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Asymptotic stability of relaxation shock profiles for hyperbolic conservation laws

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Abstract

This paper studies the asymptotic stability of traveling relaxation shock profiles for hyperbolic systems of conservation laws. Under a stability condition of subcharacteristic type the large time relaxation dynamics on the level of shocks is shown to be determined by the equilibrium conservation laws. The proof is due to the energy principle, using the weighted norms, the interaction of waves from various modes is treated by imposing suitable weight matrix.

Keywords: Asymptotic stability; Relaxation shocks; Hyperbolic system

1. Introduction

This work is concerned with stability of relaxation shock profiles for a system of conservation laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n, \quad x \in \mathbb{R}. \tag{1.1}$$

Here $f(u)$ is a given smooth vector function. The simplest discontinuous solution to (1.1) is of the form

$$u(x,t) = u_- + (u_+ - u_-)H(x - st) \tag{1.2}$$

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with $H(x)$ being the usual Heaviside function; the quantities $u_\pm$ and s are related by the Rankine–Hugoniot relation

$$f(u_+) - f(u_-) = s(u_+ - u_-) \tag{1.3}$$

and satisfy Liu's strict entropy condition (stated later).

In this paper we consider the following well-known semilinear hyperbolic relaxation system:

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + au_x &= \frac{1}{\varepsilon}(f(u) - v). \end{aligned} \tag{1.4}$$

Here u, v are unknown vector functions, $\varepsilon > 0$ is a small constant representing the rate of relaxation, and $a > 0$ is a given constant relating with the flux function f via the following stability condition:

$$|\lambda_k(u)| < \sqrt{a}, \quad k = 1, \ldots, n \tag{1.5}$$

for all u under consideration. Here $\lambda_k(u)$ denotes the kth eigenvalue of the Jacobian $Df(u)$. This system was originally introduced by Jin-Xin [5] for numerical purposes. This model possesses the key features of a more general hyperbolic relaxation system, thus serves as an ideal model problem to understand the more general ones.

Relaxation effect is known to provide a subtle dissipative mechanism for discontinuities against the destabilizing effect of nonlinear response [15]. The relaxation phenomena are important since they arise in many relevant physical situations as the kinetic theories, elasticity with memory, chemically reacting flows and the theory of linear and nonlinear waves, (see, for example, [26]). The relaxation model (1.4) can be loosely interpreted as discrete velocity kinetic equation. The relaxation parameter, ε, plays the role of the mean free path and the system models the macroscopic conservation laws. In fact when ε is small, solutions of (1.4) are expected to approach the equilibrium states $v = f(u)$. With $v = f(u)$, the first equation in (1.4) becomes (1.1). System (1.1) is called the equilibrium system of the full system (1.4). Rigorous justification of this passage from (1.4) to (1.1) has recently been performed by Serre [22].

Presumably, functions (1.2) may be approximated, as $\varepsilon \downarrow 0$, by the u-component of a family of solutions in the form of traveling waves. Taking advantage of the scaling in (1.4), we seek, under the stability condition (1.5), a family of solutions in the form

$$u(x,t) = \phi\left(\frac{x - st}{\varepsilon}\right), \quad v(x,t) = \psi\left(\frac{x - st}{\varepsilon}\right) \tag{1.6}$$

satisfying

$$(\phi, \psi)(\pm\infty) = \Gamma(u_\pm) \tag{1.7}$$

with $\Gamma(u) = (u, f(u))$ being the equilibrium states. Corresponding to the entropy shock solution (1.2) the function (ϕ, ψ) is called a "relaxation shock profile".

Our interest in this paper is the asymptotic stability of such relaxation shock profiles. Thus, we rescale system (1.4) by using $(x, t) \to (\varepsilon x, \varepsilon t)$ so that system (1.4) becomes

$$u_t + v_x = 0,$$
$$v_t + au_x = f(u) - v. \tag{1.8}$$

The behavior of the solution (u, v) of (1.4) at any fixed time as $\varepsilon \to 0^+$ is equivalent to the long time behavior of solution (u, v) to (1.8) as $t \to \infty$.

The stability problem of relaxation shock profiles for a class of 2×2 hyperbolic relaxation systems was first proposed by Liu [15]. Therein an important observation is that the admissible shock wave can be smoothed out by the relaxation effect if the subcharacteristic condition is imposed. In fact, the subcharacteristic condition plays a similar role to the CFL condition for numerical approximation. The condition of this kind requires that the limit of domain of dependence for the approximation system contains the domain of dependence for original system. The asymptotic stability of the relaxation shock profiles for the 2×2 relaxation system (i.e., the equilibrium equation is a scalar law) has been well understood in the previous studies, see for example [11,12,15,19]. For the 3×3 Broadwell model, a widely known discrete kinetic model, Szepessy and Xin [23,25] has announced the result of linear and nonlinear stability of weak shocks, extending partial results of [1,6].

The main new result of this work is to establish the stability result of weak relaxation shock profiles for general system (1.1) with $n > 1$. For this purpose we assume that the underlying hyperbolic conservation law is strictly hyperbolic. Let $\lambda_k \in \sigma(Df(u))$ be an eigenvalue in a neighborhood of some reference state u_*, $u_\pm$ are close to u_*. For each $k \in \{1, \dots, n\}$, we assume that the shock speed s of a kth wave field satisfy the Liu's strict entropy condition:

$$s = s(\rho_+) < s(\rho) \tag{1.9}$$

for ρ between 0 and ρ_+, where the parameter $\rho = l_k(u_-) \cdot (u(\rho) - u_-)$ parameterizes the kth Hugoniot curve $u(\rho) = u(\rho, u_-)$ passing through u_- and $u_+ = u(\rho_+)$.

Indicated by the conservation form of the first equation we assume that $u_0 - \phi$ is integrable on $\mathbb{R}$ and satisfies

$$\int_{\mathbb{R}} (u_0(x) - \phi(x))\, dx = h(u_+ - u_-) \quad \text{for some } h \in \mathbb{R}$$

for which there is no mass distribution in other directions than $u_- - u_+$. We should mention in passing that the mass distribution in other directions may lead to possible diffusion waves as in the viscous case [14]. Consult [2] for the study of diffusion waves under the effect of relaxation when the equilibrium system is a scalar law.

It is noted that the shifted profile $(\phi, \psi)(x - st + h)$ is also a relaxation shock wave connecting $\Gamma(u_-)$ and $\Gamma(u_+)$, and satisfies

$$\int_{\mathbb{R}} (u_0(x) - \phi(x+h))\, dx = 0.$$

Thus the conservation form of the first equation of (1.8) gives

$$\int_{\mathbb{R}} (u(x,t) - \phi(x - st + h))\, dx = 0.$$

This relaxation shock wave $(\phi, \psi)(x - st + h)$ is proved to be stable as $t \to \infty$ provided the perturbation is suitably small. Without loss of generality, we assume $h = 0$ and define

$$U(x,0) := \int_{-\infty}^{x} [u(y,0) - \phi(y)]\, dy, \quad W(x,0) := \psi(x) - v(x,0), \quad x \in \mathbb{R}. \tag{1.10}$$

We can now formulate our main result:

Theorem 1.1. *Assume the subcharacteristic condition* (1.5) *for u close to* u^*. *There are numbers* ε_0, β_0 *such that if* $|u_\pm - u_*| < \varepsilon_0$ *and* $(\phi, \psi)(x - st)$ *is a relaxation shock profile connecting* $\Gamma(u_\pm)$ *and*

$$\left\{ \sum_{\alpha=0}^{3} \int_{\mathbb{R}} |\partial_x^\alpha U(x,0)|^2\, dx + \sum_{\alpha=0}^{2} \int_{\mathbb{R}} |\partial_x^\alpha W(x,0)|^2\, dx \right\}^{1/2} \leqslant \beta_0,$$

then problem (1.8) *with initial data* $(u,v)(x,0)$ *has a unique global solution* (u,v) *satisfying*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |(u,v)(x,t) - (\phi,\psi)(x - st)| = 0.$$

The proof of this theorem employs the elementary energy method which has been extensively used in the study of the viscous shock waves (see, for example, [3,4,7,14,16,21,24]). More recently, the stability theory of strong viscous shock waves has been developed by Kreiss–Kreiss [8], Zumbrun–Howard [28] and related references therein. It would be desirable to develop the stability theory of relaxation shock profiles for system (1.1).

When we employ the energy method for the relaxation problems we meet with new difficulty at handling waves of full system as well as the equilibrium system. Actually for the hyperbolic relaxation problems, the initial disturbances propagate along the characteristic fields of the full system (1.8), whose waves will have important roles to play. Yet the equilibrium system (1.1) is expected to determine the large time relaxation dynamics under stability condition (1.5). The key is to clarify further the roles of waves for (1.8) and waves for (1.1), and to see how each set modified by the presence of the other. This work concentrates only on the semilinear system (1.8), further contribution is expected for more general relaxation systems. Consult a recent work by Yong and Zumbrun [27] for the existence of relaxation shock profiles of more general hyperbolic relaxation systems.

The weighted energy method employed in this work is in spirit similar to that in [16] for the 2×2 relaxation system. However, for system (1.8) with more components much more efforts are involved to treat the interaction of various wave modes. The main task of this paper is to establish the basic energy inequality of the form

$$||(U, W, W_z)(t)||^2 + \int_0^t ||(W, U_z)(\tau)||^2 \, d\tau + \int_0^t (U, |\phi_z| U) \, d\tau \leqslant C||(U, W, V_z)(0)||^2.$$

A combination of this estimate with some higher order estimates leads to the desired a priori estimate. To establish this basic energy inequality we proceed through several steps. First, we diagonalize the lower order wave part $U_t + f'(\phi)U_x$ as suggested by Goodman [4], and then choose suitable weighted multipliers, such that one gets a useful relaxation energy function and also a corresponding dissipative energy under the stability condition (1.5). For the contributions from the wave modes of $U_t + f'(\phi)U_x$, we use suitable weight matrix. For the shock wave associated with possibly nonconvex mode, the weight introduced by Fries [3] for viscous conservation laws can be implemented here, for other wave modes we choose individual weights for a valid control. The interaction of waves from different modes as well as from different levels (full system (1.8) and the equilibrium one (1.1)) is carefully treated by using the structure of shock waves and the choice of weight matrix. The smallness of shock strength is essentially used in the interaction estimates.

The paper is organized as follows. In Section 2 we will state the existence and properties of the relaxation shock profiles. The reformulation of the problem is made in Section 3. The key energy estimate is established in Section 4. To close the energy analysis we make the higher order estimates in Section 5.

Notations. For vectors $u, v \in \mathbb{R}^n$ and matrices $M \in \mathbb{R}^{n \times n}$, we shall use the notations:

$$\langle u, v \rangle = \sum_{i=1}^n u_i v_i = u^T v, \quad |u| = \langle u, u \rangle^{1/2}, \quad |M| = \sum_{|u|=1} \langle u, Mu \rangle^{1/2}.$$

For vector functions in L^2 spaces, the basic inner product and norm are

$$(u, v) = \int_{\mathbb{R}} \langle u(x), v(x) \rangle \, dx, \quad ||u|| = (u, u)^{1/2}.$$

Let H^k $(k\geqslant 0)$ denote the usual Sobolev's spaces with the norm

$$(u,v)_{H^k}=\sum_{j=0}^{k}(D^ju,D^jv),\quad ||u||_k=(u,u)_{H^k}^{1/2}.$$

We note $L^2=H^0$ and simply set $||\cdot||=||\cdot||_0$.

2. Properties of relaxation shocks

Let $(u,v)(x,t)=(\phi,\psi)(x-st)$ be a traveling wave solution to (1.4) connecting two constant states $(u_\pm,v_\pm)=\Gamma(u_\pm)$.

From (1.8) we see that

$$s(u_+-u_-)=v_+-v_-$$

which yields

$$f(u_+)-f(u_+)-s(u_+-u_-)=0. \tag{2.1}$$

Fixing u_-, the local structure of the set of states u_+ satisfying (2.1) is well known, see [9,13]. In some neighborhood of u_-, there exists kth Hugoniot curve $u(\rho)=u(\rho,u_-)$ passing through u_- with corresponding shock speed $s(\rho)$.

If $\lambda_k(u)$ is genuinely nonlinear and $|u_--u_+|$ is small, Liu's strict entropy condition (1.9) is equivalent to Lax's shock condition

$$\lambda_k(u_+)<s(\rho_+)<\lambda_k(u_-).$$

These entropy conditions combined with condition (1.5) ensure the existence of the relaxation shock profiles. The existence proof for weak shocks, under the assumptions of the stability theorem, is a special case of the results of Yong and Zumbrun [27]. Due to the special structure of system (1.8) the existence and some properties of the relaxation shock profiles can be easily reduced to the viscous case.

In fact, substitution of $(\phi,\psi)(x-st)$ into system (1.8) leads to the following ODE system:

$$-s\phi_z+\psi_z=0, \tag{2.2}$$

$$-s\psi_z+a\phi_z=f(\phi)-\psi, \tag{2.3}$$

$$(\phi,\psi)(\pm\infty)=\Gamma(u_\pm), \tag{2.4}$$

where $z = x - st$. Upon integration of (2.2) once we obtain

$$-s\phi + \psi = q, \quad q = -su_{\pm} + f(u_{\pm}).$$

Thus system (2.2)–(2.3) can be rewritten as

$$\phi_z = (a - s^2)^{-1}(f(\phi) - \psi), \tag{2.5}$$

$$\psi = s(\phi - u_-) + f(u_-). \tag{2.6}$$

Restriction on the curve

$$S(u, v) := \{(u, v), v = s(u - u_-) + f(u_-)\}$$

shows that the ϕ-component is nothing but a viscous shock profile for the viscosity matrix $B = (a - s^2)I_{n\times n}$. Under the subcharacteristic condition (1.5) this matrix obeys the stability condition of Majda and Pego [18]. Thus the existence of ϕ-component of the relaxation shock profiles when $|u_- - u_+|$ is sufficiently small is immediate. Defining $\psi = s(\phi - u_-) + f(u_-)$ such that (ϕ, ψ) is on the relaxation wave curve $S(u, v)$. Then the (ϕ, ψ) are the desired relaxation shock profiles.

The existence and further properties of the relaxation shock profile are summarized in

Lemma 2.1. *Given an admissible shock wave to* (1.1) *as described above and* (1.5) *is assumed. The two-point boundary value problem* (2.2)–(2.4) *has a solution when* $\varepsilon = |u_+ - u_-|$ *is sufficiently small. Moreover, such wave profile satisfies for some constant* $C>0$

$$|\phi_z| \leqslant C\varepsilon \quad \textit{for all } z \in \mathbb{R}, \tag{2.7}$$

$$|(a - s^2)\phi_{zz} - \lambda_k(\phi)\phi_z| \leqslant C|\phi_z|^2, \tag{2.8}$$

$$|\phi_{zz}| \leqslant \varepsilon C|\phi_z|, \tag{2.9}$$

$$\int_{\mathbb{R}} |\phi_z|\, dz \leqslant 2\varepsilon, \tag{2.10}$$

$$\psi(z) = s\phi(z) + f(u_{\pm}) - su_{\pm}. \tag{2.11}$$

Proof. The properties of the relaxation shock wave profile with sub-characteristic speed can be obtained in the same manner as that in [3] for viscous shock profiles. The details are omitted. □

3. Reformulation of the problem

The proof of Theorem 1.1 is based on L^2 energy estimates. We first rewrite problem (1.8) using the moving coordinate $z = x - st$. The assumptions in Theorem 1.1 imply

$$\int_{\mathbb{R}} (u(x,t) - \phi(x - st))\, dx = \int_{\mathbb{R}} (u_0 - \phi)\, dx = 0,$$

thus we shall look for the solution of the form

$$(u, v)(x,t) = (\phi, \psi)(z) + (U_z, -W)(z,t), \quad z = x - st. \tag{3.1}$$

We substitute (3.1) into (1.8), by virtue of (2.2)–(2.4), and integrate the equation (2.2) once with respect to z, the perturbation (U, W) satisfies

$$U_t - sU_z - W = 0,$$

$$-W_t + sW_z + aU_{zz} = f(\phi + U_z) - f(\phi) + W.$$

Linearization of the above system about $(\phi, \psi)(z)$ yields a closed system for (W, U)

$$U_t - sU_z = W, \tag{3.2}$$

$$W_t - sW_z - aU_{zz} + U_t + Q'(\phi)U_z = -F(\phi, U_z), \tag{3.3}$$

where

$$F(\phi, U_z) = f(\phi + U_z) - f(\phi) - f'(\phi)U_z$$

satisfies $|F| \leqslant C|U_z|^2$ for small $|U_z|$ and $Q'(\phi) = f'(\phi) - s$.

The corresponding initial data for (3.2)–(3.3) becomes

$$U(z,0) = \int_{-\infty}^{z} (u(x,0) - \phi(x))\, dx = U_0(z), \quad W(z,0) = \psi(z) - V(z,0) = W_0(z). \tag{3.4}$$

Now, we introduce the solution space of problem (3.2)–(3.4) as follows:

$$X(0,T) = \{(U,W)(z,t) : (U,W) \in C^0(0,T;H^3) \times C^0(0,T;H^2),$$

$$(U_z, W) \in L^2(0,T;H^2)\} \quad \text{with } 0 < T \leqslant +\infty.$$

By the Sobolev embedding theorem, if we set

$$N(t) = \sup_{0 \leqslant \tau \leqslant t} \{||U(\tau)||_3^2 + ||W(\tau)||_2^2\}^{1/2},$$

then

$$\sup_{z\in\mathbb{R}} \{|U|, |U_z|, |U_{zz}|, |W|, |W_z|\} \leqslant CN(t).$$

Theorem 3.1 (Local existence). *Given any $\delta_0 > 0$, there exists a constant $T_0 > 0$ depending on δ_0, such that if $(U_0, W_0) \in H^3 \times H^2$, with $N(0) < \delta_0$, then problem* (3.2)–(3.4) *has a unique solution $(U, W) \in X(0, T_0)$ satisfying*

$$N(t) \leqslant 2N(0)$$

for any $0 \leqslant t \leqslant T_0$.

Theorem 3.2 (A priori estimate). *We assume the same conditions as those in Theorem* 1.1. *Let $(U, W) \in X(0, T)$ be a solution for a positive constant T, and $|u_- - u_+|$ be sufficiently small. Then there exist positive constants δ_1 and C_1 (both are independent of T) such that if $N(T) \leqslant \delta_1$, then it holds for $0 \leqslant t \leqslant T$*

$$||U(t)||_3^2 + ||W(t)||_2^2 + \int_0^t ||(U_z, W)(\tau)||_2^2 d\tau \leqslant C_1 N(0)^2. \tag{3.5}$$

The local existence can be proved via a classical iteration scheme and a short time energy estimate, see [17] for more details. A priori estimates will be established in Sections 4 and 5. Equipped with the above theorems, we can prove Theorem 1.1 in a same manner as that in [11,12]. To make sure we sketch the procedure as follows:

Proof of Theorem 1.1. We define $\beta_0 = \min\{\frac{\delta_1}{2}, \frac{\delta_1}{2\sqrt{C_1}}\}$, and $\delta_0 = \frac{\delta_1}{2}$. The assumption in Theorem 1.1 gives $N(0) \leqslant \beta_0 \leqslant \delta_0$. Then Theorem 3.1 with $\delta_0 = \delta_1/2$ ensures a unique solution (U, W) on $[0, T_0(\delta_0)]$ satisfying $N(T_0) \leqslant 2N(0) \leqslant 2\delta_0 = \delta_1$. So, Theorem 3.2 with $T = T_0$ implies $N(T_0) \leqslant \sqrt{C_1} N(0) \leqslant \sqrt{C_1}\beta_0 \leqslant \delta_1/2$. Hence Theorem 3.1 with initial data $(U, W)(T_0)$ and $\delta_0 = \delta_1/2$ again gives a unique solution on $[T_0, 2T_0]$ satisfying $N(2T_0) \leqslant 2N(T_0) \leqslant \delta_1$. Using Theorem 3.2 on $[0, 2T_0]$ gives $N(2T_0) \leqslant \sqrt{C_1} N(0) \leqslant \delta_1/2$. Thus, repeating this continuation process, we can extend Theorem 3.1 to $T_0 = \infty$. Therefore a priori estimate (3.5) holds true for $T = \infty$. Finally, thanks to the uniform estimate (3.5) and Eq. (3.2)–(3.3), we can see that both $||U_z(t)||^2 + ||W(t)||^2$ and its time-derivatives are integrable over $t \geqslant 0$. This implies $||U_z(t)|| + ||W(t)|| \to 0$ as $t \to \infty$. Hence by Sobolev's inequality and (3.5) one has

$$\begin{aligned}
&\lim_{t\to\infty} \sup_z [|u(x,t) - \phi(x - st)|^2 + |v - \psi(x - st)|^2] \\
&\quad = \lim_{t\to\infty} \sup_z (|U_z|^2 + |W|^2) \leqslant \lim_{t\to\infty} 2[||(U_z|| \cdot ||U_{zz}) + ||W(t)|| \cdot ||W_z(t)||] \\
&\quad \leqslant 2\sqrt{C_1} N(0) \lim_{t\to\infty} [||||U_z(t)|| + ||W(t)||] = 0.
\end{aligned}$$

This completes the proof. □

4. Energy estimates

This section contains the basic energy estimate. Now we proceed into several steps.

4.1. Diagonalization

It is expected that the large-time dynamics will be determined by $U_t + Q'(\phi)U_z = 0$. To make full use of this underlying hyperbolic properties, we first diagonalize the matrix $Q'(\phi)$ by taking $L(z),\ R(z) : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $LR = I$ and $\Lambda = LQ'(\phi)R = \mathrm{diag}(\lambda_j - s)$. Following Goodman [4], we choose $l_i(z)$ and $r_j(z)$ such that they are determined by solving the following ODEs:

$$\begin{aligned}
(l_i)_z r_j &= \frac{1}{\lambda_i - \lambda_j} l_i(Q'(\phi))_z r_j, \quad i \neq j,\\
l_i(r_j)_z &= -(l_i)_z r_j, \quad i \neq j,\\
(l_k)_z r_k &= -l_k(r_k)_z = 0, \quad k = 1, \dots, n
\end{aligned} \tag{4.1}$$

with $L(0) = L(\phi(0))$ and $R(0) = R(\phi(0))$. From now on we will use this diagonalization.

Substituting $U = RV$ into (3.3) and multiplying by L with $M := LR_z$ and notations

$$V = LU, \quad P := V_t - sV_z = L(W - sMU) \quad \text{and } \mu := a - s^2, \tag{4.2}$$

one obtains

$$\begin{aligned}
&P_t - sP_z - aV_{zz} + V_t + \Lambda V_z + \Lambda MV - \mu[(MV)_z + MV_z + M^2 V_z]\\
&\quad - 2sMV_t - LF(\phi, (RV)_z) = 0.
\end{aligned} \tag{4.3}$$

The rest of this section is devoted to establishing the following key estimate:

Theorem 4.1. *Under the assumptions of Theorem* 3.2, *it holds that*

$$\|(V, P, V_z)(t)\|^2 + \int_0^t \|(P, V_z)(\tau)\|^2\, d\tau + \int_0^t (V, |\phi_z| V)\, d\tau \leqslant C\|(V, P, V_z)(0)\|^2 \tag{4.4}$$

for $t \in [0, T]$.

Here and in the sequel we use "C" to denote a generic constant, independent of x and t. C may still depend on general data of the problem ($f, u_\pm, a$, etc.).

4.2. Weighted multipliers

We begin with the construction of energy functions by suitable multipliers.

Let $w = \mathrm{diag}(w_1, \ldots, w_n)$ be a weight function yet to be determined and

$$\Lambda_0 := \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Multiplying (4.3) by $V^T w$, and integrating $\int dz$, we obtain

$$\begin{aligned}&(wV, P_t - sP_z - aV_{zz} + V_t + \Lambda V_z + \Lambda MV)\\&\quad + (wV, -\mu[(MV)_z + MV_z + M^2V_z] - 2sMV_t - LF) = 0.\end{aligned}$$

Using $P = V_t - sV_z$ and integration by parts, we have

$$\begin{aligned}(wV, P_t - sP_z) &= (wV, P)_t - (wV_t, P) + s(w_zV + wV_z, P)\\&= (V, wP)_t - (P, wP) + (P, sw_zV)\\&= \frac{d}{dt}\left[(V, wP) + \frac{1}{2}(V, sw_zV)\right]_t - (P, wP) + \frac{s^2}{2}(V, w_{zz}V)\end{aligned}$$

and

$$(wV, -aV_{zz}) = a(V_z, wV_z) - \frac{a}{2}(V, w_{zz}V),$$

$$(wV, V_t + \Lambda V_z + \Lambda MV) = \tfrac{1}{2}(V, wV)_t - \tfrac{1}{2}(V, (w\Lambda)_zV) + (V, w\Lambda MV).$$

A combination of above identities yields

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}(V, wV) + (V, wP) + \frac{1}{2}(V, sw_zV)\right] - (P, wP) + a(V_z, wV_z) + (V, A(z)V)\\&\quad - \mu(V, w_zMV) + (wV, -\mu[(MV)_z + MV_z + M^2V_z] - 2sMV_t - LF) = 0, \qquad (4.5)\end{aligned}$$

where

$$A(z) = -\frac{\mu}{2}w_{zz} - \frac{1}{2}(w\Lambda)_z + (\mu w_z + w\Lambda)M. \qquad (4.6)$$

To make (4.5) useful we further multiply $2P^T w$ and integrate $\int dz$ by parts, after a suitable grouping of terms, we arrive at

$$\begin{aligned}&\frac{d}{dt}[(P, wP) + a(V_z, wV_z)] + 2\left(P, \left(w + \frac{s}{2}w_z\right)P\right) + as(V_z, w_zV_z) + 2(P, \Lambda_0 wV_z)\\&\quad + 2a(P, w_zV_z) + 2(P, w\Lambda MV) + 2(wV, -\mu[(MV)_z + MV_z + M^2V_z]\\&\quad - 2sMV_t - LF) = 0. \qquad (4.7)\end{aligned}$$

Now we define the following energy functions:

$$E_1(t) = \tfrac{1}{2}(V, wV) + (V, wP) + \tfrac{1}{2}(V, sw_z V) + (P, wP) + a(V_z, wV_z),$$

$$E_2(t) = (P, (w + sw_z)P) + 2(P, (\Lambda_0 w + aw_z)V_z) + a(V_z, wV_z),$$

$$G_1(t) = (V, A(z)V).$$

Then a combination of (4.5) and (4.7) gives

$$\frac{d}{dt}E_1(t) + E_2(t) + G_1(t) + G_2(t) = 0, \tag{4.8}$$

where

$$G_2(t) = 2(P, w\Lambda MV) - 2a(V + 2P, wMV_z) - 2\mu(V + 2P, w(M_z V + M^2 V)) - \mu(V, (wM)_z V + M^2 V) - 2s(V + 2P, wMP) - (V + 2P, wLF).$$

Here the function $E_1(t)$ is called the *relaxation energy* which is bounded by the initial one. $E_2(t)$ will be positive under the relaxation stability condition (1.5), the dissipative energy of this kind makes the small perturbations decay to zero. The key is to carefully choose weight matrix w such that $G_1(t)$ is also positive. This term reflects the contribution from each wave modes. The interaction of waves from various modes is contained in $G_2(t)$. The terms in G_2 can be estimated by using the properties of ϕ and the weight w, and will be majored by the $E_2(t) + G_1(t)$ for weak shocks.

4.3. Weight matrix

Now we choose w such that there exists a constant $c_0 > 0$

$$G_1(t) \geqslant c_0(V, |\phi_z| V). \tag{4.9}$$

For this purpose one needs to separate the principle wave mode from the other wave modes.

For $A(z) = (a_{ij})(z)$, we write $G_1(t)$ as

$$G_1(t) = a_{kk} V_k^2 + \sum_{i \neq k} a_{ii} V_i^2 + \sum_{i \neq j} a_{ij} V_i V_j.$$

First

$$a_{kk} = -\tfrac{1}{2}[\mu w_{kz} + (\lambda_k - s) w_k]_z.$$

If λ_k is monotone (i.e., the kth field is genuinely nonlinear), then the principle wave is compressive in the sense that $\lambda_{kz} < 0$. In this case we simply take $w_k = 1$ and find that

$$a_{kk} = -\tfrac{1}{2}\lambda_{kz} > 0.$$

For the shocks associated with possibly nonconvex mode, a weight introduced in [3] can be used and written as

$$w_k = \frac{2}{a - s^2}\int_z^{+\infty} e^{\int_z^y \frac{\lambda_k(\xi)-s}{a-s^2}\,d\xi}\left[\int_{x_0}^y |\phi_z|\,d\xi\right]dy \tag{4.10}$$

for suitable x_0 which is chosen to fix a location on the shock curve. Moreover, this weight satisfies

$$|w_k(\lambda_k - s) + \mu w_{kz}| \leqslant 4\varepsilon, \tag{4.11}$$

$$|\mu(w_k\phi_z)_z| \leqslant 4\varepsilon|\phi_z|, \tag{4.12}$$

$$|\mu w_k \phi_z| \leqslant 8\varepsilon^2, \tag{4.13}$$

$$|a_{kk}| = |\phi_z| \tag{4.14}$$

for $\varepsilon = |u_+ - u_-|$ sufficiently small, see [3]. For the other modes $i \neq k$, we have

$$a_{ii} = -\tfrac{1}{2}[\mu w_{izz} + (\lambda_i - s)w_{iz}].$$

In order to control the corresponding terms we choose the weight

$$w_i(z) = |\lambda_i - s|^{-1} e^{-\int_{-\infty}^z \frac{C_i|\phi_z|}{\lambda_i(\xi)-s}\,d\xi}, \quad i \neq k$$

with $C_i > 0$ being constants to be determined.

Lemma 4.2. *For $\varepsilon = |u_+ - u_-|$ sufficiently small and given $c_1 > 0$, there exists $C_i > 0$ such that*

$$-\tfrac{1}{2}(\mu w_{iz} + (\lambda_i - s)w_i)_z \geqslant c_1|\phi_z|w_i,$$

$$|\mu w_{iz} + w_i(\lambda_i - s)| \leqslant C w_i(1 + C_i|\phi_z|).$$

Proof. Writing $\lambda_i - s$ as λ_i for notational convenience. For $i \neq k$ we compute

$$-(\lambda_i w_i)_z = -\mathrm{sgn}(\lambda_i)(w_i|\lambda_i|)_z = e^{-\int_{-\infty}^z \frac{C_i|\phi_z|}{\lambda_i}} \cdot \frac{C_i|\phi_z|}{\lambda_i} = C_i|\phi_z|w_i(z). \tag{4.15}$$

Also

$$w_{iz}(z) = -\lambda_i^{-1}(C_i|\phi_z| + \lambda_{iz})w_i$$

which when combined with the fact $|\lambda_{iz}| \leqslant \sup|\nabla\lambda_i||\phi_z| \leqslant C|\phi_z|$ yields

$$|w_{iz}| \leqslant C(1 + C_i)w_i|\phi_z|.$$

Further one computes

$$w_{izz} = -(\lambda_i)^{-1}[C_i|\phi_z|_z + \lambda_{izz}]w_i - (\lambda_i)^{-1}[C_i|\phi_z| + 2\lambda_{iz}]w_{iz}.$$

Using the above estimate on w_{iz} and the following facts

$$|\phi_z|_z = \langle \phi_z, \phi_{zz} \rangle / |\phi_z| \leqslant |\phi_{zz}| \leqslant C\varepsilon|\phi_z|$$

and

$$\lambda_{izz} \leqslant \varepsilon C|\phi_z|,$$

one gets

$$|w_{izz}| \leqslant |\phi_z|w_i[(1 + C_i)\varepsilon C + (1 + C_i)^2 C|\phi_z|].$$

This combined with (4.15) leads to

$$-\mu w_{izz} - (\lambda_i w_i)_z \geqslant |\phi_z|w_i[C_i - (1 + C_i)\varepsilon C - (1 + C_i)^2 C|\phi_z|] \geqslant c_1|\phi_z|w_i$$

for given c_1 we can always choose a suitable C_i such that the above holds if ε is sufficiently small. Lemma 4.1 is thus proved. □

Using this lemma and choosing C_i appropriately, we bound the $G_1(t)$ from below. Noting that for $i \neq k$

$$|a_{ij}| = |(\mu w_{iz} + \lambda_i w_i)m_{ij}| \leqslant Cw_i(1 + C_i)|\phi_z|^2 + Cw_i|\phi_z|$$

and

$$|a_{kj}| = |(\mu w_{kz} + \lambda_k w_k)m_{kj}| \leqslant \varepsilon C|\phi_z|, \quad j \neq k,$$

thus we have

$$
\begin{aligned}
I &= \sum_{i \neq j} a_{ij} V_i V_j \\
&\geqslant -\sum_{i \neq k} C w_i (1 + C_i)^2 |\phi_z|^2 |V_i|^2 - \frac{1}{8} |\phi_z| |V_k|^2 \\
&\quad - \sum_{i=1}^{n} \varepsilon C |\phi_z| |V_i|^2 - \sum_{i \neq k} C w_i |\phi_z| |V_i|^2.
\end{aligned}
$$

A combination of the above estimates leads to

$$
G_1(t) = a_{kk} V_k^2 + \sum_{i \neq k} a_{ii} V_i^2 + \sum_{i \neq j} a_{ij} V_i V_j \tag{4.16}
$$

$$
\begin{aligned}
&\geqslant \left[1 - \frac{1}{8} - \varepsilon C\right] |\phi_z| |V_k|^2 + \sum_{i \neq k} [C_i - (1 + C_i)\varepsilon C - C(1 + C_i)^2 |\phi_z| \\
&\quad - C(1 + C_i)^2 |\phi_z|^2 - C] w_i |\phi_z| V_i^2 \\
&\geqslant \frac{1}{8} |\phi_z| |V_k|^2 + \frac{1}{2} \sum_{i \neq k} w_i |\phi_z| |V_i|^2 \geqslant c_0 \langle V, |\phi_z| V \rangle.
\end{aligned} \tag{4.17}
$$

Here we have taken suitably large C_i and then sufficiently small ε to recover the desired estimate claimed in (4.9).

4.4. *Dissipative energy*

Before continuing the argument, it is necessary to show the positivity of $E_2(t)$ under stability condition (1.5).

Lemma 4.3. *Assume* ε *is sufficiently small and* (1.5) *holds, then we have*

$$
E_2(t) \geqslant \tfrac{1}{2}[(P, wP) + a(V_z, wV_z)].
$$

Proof. Rewrite $E_2(t)$ as

$$
\begin{aligned}
E_2(t) = &\sum_{i \neq k} [(w_i + s w_{iz}) P_i^2 + 2(\lambda_i w_i + \mu w_{iz}) P_i (V_i)_z + a w_i (V_{iz})^2] \\
&+ [(w_k + s w_{kz}) P_k^2 + 2(\lambda_k w_k + \mu w_{kz}) P_k (V_k)_z + a w_k (V_{kz})^2].
\end{aligned}
$$

By the choice of w_i we find that

$$
|\alpha_i| := |w_i'/w_i| \leqslant C|\phi_z| \leqslant C\varepsilon
$$

is small. Thus by the stability condition (1.5) the first part in $E_2(t)$ is estimated as

$$\sum_{i\neq k} w_i(P_i, V_{iz})\begin{pmatrix} 1+s\alpha_i & \lambda_i+\mu\alpha_i \\ \lambda_i+\mu\alpha_i & a \end{pmatrix}\begin{pmatrix} P_i \\ V_{iz} \end{pmatrix} \geqslant \frac{1}{2}\sum_{i\neq k}[P_i^2 + a|V_{iz}|^2].$$

For $i = k$ one has

$$|\mu w_{kz}| = \left| -2\int_{x_0}^{z} |\phi_z|\, dz - \lambda_k w_k \right| \leqslant 4\varepsilon + C\varepsilon w_k,$$

also

$$w_k + s w_{kz} \geqslant \tfrac{1}{2} w_k$$

for ε sufficiently small. A combination of these with the fact

$$|\mu w_{kz} + \lambda_k w_k| \leqslant 4\varepsilon$$

leads to

$$[(w_k + s w_{kz})P_k^2 + 2(\lambda_k w_k + \mu w_{kz})P_k(V_k)_z + a w_k (V_{kz})^2] \geqslant \tfrac{1}{2} w_k [P_k^2 + a V_{kz}^2].$$

This proves Lemma 4.3. □

Remark 1. For the scalar law, the weakness of shocks implies the subcharacteristic condition as shown here for the principle wave mode $i = k$. But for system case the stability condition (1.5) is essentially needed for controlling the other wave modes.

4.5. Interaction estimates for $G_2(t)$

The next step in the argument is to estimate the terms grouped in $G_2(t)$ such that it can be majored by

$$\tfrac{1}{2}[(P, WP) + a(V_z, WV_z) + c_0(V, |\phi_z|V)].$$

Before going further, we collect some estimates which will be used repeatedly.

For

$$M = LR_z = (m_{ij})_{n\times n} \quad \text{and} \quad w = \text{diag}(w_1, \dots, w_n),$$

- $m_{ii} = 0,\ i = 1, \dots, n,\ |m_{ij}| \leqslant C|\phi_z|$,
- $\frac{1}{C} \leqslant w_i \leqslant C$ for $C > 1,\ i = 1, \dots, n$,
- $|w_i m_{ij}| \leqslant C\varepsilon$,
- $|(w_i m_{ij})_z| \leqslant \varepsilon C|\phi_z|$,
- $|w_i m_{ij}^2| \leqslant C\varepsilon$ for $i \neq j$.

These estimates can be checked through the choice of L in (4.2) and of weight matrix w defined in Section 4.3.

Equipped with these basic facts, we turn to the estimate of G_2 (still denoting $\lambda_i - s$ by λ_i henceforth).

$$\begin{aligned}\langle 2P, w\Lambda MV\rangle &= \sum_{j\neq k} 2P_k w_k \lambda_k m_{kj} V_j + 2\sum_{i\neq j, i\neq k} P_i w_i \lambda_i m_{ij} V_j\\ &\geqslant -\sum_{j\neq k}\left[\frac{w_k}{8(n+1)} P_k^2 + Cw_k|\lambda_k|^2|\phi_z|^2 V_j^2\right]\\ &\quad -\sum_{i\neq j, i\neq k}\left[\frac{w_i}{8(n+1)} P_i^2 + Cw_i|\phi_z|^2|V_j|^2\right]\\ &\geqslant -\frac{1}{8}\langle P, wP\rangle - \varepsilon C\langle V, |\phi_z| V\rangle.\end{aligned}$$

The second term is

$$-2a(V+2P, wMV_z) = -2a(V, wMV_z) - 4a(P, wMV_z),$$

where

$$\begin{aligned}&-2a(V, wMV_z)\\ &= -2a\int_{\mathbb{R}}\left[\sum_{i\neq j, i\neq k} w_i m_{ij} V_i V_{jz} + \sum_{j\neq k} w_k m_{kj} V_k V_{jz}\right] dz\\ &= -2a\int_{\mathbb{R}}\left[\sum_{i\neq j, i\neq k} w_i m_{ij} V_i V_{jz} - \sum_{j\neq k} (w_k m_{kj})_z V_k V_j - \sum_{j\neq k} w_k m_{kj} V_{kz} V_j\right] dz\\ &\geqslant -\int_{\mathbb{R}}\sum_{i\neq j}\left\{\frac{aw_j}{8(n+1)}|V_{jz}|^2 + C\frac{w_i^2}{w_j} m_{ij}^2 V_i^2\right\}\\ &\quad + \sum_{j\neq k}\left\{\varepsilon C|\phi_z|(V_k^2 + V_j^2) + \frac{a}{8(n+1)} w_k V_{kz}^2 + Cw_k m_{kj}^2 V_j^2\right\} dz\\ &\geqslant -\frac{a}{4}(V_z, wV_z) - \varepsilon C(V, |\phi_z| V),\end{aligned}$$

where we have used $|w_k\phi_z| \leqslant C\varepsilon$ and $|(w_k m_{kj})_z| \leqslant \varepsilon C|\phi_z|$. Further using $C^{-1} \leqslant w_i/w_j \leqslant C$ we get

$$\begin{aligned}-4a(P, wMV_z) &= -4a\sum_{i\neq j} w_i m_{ij} P_i V_{jz}\\ &\geqslant -4a\sum_{i\neq j} |P_i\sqrt{w_i}|C|\phi_z||\sqrt{w_i}/\sqrt{w_j}|V_{jz}|\sqrt{w_j}\\ &\geqslant -C\varepsilon[(P, wP) + a(V_z, wV_z)].\end{aligned}$$

Setting $\tilde{M} := M_z + M^2$, by the properties of $R(z)$, one can show that

$$|\tilde{m}_{ij}| \leqslant C|\phi_z|^2.$$

This combined with $|w_i m_{ij}| \leqslant C\varepsilon$ yields

$$-2\mu(P, w(M_z + M^2)V) = -2\mu(P, w\tilde{M}V) \geqslant -C\varepsilon[(P, wP) + (V, |\phi_z|V)].$$

The next term is

$$\begin{aligned} -\mu(V, (wM)_z V + wM^2 V) &\geqslant -\mu \sum_{i,j} |(w_i m_{ij})_z V_i V_j| - \mu \sum_{ij} C w_i |\phi_z|^2 |V_i||V_j| \\ &\geqslant -C\varepsilon(V, |\phi_z|V). \end{aligned}$$

Using the bounds $C^{-1} \leqslant w_i \leqslant C$, one obtains by choosing δ suitably small

$$\begin{aligned} -2s(V, wMP) - 4s(P, wMP) &\geqslant -2|s| \sum_{i \neq j} |w_i m_{ij} V_i P_j| - 4|s| \sum_{i \neq j} w_i |m_{ij}||P_i||P_j| \\ &\geqslant -4|s| \sum_{i \neq j} [\delta w_j P_j^2 + \delta^{-1}(w_i m_{ij})^2 / w_j V_i^2 \\ &\quad + |\phi_z| w_i^2 P_i^2 + |\phi_z| \frac{w_i}{w_j} P_j^2 w_j^2] \\ &\geqslant -(1/16 + C\varepsilon)(P, wP) - C\varepsilon(V, |\phi_z|V). \end{aligned}$$

Finally by the definition of F we find that

$$|F| \leqslant C(|V_z|^2 + |\phi_z||V|^2)$$

and thus we have

$$\begin{aligned} |-(V + 2P, wLF)| &\leqslant C \sup_{z \in \mathbb{R}} |(V + 2P)| \int_{\mathbb{R}} w(|V_z|^2 + |\phi_z||V|^2)\, dz \\ &\leqslant CN(t)(V, |\phi_z|V) + CN(t)(V_z, wV_z). \end{aligned}$$

Collecting all the estimates above and the estimates for $E_2 + G_1$, we get

$$\begin{aligned} &\frac{d}{dt} E_1(t) + \left[\frac{1}{2} - \frac{1}{8} - C\varepsilon - \frac{1}{16}\right](P, wP) + \left[\frac{a}{2} - \frac{a}{4} - C\varepsilon - CN(t)\right](V_z, wV_z) \\ &\quad + [c_0 - \varepsilon C - CN(t)](V, |\phi_z|V) \leqslant 0. \end{aligned}$$

Now we choose ε and $N(t)$ sufficiently small such that

$$\frac{d}{dt} E_1(t) + \frac{1}{4}[(P, wP) + a(V_z, wV_z) + c_0(V, |\phi_z|V)] \leqslant 0.$$

Integration over t yields

$$||(V,P,V_z)(t)||^2+\int_0^t ||(P,V_z,\sqrt{|\phi_z|}V)(\tau)||^2 d\tau \leqslant C||(V,P,V_z)(0)||^2. \tag{4.18}$$

This concludes the proof of Theorem 4.1.

5. Higher estimates

Now we conclude the proof of Theorem 3.2 by providing the necessary estimates on the derivatives of (P,V). Having the basic L^2 energy estimate, Theorem 4.1, at hand, one can establish the higher estimates quite easily as we show below.

Applying D^l to (4.3) for $1\leqslant l\leqslant 2$ with new notations $V_l=D^lV$ and $P_l=D^lP$ we have

$$\begin{aligned}(P_l)_t - s(P_l)_z - a(V_l)_{zz} + (V_l)_t + \Lambda(V_l)_z + D^l(\Lambda V_z) - \Lambda(D^lV)_z + D^l(\Lambda MV)\\ - \mu D^l[(MV)_z + MV_z + M^2V_z] - 2sD^l(MV_t) - D^l(LF(\phi,U_z)) = 0.\end{aligned} \tag{5.1}$$

Multiplying on the left by $(V_l+2P_l)^T$, and integrating over $\mathbb{R}$, one obtains

$$\frac{d}{dt}E_{1l}+E_{2l}+Q_l=0, \tag{5.2}$$

where for $\Lambda_0=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$

$$E_{1l}(t)=\tfrac{1}{2}(V_l,V_l)+(V_l,P_l)+(P_l,P_l)+a(V_{l+1},V_{l+1}),$$

$$E_{2l}(t)=(P_l,P_l)+2(P_l,\Lambda_0V_{l+1})+a(V_{l+1},V_{l+1}),$$

$$\begin{aligned}Q_l(t)=(V_l+2P_l, D^l(\Lambda MV)+D^l(\Lambda V_z)-\Lambda(D^lV)_z\\ -\mu D^l[(MV)_z+MV_z+M^2V]-2sD^l(MV_t)-D^l(LF)).\end{aligned}$$

Using the stability condition (1.5) we find that both E_{1l} and E_{2l} are positive. It remains to estimate Q_l for $1\leqslant l\leqslant 2$ by utilizing the following Morser-type inequalities (see, for example [17, Proposition 2.1, p. 43])

(1) For $h,g\in H^s\cap L^\infty$ and $|l|\leqslant s$

$$||D^l(hg)||\leqslant C_l(|Dh|_{L^\infty}||D^sg||+||g||_\infty||D^sh||).$$

(2) For $h\in H^s, Dh\in L^\infty, g\in H^{s-1}\cap L^\infty$ and $|l|\leqslant s$

$$||D^l(hg)-hD^l(g)||\leqslant C_s(|Dh|_\infty||D^{s-1}g||+|g|_{L^\infty}||D^sh||).$$

Now we estimate $\sum_{1\leqslant l\leqslant 2}|Q_l|$ term by term. The first one is

$$\begin{aligned}
&\sum_{1\leqslant l\leqslant 2}\left(V_l+2P_l,\quad \sum_{k=0}^{l} C_l^k D^k(\Lambda M)D^{l-k}V\right)\\
&\quad\leqslant \sum_{1\leqslant l\leqslant 2}2^l\sum_{k=0}^{l}||D^k(\Lambda M)||_{L^\infty}|(V_l+2P_l,V_{l-k})|\\
&\quad\leqslant \delta(V,|\phi_z|V)+C\varepsilon[||(P_z,V_z)||_1^2]
\end{aligned}$$

with $\delta>0$ suitably small. Further, since $||D^l\Lambda||\leqslant C\varepsilon$ for $l\geqslant 1$, we have

$$\begin{aligned}
&\left|\sum_{1\leqslant l\leqslant 2}(V_l+2P_l,D^l(\Lambda V_z)-\Lambda(D^lV)_z)\right|\\
&\quad\leqslant \sum_{1\leqslant l\leqslant 2}||V_l+2P_l||\,||D^l(\Lambda V_z)-\Lambda(D^lV)_z||\\
&\quad\leqslant \sum_{1\leqslant l\leqslant 2}C[||V_l||+||P_l||](|D\Lambda|_{L^\infty}||D^lV||+||V_z||_{L^\infty}||D^l\Lambda||)\\
&\quad\leqslant C(\varepsilon+N(t))[||(P_z,V_z)||_1^2].
\end{aligned}$$

Further we have for suitably small δ

$$\begin{aligned}
&\sum_{1\leqslant l\leqslant 2}(V_l+2P_l,-\mu D^l[(MV)_z+MV_z+M^2V])\\
&\quad=\sum_{1\leqslant l\leqslant 2}\Bigg(V_l+2P_l,-\mu\sum_{k=0}^{l+1}C_{l+1}^kD^kMV_{l+1-k}-\mu\sum_{k=0}^{l}C_l^kD^kMV_{l+1-k}\\
&\qquad-\mu\sum_{k=0}^{l}C_l^kD^k(M^2)V_{l-k}\Bigg)\\
&\quad\leqslant\sum_{1\leqslant l\leqslant 2}2^{l+1}||V_l+2P_l||\Bigg(\sum_{k=0}^{l+1}||D^kM||_\infty||V_{l+1-k}||\\
&\qquad+\mu\sum_{k=0}^{l}|D^kM|_\infty||V_{l+1-k}||+\mu\sum_{k=0}^{l}|D^kM^2|_\infty||V_{l-k}||\Bigg)\\
&\quad\leqslant C\varepsilon\sum_{1\leqslant l\leqslant 2}\left[||V_l||^2+\sum_{k=0}^{l}||V_{l+1-k}||^2+||P_l||^2\right]+\delta(V,|\phi_z|V)\\
&\quad\leqslant C\varepsilon||(P_z,V_z)||^2+\delta(V,|\phi_z|V).
\end{aligned}$$

Similarly, one gets

$$\sum_{1\leqslant l\leqslant 2}(V_l+2P_l,-2sD^l(MV_z))$$
$$=\sum_{1\leqslant l\leqslant 2}(V_l+2P_l,-2sD^l(MP)-2s^2D^l(MV_z))$$
$$\leqslant C\varepsilon[||V_z||_2^2+||P_z||_1^2]+\delta(V,|\phi_z|V).$$

Since

$$LF\leqslant C(|V_z|^2+|\phi_z||V|^2),$$

$$|D^l(LF)|=|\sum_{k=0}^{l}C_l^kD^kLD^{l-k}F|\leqslant C[|V_z|^2+|\phi_z||V|^2+|V_{zz}|^2+|(V_z,V_{zzz})|].$$

Thus

$$\left|\sum_{1\leqslant l\leqslant 2}(V_l+2P_l,-D^l(LF)\right|\leqslant CN(t)||(V_z,P)||_2^2.$$

A combination of above estimates gives

$$\sum_{1\leqslant l\leqslant 2}|G_l|\leqslant C\varepsilon[||(P_z,V_z)||_1^2]+\delta(V,|\phi_z|V)+N(t)||(V_z,P)||_2^2.$$

Noting that

$$\sum_{1\leqslant l\leqslant 2}E_{1l}(t)\sim||(V,P,V_z)_z||_1^2,$$
$$\sum_{1\leqslant l\leqslant 2}E_{2l}(t)\geqslant\frac{1}{4}||(P,V_z)_z||_1^2.$$

Recalling the basic estimate in Theorem 4.1, we obtain

$$||(V,P,V_z)(t)||_2^2+(1-C\varepsilon-CN(t))\int_0^t||(P,V_z)(\tau)||_2^2d\tau+(c_0-\delta)\int_0^t(V,|\phi_z|V)\,d\tau$$
$$\leqslant C||(V,P,V_z)(0)||_2^2.$$

Taking ε, $N(t)$ and δ sufficiently small such that the coefficients $(\cdots)$ above are all positive and then returning to the original variables (U,W), we prove Theorem 3.2 immediately.

Added in proof: In the years since this paper was first submitted (March 15, 1999, see [10] for its preprint), I have learned of several new results on stability of relaxation shocks for Jin-Xin system (1.8), consult [20] and references therein for the new development along this line.

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