989</td>
<td>3.56E-02</td>
<td>0.9798</td>
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<tr>
<td>640</td>
<td>4.58E-02</td>
<td>0.9948</td>
<td>1.79E-02</td>
<td>0.9902</td>
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</tbody>
</table>

Table 5
The $L^1$ and $L^\infty$ errors for linear advection, Example 5.3, using $N$ equally spaced cells for global second order AE scheme at $T = 1$, when the solution is continuous.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.47E-01</td>
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<td>5.26E-02</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>1.91E-02</td>
<td>3.0444</td>
<td>7.16E-03</td>
<td>2.9824</td>
</tr>
<tr>
<td>80</td>
<td>3.33E-03</td>
<td>2.5656</td>
<td>1.28E-03</td>
<td>2.5333</td>
</tr>
<tr>
<td>160</td>
<td>7.24E-04</td>
<td>2.223</td>
<td>2.81E-04</td>
<td>2.2036</td>
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<tr>
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<tr>
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<td>4.28E-05</td>
<td>2.0204</td>
<td>1.68E-05</td>
<td>2.0159</td>
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</tbody>
</table>

We plot the density, velocity, and pressure profiles for all four types of initial data in the rest of the paper.

For Lax initial data, we choose CFL number 0.5 and $\Delta t = 0.8\epsilon$. The reference solution is calculated using the ENO scheme with local LF numerical flux (see [35, 36]), with 5120 grid points with $\epsilon = 0.1\Delta x$ and $\Delta t = 0.95\epsilon$. Figure 6 makes a comparison between second order (density) and third order global AE and local AE1 schemes. We can see that, in general, AE1 captures the corners much better than the AE scheme, and the resolution of discontinuities is much sharper in third order schemes as compared to second order.

For the Sod, Osher–Shu, and Woodward–Colella initial data, the reference solution is calculated using the third order ENO scheme with local LF numerical flux and with 2560 grid points.

For Sod initial data, we choose CFL number 0.5 and $\Delta t = 0.8\epsilon$. Figure 7 shows the density, velocity, and pressure profiles at the final time for both second and third order AE1 schemes. As in the case of Lax initial data, the resolution of the third order scheme is better than that of the second order scheme.

In the Osher–Shu problem, there are several extrema in the smooth regions which present a good case for examination of the accuracy of shock-capturing. In this case, we choose CFL number 0.6 and $\Delta t = 0.8\epsilon$. The density, velocity, and pressure profiles for third order AE, AE1 with 400 grid points are given in Figure 8. The final time $T = 1.8$. We note that with 400 grid points, the difference between the local and global schemes in third order is not significant. Figure 9 gives plots for second and third order AE1 schemes with 800 grid points. We can see that a lot of peaks are captured in the third order but not in the second order scheme.
Fig. 6. Plots for the Euler equation with Lax initial data on $[0, 1]$, $T = 0.16$, $\Delta x = \frac{1}{200}$, $\Delta t = 0.8\epsilon$, second and third order schemes.
The Woodward–Colella problem involves the interaction of two blast waves. The initial data given presents one shock at $x = 0.1$ and the other at $x = 0.9$. The boundaries at $x = 0$ and $x = 1$ are solid walls with a reflective boundary condition. After a certain time, these two shocks collide with each other. At the final time step of $t = 0.038$, the flow field involves two shocks and three contact discontinuities. For numerical computations with Woodward–Colella initial data, we choose $CFL = 0.3$ and $\Delta t = 0.8\epsilon$. Figure 10 show the density, velocity, and pressure profiles for both second and third order local AE1 schemes for time $T = 0.01$, and Figure 11 shows the third order AE1 scheme for time $T = 0.03, 0.038$. 

Fig. 7. Plots for the Euler equation with Sod initial data on $[0,1]$, $T = 0.1644$, $\Delta t = 0.8\epsilon$, $\Delta x = \frac{1}{200}$, AE1 scheme.
Example 5.5. Explosion problem for the Euler equation. The two-dimensional Euler equation can be written as

$$
\phi_t + f(\phi)_x + g(\phi)_y = 0, \quad \phi = (\rho, \rho u, \rho v, E)^T, \quad p = (\gamma - 1) \left[ E - \frac{\rho}{2}(u^2 + v^2) \right], \\
f(\phi) = (\rho u, \rho u^2 + p, \rho uv, u(E + p))^T, \quad g(\phi) = (\rho v, \rho uv, \rho v^2 + p, v(E + p))^T,
$$

where $\gamma = 1.4$. This example, chosen from [39], consists of a high density and high pressure region inside a bubble of radius 0.4 centered at the origin and a low density

Fig. 8. Plots for the Euler equation with Osher–Shu initial data on $[-5, 5], T = 1.8, \Delta t = 0.8\epsilon, \Delta x = \frac{1}{40}$, third order scheme.
AE SCHEMES FOR HYPERBOLIC CONSERVATION LAWS

Fig. 9. Plots for the Euler equation with Osher–Shu initial data on $[-5, 5]$, $T = 1.8$, $\Delta t = 0.8\epsilon$, $\Delta x = \frac{1}{180}$, AE1 scheme.

and pressure region outside the bubble which causes the explosion. The initial data is

$$(p, \rho, u, v) (0) = \begin{cases} 
(1, 1, 0, 0), & x^2 + y^2 < 0.16, \\
(0.1, 0.125, 0, 0), & x^2 + y^2 \geq 0.16.
\end{cases}$$

The solution is computed at time $T = 0.25$. In our numerical experiments, we use constant extension boundary conditions on all four walls. The CFL number used is 0.4 and time step is taken as $\Delta t = 0.95\epsilon$. 
We use the general minmod limiter [33] with \( \theta = 1.5 \),

\[
\begin{align*}
\hat{s}_{k,l} &= \minmod \left\{ \theta \frac{\Phi_{k+1,l} - \Phi_{k,l}}{\Delta x}, \Phi_{k+1,l} - \Phi_{k-1,l}, \theta \frac{\Phi_{k,l} - \Phi_{k-1,l}}{\Delta x} \right\}, \\
\hat{s}^\prime_{k,l} &= \minmod \left\{ \theta \frac{\Phi_{k,l+1} - \Phi_{k,l}}{\Delta x}, \Phi_{k,l+1} - \Phi_{k-1,l+1}, \theta \frac{\Phi_{k,l} - \Phi_{k,l-1}}{\Delta x} \right\},
\end{align*}
\]

for the numerical derivative calculations. This \( \theta \)-dependent family of limiters with \( \theta \in (1, 2) \) is known to be less dissipative than the case \( \theta = 1 \). In this test, the solution exhibits a circular shock and circular contact discontinuity moving away from the...
center of the circle and circular rarefaction wave moving in the opposite direction, a complex wave pattern emerging as time evolves. We compute this two-dimensional bubble explosion solution by second order AE scheme with mesh $\Delta x = \Delta y = \frac{1}{800}$, $CFL = 0.5$, and $\Delta t/\epsilon = 0.95$. The isolines of the density and pressure are given in Figure 12 (left).

A reference solution for this problem with cylindrical symmetry can be produced by the one-dimensional inhomogeneous system

$$U_t + F(U)_r = S(U),$$

Fig. 11. Plots for the Euler equation with Woodward-Colella initial data on $[0, 1]$, $\Delta x = \frac{1}{800}$, $\Delta t = 0.8\epsilon$, third order AE1 scheme.
where \( r > 0 \) denotes the radial variable,

\[
U = \begin{pmatrix} \rho \\ \rho \xi \\ \frac{\rho \xi^2 + p}{\xi(E + p)} \end{pmatrix}, \quad F = \begin{pmatrix} \rho \xi \\ \rho \xi^2 + p \\ \frac{\rho \xi^2}{\xi(E + p)} \end{pmatrix}, \quad S = -\frac{1}{r} \begin{pmatrix} \rho \xi \\ \rho \xi^2 \\ \frac{\rho \xi}{\xi(E + p)} \end{pmatrix},
\]

with \( p = 0.4(E - \frac{1}{2} \rho \xi^2) \). Here \( \xi \) is the velocity in radial direction. The initial data can be rewritten into the Sod-type initial data for \( r \in [0, 1] \),

\[
(p, \rho \xi, E) = \left\{ \begin{array}{ll}
(1, 0, 2.5), & r < 0.4, \\
(0.125, 0, 0.25), & 0.4 < r < 1.
\end{array} \right.
\]

We compute this reference solution by the second order global AE scheme with 2560 points, \( CFL = 0.4 \), and \( \Delta t/\epsilon = 0.9 \). Figure 12 (right) shows the comparison of density and pressure as functions of \( r \) at the output time \( T = 0.25 \). Performance of the AE scheme for this explosion problem is clearly demonstrated.

**6. Concluding remarks.** In this work we have developed several AE schemes based on the alternating evolution approximation to nonlinear hyperbolic conservation laws. The AE system captures the exact solution when initially both components are chosen as the given initial data for the conservation laws. Such a feature allows for a sampling of two components over alternating grids. High order accuracy is achieved
by a combination of high order nonoscillatory polynomial reconstruction from the obtained averages and a matching Runge–Kutta solver in time discretization. Local AE schemes are made possible by letting the scale parameter $\epsilon$ reflect the local distribution of nonlinear waves. The AE schemes have the advantage of easier formulation and implementation, as well as efficient computation of the solution. For the first and second order local AE schemes applied to scalar conservation laws, we have proved both the maximum principle and total variation diminishing property. Numerical tests for both scalar conservation laws and compressible Euler equations presented in this work have demonstrated the high order accuracy and capacity of these AE schemes. Extension to multidimensional problems is straightforward; the derived first and second order AE schemes have been included and tested using both a linear transport problem and an explosion problem for the two-dimensional Euler equation.

We note that the development of higher order methods that could potentially yield better accuracy or reduced computational costs is a subject of ongoing research. One of the most promising approaches is the discontinuous Galerkin (DG) method, for which the theoretical basis has been provided in a series of papers by Cockburn and coworkers [4, 3, 2, 5]. Our AE schemes are simple to implement and have lower computational costs. In terms of accuracy, the advantage is still for the DG methods, and future work will be dedicated to making the AE scheme more accurate by exploiting compact high order approximations.

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