Contents

Relaxation Dynamics, Scaling Limits and Convergence of Relaxation Schemes

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Summary. Relaxation dynamics, scaling limits, and relaxation schemes are three main topics on hyperbolic relaxation problems that, remarkably, can be well understood with one model equation. The criterion that leads to desired results for the three problems is the so called "sub-characteristic condition". The criterion of this nature is also pivotal in the study of general hyperbolic relaxation problems.

In this article we review the recent research development in hyperbolic relaxation problems. The emphasis is on contributions associated with our own project within ANumE priority research program. We will first review some basic properties and notions for hyperbolic relaxation problems, and then focus our investigation on three main topics associated with the underlying relaxation model: relaxation dynamics, scaling limits as well as convergence theory of relaxation schemes.

1 Introduction

This article is written on the occasion that a special book on the ANumE program (a priority research program on Analysis and Numerics for Conservation Laws) is being published, and serves as a review article on a few selected topics on hyperbolic relaxation problems. The emphasis is mainly on our research contributions in these topics associated with the ANumE research project, entitled "Stability in Hyperbolic Systems with Relaxation", DFG grant Wa 633/11-1. Following the main theme of this article, we restrict the discussion to our results on basic models which contribute to an understanding of relaxation dynamics, scaling limits as well as the relaxation schemes. We also include some earlier complementary results of the author in hopes to provide the reader with a more complete picture. However we make no attempt to summarize the extensive literature to the vast amount of special and intrinsic analysis for various hyperbolic relaxation problems in recent years.

Historically the program on hyperbolic relaxation problems was pioneered by Whitham from examining interactions of hyperbolic waves of different orders in one single equation

$$u_t + cu_x + \epsilon(u_{tt} - a^2 u_{xx}) = 0, \tag{1}$$

in which first and second order waves are present simultaneously, see [85, Chapter 10]. Indeed if the lower order terms were absent ($\epsilon = \infty$), the general solution would be

$$u = u_1(x - at) + u_2(x + at).$$

Conversely, if the high order waves were absent ($\epsilon = 0$), the solution would be

$$u = u_0(x - ct).$$

It turns out that both kinds of wave play important roles, and there are important interaction effects between the two. The higher order waves carry the "first signal" with speed a, but the "main disturbance" travels with the lower order waves at speed c.

One important observation by Whitham is the following stability criterion

$$-a < c < a, \tag{2}$$

which ties in nicely with the ideas on propagation. The relevant ideas can be formally taken over to the nonlinear situation ($c \to f'(u)$), and the condition (2) becomes

$$-a < f'(u) < a. \tag{3}$$

The equation (1) with c replaced by $f'(u)$ gives

$$u_t + f(u)_x + \epsilon(u_{tt} - a^2 u_{xx}) = 0, \tag{4}$$

which can be compared to the widely accepted viscous approximation

$$u_t + f(u)_x = \epsilon u_{xx}.$$

Also the equation (4) when written into a system of first order PDEs leads to

$$\left.\begin{aligned} u_t + v_x &= 0, \\ v_t + a^2 u_x &= \tfrac{f(u)-v}{\epsilon}. \end{aligned}\right\} \tag{5}$$

Nonlinear analysis of hyperbolic relaxation problems began with Liu [38] for a more general 2×2 relaxation system (than (5)), in which the condition of the type (3) is identified as the "sub-characteristic condition". The system analyzed in [38] belongs to the following class

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \sum_{j=1}^{d} F_j(u, v)_{x_j} = \begin{pmatrix} 0 \\ q(u, v) \end{pmatrix} \tag{6}$$

with the unknown function $U = (u, v) \in R^m$, and the given flux functions F_i, $q(u, v)$ is a given vector valued smooth function. Systems of form (6) occur in a large number of applications involving various non-equilibrium processes.

They also occur as approximations to systems in various applications, see e.g. [6, 23, 24, 32, 69, 76].

As the simplest model in the class of (6), the system (5) has become quite popular after Jin and Xin proposed to use it as a numerical device to compute the entropy solution for hyperbolic conservation laws, see [24]. In particular this model possesses the key features of a more general hyperbolic relaxation system, thus serves as an ideal model problem to understand the more general ones.

The requirement (3) is shown to be essential for relaxation model (5) to enjoy the stability property. Actually Leveque and Wang [48] showed that if (3) is violated, the instability may, and actually does occur even for linear flux $f = cu$. For more general hyperbolic relaxation systems first successful attempts to identify necessary stability conditions started with Chen, Liu and Levermore in [12], and independently with Yong in [90]. A more recent account on basic structures of hyperbolic relaxation systems can be found in [91]. For better understanding of conditions of such nature more contributions would be certainly desirable. Among others along this line, let me mention Boillat & Ruggeri [9], Zeng [94], Bouchut [6], Yong [91] and Hanouzet and Natalini [19]. Also consult [15, 14] for stability conditions in some balance laws.

The study of problems in this area suggests the whole range of the problem from the microscopic kinetic equations, say Boltzmann equation, to the macroscopic systems, say Euler/Navier-Stokes equations. Therefore it is not surprising that, in recent years, hyperbolic relaxation problems have received a considerable attention, see e.g., [4, 12, 20, 24, 32, 45, 38, 70, 76].

Our research project on hyperbolic relaxation problems has been conducted in developing stability and convergence theory for a class of hyperbolic relaxation systems, see e.g. [34, 35, 36, 37, 42, 51, 52, 53, 60]. The main theme for underlying relaxation system is to examine the stability of solutions in terms of the delicate balance of the nonlinear convection and the relaxation forcing. Our main emphasis in this project is to understand the relaxation dynamics and see how this mechanism affects the stability of the hyperbolic systems. Thus, we seek both the justification of stable relaxation wave patterns and the convergence analysis of approximate solutions in light of the competition of relaxation and the nonlinear convection.

Relaxation dynamics, scaling limits, and relaxation schemes are three main topics for hyperbolic relaxation problems that, remarkably, can be well understood with the model equation (5). The "sub-characteristic condition" (3) plays a pivotal role in the study of above three topics. Therefore we shall review our research results mainly associated with the model (5), while knowing that some of these underlying issues for more general relaxation systems remain to be unveiled.

The rest of this article is organized as follows. In the second section, some fundamental notions and properties about the relaxation will be discussed. We will mostly focus on the model (5) though some of what we present can be easily carried over to some more general hyperbolic relaxation systems. Sect. 3

is devoted to the study of long time solution behaviors driven by the relaxation dynamics. Depending on the pattern of the initial data we will meet the large time diffusion waves, asymptotic towards refraction waves, nonlinear stability of relaxation shocks as well as the stability of relaxation boundary layers. Scaling limits will be addressed in Sect. 4, where results on zero relaxation limit and diffusive limit are reviewed, respectively. Finally convergence results of relaxation schemes will be described in Sect. 5.

2 An Overview of Relaxation Basics

The goal of this section is to remind the reader of some basic structure properties of hyperbolic system with relaxation, which will be needed in following sections.

We start with a hyperbolic balance law of the form

$$\left.\begin{array}{l} u_t + v_x = 0, \quad (x,t) \in \mathsf{IR} \times \mathsf{IR}^+ \\ v_t + a^2 u_x = f(u) - v, \end{array}\right\} \tag{7}$$

with $f = f(u)$ being a given smooth function. The variables u and v are the unknowns, $a > 0$ is a given constant.

Equilibrium Manifold

We are interested in the large time behavior of solutions developed by the relaxation dynamics starting with a certain class of initial data. Intuitively if some stable criteria are met the solution should be attracted to the equilibrium states or to wave patterns connecting equilibrium states. The set of the equilibrium states of a relaxation system for which the source term vanishes is named as the **equilibrium manifold**. For (7) such manifold can be described as

$$\Gamma(u) := \{(u,v); \quad v = f(u)\}. \tag{8}$$

A simple check shows that $\Gamma(u)$ is a stable manifold in the sense that for any constant initial data $(u,v) = (\alpha, \beta) \in \mathsf{IR}^2$, the corresponding solution

$$(u,v) = (\alpha, \quad \beta e^{-t} + f(\alpha)(1 - e^{-t})),$$

will approach $\Gamma(\alpha)$ exponentially as t goes to infinity, independent of the choice of β.

Scaling Laws

In the relaxation system the solution behavior clearly relies upon the competition of the convection and the driving force from the source term. Under different scalings such competition could drive the solution towards different states. Two typical scalings are: the hyperbolic scaling and the parabolic scaling.

(a) **Hyperbolic scaling**. Under the hyperbolic scaling

$$(t, x) \to \left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$$

the system (7) takes the form (5) in Sect. 1, i.e.,

$$\left.\begin{aligned} u_t + v_x &= 0, \\ v_t + a u_x &= \tfrac{1}{\epsilon}(f(u) - v). \end{aligned}\right\} \tag{9}$$

Here ϵ is a positive constant representing the rate of relaxation. Relaxation effect is known to provide a subtle dissipative mechanism for discontinuities against the destabilizing effect of nonlinear response (see, for example, Liu [38]). The relaxation model (9) can be loosely interpreted as a discrete velocity kinetic equation. The relaxation parameter, ϵ, plays the role of the mean free path and the limiting system (as $\epsilon \downarrow 0$) models the macroscopic conservation laws.

(b) **Parabolic scaling.** Under the parabolic scaling

$$(t, x) \to \left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}\right), \quad (u, v) \to (\epsilon u, \epsilon^2 v)$$

the system (7) takes the form

$$\left.\begin{aligned} u_t + v_x &= 0, \\ \epsilon^2 v_t + a^2 u_x &= f(u) - v, \end{aligned}\right\} \tag{10}$$

where we have formally replaced the scaled function $f(\epsilon u)$ by $\epsilon^2 f(u)$, see [21, 22, 42]. One of important tasks in the study of relaxation problems is to justify the scaling limits in various regimes, depending on the relative size of physical scales. The limiting process under the hyperbolic scaling is usually called **the zero relaxation limit**, which provides a passage in parallel to that from the Boltzmann equation to the Euler equation. In contrast, the limit under parabolic scaling is connected to **the diffusive kinetic limit** of the Boltzmann equation to the Navier-Stokes equation.

Further insight about this statement may be obtained via the **Chapman-Enskog expansion**, an expansion on the differential operator in terms of the small relaxation parameter. With this procedure one can also discover some necessary structure conditions for both scaling limits.

Sub-characteristic condition

The leading order of the relaxation system (9) gives

$$u_t + f(u)_x = 0, \tag{11}$$

which shows the role of (9) as a new way of regularizing the hyperbolic conservation law (11).

The Chapman-Enskog expansion on (9), up to $O(\epsilon)$ order, yields

$$u_t + f(u)_x = \epsilon[(a^2 - f'(u)^2)u_x]_x\,. \tag{12}$$

Clearly, in order to guarantee the dissipative nature of this convection-diffusion equation it is necessary to require $a^2 - f'(u)^2 > 0$, i.e., (3). Note that the sub-characteristic condition (3) plays a similar role to the CFL condition for numerical approximations. The condition of this nature requires that the limit of domain of dependence for the approximation system contains the domain of dependence for original system.

Indeed, under this condition, the rigorous passage from (9) to (11) has been justified, see e.g. [68].

Diffusive sub-characteristic condition.

Using the Chapman-Enskog expansion on the diffusive scaled model (10), one obtains

$$u_t + f(u)_x = [(a^2 - \epsilon^2 f'^2)u_x]_x + a^2\epsilon^2[f(u)_{xx} + f'(u)u_{xx}]_x - a^2\epsilon^2 u_{xxxx}, \tag{13}$$

and the formal limit system reads

$$u_t + f(u)_x = a^2 u_{xx}. \tag{14}$$

Obviously, to ensure the dissipative nature of (13), one just needs a weakened characteristic condition

$$-a < \epsilon f' < a. \tag{15}$$

In [22] this condition is called **the diffusive sub-characteristic condition**, under which we rigorously justified that (14) is the limit of (10) as $\epsilon \to 0$, see Theorem 4.3 to be presented in Sect. 4.2.

In the study of the solution behavior to hyperbolic relaxation systems there are two different kinds of asymptotic limits. The first one is the large time behavior of solutions if we look at the long–time effects of relaxation, called **the relaxation dynamics**. A different behavior is found when dealing with **scaling limits** in various regimes, which are essentially driven by equilibrium equations (11) or (14).

We summarize this discussion in stating that large time dynamics and the scaling limits are related issues where a key role is played by the sub-characteristic condition (3), under which we establish rigorous theorems for various settings. Moreover these basic properties for the relaxation system (7) are shared by most of the general relaxation systems even though structure conditions appear more complicated in general contexts, see e.g. [12].

This leads us first to the discussion on relaxation dynamics as described in the next section.

3 Relaxation Dynamics

This section is devoted to the time-asymptotic behavior of solutions of (7) subject to various class of initial (and boundary if applicable) data. According to Whitham's observation for linear problem (1), the striking feature of the relaxation dynamics would be the interaction of the waves of the equilibrium equation and the waves of the full relaxation system. Actually in hyperbolic relaxation systems, initial disturbances often propagate along the characteristics of the full system, whose waves will have important roles to play. Yet the equilibrium system is expected to determine the large time relaxation dynamics if some structure conditions are met.

The large time solution behavior will certainly rely on the basic pattern of the initial data, which will be detailed in following subsections when the model (7) is being used. The key in dealing with different cases is to clarify the roles of each wave set, and to see how each set is modified by the presence of the other.

3.1 Long-time diffusive behavior

Consider the relaxation system (7) with $f(u) = \alpha u^2/2$, $\alpha > 0$, subject to the initial data

$$(u, v) = (u_0, v_0) \quad \text{in} \quad \mathsf{IR} \times \{0\}, \tag{16}$$

where

$$u_0, v_0 \in L^1(\mathsf{IR}) \cap L^\infty(\mathsf{IR}), \quad u_0 \geq a|v_0| \quad a.e. \;\; x \in \mathsf{IR}. \tag{17}$$

Solutions of (7), (16) with L^1 initial data satisfy the following property

$$\int_{\mathsf{IR}} u(x,t)dx = \int_{\mathsf{IR}} u_0(x)dx = m, \quad \forall\, t \geq 0. \tag{18}$$

We assume that $m > 0$ and a is large enough such that the sub-characteristic condition (10) holds for $u \in I$, where I is a bounded interval of range of the component u, see [42].

Since $u_0(x) \to 0$ as $|x| \to \infty$ and the mass $m > 0$, it follows from Sect. 2 that the sub-characteristic condition may strengthen the diffusive effect hidden in the relaxation dynamics. Thus, in the long time behavior, the hyperbolic system (7) is expected to approach the convection-diffusion equation (14).

With Roberto Natalini in [42], we showed that the large time attractor of the solution to (7) subject to the above initial data is indeed a diffusion wave profile to (14). More precisely, let $\theta_m : \mathsf{IR} \to \mathsf{IR}$ be the fundamental solution for the convection diffusion equation (14), starting with a Dirac mass $m\delta(x)$. Our result in [42] can be summarized in

Theorem 3.1 *[42] Assume the sub-characteristic condition (3) be satisfied. Let (u, v) be a solution of (7), (16), with $f = \alpha \frac{u^2}{2}$, for bounded initial data*

$(u_0, v_0) \in L^1(\mathrm{IR})^2$ *carrying a finite mass* $m = \int_{\mathrm{IR}} u_0 dx > 0$. *Assume that there exists a constant* $K > 0$ *such that* $|v_0| \le a u_0$ *and* $a u_{0x} \pm v_{0x} \le K$ *for* $x \in \mathrm{IR}$. *Then for every* $p \in [1, \infty)$

$$\lim_{t\to\infty} t^{\frac{p-1}{2p}} \|u(\cdot,t) - \theta_m(\cdot,t)\|_{L^p(\mathrm{IR})} = 0. \tag{19}$$

Some remarks are in order.

1. The diffusive phenomena in the relaxation system was first observed by Liu in [38], via the Chapman-Enskog expansion for a class of 2×2 hyperbolic relaxation system. Later, Chern [10] showed that the global smooth solution of relaxation system approaches a diffusive wave $\theta(x,t)$ in the L^p–norm at a certain rate if the initial data are sufficiently small. Respect to Chern's result, the main achievement of Theorem 3.1 consists in removing the smallness on initial data by using a quite different approach. This seems remarkable for hyperbolic relaxation problems since the dissipation from the relaxation effect is known to be weaker than that in the viscous approximation.

2. Our approach is due to a parabolic scaling which has led to (10). With this argument the investigation of the asymptotic behavior of the solution $\{u, v\}$ can be reduced to studying the convergence of the family $\{u^\epsilon, v^\epsilon\}$, as $\epsilon \to 0$. The main difficulty is to obtain the appropriate a priori bounds on the family $\{u^\epsilon\}$. One of our main tools for doing that is the following entropy-type estimates

$$\frac{\partial u}{\partial x} \le \frac{C}{t}, \quad \|u\|_{L^\infty(\mathrm{IR})} \le C\frac{m}{\sqrt{t}}, \quad t > 0.$$

The scaling arguments of such nature have been widely used in studying the large time behavior of solutions for equations of parabolic type, see for instance [17, 25]. We want to stress here the main phenomena that, in the long time behavior, the hyperbolic system (7) is driven by an equation which is different in type, e.g.: parabolic.

3. The asymptotic rate obtained in Theorem 3.1 is sharp, and it implies that $t^{\frac{p-1}{2p}}\theta$ is the L^p attractor of $t^{\frac{p-1}{2p}}u$.

4. For more general flux functions similar results can be established by exploiting the same scaling argument. Actually it is enough to assume that $f(u)$ behaves like $|u|^{q-1}u$ for $q \ge 2$ and for u small enough. For $q > 2$ the asymptotic behavior will be driven by a simple heat equation, see [42] for details.

3.2 Asymptotic towards rarefaction waves

In this section and in the following one, our discussion about large time relaxation dynamics will be continued. Here we assume that the initial data are asymptotically approaching equilibrium states i.e,

$$(u, v)(x, 0) = (u_0, v_0)(x) \to \Gamma(u_\pm) \quad \text{as} \quad x \to \pm\infty, \tag{20}$$

where $u_\pm$ are constants satisfying $u_- < u_+$. The complementary situation, when $u_- > u_+$, and the problem (7), (20) admits relaxation shock profiles, will be dealt with in Sect. 3.3. Again we assume the sub-characteristic condition (3) holds in a bounded interval $u \in I$ such that the global existence of the solution (u, v), as well as the L^1 contraction property are secured, see [35].

Our goal is again to show the large time behavior of solutions and to measure their asymptotic rates to the large time wave profiles. The time asymptotic behavior of the solutions to (7), (20) is related to that of the Riemann problem for the equilibrium conservation law:

$$
\begin{aligned}
r_t + f(r)_x = 0, \quad (x,t) \in \mathrm{IR} \times \mathrm{IR}^+, \\
r(x,0) = u_\pm, \quad \text{for} \pm x > 0.
\end{aligned} \tag{21}
$$

Its entropy solution $r(x,t)$ is called the rarefaction wave, given explicitly by

$$
r(x,t) = \begin{cases} u_-, & x < f'(u_-)t \\ (f')^{-1}(\frac{x}{t}), & f'(u_-)t \le x \le f'(u_+)t. \\ u_+, & x > f'(u_+)t \end{cases} \tag{22}
$$

Compared to the relaxation shock waves to be discussed in Sect. 3.3, the rarefaction wave is time-varying. One does not know what the exact large time wave profile is when trying to prove it stable. The rarefaction stability results depend strongly on how well one defines the approximate profile. Therefore in [35] we proposed a notion of the relaxation rarefaction profile in the following way: for any constant $\gamma > 0$, there exists $t_0 > 0$ such that

$$
0 \le r(x, t_0)_x \le \gamma.
$$

The u-component of the solution to (7) with initial data $\Gamma(r(x,t_0))$ is called **the relaxation rarefaction profile**. In fact this solution is shown to be monotone non-decreasing in space and flatten out in a way as the rarefaction wave does.

Such relaxation wave profile is shown to behave like the usual rarefaction wave for the equilibrium conservation laws, and serves as an $L^p (p > 1)$ attractor for a large class of initial data in $L^\infty + L^1 \cap H^1$.

Theorem 3.2 *[35] Let* $(u_0, v_0) \in \Gamma(r(x,t_0)) + L^1(\mathrm{IR}) \cap H^1(\mathrm{IR})$ *be the initial data, and* $(\bar{u}, \bar{v})$ *be the relaxation rarefaction profile as described above. Suppose that the stability condition (3) holds. Then there exists* $C > 0$ *such that*

$$
\|u(t) - \bar{u}(t)\|_{L^p} + \|v(t) - \bar{v}(t)\|_{L^p} \le C(1+t)^{-(1/2)+(1/2p)}, \quad \forall t \ge 0.
$$

An immediate consequence of the above theorem for the case $u_+ = u_-$ is

$$
\|(u,v)(t) - \Gamma(u_-)\|_{L^p} \le C(1+t)^{-(1/2)+(1/2p)}, \quad \forall t \ge 0.
$$

One may also ask how fast the solution converges to the rarefaction wave (22). The answer provided in [35] is the sharp estimate

$$\|u(\cdot,t) - r(\cdot, t+t_0)\|_{L^p} \le CN(t)^{(1/2)+(1/2p)}(t+t_0)^{-(1/2)+(1/2p)}, \qquad (23)$$

where

$$N(t) := \|\bar{u}(\cdot,t) - r(x, t+t_0)\|_{L^1}, \quad t \ge 0.$$

If $N(t) \sim 1 + \ln(t+t_0)$ holds, then the above estimate (23) is consistent with the result obtained in [40] for viscous case.

We use both the L^1-contraction and the time-weighted L^2 energy approach in proving the above results, for which the regularity estimate for u-component plays essential role and with such regularity we were able to investigate the global solution behavior of the relaxation system starting with the general large data, as described in Theorem 3.2.

Consult [38, 39] for stability results of rarefaction wave profiles of some hyperbolic relaxation systems.

3.3 Nonlinear stability of relaxation shocks

As pointed out earlier, if $u_- > u_+$, the problem (7), (20) is expected to approach a travelling shock profile.

Actually under the stability condition (3), the relaxation system (7) admits a smooth travelling wave solution of the form

$$(u,v)(x,t) = (\phi,\psi)(x-st), \quad (\phi,\psi)(\pm\infty) = \Gamma(u_\pm) \qquad (24)$$

where the shock speed $s = [f(u_+) - f(u_-)]/(u_+ - u_-)$, see e.g. [50]. In [37] the travelling wave (ϕ,ψ) is named as **the relaxation shock profile**. Our main interest is to investigate the asymptotic stability of relaxation shock profiles.

- **Stability of relaxation shocks for 1-D scalar law**.

In a series of works [50, 55, 56] joint with Wang, Yang and Woo, we studied the stability of relaxation shock profiles for some 2×2 hyperbolic relaxation models. In these papers the stability of strong shock profiles with possibly non-convex flux were established via the weighted energy method. The selection of the weight depends on the underlying shock profile and the flux function f. We refer to these papers for the full story, a decay rate result from [55] is summarized in the following

Theorem 3.3 *[55]. Let (ϕ,ψ) be a non-degenerate ($f'(u_+) < s < f'(u_-)$) relaxation shock profile connecting $\Gamma(u_\pm)$, determined by $\int_{\mathbb{R}}(u_0(x) - \phi(x))dx = 0$. Let (u,v) be a global solution to (7) subject to initial data, having asymptotic rate $|x|^{-\alpha/2}$ to (ϕ,ψ) as $|x| \to \infty$ in the following sense*

$$\int_{\mathbb{R}} (1+x^2)^{\frac{\alpha}{2}} \left| \left(\int_{-\infty}^{x} (u_0 - \phi)dx, u_0 - \phi, v_0 - \psi(x)\right)\right|^2 dx \le C. \qquad (25)$$

Then

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(\phi,\psi)(x-st)|\le Const(1+t)^{-\frac{\alpha}{2}}.$$

1. In [55] the decay rate for non-integer α was obtained as $t^{-\alpha/2+\epsilon}$. Actually this rate can be improved by removing ϵ, see the discrete version to be reviewed in Sect. 5.3. If the relaxation shock is degenerate ($s = f'(u_+)$ or $s = f'(u_-)$), the decay rate obtained in [55] is $t^{-\alpha/4}$ provided that an additional spatial decay rate on the degenerate side is imposed.
2. These results reveal that the information on the decay rate can be transferred from space to time.
3. The L^1 stability of the relaxation shocks for (7) was first obtained in [64], which when combined with our result in [42] stated in Sect. 3.1 may lead to a definitive statement: the underlying relaxation shock stands as an L^1 large time attractor; Consult [79] for an independent argument.
4. These decay rates are shown to be the same as those for the viscous conservation laws obtained by Matsumura and Nishihara in [63].

- **The equilibrium equation is a $n \times n$ system**

If the limit system (11) is a $n\times n$ system, the stability issue becomes quite subtle. In such situation there are more wave fields for equilibrium system as well as for the full relaxation system, much more effort needs to be involved to handle the interaction of various wave modes.

Let $\lambda_k \in \sigma(Df(u))$, spectrum of Df, be a simple eigenvalue in a neighborhood of some reference state u_*, $u_\pm$ are close to u_*. For each $k \in \{1,\cdots,n\}$, we assume that the shock speed s satisfies the strict Liu's entropy condition

$$s = s(\rho_+) < s(\rho) \tag{26}$$

for ρ between 0 and ρ_+, where the parameter $\rho = l_k(u_-)\cdot(u(\rho)-u_-)$ parameterizes the k-th Hugoniot curve $u(\rho) = u(\rho, u_-)$ passing through u_- and $u_+ = u(\rho_+)$.

When restricting ourselves to the non-degenerate shocks, i.e., $\lambda_k(u_+) < s < \lambda_k(u_-)$, we recorded here a main result obtained in [37].

Theorem 3.4 *[37] Assume the sub-characteristic condition for* $\lambda \in \sigma(Df(u))$

$$|\lambda| < a, \quad |u-u^*| < \epsilon_0. \tag{27}$$

There are numbers $\epsilon_0, \quad \beta_0 > 0$ *such that if* $|u_\pm - u_*| < \epsilon_0$ *and* $(\phi,\psi)(x-st)$ *is a relaxation shock profile connecting* $\Gamma(u_\pm)$

$$u(x,0) = \phi(x) + U_x(x,0), \quad v(x,0) = \psi(x) - W(x,0) \tag{28}$$

with

$$\int_{\mathbb{R}}|\partial_x^\alpha U(x,0)|^2dx + \int_{\mathbb{R}}|\partial_x^{\alpha-1}W(x,0)|^2dx \le \beta_0, \quad \alpha = 0,1,2,$$

then

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(\phi,\psi)(x-st)|=0.$$

1. Note that the present result restricts to perturbations with zero total mass in u-component, so there are no diffusion waves. We refer to [10, 42] for the decay estimate of diffusion waves when the equilibrium system is a scalar law .
2. The stability of relaxation shocks for the 2×2 system has been well understood in previous studies. The results for 3×3 Broadwell model can be found in [13, 29, 81]. It would be interesting to extend our stability result to more general relaxation systems, at least to those the existence of relaxation shocks has been established by Yong and Zumbrun [93].
3. We refer to [67] for an elegant pointwise analysis of relaxation shocks and references therein for the recent development along this line.

- **2-D relaxation shock fronts.**

The stability issue of relaxation shock fronts in multi-D case is much more delicate. First the wave front will not only translate in the heading direction but is forced to reshape in the transverse direction. In [36] we prove nonlinear stability of planar shock fronts for certain relaxation system in two spatial dimensions. If the sub-characteristic condition is assumed and the initial perturbation is sufficiently small, even though the mass carried by initial perturbations is not necessarily finite, then the solution converges to a shifted planar shock front solution as time $t\uparrow\infty$. The asymptotic phase shift of shock fronts is in general non-zero and governed by a similarity solution to heat equation. The asymptotic decay rate to the shock front is proved to be $t^{-1/4}$ in $L^\infty(\mathbb{R}^2)$ without imposing extra decay rate in space for initial perturbations.

To make things more precise we follow [36] and consider a simple example of Jin-Xin's 2-D relaxation system

$$\begin{aligned}
u_t+v_{1x}+v_{2y}&=0, \qquad (t,x,y)\in\mathbb{R}^+\times\mathbb{R}^2,\\
v_{1t}+a^2u_x&=f(u)-v_1,\\
v_{2t}+b^2u_y&=-v_2.
\end{aligned}\tag{29}$$

The unknowns u, v_1, v_2 belong to $\mathbb{R}$, the function $f=f(u)$ is in C^2, and $a, b>0$ are fixed constants satisfying the sub-characteristic condition (3). The equilibrium manifold reads

$$\Gamma_2(u)=\{(u,v_1,v_2),\quad v_1=f(u),\quad v_2=0\}.$$

The initial data are asymptotically constants as $x\to\pm\infty$, i.e.,

$$(u,v_1,v_2)(0,x,y)=(u_0,v_{10},v_{20})(x,y)\to\Gamma_2(u_\pm),\quad x\to\pm\infty \tag{30}$$

with $u_\pm$ being two given constants such that $u_->u_+$.

A planar relaxation shock front is a travelling wave solution to (29) of the form $(u, v_1, v_2) = (U, V_1, V_2)(z)$, $z = x - st$, connecting $\Gamma_2(u_\pm)$. Its existence is ensured by the sub-characteristic condition (3) combined with the Oleinik entropy condition i.e., $f(u) - f(u_\pm) < s(u - u_\pm)$ for $u_+ < u < u_-$. We focus on the stability of the relaxation shock front with perturbation carrying infinite mass, i.e.,

$$\int_{\mathbb{R}^2} (u_0(x, y) - U(x))dxdy = \infty,$$

in such situation the asymptotic state is shown to be

$$(U, V_1, V_2)(x - st + d(t, y)),$$

and the effective phase shift $d(t, y)$ in the shock front may have different values at $y = \pm\infty$.

In fact the location of the shock front may be determined by solving a wave equation of $d(t, y)$, which satisfies

$$d(t, y) = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} [u(t, x, y) - U(x)]dx,$$

for all y, t. The effective phase shift $d(t, y)$ is time-asymptotically governed by a solution to the heat equation

$$d_t = b^2 d_{yy}.$$

The asymptotic ansatz can be actually given by

$$\rho(y) = d_- \int_{\frac{y}{b\sqrt{4\pi}}}^{+\infty} e^{-\pi z^2} dz + d_+ \int_{-\infty}^{\frac{y}{b\sqrt{4\pi}}} e^{-\pi z^2} dz \to d_\pm \quad \text{as} \quad y \to \pm\infty, \tag{31}$$

$m_1 := \int_{\mathbb{R}} d_t(0, y)dy$ and $\theta(y)$ is a smooth function with compact support and unit integral.

Theorem 3.5 *[36] Let (u, v_1, v_2) be a global solution to (29) subject to initial data being a small perturbation of the relaxation shock front $(U, V_1, V_2)(x, y)$, with $(\int_{-\infty}^{x}(u_0 - U)dx, u_0 - U, v_{10} - V_1, v_{20})$ having decay rate $|x|^{-\alpha/2}$ as $|x| \to \infty$ in the sense of (25). Then the following convergence rate estimate holds*

$$\sup_{\mathbb{R}^2} \left| (u, v_1, v_2)(t, x, y) - (U, V_1, V_2)(x - st + \rho(\frac{y + y_0}{\sqrt{t+1}})) \right| \le C(1+t)^{-\min\{1, \frac{\alpha}{2} + \frac{1}{4}\}}.$$

1. This result indicates that, as $t \to \infty$, the wave equation has a parabolic structure through the effective phase shift d.
2. Theorem 3.5 suggests that the decay rate of perturbations could not be faster than t^{-1} even if a stronger localization of perturbation may be imposed. However the decay rate is always not slower than $t^{-1/4}$. This is in sharp contrast to the one dimensional theory which we have seen in Theorem 3.3.
3. We refer to [59] for a stability result on weak shock front to (29) in the presence of transverse force $g(u)$, i.e., with $g(u) - v_2$ on the right of the third equation in (29). In order to control the effect from $g(u)$, the convexity of f plays essential role in the stability analysis of [59].

3.4 Stability of relaxation boundary layers

We now return to the 1-D model (7) and discuss boundary layers under relaxation. Sub-characteristic condition (3) has been proved to be necessary and also sufficient for the global-wellposedness of the initial value problem for (7). One would expect some additional criterion to ensure stable boundary layers. Such requirement is complicated by the interaction of two sets of waves at the boundary. With Wen-An Yong in [60] we proved the time-asymptotic boundary layer under a dissipative boundary condition.

We consider (7) on the quarter-plane $x, t \geq 0$ subject to initial data (u_0, v_0) for $x \geq 0$. Since the coefficient matrix of the vector $(u_x, v_x)^\top$ in (7)

$$A = \begin{pmatrix} 0 & 1 \\ a^2 & 0 \end{pmatrix}$$

has only one positive eigenvalue at $x = 0$, it is well known, [26], that one relation of boundary data:

$$B(u(0,t), v(0,t)) = 0 \tag{32}$$

should be given with $B = B(u, v)$ satisfying

$$B_u + aB_v \neq 0, \tag{33}$$

where $(1, a)^\top$ is the right eigenvector associated with the positive eigenvalue of the coefficient matrix of (7).

The asymptotic behavior has been studied for some special cases of boundary data, say $B_v = 0$, by several authors, for instance [44, 66, 74, 75]. Despite the known results concerning IBVP, one may still wonder what is a natural condition on B in order that IBVP can have globally stable boundary layer. Our answer provided in [60] is the dissipative stability condition

$$B_u \neq 0 \quad \text{and} \quad B_u B_v \geq 0, \tag{34}$$

with which we came up with the following

Theorem 3.6 *[60] Let $(U, V)(x)$ be a bounded steady solution to IBVP and $f'(u_+) < 0$. Assume (3) and (34) hold. Then there exists a positive constant $\delta > 0$ such that if*

$$\| \int_x^{+\infty} (u_0 - U)(x)dx \|_{H^3(\mathrm{IR}^+)} + \|(v_0 - V)(x)\|_{H^2(\mathrm{IR}^+)} + |B(u_+, f(u_+))| < \delta,$$

then the IBVP (7), (32) has a unique global solution $(u, v)(x, t)$ satisfying

$$\lim_{t \to \infty} \sup_{x \in \mathrm{IR}^+} |(u, v)(x, t) - (U, V)(x)| = 0.$$

See [60] for details and we conclude this section by two remarks.
1. Note that different from the initial value problem(IVP), for the IBVP the total mass in u-component is not preserved, but changing with the boundary flow.
2. The condition of the type (34) seems also necessary, as evidenced by previous studies on zero relaxation limit problems with boundaries, see e.g. [27, 73, 86, 92]. For boundary layer analysis associated with Broadwell model see [58]; for more general Jin-Xin model see [88, 89].
3. Consult [16] for kinetic approximation of a boundary problem for conservation laws.

4 Scaling Limits

Scaling limits represent sharper physical modelling of macroscopic equations, the prototypical example being the hydrodynamic limit problem for the Boltzmann equation. So one of the central issues in the hyperbolic relaxation problems is to justify the limiting process in various scaling regimes.

4.1 The zero relaxation limit

In a relaxation limit process the primary system describes the physical dynamics on a finer scale than the system of conservation laws. As we pass to the relaxation limit fine scale features are often lost and we recover the hyperbolic conservation laws.

The relaxation approximation to conservation laws is in spirit close to the description of the hydrodynamic equations by the detailed microscopic evolution of gases in kinetic theory. The rigorous theory of kinetic approximation for solutions with shocks is well developed when the limit equation is scalar. For works using the continuous velocity kinetic approximation, see Giga and Miyakawa [18], Lions, Perthame and Tadmor [45] and Perthame and Tadmor [76], for discrete velocity approximation of entropy solutions to multidimensional scalar conservation laws see Natalini [69], Katsoulakis and Tzavaras [32].

For the scaled relaxation system (9), as $\epsilon \downarrow 0$, the u-component is expected to converge to the entropy solution of scalar conservation laws (11).
We make the following assumptions:

(H$_1$) $u_0^\epsilon(x) := u_0(x), \qquad v_0^\epsilon = f(u_0(x)) + K(x)\omega(\epsilon)$
where $K \in L^\infty \cap L^1(\mathrm{IR}) \cap BV(\mathrm{IR})$, $\omega : [0,\infty[\to [0,\infty[$ is continuous and $\omega(0) = 0$.
(H$_2$) the flux function f is a C^1 function with $f(0) = f'(0) = 0$;
(H$_3$) the initial data satisfy $(u_0^\epsilon, v_0^\epsilon) \in L^1(\mathrm{IR}) \cap L^\infty(\mathrm{IR}) \cap BV(\mathrm{IR})$.

Equipped with assumptions made in (H_1)-(H_3), it has been proved that, as $\epsilon \to 0^+$, the solution sequence to (9) converges strongly to the unique entropy solution of (11), see Natalini [68], also [46] when the equilibrium equation is a $n \times n$ system.

Our interest is more than convergence, and we want also detect the convergence rates. With Gerald Warnecke in [49], we have obtained sharp estimates.

Theorem 4.1 *[49] Consider the relaxation system (9), subject to* $L^\infty(\mathrm{IR}) \cap BV(\mathrm{IR})$*-perturbed initial data satisfying* (H_1)-(H_3). *Then the global solution* (u^ϵ, v^ϵ) *converges to* $(u, f(u))$ *as* $\epsilon \downarrow 0$ *and the following error estimates hold,*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_1 \le \mathrm{C}_T\sqrt{\epsilon}, \tag{35}$$

$$\|v^\epsilon(\cdot,t) - f(u^\epsilon(\cdot,t))\|_1 \le C_T[\mathrm{e}^{-\frac{t}{\epsilon}}\omega(\epsilon) + \epsilon(1-e^{-\frac{t}{\epsilon}})], \quad 0 \le t \le T. \tag{36}$$

Some remarks are in order.
1. Our estimates are sharp and (36) reflects two sources of error: the initial error of size $\omega(\epsilon)$ and the relaxation error of order ϵ.
2. *It is striking that the effect of initial error persists only for a short time of order* ϵ*! Beyond this time the non-equilibrium solution approaches the equilibrium state at an exponential rate.*
3. We would like to mention that an analogous result for a class of relaxation systems was obtained by Kurganov and Tadmor [31] by using the Lip′-framework initiated by Nessyahu and Tadmor [72]. But their argument uses the convexity of the flux function. For the case of a possibly nonconvex flux function f, our work uses Kuznetzov-type error estimates, see [8, 33]. We also refer to [82, 83] for first order convergence rates when piecewise smooth solutions with finitely many discontinuities are to be computed with the assumption of convex fluxes $f(u)$.

4.2 The diffusive scaling limit

We now turn to discuss the diffusive scaling limit. Consider the system (10), which is obtained from (7) under a parabolic scaling as stated in Sect. 2. The limit equation is expected not to be the scalar conservation law (11) but the convection diffusion equation (14).

With Shi Jin in [21], we rigorously justify the diffusive limit even the relaxation shocks are present in the solution. To state our result more precisely we make the following assumptions on initial data:

(B_1) $u_\pm$ generate a travelling wave (U^ϵ, V^ϵ) of (10).
(B_2) $\|\int_{-\infty}^x (u^\epsilon - U^\epsilon)(y)dy\|_{H^3} + \|\epsilon(v_0^\epsilon - V^\epsilon)(x)\|_{H^2} \le c_1$,
(B_3) $\|(v_0^\epsilon - V^\epsilon - s(u_0^\epsilon - U^\epsilon)\|_{H^2} \le c_2$ and

$$\frac{1}{\epsilon}\|v_0^\epsilon - f(u_0^\epsilon) + a\partial_x u_0^\epsilon\|_{H^1} \le c_3,$$

(B_4) $\int_R (u_0^\epsilon - u_0)(x)dx = 0$ and, as $\epsilon \to 0$,

$$u_0^\epsilon - U^\epsilon \to u_0 - U^0\,, \qquad v_0^\epsilon - V^\epsilon \to v_0 - V^0\,, \qquad \text{in} \quad H^2 \quad \text{as} \quad \epsilon \to 0.$$

In these assumptions, c_1, c_2, c_3 are positive constants independent of ϵ. Under these conditions, we have

Theorem 4.2 *[21] Assume $0 < \epsilon \le 1$, f is a smooth convex function satisfying the sub-characteristic condition (3) for $u \in (u_+, u_-)$ and $(B_1) - (B_4)$ hold, then there exists an $u \in C^1([0,T] \times \mathbb{R})$ for any $T \ge 0$ such that*

$$u^\epsilon - U^\epsilon \to u(x,t) - U^0 \quad in \quad C^0([0,T]; H^{2-\delta_1}_{loc}) \quad for\ any \quad \delta_1 > 0,$$

$$v^\epsilon - V^\epsilon \to v(x,t) - V^0 \quad in \quad C^0([0,T]; H^{2-\delta_1}_{loc}),$$

and the limit function (u, v) solves the relaxed parabolic problem:

$$u_t + f(u)_x = a^2 u_{xx}, \quad u|_{t=0} = u_0(x)$$

with $v = f(u) - au_x$.

¿From the discussion in Sect. 2 we know that the requirement of (3) for diffusive limit is too strong. Using a refined energy argument we were finally able to prove the same result but under a weaker requirement (15).

Theorem 4.3 *[22] Assume that there exists a constant $\epsilon_0 > 0$ such that*

$$0 < \epsilon \le \epsilon_0, \qquad a > \epsilon |f'| \quad \text{for} \;\; u \in (u_+, u_-)\,, \tag{37}$$

and $(B_1) - (B_4)$ hold, then the convergence result in Theorem 4.2 still holds if $|u_+ - u_-| \le \beta$ for some $\beta > 0$.

1. Our arguments also provide a clear picture of the large time behavior of solutions for both original system and the reduced equation.
2. The diffusive scaling introduced in (10) is very typical in many important physical problems, for example, in transport equation in diffusive regime [5, 41, 77], in kinetic equations near incompressible Navier-Stokes regimes [3, 11], in hyperbolic balance laws [62] and in nonlinear parabolic equation [7].
3. For similar diffusive limits justified by using L^1 compactness tools, see e.g. [43, 47, 65]. However the energy method we exploited is capable of generalizations to system case, see e.g. [61].

5 Convergence of Relaxation Schemes

Relaxation schemes to be considered here are a class of non-oscillatory numerical schemes for systems of conservation laws proposed by Jin and Xin [24]. These schemes provide a new way of perturbing, even regularizing, systems of conservation laws and approximating their solutions. The computational results that are available, see e.g. [24] as well as Aregba-Driollet and Natalini [2], indicate that the relaxed schemes obtained in the limit $\epsilon \to 0$ provide a quite promising class of new schemes. We point out that the main assets of these schemes are that they neither require the computation of the Jacobians of fluxes for the conservation laws nor the use of Riemann-solvers. This important property is shared by other schemes such as for instance the high resolution central schemes introduced by Nessyahu and Tadmor [71], see also Kurganov and Tadmor [30] for references therein on recent developments.

To make things more precise we want to approximate the scalar conservation law (11). We choose a time step Δt, a spatial mesh size Δx, a parameter a which will be related to the characteristic speed of the conservation law and a small relaxation parameter $\epsilon > 0$. From these we define the mesh ratio $\lambda = \frac{\Delta t}{\Delta x}$, the CFL parameter $\mu = a\lambda \in]0,1[$, and the scale parameter $k = \frac{\Delta t}{\epsilon}$. The mesh is given by the points $(j\Delta x, n\Delta t)$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. The approximate solution takes the discrete values u_j^n at the mesh points. Further, relaxation schemes involve the discrete relaxation fluxes v_j^n. We want to focus on the following *semi-implicit relaxation scheme*

$$\left.\begin{array}{l} u_j^{n+1} - u_j^n + \frac{\lambda}{2}(v_{j+1}^n - v_{j-1}^n) - \frac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0, \quad j \in \mathbb{Z}, \quad n \in \mathbb{N} \\ v_j^{n+1} - v_j^n + \frac{a^2\lambda}{2}(u_{j+1}^n - u_{j-1}^n) - \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) = -k[v_j^{n+1} - f(u_j^{n+1})]. \end{array}\right\} \tag{38}$$

The discrete initial data are given by averaging the initial data u_0 over mesh cells $I_j = \left](j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x\right[$, i.e. taking

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx, \qquad \text{and setting } \; v_j^0 \sim f(u_j^0). \tag{39}$$

5.1 Optimal convergence rates

With Gerald Warnecke in [49], we obtained a global error estimate for the relaxation scheme (38) approximating the scalar conservation law (11). The key idea is to decompose the error into a relaxation error and a discretization error. Including an initial error $\omega(\epsilon)$ we thus obtained the rate of convergence of $\sqrt{\epsilon}$ in L^1 for the relaxation step, as stated in Sect. 4.1. In the discretization step a sharp convergence rate of $\sqrt{\Delta x}$ in L^1 is obtained. These rates are independent of the choice of initial error $\omega(\epsilon)$. Thereby, we obtain the order 1/2 for the total error. Also our estimate is independent of the type of nonlinearity.

Theorem 5.1 *[51] Let u be the entropy solution of scalar conservation law (11) satisfying the initial data u_0, and let (u^N, v^N) be a piecewise constant representation of the data $(u_i^N, v_i^N)_{i\in\mathbb{N}}$ generated by relaxation scheme (38). Then, for any fixed $T = N\Delta t > 0$, there exists a constant C_T, independent of Δx, Δt and ϵ such that*

$$\|u^N - u(\cdot, T)\|_1 \le C_T\left[\sqrt{\epsilon} + \sqrt{\Delta x}\right].$$

1. The total error bound is proved to be independent of the initial error $\omega(\epsilon)$, which provides a theoretical support for the relaxation scheme that one could

choose any initial $v_0(x)$, not necessarily on the equilibrium manifold $\Gamma(u)$.
2. Taking $\epsilon = 0$ in discretization step, we immediately recover the optimal convergence rate of order 1/2 for monotone schemes of scalar conservation laws , see Sabac [78], Tang and Teng [84].
3. Consult [87] for convergence analysis of second order relaxation schemes. For convergence results on some other relaxation schemes, see e.g. [1, 28, 80].

5.2 Lip^+ stability and error estimates

As is well known if the flux function f is strictly convex, say $f''(u) \geq \alpha > 0$, the entropy solution to the scalar conservation laws enjoys Oleinik's one-sided regularity estimate

$$u_x \leq \frac{1}{\alpha t + \|u_0\|^{-1}_{Lip^+}}, \quad \|w\|_{Lip^+} := \text{esssup}_{x\neq y}\left(\frac{w(x)-w(y)}{x-y}\right)^+, \quad (\cdot)^+ = max(\cdot, 0).$$

The natural question is whether one can obtain a better estimate for the relaxation scheme (38) with convex flux. With Wang and Warnecke, we provided a definite answer in [52] for (38).

We first established the discrete lip^+-stability for the relaxation scheme (38), which is the heart of our effort and the proof is quite elegant. Such lip^+ stability says that if $\|u_0\|_{Lip^+} = L < \infty$, then

$$u_j^n - u_{j-1}^n \leq 2L\Delta x \quad \text{for} \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{40}$$

Equipped with (40) we obtained global error estimates in the spaces $W^{s,p}$ for $-1 \leq s \leq 1/p$, $1 \leq p \leq \infty$ and point-wise error estimates for the approximate solution obtained by the relaxation scheme (38). The proof uses the framework introduced by Nessyahu and Tadmor [72, Corollary 2.2, 2.4]. The resulting error estimates are summarized as follows.

Theorem 5.2 *[52] Consider the convex scalar conservation law (11) with Lip^+-bounded initial data u_0. Then the relaxation scheme (38) with discrete initial data $(u_j^0, f(u_j^0))_{j\in\mathbb{Z}}$ converges. The piecewise linear interpolants $u^{\Delta,\epsilon}$ satisfy the convergence rate estimates*

$$\|u^{\Delta,\epsilon}(\cdot,T) - u(\cdot,T)\|_{W^{s,p}} \leq C_T(\Delta x + \epsilon)^{\frac{1-sp}{2p}}, \quad \text{for} \quad -1 \leq s \leq \frac{1}{p},\ 1 \leq p \leq \infty, \tag{41}$$

as well as

$$|u^{\Delta,\epsilon}(x,T) - u(x,T)| \leq \text{Const}_{x,T}(\Delta x + \epsilon)^{\frac{1}{3}}, \tag{42}$$

with

$$\text{Const}_{x,T} \sim 1 + \left|u_x(\cdot,T)\right|_{L^\infty(x-(\Delta x+\epsilon)^{1/3},\ x+(\Delta x+\epsilon)^{1/3})}.$$

Remarks:
1. When $(s,p) = (-1,1)$ the error estimate (41) turns into the Lip' error estimate

$$\|u^{\Delta,\epsilon}(\cdot,t)-u(\cdot,t)\|_{Lip'(\mathbb{R})} \le O(\epsilon+\Delta x).$$

2. When $(s,p)=(0,1)$ the error estimate (41) yields an L^1-convergence rate of order $O(\sqrt{\Delta x+\epsilon})$ which is consistent with the result in Theorem 5.1 for conservation laws with possibly non-convex flux functions.
3. In this context we also established a remarkable *discrete maximum principle* for (38) in the sense that if $b_1 \le u_0(x) \le b_2$ and $v_0 = f(u_0)$ then

$$b_1 \le u_j^n \le b_2 \quad \text{for} \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}.$$

5.3 Convergence to discrete relaxation shocks

Discrete shock profiles epitomize the propagation of solutions and structure properties of shocks in numerical solutions. The study of discrete shocks for conservative schemes have attracted much attention. Among researchers who joined this endeavor, let us list Jennings, Majda and Ralston, Michelson, Smyrlis, J.-G. Liu and Xin, T.P. Liu and Yu, Fan, Ying $\cdots$. For relaxation schemes of the type (38) together with J. Wang and T. Yang in [34, 54, 57], we have obtained a series of positive results on discrete relaxation shocks.

More precisely we considered the existence, the asymptotic stability and the decay rate of discrete relaxation shocks for relaxation schemes including (38), these results develop the stability theory of discrete relaxation shock profiles. To see the flavor a result in [34] is recorded here.

Theorem 5.3 *Assume that the CFL condition $0<\mu<1$, and the sub-characteristic condition (3) hold. Let $(U_j,V_j)_{j\in\mathbb{Z}}$ be a stationary discrete relaxation shock wave connecting $\Gamma(u_\pm)$. Assume that*

$$\sum_{j\in\mathbb{Z}} (u_j^0 - U_j) = 0$$

and for some $\alpha>0$ and some $\delta>0$

$$\sum_{j\in\mathbb{Z}} \left[(1+j^2)^{\alpha/2+1}|u_j^0-U_j|^2 + (1+j^2)^{\alpha/2}|v_j^0-V_j|^2\right] \le \delta.$$

Then the unique global solution $(u_j^n,v_j^n)_{j\in\mathbb{Z}}$ to (38) with the initial data $(u_j^0,v_j^0)_{j\in\mathbb{Z}}$ satisfies

$$\sup_j |(u_j^n,v_j^n)-(U_j,V_j)| \le C(1+nh)^{-\alpha/2}\sqrt{\delta}, \quad n\ge 0,$$

provided λ is suitably small, and the scale parameter $k=\Delta/\epsilon \in \mathbb{R}^+$.

This result shows that there is a relation between the spatial decay assumed of the initial perturbation and the rate of decay in time. In this sense the theorem exhibits the transformation of spatial decay into temporal decay.

5.4 The R-W model of Boltzmann equation

Under the sub-characteristic condition (3), the relaxation model (7) as a balance law enjoys the quasi-monotone property, which plays essential role in the analysis of this model.

The R-W model for the Boltzmann equation is, however, a typical model which does not share such a quasi-monotone property. With Wang and Warnecke in [53], we proposed a splitting scheme for such RW-model and proved the uniform convergence as both relaxation parameter and mesh size tend to zero; the convergence rate (local and global) is recovered following Tadmor's Lip' theory. One of the new elements in this study is that there is no monotonicity for this model, therefore a more careful and intrinsic analysis of its structure becomes essential. See [53] for details.

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