ANALYSIS OF DIRECT DISCONTINUOUS GALERKIN METHODS FOR MULTI-DIMENSIONAL CONVECTION-DIFFUSION EQUATIONS

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Abstract. We provide a framework for the analysis of the direct discontinuous Galerkin (DDG) methods for multi-dimensional convection-diffusion equations subject to various boundary conditions. A key tool is the global projection constructed by the DDG scheme applied to an associated elliptic problem. Such projection is well-defined for a class of diffusive flux parameters, and the optimal projection error in L^2 is obtained with an arbitrary locally regular partition of the domain and for an arbitrary degree of polynomials. This results in the optimal L^2 error for the DDG method to the elliptic problem, and further leading to the optimal L^2 error for the DDG method to the convection-diffusion problem.

1. Introduction

In this article, we present a unified analysis of the direct discontinuous Galerkin (DDG) method for multi-dimensional convection-diffusion problems, as an extension of [33]. To demonstrate the main ideas, we will focus on the model equation

$$\partial_t U + \nabla \cdot f(U) = \Delta U \tag{1}$$

with nonlinear smooth vector flux $f(U)$ given, posed on a convex, bounded domain in $\mathbb{R}^d (d > 1)$, subject to different types of boundary conditions.

The DDG method was introduced in [37] for diffusion, refined with interface corrections in [38], and has since been extended to multi-dimensional settings as well as equations with nonlinear diffusion, for which extensive numerical tests have shown the optimal $(k+1)$th order of accuracy for polynomial elements of degree k. However, the optimal L^2 error estimate for multi-dimensional arbitrary grids has not been available. For one-dimensional uniform grids, the optimal L^2 error $O(h^{k+1})$ was obtained in [33] by using an explicitly constructed global projection as well as its approximation property. The extension to multi-dimensional Cartesian grids was made possible when the approximation space is chosen as tensor products of piecewise polynomials. In this work we consider arbitrary shape-regular grids in any dimensional space with standard approximation spaces of piecewise polynomials. One main tool is the global projection defined by the same DDG method for an associated elliptic problem. The projection error gives the optimal error for the DDG method to both the elliptic and parabolic problems. This methodology may well be applied to other DG methods as long as some structure conditions are met.

The discontinuous Galerkin (DG) method we discuss in this paper is a class of finite element methods, using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. One main advantage of the DG method was the flexibility afforded by local approximation spaces combined with the suitable design of numerical fluxes crossing cell interfaces. It was first designed by Reed and Hill [45] and has been quite successful for solving first order partial differential equations (PDEs) such as hyperbolic conservation laws

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[21, 17, 15, 24]. However, the application of the DG method to diffusion problems has been known to be a challenging task because of the subtle difficulty in selecting appropriate numerical fluxes for the solution gradient. There have been several DG methods suggested in the literature to solve the problem, including the method originally proposed by Bassi and Rebay [5] for compressible Navier-Stokes equations, its generalization called the local discontinuous Galerkin (LDG) method introduced in [22] by Cockburn and Shu and further studied in [10, 14, 16, 9]; its improved version called the compact DG (CDG) method [44], as well as the method introduced by Baumann-Oden [6, 41]. See also the earlier works [2, 4, 50, 1] on so called *interior penalty* (IP) methods, and [3] for the unified analysis of DG methods for elliptic problems, as well as a comparison of the performance of various DG schemes from a practical perspective [8]. More general information about DG methods for elliptic, parabolic, and hyperbolic PDEs can be found in the recent books and lecture notes, see, e.g., [28, 47, 49].

The idea of DDG methods for higher order PDEs, such as the convection diffusion equation (1), is to directly force the weak solution formulation of the PDE into the DG function space for both the numerical solution and test functions. Unlike the LDG method, the DDG method does not introduce any auxiliary variables or rewrite the original equation into a larger first order system. The main novelty in the DDG schemes proposed in [37, 38] lies in numerical flux choices for the solution gradient, which involve higher order derivatives evaluated crossing cell interfaces, motivated by a trace formula for the derivatives of the solution to the heat equation [37]. With this choice, the obtained schemes are provably stable and optimally convergent as well as superconvergent for $\beta_1 \neq 0$ [33, 19] (β_1 is the involved method parameter). The DDG method has the advantage that it is compact in the sense that only the degrees of freedom belonging to neighboring elements are connected in the discretization, meaning lower storage requirements and higher computational performance than non-compact schemes. However, for arbitrary grids one needs to be careful in choosing the flux parameters since they can depend on both the mesh and the approximation order. Nevertheless, the DDG method has been successfully extended to various application problems, including linear and nonlinear Poisson equations [27, 54], the Schrödinger equation [30], several Fokker-Planck type equations [39, 40, 34, 35], and the three dimensional compressible Navier-Stokes equation [25]. We note that in the literature there are other works also featuring the direct DG discretization, such as those by Van Leer and Nomura in [31], Gassner et al. in [26], and Cheng and Shu in [23].

Obtaining error estimates for various DG methods has been a main subject of research. Here we mention only some results on the *a priori* error estimates for several DG methods when applied to convection-diffusion equations. For smooth solutions of scalar hyperbolic equations, the L^2 error estimate of $O(h^{k+1/2})$ can be obtained for the most general situation [29, 43], where h is the mesh size, and k the degree of polynomial approximations. However in many cases the optimal $O(h^{k+1})$ error bound can be proved [32, 46, 12]. The first error estimate of order $O(h^k)$ in the energy norm for the LDG method to linear convection-diffusion equations was obtained in [22]. With a particular numerical flux, the optimal convergence rate of order $O(h^{k+1})$ was obtained in [7, 10, 11], further obtained in [53] for equations with nonlinear convection; the use of generalized Gauss-Radau projections in the error analysis of the LDG method for linear convection-diffusion equations is found in [20]. For the symmetric interior penalty (SIP) method, it can be proved that for large enough penalty parameter, the method is stable and has optimal $O(h^{k+1})$ order convergence in L^2 [50, 1]. For the non-symmetric interior penalty (NIP) method of Baumann and Oden [6, 41], it is stable and convergent, with a suboptimal $O(h^k)$ order of L^2 errors for even k; however, the optimal error estimate for quadratic polynomials was obtained by Rivière and Wheeler [48] when applied to nonlinear convection-diffusion equations. Suboptimal L^2 error estimates are given in [23] for the ultra weak DG method introduced therein. For a unified error analysis of a class of DG methods applied to the elliptic problem, we refer to [3].

For error estimates of fully discrete DG schemes to hyperbolic conservation laws with third order Runge-Kutta time discretization we refer to [58, 59]. The error estimate for the fully discrete DG algorithm to solve convection-diffusion equations is more recent, see, e.g. [52, 51] for the LDG method coupled with a third order Runge–Kutta time discretization. For the optimal L^2 error of order $O(h^{k+1})$ for the AEDG scheme to one-dimensional linear convection-diffusion equations, we refer to [36].

For the DDG method, the first *a priori* error estimate of order $O(h^k)$ in the energy norm was obtained in [38]. The accurate recovery algorithm of the normal derivatives presented in [27] provides a set of effective choices of scheme parameters in the DDG diffusive flux. The optimal $O(h^{k+1})$ error in L^2 was first obtained in [33] using a special global projection, dictated by the form of the DDG numerical flux, when using Cartesian grids.

This article presents both an a priori error analysis and and analysis if the practical implications of selection of the method parameters for the DDG method on arbitrary multi-dimensional shape-regular grids.

A main tool is the global projection defined through the DDG scheme of the form $A(w_h, v) = R(v)$ for the elliptic problem $-\Delta W = G$ subject to the same boundary condition as that for (1) – here R is a linear operator depending on G and the boundary data. The projection is shown to be well-defined for $\beta_0 > \beta^*$ which is the sufficient condition for the L^2-stability of the semi-discrete DDG scheme to $\partial_t U = \Delta U$. The optimal L^2 error estimate follows from both the stability estimate and the projection error. One main task goes to the estimate of the projection error, which is carried out in two steps: first we introduce a DDG energy norm, involving terms from boundary contributions, with which we are able to obtain the projection error of order $O(h^k)$ in the same energy norm. We further obtain the optimal L^2 error using a duality argument carefully adapted to the case with jumps of second order derivatives along the interface normal, these together leading to the desired optimal projections error – also the optimal error estimate for the elliptic problem. This part of analysis when $\beta_1 = 0$ is standard (see, e.g. [1, 3]), and our effort is devoted to the careful adaptation to the case when $\beta_1 \neq 0$. For the DDG method to equation (1), we first obtain the error equation and introduce the global projection PU defined by $A(PU, v) = R(v)$ with $R(v) := A(U, v)$, we then split the error into two parts by using the global projection: $u - PU$ and $PU - U$, which enables us to control both cell integrals and the inter-element jump terms simultaneously. The nonlinear convection is treated in a similar fashion than in [33].

The main conclusion of this paper is as follows: for the DDG scheme to multidimensional linear convection-diffusion equations, and the Poisson equation, subject to the Dirichlet boundary condition or the Dirichlet boundary condition mixed with the Neumann or Robin boundary condition, we prove the optimal oder $O(h^{k+1})$ of the L^2 error, using polynomials of degree k over arbitrary shape-regular meshes. For the linear diffusion with nonlinear convection, we only obtain sub-optimal order $h^{k+1/2}$ due to lack of control on the part of the boundary on which either the Neumann or Robin boundary condition is imposed.

The paper is organized as follows. In §2 we set up the multi-dimensional discrete settings, and present the DDG scheme for (1) with consistent and conservative DDG numerical fluxes. In §3, we quantify the admissible parameter set for the DDG diffusive flux to ensure that the projection defined by the same DDG method to the elliptic problem is well-defined. We study the properties of consistency, boundedness, and stability in ways typically found in the error analysis for linear elliptic equations (see, e.g., [3]). The optimal L^2 projection error is carefully estimated. In §4 we present the L^2-error estimate of the numerical solution to linear diffusion with nonlinear convection. The extension to various boundary conditions is discussed in §5. Concluding remarks are given in §6.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. We use the notation $A \lesssim B$ to indicate that A can be bounded by B multiplied by a constant independent of the mesh size h. $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$.

Note: this article was first written in Spring 2017 in order to address some convergence issues of the DDG method when applied to compressible viscous fluid problems [25, 18], in particular on the choice of the method parameters for different boundary conditions. Recently the DDG method has been applied to incompressible viscous fluid problems, see e.g., [56], and models with anisotropic diffusivity, see [55].

2. The DDG methods

For the sake of simplicity, we begin with the model problem

$$\partial_t U + \nabla \cdot f(U) = \Delta U + S, \quad [0,T] \times \Omega =: Q_T, \tag{2a}$$

$$U = 0, \quad [0,T] \times \partial\Omega, \tag{2b}$$

$$U(x,0) = U^{\text{in}}(x), \tag{2c}$$

where $x = (x^1, \cdots, x^d)$, Ω is a bounded and convex domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, $f : \mathbb{R} \to \mathbb{R}^d$ is a smooth vector function, and S a given function in $L^2(Q_T)$.

We also consider the following elliptic problem

$$-\Delta W = G \text{ in } \Omega, \quad W = 0 \text{ on } \partial\Omega, \tag{3}$$

where $G \in L^2(\Omega)$ is a given function. A detailed discussion of extensions to non-homogeneous boundary conditions is postponed to §5. We should point out that for the error estimate of the DDG method to (2), the numerical solution of (3) by the same DDG method will be essentially used.

2.1. Discrete settings.

Let Ω be subdivided into shape-regular elements K (which can be a triangle in $2D$, or a tetrahedron in $3D$). To place the approximation into the DDG framework, we multiply the parabolic equation (2a) by a test function v, integrating formally on $K \subset \Omega$, to get

$$\int_K \partial_t U v dx + \int_K \nabla U \cdot \nabla v dx - \int_{\partial K} \partial_n U v ds - \int_K f(U) \cdot \nabla v dx + \int_{\partial K} f \cdot n v ds = \int_K S v dx, \tag{4}$$

where n is an outward normal to the boundary of K, denoted by ∂K. This weak formulation serves as a base for constructing the DDG method.

Before doing that, we need to introduce the finite element spaces associated with the triangulation of the domain Ω; as usual, we assume the mesh is conforming and the subdivision is regular; that is, there is a constant independent of mesh size such that

$$\frac{h_K}{\rho_K} \le \sigma \quad \forall K,$$

where h_K is the diameter of K, ρ_K denotes the diameter of the maximum ball included in K. The resulting division is denoted by $\mathcal{T}_h = \{K\}$, and h denotes the characteristic length of all the elements of $\mathcal{T}_h$. Let Γ_h denote the set of all edges (faces) of the subdivision $\mathcal{T}_h$, Γ_h^0 and Γ_h^∂ all internal edges and boundary edges, respectively.

As an appropriate functional setting, we denote by $H^m(\mathcal{T}_h)$ the space of functions on Ω whose restriction to each element K belongs to the Sobolev space $H^m(K)$, hence the traces of functions in $H^m(\mathcal{T}_h)$ are double-valued on $\Gamma_h^0 := \Gamma_h \backslash \partial\Omega$ and single-valued on $\Gamma_h^\partial = \partial\Omega$. As a subset of $H^m(\mathcal{T}_h)$, for any m, we set our finite element space

$$V_h = \left\{ v \in L^2(\Omega) \colon \forall K \in \mathcal{T}_h,\ v|_K \in P^k(K) \right\},$$

where $P^k(K)$ is the space of polynomial functions of degree at most $k \geq 1$ on K. With such a discrete space, formulation (4) has to be refined by including some interface corrections. We next introduce notations that will help us to define interface corrections. Let e denote a common edge (face) $e \in \Gamma_h^0$, shared by K_1 and K_2, and n be the normal vector to e oriented from K_1 to K_2. Define the average $\{w\}$ and the jump $[w]$ of w on e as follows:

$$\{w\} = \frac{1}{2}(w|_{K_1} + w|_{K_2}), \quad [w] = w|_{K_2} - w|_{K_1} \quad \forall e \in \partial K_1 \cap \partial K_2.$$

For e in the set of boundary edges, each w has a uniquely defined restriction on e; we set

$$[w] = -w, \quad \{w\} = \frac{1}{2}w,$$

since on the boundary face, n is taken as the outward unit normal to the domain boundary. For shape regular meshes there exists C_1 independent of h such that

$$C_1^{-1}h \leq h_e \leq C_1 h, \tag{5}$$

where h_e is the characteristic length of the face e.

2.2. **Scheme formulation.** The semi-discrete DDG formulation for (2) is to find $u_h \in V_h$ such that for all $K \in \mathcal{T}_h$,

$$\begin{aligned} &\int_K \partial_t u_h v dx + \int_K \nabla u_h \cdot \nabla v dx - \int_{\partial K} \left(\widehat{\partial_n u_h} v + (u_h - \widehat{u_h}) \partial_n v \right) ds \\ &\quad - \int_K f(u_h) \cdot \nabla v dx + \int_{\partial K} \widehat{f^n} v ds = \int_K S v dx, \quad \forall v \in V_h, \end{aligned} \tag{6}$$

where the numerical fluxes $\widehat{\partial_n u_h}$ and $\widehat{u_h}$ are approximations to $\partial_n u$ and u on the boundary of K with outward normal unit vector n to ∂K given as follows

$$\widehat{u_h} = \{u_h\}, \quad \widehat{\partial_n u_h} = \frac{\beta_0}{h_e}[u_h] + \{\partial_n u_h\} + \beta_1 h_e [\partial_n^2 u_h] \quad \text{on} \quad e \in \Gamma_h^0, \tag{7a}$$

$$\widehat{u_h} = 0, \quad \widehat{\partial_n u} = \frac{-\beta_0 u_h}{h_e} + \partial_n u_h \quad \text{on} \quad e \in \partial\Omega. \tag{7b}$$

The diffusive flux of form (7a) was first introduced by Liu and Yan in [38],

The numerical flux $\widehat{f^n}$ for nonlinear convection can be chosen as a monotone flux:

$$\widehat{f^n} = J(u^-, u^+), \tag{8}$$

which is Lipschitz continuous in its arguments, consistent with $f^n := f(u) \cdot n$ in the sense that $J^n(u, u) = f^n(u)$, and J^n is non-decreasing in u^- and non-increasing in u^+,

$$\widehat{f^n} = J(u_h^-, u_h^+) \quad \text{on} \quad e \in \Gamma_h^0, \tag{9a}$$

$$\widehat{f^n} = J(u_h^-, 0) \quad e \in \partial\Omega. \tag{9b}$$

The initial data for the ODE system (6) is chosen as the piecewise L^2 projection of $U^{\text{in}}(x)$, defined by

$$\int_K u_h(x, 0) v dx = \int_K U^{\text{in}}(x) v dx, \quad \forall v \in V_h. \tag{10}$$

To analyze the above DDG scheme, we sum (6) over all elements to obtain the following global formulation: find $u_h \in V_h$ such that

$$\langle \partial_t u_h, v\rangle + A(u_h, v) + F(u_h, v) = L(v), \ \forall v \in V_h, \tag{11}$$

where $\langle \cdot, \cdot\rangle$ denotes the L^2 inner product over Ω, and

$$A(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx + \sum_{e\in\Gamma_h^0} \int_e (\widehat{\partial_n u}[v] + [u]\{\partial_n v\}) ds \tag{12a}$$
$$+ \sum_{e\in\Gamma_h^\partial} \int_e \left(\frac{\beta_0 uv}{h_e} - u\partial_n v - v\partial_n u \right) ds,$$

$$F(u,v) = -\sum_{K\in\mathcal{T}_h} \int_K f(u)\cdot \nabla v \, dx - \sum_{e\in\Gamma_h^0} \int_e \widehat{f^n}[v] ds + \int_{\partial\Omega} \widehat{f^n} v ds, \tag{12b}$$

$$L(v) = \langle S, v\rangle. \tag{12c}$$

Note that at each cell interface the normal vector n is chosen oriented from one cell to its neighbor, and the resulting scheme (11) is independent of the orientation of n. For any functions $u \in H^3(\mathcal{T}_h)$ and $v \in H^3(\mathcal{T}_h)$, (12a) defines a bilinear operator $A(u,v)$. $F(u,v)$ is nonlinear in u and linear in v, unless $f(u)$ is linear; $L(v)$ is a linear operator.

Likewise, the DDG formulation for the corresponding elliptic problem (3) can be described as: find $w_h \in V_h$ such that

$$A(w_h, v) = R(v), \quad \forall v \in V_h, \tag{13}$$

where R is a linear operator defined by

$$R(v) = \langle G, v\rangle.$$

Both (11) and (13) are well defined once a proper parameter pair (β_0, β_1) is taken.

Remark 2.1. We pause to look at whether the DDG scheme (13) fits the unified framework by Arnold et al [3] for the Poisson equation (3). The flux formulation in [3] is based on the first order system

$$\sigma = \nabla W, \quad -\nabla\cdot\sigma = G \ \text{ in } \Omega, \quad W = 0 \text{ on } \partial\Omega,$$

and takes the form

$$\int_K \sigma_h \cdot \tau dx = -\int_K w_h \nabla\cdot\tau dx + \int_{\partial K} \widehat{w}_K n_K \cdot \tau ds, \tag{14a}$$

$$\int_K \sigma_h \cdot \nabla v dx = \int_K G v dx + \int_{\partial K} \widehat{\sigma}_K \cdot n_K v ds, \tag{14b}$$

where in [3] n_K is the outward normal unit vector to ∂K, and the numerical fluxes $\widehat{\sigma}_K$ and $\widehat{w}_K$ are approximations to $\sigma = \nabla W$ and W, respectively, on the boundary of K. Let $\tau = \nabla v$, we may reduce the above to

$$\int_K \nabla w_h \cdot \nabla v dx + \int_{\partial K} (\widehat{w}_K - w_h) n_K \cdot \nabla v ds - \int_{\partial K} \widehat{\sigma}_K \cdot n_K v ds = \int_K G v dx,$$

which may be used to obtain the flux formulation of the DDG scheme, with

$$\widehat{w}_K = \widehat{w}_h, \quad \widehat{\sigma}_K \cdot n_K = \widehat{\partial_n w_h}$$

as defined in (7a). However, the flux for solution gradient using $[\partial_n^2 w_h]$ is not covered in Table 3.1 of [3]. The primal formulation in [3] is obtained by summing the flux formulation over all elements, expressing σ_h solely in terms of w_h via lifting operators. Our primal formulation (13) can also be put in the abstract form in [3, Equation (3.11)], but is not covered in Table 3.2 of [3].

2.3. **On numerical flux $\widehat{f^n}$.** A well-known numerical flux is the so-called E-flux (see [42]) as defined by

$$\operatorname{sign}(u^+ - u^-)(J(u^-, u^+) - f^n(u)) \le 0 \tag{15}$$

for all u between u^- and u^+, including the monotone flux. A celebrated monotone flux is the Lax-Friedrich flux of the form

$$\widehat{f^n} = \{f^n(u)\} - \frac{\sigma}{2}[u] \quad \text{on} \quad e \in \Gamma_h, \tag{16}$$

where $f^n(u) := f(u) \cdot n$, and σ is chosen as $\sigma = \max|f'|$ for all u between u^- and u^+. For any piecewise smooth function $u \in L^2$, on any cell interface we define

$$\alpha^n(\hat{f}; \xi) := \begin{cases} [u]^{-1}(f(\xi) \cdot n - \widehat{f^n}), & \forall \xi \in (\min(u^-, u^+), \max(u^-, u^+)), \\ \frac{1}{2}\max|f'|, & \text{if } [u] = 0, \end{cases} \tag{17}$$

where $\widehat{f^n} \equiv J(u^+, u^-)$ is the numerical flux consistent with $f^n(u)$.

One can verify that for monotone fluxes,

$$\alpha^n(\widehat{f^n}, \xi) \ge 0, \quad \forall \xi \in (\min(u^-, u^+), \max(u^-, u^+)), \tag{18}$$

and bounded from above for any $(u^-, u^+) \in \mathbb{R}^2$. This will be used in our error analysis in §4. We refer to [58, Lemma 3.1] for some refined bounds of such quantity.

3. Error estimates for the elliptic problem

In this section, we discuss several properties of the bilinear operator $A(\cdot, \cdot)$ and the approximation properties of the space V_h with respect to an appropriate norm, and then carry out the error analysis for (13). The main result is stated here.

Theorem 3.1. *Let w_h be obtained from the DDG scheme (13) subject to numerical fluxes (7), for elliptic problem (3), and W be the smooth solution of (3). Then there exists a constant $\beta_0^* > 0$ such that if $\beta_0 > \beta_0^*$ the following error estimate holds,*

$$\int_\Omega (W(x) - w_h(x))^2 dx \le Ch^{2(k+1)}, \tag{19}$$

where C depends solely on $|W|_{k+1,\Omega}$, but is independent of h.

3.1. **Consistency.** It can be verified that the DDG formulation (13) is consistent in the sense that if the weak solution to (3) satisfies $W \in H^{s+1}(\Omega)$ for $s > 3/2$, then

$$A(W, v) = \langle G, v \rangle, \quad \forall v \in V_h. \tag{20}$$

In fact, for such W the jumps $[W] = 0$ and $[\partial_n W] = [\partial_n^2 W] = 0$ a.e. on the interior faces, and $[W] = [\partial_n W] = 0$ a.e. on boundary faces, hence for all $v \in V_h$,

$$\begin{aligned} A(W, v) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla W \cdot \nabla v \, dx + \sum_{e \in \Gamma_h^0} \int_e (\partial_n W[v]) ds \\ &\quad + \sum_{e \in \Gamma_h^\partial} \int_e \left(\frac{\beta_0 W v}{h_e} - W \partial_n v - v \partial_n W \right) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K (-\Delta W) v \, dx = \langle -\Delta W, v \rangle. \end{aligned} \tag{21}$$

3.2. **Coercivity.** We define the energy norm

$$\|v\|_E = \left(\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 dx + \sum_{e\in\Gamma_h^0} \int_e \frac{\beta_0}{h_e}[v]^2 ds + \sum_{e\in\Gamma_h^\partial} \int_e \frac{\beta_0}{h_e} v^2 ds \right)^{1/2},$$

and set

$$\beta_0^* := \max\{\Gamma_d(\beta_1), 2\Gamma_d(0)\},$$

where $\Gamma_d(\beta_1)$ is a constant independent of h, yet it is required to satisfy the following

$$\Gamma_d(\beta_1) \geq \frac{1}{2} \sup_{v\in P^k(K)} \frac{h_e \int_{\partial K}(\partial_n v - \beta_1 h_e \partial_n^2 v)^2 ds}{\int_K |\nabla v|^2 dx}, \quad \forall K \in \mathcal{T}_h.$$

Then we have the following result.

Lemma 3.2. *If $\beta_0 > \beta_0^*$, then there exists $\gamma \in (0,1)$ such that*

$$A(v,v) \geq \gamma \|v\|_E^2, \quad \forall v \in V_h. \tag{22}$$

Proof. A direct calculation gives

$$\begin{aligned} A(v,v) &= \sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 dx + \sum_{e\in\Gamma_h^0} \int_e \left(\widehat{\partial_n v} + \{\partial_n v\}\right)[v] ds + \sum_{e\in\Gamma_h^\partial} \int_e \left(\beta_0 h_e^{-1} v^2 - 2\partial_n v v\right) ds \\ &= \|v\|_E^2 + \sum_{e\in\Gamma_h^0} \int_e \left(2\{\partial_n v\} + \beta_1 h_e [\partial_n^2 v]\right)[v] ds + \sum_{e\in\Gamma_h^\partial} \int_e (-2\partial_n v v)\, ds. \end{aligned}$$

Note for $e = \partial K_1 \cap \partial K_2 \in \Gamma_h^0$, we use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} \int_e \left(2\{\partial_n v\} + \beta_1 h_e [\partial_n^2 v]\right)[v] ds &= \int_{e\in\partial K_1} \left(\partial_{n_1} v_1 - \beta_1 h_e \partial_{n_1}^2 v_1\right)(v_2 - v_1) ds \\ &\quad + \int_{e\in\partial K_2} \left(\partial_{n_2} v_2 - \beta_1 h_e \partial_{n_1}^2 v_2\right)(v_1 - v_2) ds \\ &\leq (1-\gamma) \int_e \frac{\beta_0 [v]^2}{h_e} ds + \sum_{i=1}^2 \int_{e\in\partial K_i} \frac{h_e}{2\beta_0(1-\gamma)} (\partial_n v - \beta_1 h_e \partial_n^2 v)^2 ds, \end{aligned}$$

and for $e = \partial K \cap \partial\Omega_D$,

$$\begin{aligned} \int_e (-2\partial_n v v)\, ds &= \int_{e\in\partial K} 2\,(\partial_n v)\,(0 - v) ds \\ &\leq (1-\gamma) \int_e \frac{\beta_0 v^2}{h_e} ds + \int_{e\in\partial K} \frac{h_e}{\beta_0(1-\gamma)} (\partial_n v)^2 ds. \end{aligned}$$

Hence counting all terms involved we obtain

$$\begin{aligned} A(v,v) &\geq \gamma \|v\|_E^2 + (1-\gamma) \left[\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 dx - \frac{1}{2\beta_0(1-\gamma)^2} \sum_{e\in\Gamma_h^0} \int_e h_e (\partial_n v - \beta_1 h_e \partial_n^2 v)^2 ds \right. \\ &\qquad \left. - \frac{1}{\beta_0(1-\gamma)^2} \sum_{e\in\partial\Omega_D} \int_e h_e (\partial_n v)^2 ds \right] \\ &\geq \gamma \|v\|_E^2, \end{aligned}$$

provided

$$\beta_0(1-\gamma)^2 \geq \sup_{v\in \tilde{V}_h} \frac{\frac{1}{2}\sum_{e\in\Gamma_h^0}\int_e h_e(\partial_n v - \beta_1 h_e \partial_n^2 v)^2 ds + \sum_{e\in\partial\Omega_D}\int_e h_e(\partial_n v)^2 ds}{\sum_{K\in\mathcal{T}_h}\int_K |\nabla v|^2 dx},$$

where $\tilde{V}_h := \{v \in V_h, \sum_{K\in\mathcal{T}_h}\int_K |\nabla v|^2 dx \neq 0\}$.

This can be ensured if we take $\gamma = 1 - \sqrt{\frac{\beta_0^*}{\beta_0}}$ with

$$\beta_0^* \geq \frac{1}{2}\sup_{v\in P^k(K)} \frac{h_e\int_{\partial K}(\partial_n v - \beta_1 h_e \partial_n^2 v)^2 ds}{\int_K |\nabla v|^2 dx}, \quad \partial K \cap \partial\Omega_D = \emptyset,$$

and on boundary edges

$$\beta_0^* \geq \sup_{v\in P^k(K)} \frac{h_e\int_{\partial K}(\partial_n v)^2 ds}{\int_K |\nabla v|^2 dx}, \quad \partial K \cap \partial\Omega_D \neq \emptyset.$$

Here we simply assume that the integral on each face of K over $\int_K |\nabla v|^2 dx$ shares same upper bound. □

Existence and uniqueness: Since the DDG formulation (13) is a square system of linear equations in finite dimension, it suffices to show uniqueness of the solution. Any two solutions w_{1h}, w_{2h} will satisfy

$$A(w_{1h} - w_{2h}, v) = 0, \quad \forall v \in V_h,$$

this when combined with the coercive bound with $v = w_{1h} - w_{2h}$ ensures that $\|v\|_E^2 = 0$, hence $v \equiv 0$, that is $w_{1h} \equiv w_{2h}$.

3.3. **Boundedness.** To bound the bilinear operator $A(\cdot,\cdot)$ from above, we adopt an extended space of V_h defined by

$$V(h) = V_h + H^s(\Omega) \cap H_0^1(\Omega) \subset H^{\min\{3,s\}}(\mathcal{T}_h),$$

and define the following seminorms and norms for $v \in V(h)$:

(23a) $$|v|_{m,h}^2 := \sum_{K\in\mathcal{T}_h} |v|_{m,K}^2, \ |v|_*^2 := \sum_{e\in\Gamma_h^0} h_e^{-1}\|[v]\|_{0,e}^2 + \sum_{e\in\Gamma_h^\partial} h_e^{-1}\|v\|_{0,e}^2,$$

(23b) $$\|v\|_{DG}^2 = \sum_{m=1}^{\min\{s,3\}} h^{2m-2}|v|_{m,h}^2 + \beta_0|v|_*^2.$$

Here we take $s = 2$ when $k = 1$ in V_h, and $s \geq 3$ for $k > 1$. The norm (23) is the natural one for obtaining boundedness of the bilinear form $A(\cdot,\cdot)$. On the other hand, the energy norm which can now be written as

(24) $$\|v\|_E := (|v|_{1,h}^2 + \beta_0|v|_*^2)^{1/2}$$

is the natural one for analyzing the stability of the DDG method. Note that both (23b) and (24) define norms, not just seminorms, on $V(h)$.

We collect a few very basic inequalities, in which the bounding coefficients are easy to figure out in one dimension, yet often more involved in the case of several dimensions.

(1) Note that if $w \in H^3(K)$ and e is an edge of K, we have the following trace inequality for $i = 0, 1, 2$,

(25) $$\|\partial_n^i w\|_{0,e}^2 \leq C(h_e^{-1}|w|_{i,K}^2 + h_e|w|_{i+1,K}^2),$$

where the constant C can depend on several geometric features of K, but it does not depend on the size of K and e.

(2) Inverse inequality. In a finite dimensional space, all norms are equivalent. For every polynomial of degree $\leq k$, there exists C depending on k such that

$$|v|_{s,K}^2 \leq Ch^{-2(s-m)}|v|_{m,K}^2 \quad \text{for integers } s, m \text{ with } s > m. \tag{26}$$

Moreover, for any function $v \in V_h$, the following inverse inequalities hold

$$\|v\|_{\Gamma_h} \leq Ch^{-1/2}\|v\|, \tag{27a}$$

$$\|v\|_\infty \leq Ch^{-d/2}\|v\|, \tag{27b}$$

where d is the spatial dimension, and $\|v\|_{\Gamma_h}^2 := \sum_{e\in\Gamma_h}\int_e v^2 ds$. For more details of these inverse properties, we refer to [13].

Restricted to $v \in V_h$, both energy norm and the DG norm are equivalent:

$$\|v\|_E \leq \|v\|_{DG} \leq C_1\|v\|_E. \tag{28}$$

as can be shown using inverse inequality (26).

We now show that the bilinear operator $A(\cdot,\cdot)$ is bounded with respect to the norm $\|\cdot\|_{DG}$, that is,

$$A(u,v) \leq C_A\|u\|_{DG}\cdot\|v\|_{DG} \quad \forall u, v \in V(h). \tag{29}$$

For $v \in V(h)$ and w well defined on e, we have

$$\sum_{e\in\Gamma_h}\int_e [v]w \leq \left(\sum_{e\in\Gamma_h} h_e^{-1}\|[v]\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\Gamma_h} h_e\|w\|_{0,e}^2\right)^{1/2}.$$

If we take $w = [u], \{\partial_n u\}$ and $[\partial_n^2 u]$, and use (25), respectively, we can obtain the following estimates

$$\sum_{e\in\Gamma_h} h_e^{-1}\int_e [v][u]ds \leq |v|_*|u|_* \leq \beta_0^{-1}\|v\|_E\|u\|_E \leq C\|v\|_{DG}\|u\|_{DG},$$

$$\sum_{e\in\Gamma_h}\int_e [v]\{\partial_n u\}ds \leq C|v|_*(|u|_{1,h}^2 + h_e^2|u|_{2,h}^2)^{1/2} \leq C\|v\|_{DG}\|u\|_{DG},$$

$$\sum_{e\in\Gamma_h} h_e\int_e [v][\partial_n^2 u]ds \leq C|v|_*(h_e^2|u|_{2,h}^2 + h_e^4|u|_{3,h}^2)^{1/2} \leq C\|v\|_{DG}\|u\|_{DG}.$$

Hence

$$\sum_{e\in\Gamma_h}\int_e [v]\widehat{(\partial_n u)}ds \leq C\|v\|_{DG}\|u\|_{DG};$$

and also

$$\sum_{e\in\Gamma_h}\int_e [u]\{\partial_n v\}ds \leq C\|v\|_{DG}\|u\|_{DG}.$$

Similarly we can bound other boundary terms in (12a). The integral terms in (12a) is bounded by $|u|_{1,h}|v|_{1,h} \leq C\|u\|_{DG}\|v\|_{DG}$. All the above together ensures the upper bound claimed in (29).

3.4. **Error analysis for elliptic problems.** We proceed to prove error estimates for the DDG method (13) to the elliptic problem. This splits into two steps: (i) error in the energy norm; and (ii) recovery of the optimal L^2 error.

Step 1. Optimal error in energy norm.

From consistency (20) and scheme (13) it follows

$$A(w_h - W, v) = 0 \quad \forall v \in V_h. \tag{30}$$

Using this relation we are able to prove the following.

Lemma 3.3. *The error estimate in the DG norm is*

$$\|W - w_h\|_{DG} \leq Ch^k |W|_{k+1,\Omega},$$

for $k \geq 1$, where C is independent of h.

Proof. Let $u \in V_h$ be specified later. Then, using coercivity (22), orthogonality (30), and boundedness (29), we have

$$\begin{aligned}\gamma\|u - w_h\|_E^2 &\leq A(u - w_h, u - w_h) = A(u - W, u - w_h) \\ &\leq C_A\|u - W\|_{DG}\|u - w_h\|_{DG}.\end{aligned}$$

Note that $\|u - w_h\|_{DG} \leq C_1\|u - w_h\|_E$ by (28), then

$$\|u - w_h\|_{DG} \leq \frac{C_A C_1^2}{\gamma}\|u - W\|_{DG}.$$

By the triangle inequality,

$$\|W - w_h\|_{DG} \leq \|W - u\|_{DG} + \|u - w_h\|_{DG} \leq (1 + \gamma^{-1}C_A C_1^2)\|W - u\|_{DG}.$$

Minimization of the error on the right hand side over all $u \in V_h$ leads to

$$\|W - w_h\|_{DG} \leq C \inf_{u \in V_h} \|W - u\|_{DG}. \tag{31}$$

Let W_I be a local approximation to W in each cell with the optimal error as

$$\|W_I - W\|_{s,K} \leq Ch^{k+1-s}|W|_{k+1,K} \quad \forall K \in \mathcal{T}_h, s = 0, 1, 2, \tag{32}$$

where C depends only on k and the shape of K. With this we will have

$$\|W - w_h\|_{DG} \leq C\|W - W_I\|_{DG} \lesssim h^k|W|_{k+1,\Omega}, \tag{33}$$

as desired; the last step involves bounding each boundary term in $\|\cdot\|_{DG}$ using (25), and then (32). □

Step 2. Error in L^2 norm. We proceed to show how to obtain the optimal order of L^2-error estimates by using the usual duality argument.

Lemma 3.4. *If $W \in H^s(\Omega)$ with $s \geq \min\{k, 2\} + 1$, then the error estimate in L^2 norm can be recovered from that in the DG norm:*

$$\|W - w_h\| \lesssim h\|W - w_h\|_{DG}. \tag{34}$$

Proof. Set $\theta = W - w_h \in L^2(\Omega)$. As usual, we define the auxiliary function ψ as the solution of the adjoint problem

$$-\Delta\psi = \theta \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega. \tag{35}$$

As Ω is convex, elliptic regularity ensures that $\psi \in H^2(\Omega)$, and

$$|\psi|_{2,\Omega} \leq C\|\theta\|_{0,\Omega}, \tag{36}$$

where C depends only on the domain Ω. On the other hand

$$\|\theta\|_{0,\Omega}^2 = -\int_\Omega \Delta\psi\theta dx.$$

This upon further integration by parts on each element yields

$$\begin{aligned}
\|\theta\|_{0,\Omega}^2 &= \sum_{K\in\mathcal{T}_h}\left(\int_K \nabla\theta\cdot\nabla\psi dx - \int_{\partial K}\theta\partial_n\psi ds\right)\\
&= \sum_{K\in\mathcal{T}_h}\int_K \nabla\theta\cdot\nabla\psi dx + \sum_{e\in\Gamma_h^0}\int_e([\theta]\partial_n\psi)\,ds - \sum_{e\in\Gamma_h^\partial}\int_e(\theta\partial_n\psi)\,ds\\
&= \sum_{K\in\mathcal{T}_h}\int_K \nabla\theta\cdot\nabla\psi\,dx + \sum_{e\in\Gamma_h^0}\int_e(\widehat{\partial_n\theta}[\psi] + [\theta]\{\partial_n\psi\})ds\\
&\quad + \sum_{e\in\Gamma_h^\partial}\int_e\left(\frac{\beta_0\psi\theta}{h_e} - \psi\partial_n\theta - \theta\partial_n\psi\right)ds = A(\theta,\psi),
\end{aligned}$$

where we have used the fact that $[\psi]=0, \quad \partial_n\psi = \{\partial_n\psi\}$ for $e\in\Gamma_h^0$ and $\psi=0$ for $e\in\Gamma_h^\partial$. Note that the regularity assumption $W\in H^{\min\{k,2\}+1}(\Omega)$ is sufficient for well defining $\widehat{\partial_n\theta}$, since in the case $k=1$, no need to use the term $[\partial_n^2\theta]$. By subtracting the orthogonality relation (30), i.e.,

$$A(\theta, v) = 0, \quad \forall v\in V_h,$$

we obtain

$$\begin{aligned}
\|\theta\|_{0,\Omega}^2 &= A(\theta,\psi - v)\\
&= \sum_{K\in\mathcal{T}_h}\int_K \nabla\theta\cdot\nabla(\psi - v)\,dx + \sum_{e\in\Gamma_h^0}\int_e(\widehat{\partial_n\theta}[\psi - v] + [\theta]\{\partial_n\psi - \partial_n v\})ds\\
&\quad + \sum_{e\in\Gamma_h^\partial}\int_e\left(\frac{\beta_0(\psi - v)\theta}{h_e} - (\psi - v)\partial_n\theta - \theta(\partial_n\psi - \partial_n v)\right)ds\\
&\le C\|\theta\|_{DG}\left(\sum_{m=1}^2\sum_{K\in\mathcal{T}_h} h^{2m-2}|\psi - v|_{m,K}^2 + \beta_0|\psi - v|_{\Gamma_h}^2\right)^{1/2}\\
&\le C\|\theta\|_{DG}\left(\sum_{m=1}^2\sum_{K\in\mathcal{T}_h} h^{2m-2}|\psi - v|_{m,K}^2\right)^{1/2}.
\end{aligned}$$

Here we have used the more precise upper bound on $\psi - v$, instead of $\|\psi - v\|_{DG}$ as in (29).

We now take a piecewise linear approximation $v=\psi_I\in V_h$ to ψ in each cell so that

$$\|\psi_I - \psi\|_{s,K} \le Ch^{2-s}|\psi|_{2,K} \quad \forall K\in\mathcal{T}_h, s=1,2,$$

where C depends only on the geometry of K. Putting the above estimates together and then using (36) we obtain

$$\|\theta\|_{0,\Omega}^2 \le Ch\|\theta\|_{DG}|\psi|_{2,\Omega} \lesssim h\|\theta\|_{DG}\|\theta\|_{0,\Omega},$$

leading to the desired estimate (34). □

4. Error analysis for DDG methods to parabolic problems

This section is devoted to the error estimate for the DDG scheme (11), i.e.,

$$\langle u_{ht}, v\rangle + A(u_h, v) + F(u_h, v) = L(v).$$

4.1. Error estimates.
The coercivity of $A(\cdot,\cdot)$ also yields the following error estimate.

Theorem 4.1. *Let u_h be the solution to the semi-discrete DDG scheme (11) subject to diffusive fluxes (7) with $\beta_0 > \beta_0^*$ and monotone convective flux, for problem (2) using polynomial elements of degree k, and U be a smooth solution of (2). If $k > \frac{d}{2}$ and h is suitably small, then*

$$\int_\Omega (u_h(x,t) - U(x,t))^2 dx \le Ch^{2(k+1)}, \quad t \in [0,T], \tag{37}$$

where C depends solely on $|(U, \partial_t U)|_{k+1,\Omega}$, $|f'(U)|, |f''(U)|$, T and data given, but is independent of h.

Remark 4.1. Note that for purely diffusion problem with no convection (i.e., $f(u) = 0$) or with linear diffusion $f(u) = cu$, the restriction $k > d/2$ is unnecessary. For nonlinear convection, this restriction suggests that one should use polynomials of degree $k > 1$ for $d = 2, 3$.

Proof. Let $w_h \in V_h$ be obtained from

$$A(w_h, v) = A(U, v) \quad \forall v \in V_h. \tag{38}$$

The existence and uniqueness of w_h is ensured if $\beta_0 > \beta_0^*$ (see §3.2). Moreover, from Theorem 3.1 we have

$$\|\epsilon\|_E \le \|\epsilon\|_{DG} \le Ch^k|U|_{k+1}, \quad \|\epsilon\| \le Ch^{k+1}|U|_{k+1}, \tag{39}$$

where $\epsilon := w_h - U$. By consistency, scheme (11) remains valid when u_h is replaced by U, that is

$$\langle \partial_t U, v\rangle + A(U, v) + F(U, v) = L(v).$$

This when subtracted from (11) gives the error equation

$$\langle \partial_t(U - u_h), v\rangle + A(U - u_h, v) + F(U, v) - F(u_h, v) = 0. \tag{40}$$

Set

$$\xi := w_h - u_h, \tag{41}$$

so that the total error $U - u_h = \xi - \epsilon$, where ξ is to be estimated, and the bound of ϵ is already known from (39). Take $v = \xi$ in (40) and use (38), i.e., $A(\epsilon, \xi) = 0$, to obtain

$$\langle \partial_t \xi, \xi\rangle + A(\xi, \xi) = \langle \partial_t \epsilon, \xi\rangle - H, \tag{42}$$

where

$$H := F(U, \xi) - F(u_h, \xi).$$

This with (22) leads to

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \gamma\|\xi\|_E^2 \le \|\partial_t \epsilon\|\|\xi\| + |H|. \tag{43}$$

It is left to further estimate H. Here we extend the estimates in [33, Section 6.4]. Take the average of $u_h|_{\partial K_i}$ as a reference value $\{u_h\}$, hence

$$H = \sum_{K \in \mathcal{T}_h} \int_K (f(U) - f(u_h)) \cdot \nabla \xi dx + \int_{\Gamma_h} (f^n(U) - f^n(\{u_h\}))[\xi] ds$$
$$+ \int_{\Gamma_h} (f^n(\{u_h\}) - \widehat{f^n})[\xi] ds, \tag{44}$$

where we used the notation $f^n(\eta) = f(\eta)\cdot n$. Using the Taylor expansions for $f = (f_1, \cdots, f_d)$,

$$f_i(U) - f_i(u_h) = f_i'(U)(\xi - \epsilon) - \frac{f_i''}{2}(\xi - \epsilon)^2, \tag{45}$$

$$f_i(U) - f_i(\{u_h\}) = f_i'(U)(\{\xi\} - \{\epsilon\}) - \frac{\widetilde{f}_i''}{2}(\{\xi\} - \{\epsilon\})^2, \tag{46}$$

where f_i'' and $\widetilde{f}_i''$ are the mean values, we regroup terms in H so that

$$H = H_1 + H_2 + H_3 + H_4,$$

where

$$\begin{aligned}
H_1 &= \sum_{K\in\mathcal{T}_h}\int_K \xi f'(U)\cdot\nabla\xi dx + \int_{\Gamma_h} n\cdot f'(U)\{\xi\}[\xi]ds,\\
H_2 &= -\left(\sum_{K\in\mathcal{T}_h}\int_K \epsilon f'(U)\cdot\nabla\xi dx + \int_{\Gamma_h} n\cdot f'(U)\{\epsilon\}[\xi]ds\right),\\
H_3 &= -\frac{1}{2}\left(\sum_{K\in\mathcal{T}_h}\int_K (\xi-\epsilon)^2 f''\cdot\nabla\xi dx + \int_{\Gamma_h} (n\cdot\widetilde{f}''(\{\xi\} - \{\epsilon\})^2[\xi]ds\right),\\
H_4 &= -\int_{\Gamma_h}\alpha^n(\hat{f};\{u_h\})[\xi]^2 ds + \int_{\Gamma_h}\alpha^n(\hat{f};\{u_h\})[\epsilon][\xi]ds.
\end{aligned}$$

For the H_1 term, a simple integration by parts gives

$$\begin{aligned}
H_1 &= \int_{\Gamma_h} n\cdot f'(U)\{\xi\}[\xi]ds - \frac{1}{2}\int_{\Gamma_h} n\cdot f'(U)[\xi^2]ds - \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K f''(U)\cdot\nabla U\xi^2 dx\\
&= -\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K f''(U)\cdot\nabla U\xi^2 dx \leq C\|\xi\|^2,
\end{aligned}$$

where C depends on $|f''(U)|$ and $|\nabla U|_\infty \lesssim \|U\|_{k+1}$ for $k > d/2$. Note that on Γ_h^∂, we used $\{\xi\}[\xi] = -\frac{1}{2}\xi^2 = \frac{1}{2}[\xi^2]$. Using Young's inequality and the trace inequality (25a) we obtain

$$\begin{aligned}
H_2 &\leq \frac{\gamma}{6}\sum_{K\in\mathcal{T}_h}\int_K |\nabla\xi|^2 dx + \frac{3}{2\gamma}\sum_{K\in\mathcal{T}_h}\int_K (|f'(U)|\epsilon)^2 dx\\
&\quad + \frac{\gamma}{6}\int_{\Gamma_h}\beta_0\frac{[\xi]^2}{h_e}ds + \frac{3}{2\beta_0\gamma}\int_{\Gamma_h} h_e(f'(U)\cdot n\{\epsilon\})^2 ds\\
&\leq \frac{\gamma}{6}\|\xi\|_E^2 + C(\|\epsilon\|^2 + h^2\|\epsilon\|_{DG}^2)\\
&\leq \frac{\gamma}{6}\|\xi\|_E^2 + Ch^{2k+2},
\end{aligned}$$

where (39) has been used, with C depending on $|f'(U)|$. We now deal with the higher order terms. For H_3, we have

$$
\begin{aligned}
H_3 &\le \frac{\gamma}{6}\sum_{K\in\mathcal{T}_h}\int_K |\nabla\xi|^2dx + \frac{3}{8\gamma}\sum_{K\in\mathcal{T}_h}\int_K (|f''|(\xi-\epsilon)^2)^2dx \\
&\quad + \frac{\gamma}{6}\int_{\Gamma_h}\beta_0\frac{[\xi]^2}{h_e}ds + \frac{3}{8\beta_0\gamma}\int_{\Gamma_h} h_e(|\widetilde{f}''|\{\xi-\epsilon\}^2)^2ds \\
&\le \frac{\gamma}{6}\|\xi\|_E^2 + C\|\xi-\epsilon\|_\infty^2(\|\xi-\epsilon\|^2 + h^2\|\nabla(\xi-\epsilon)\|^2) \\
&\le \frac{\gamma}{6}\|\xi\|_E^2 + C\|\xi-\epsilon\|_\infty^2(\|\xi\|^2 + h^2\|\nabla\xi\|^2 + \|\epsilon\|^2 + h^2\|\nabla\epsilon\|^2) \\
&\le \frac{\gamma}{6}\|\xi\|_E^2 + C\|\xi-\epsilon\|_\infty^2(\|\xi\|^2 + h^{2k+2}),
\end{aligned}
$$

where (26) and (39) have been used in the last inequality, with C depending on $|f''(U)|$. For the monotone flux (8), the first term in H_4 is non-positive; using the Young inequality, we bound the last term further by

$$
\begin{aligned}
H_4 &\le \frac{\gamma}{6}\int_{\Gamma_h}\beta_0\frac{[\xi]^2}{h_e}ds + \frac{3}{2\beta_0\gamma}\int_{\Gamma_h}\left(h_e(\alpha^n)^2[\epsilon]^2\right)ds \\
&\le \frac{\gamma}{6}\int_{\Gamma_h}\beta_0\frac{[\xi]^2}{h_e}ds + C(\|\epsilon\|^2 + h^2\|\epsilon\|_{DG}^2) \\
&\le \frac{\gamma}{6}\int_{\Gamma_h}\beta_0\frac{[\xi]^2}{h_e}ds + Ch^{2k+2},
\end{aligned}
$$

with C depending on $|f'(U)|$. These estimates together lead to

$$
H \le \frac{\gamma}{2}\|\xi\|_E^2 + C(1+\|\xi-\epsilon\|_\infty^2)(\|\xi\|^2 + h^{2k+2}).
$$

Substitution of this into the equality (43) yields

$$
\frac{d}{dt}\|\xi\|^2 \le C(1+\|\xi-\epsilon\|_\infty^2)(\|\xi\|^2 + h^{2k+2}). \tag{47}
$$

Using the approximation results (39), (27), we have for small h,

$$
\|\xi-\epsilon\|_\infty \le \|\xi\|_\infty + \|\epsilon\|_\infty \le C(h^{-d/2}\|\xi\| + h^{k+1-\frac{d}{2}}) \le C(h^{-d/2}\|\xi\| + h),
$$

for $k > \frac{d}{2}$. Here we have used (39) with the inequality $\|\epsilon\|_\infty \le C(h^{-d/2}\|\epsilon\| + h^{1-d/2}\|\nabla\epsilon\|)$. Hence (47) reduces to

$$
\frac{d}{dt}\|\xi\|^2 \le C(1+h^{-d}\|\xi\|^2)(\|\xi\|^2 + h^{2k+2}). \tag{48}
$$

Set $B = \|\xi\|^2/h^{2k+2}$, so that for $h \le 1$ we have

$$
\frac{dB}{dt} \le C(h^2B+1)(B+1), \tag{49}
$$

where we have used $h^{2k-d} \le 1$. Note that at $t=0$ we have

$$
\xi(x,0) = w_h(x) - u_h(x,0) = \epsilon(x,0) + U^{\rm in}(x) - u_h(x,0),
$$

hence $\|\xi(\cdot,0)\|^2 \le C_0h^{2k+2}$ by (39) and the L^2-projection error, with C_0 depending on $|U^{\rm in}|_{k+1,\Omega}$. Thus $B_0 := \|\xi(\cdot,0)\|^2/h^{2k+2} \le C_0$. Integration of (49) gives

$$
\frac{B(t)+1}{h^2B(t)+1} \le \frac{B_0+1}{h^2B_0+1}e^{(1-h^2)Ct} \le (B_0+1)e^{CT} =: C_1
$$

for $t \in [0, T]$. Hence we have

$$B(t) \leq \frac{C_1 - 1}{1 - C_1 h^2} \leq 2(C_1 - 1) =: C^*,$$

if h is suitably small so that $h \leq \frac{1}{\sqrt{2C_1}} < 1$. Hence $\|\xi\|^2 \leq C^* h^{2k+2}$, which when combined with the triangle inequality $\|u_h - U\| \leq \|\xi\| + \|\epsilon\|$ leads to the desired error estimate (37). □

5. Non-homogeneous boundary conditions

The previous analysis can be extended to problems with non-homogeneous boundary conditions. For example, consider the Poisson equation with a boundary condition of form

$$-\Delta W = G, \quad \text{in } \Omega, \tag{50a}$$

$$W = g_D, \quad \text{on } \partial\Omega_D, \tag{50b}$$

$$\frac{\partial W}{\partial n} + \alpha W = g_N, \quad \text{on } \partial\Omega_N, \tag{50c}$$

where $\alpha \geq 0$ is given, Ω is a bounded domain with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$. Furthermore, we assume that the measure of $\partial\Omega_D$ is not zero if $\alpha = 0$ so uniqueness of the continuous problem is alway ensured.

On $\Gamma_h^\partial = \partial\Omega$, n is selected to be the usual outward unit normal to the boundary, and we define the boundary flux as follows:

$$\widehat{w_h} = g_D, \quad \widehat{\partial_n w_h} = \beta_0 h_e^{-1}(g_D - w_h) + \partial_n w_h \quad \text{on } e \in \partial\Omega_D, \tag{51a}$$

$$\widehat{w_h} = w_h, \quad \widehat{\partial_n w_h} = g_N - \alpha w_h \quad \text{on } e \in \partial\Omega_N. \tag{51b}$$

Hence the DDG method for (50) is to find $w_h \in V_h$ so that for all $v \in V_h$,

$$A(w_h, v) = L(v), \quad v \in V_h, \tag{52}$$

where the corresponding diffusive bilinear form $A(w, v)$, and the linear operator $L(v)$ are, respectively, given by

$$\begin{aligned} A(w, v) = &\sum_{K \in \mathcal{T}_h} \int_K \nabla w \cdot \nabla v dx + \sum_{e \in \Gamma_h^0} \int_e \left(\widehat{\partial_n w}[v] ds + \{\partial_n v\}[w] \right) ds \\ &+ \int_{\partial\Omega_D} \left((\beta_0 h_e^{-1} w - \partial_n w)v - \partial_n v w \right) ds + \int_{\partial\Omega_N} \alpha w v ds, \end{aligned} \tag{53a}$$

$$L(v) = \sum_{K \in \mathcal{T}_h} \int_K G v dx + \int_{\partial\Omega_D} \left(\frac{\beta_0}{h_e} v - \partial_n v \right) g_D ds + \int_{\partial\Omega_N} g_N v ds. \tag{53b}$$

Recall that, for shape-regular meshes, the basic ingredients in the previous error analysis for the elliptic equation subject to the homogeneous boundary condition, include (i) the continuity of $A(\cdot, \cdot)$ in $V(h)$, (ii) coercivity of $A(\cdot, \cdot)$ in V_h, (iii) best approximation in $DG-$norm $\inf_{v \in V_h} \|u - v\|_{DG}$.

Among these three properties, (i) and (iii) remain the same, only coercivity of $A(v, v)$ needs to be verified. A similar calculation to the proof of Lemma 3.2 still gives

$$A(v, v) \geq \gamma \|v\|_E^2,$$

for some $0 < \gamma < 1$, provided

$$\beta_0 > \Gamma_d(\beta_1) \tag{54}$$

on each interior interface, and

$$\beta_0 > 2\Gamma_d(0) \tag{55}$$

on $\partial\Omega_D$. Note that here the energy norm involves boundary contributions from both Ω_D and Ω_N defined by

$$\|v\|_E = \left(\sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 dx + \sum_{e\in\Gamma_h^0} \int_e \frac{\beta_0}{h_e}[v]^2 ds + \int_{\partial\Omega_D} \frac{\beta_0}{h_e} v^2 ds + \frac{\alpha}{\gamma} \int_{\partial\Omega_N} v^2 ds \right)^{1/2}.$$

Theorem 5.1. *Let w_h be the solution to DDG scheme (52) subject to (7) on interior interfaces, for problem (50) using polynomial elements of degree k, and W be the smooth solution of (50). Then there exists $\beta_0^* > 0$ such that if $\beta_0 > \beta_0^*$ the following error estimate holds,*

$$\int_\Omega (W(x) - w_h(x))^2 dx \le Ch^{2(k+1)}, \tag{56}$$

where C depends solely on $|W|_{k+1,\Omega}$, but is independent of h.

We further look at the time-dependent parabolic problem with the same boundary condition,

$$\partial_t U + \nabla\cdot f(U) = \Delta U + S \quad \text{in } [0,T]\times\Omega, \tag{57a}$$

$$U = g_D \quad \text{on } [0,T]\times\partial\Omega_D, \tag{57b}$$

$$\frac{\partial U}{\partial n} + \alpha U = g_N \quad \text{on } [0,T]\times\partial\Omega_N, \tag{57c}$$

$$U(x,0) = U^{\text{in}}(x). \tag{57d}$$

The DDG scheme for this problem is to find $u_h \in V_h$ such that

$$\langle \partial_t u_h, v\rangle + A(u_h, v) + F(u_h, v) = L(v), \ \forall v \in V_h, \tag{58}$$

where $A(u_h, v)$ is as in (53a), $F(u_h, v)$ as in (12b), and

$$L(v) = \sum_{K\in\mathcal{T}_h} \int_K S v dx + \int_{\partial\Omega_D} \left(\frac{\beta_0}{h_e} v - \partial_n v\right) g_D ds + \int_{\partial\Omega_N} g_N v ds, \tag{59}$$

subject to same numerical flux $J(u^-, u^+)$ for convection on interior faces, and $J(u^-, g_D)$ for $e \in \partial\Omega_D$, as well as

$$\widehat{f^n} = f^n(u_h) \quad \text{on} \quad e \in \partial\Omega_N. \tag{60}$$

The initial data for (58) is same as that defined in (10).

The proof of Theorem 4.2 can be carried over to the present case, still leading to (42), i.e.,

$$\langle \partial_t \xi, \xi\rangle + A(\xi,\xi) = \langle \partial_t \epsilon, \xi\rangle - H, \tag{61}$$

where

$$H := F(U,\xi) - F(u_h,\xi).$$

In H, terms on Γ_h^∂ now split into two parts, denoted by

$$H_D + H_N = \int_{\partial\Omega_D} (\widehat{f^n} - f^n(g_D))\xi ds + \int_{\partial\Omega_N} (\widehat{f^n} - f^n(U))\xi ds.$$

The estimate of H_D can be done the same way as that when $\partial\Omega_D = \partial\Omega$. We only need to estimate H_N. A simple estimate gives

$$\begin{aligned}|H_N| &\le C\|U-u_h\|_{0,\partial\Omega_N}\|\xi\|_{0,\partial\Omega_N}\\ &\le C(\|\xi\|_{0,\partial\Omega_N}+\|\epsilon\|_{0,\partial\Omega_N})\|\xi\|_{0,\partial\Omega_N}.\end{aligned}$$

Note that using (25) and (27a), respectively, we obtain

$$\|\epsilon\|_{0,\partial\Omega_N} \le C(h^{-1}\|\epsilon\|^2 + h\|\nabla\epsilon\|^2), \quad \|\xi\|^2_{0,\partial\Omega_N} \le Ch^{-1}\|\xi\|^2.$$

These together with (39) yield

$$|H_N| \le C(h^{-1}\|\xi\|^2 + h\|\epsilon\|^2_{DG}) \le Ch^{-1}(\|\xi\|^2 + h^{2k+2}).$$

Putting all the estimates for H into (61) we have

$$\frac{d}{dt}\|\xi\|^2 \le C(h^{-1}+h^{-d}\|\xi\|^2)(\|\xi\|^2+h^{2k+2}), \tag{62}$$

from which we can only obtain $\|\xi\| \le Ch^{k+1/2}$.

In summary we have the following result.

Theorem 5.2. *Let u_h be the solution to the semi-discrete DDG scheme (58), using polynomial elements of degree k, and U be a smooth solution of (57). If $k > \frac{d}{2}$ and h is suitably small, then*

$$\int_\Omega (u_h(x,t) - U(x,t))^2 dx \le Ch^{(2k+2)}, \tag{63}$$

in the case $|\partial\Omega_N| = 0$ or $f = 0$; otherwise we have only the sub-optimal order $O(h^{(k+1/2)})$. Here C depends solely on T, $|(U, \partial_t U)|_{k+1}$, $|f'(U)|, |f''(U)|$, but is independent of h.

Remark 5.1. Note that the sharp estimate of terms on $\partial\Omega_D$, as analyzed in the proof of Theorem 4.2, depends on two factors: (i) using a monotone numerical flux; and (ii) the control term $\frac{\beta_0}{h_e}\int_{\partial\Omega_D}|\xi|^2 ds$ induced from $A(\xi,\xi)$. The order of accuracy deterioration in the case $|\partial\Omega_N| \ne 0$ and f is nonlinear comes from the lack of a control term on $\partial\Omega_N$. This suboptimal rate has not been observed in numerical tests. It would be interesting to improve the estimate with possible new techniques.

6. On model parameters

In this section, we discuss admissible choices of (β_0, β_1) in the DDG schemes. The obtained theoretical bound may be relaxed as

$$\beta_0 > \Gamma_d(\beta_1) \quad e \in \Gamma_h^0, \tag{64a}$$

$$\beta_0 > 2\Gamma_d(0) \quad e \in \partial\Omega_D, \tag{64b}$$

where

$$\Gamma_d(\beta_1) \ge \frac{1}{2}\sup_{v\in P^k(K)} \frac{h_e\int_{\partial K}(\partial_n v - \beta_1 h_e \partial_n^2 v)^2 ds}{\int_K |\nabla v|^2 dx}, \quad \partial K \cap \partial\Omega = \emptyset, \tag{65a}$$

$$\Gamma_d(0) \ge \frac{1}{2}\sup_{v\in P^k(K)} \frac{h_e\int_{\partial K}(\partial_n v)^2 ds}{\int_K |\nabla v|^2 dx}, \quad \partial K \cap \partial\Omega_D \ne \emptyset, \tag{65b}$$

where the right hand sides clearly depend on the characteristic face length h_e, the cell K and its immediate neighbors. A uniform bound for all faces can be more restrictive than the admissible ones. Nevertheless, they are sufficient rather than necessary conditions to ensure optimal order of accuracy of numerical solutions. Therefore, in practice, this parameter pair is conveniently chosen as some admissible ones. Here we discuss how to make these choices.

On uniform meshes, h_e can be easily defined as the cell size h, for which we can identify a globally defined pair (β_0, β_1) for all interior faces and boundary faces, respectively.

Admissible pair (β_0, β_1): For one-dimensional uniform grids, a change of variables using $\partial_x v(x) = w(\xi)$ with $x = x_j + \frac{h}{2}\xi$ for $\xi \in [-1, 1]$ when $x \in I_j = [x_{j-1/2}, x_{j+1/2}]$, gives

$$\Gamma_1(\beta_1) \geq \sup_{w \in P^{k-1}[-1,1]} \frac{(w - 2\beta_1 \partial_\xi w)^2(1) + (w + 2\beta_1 \partial_\xi w)^2(-1)}{\int_{-1}^{1} |w(\xi)|^2 d\xi}.$$

As shown in [33, Lemma 3.1] it suffices to take

$$\Gamma_1(\beta_1) = 2\sum_{i=1}^{k} |\psi_i(1) - 2\beta_1 \partial_\xi \psi_i(1)|^2,$$

where $\{\psi_i\}_{i=1}^k$ are the normalized Legendre polynomials. This upon further calculation gives

$$\Gamma_1(\beta_1) = k^2 \left(1 - \beta_1(k^2 - 1) + \frac{\beta_1^2}{3}(k^2 - 1)^2\right).$$

Hence the admissible pair (β_0, β_1) can be chosen to satisfy

$$\beta_0 > \Gamma_1(\beta_1) \quad \text{at } x_{j+1/2},\ j = 1, \cdots, N-1, \tag{66a}$$

$$\beta_0 > 2k^2 \quad \text{at } x_{1/2} \text{ or } x_{N+1/2}, \tag{66b}$$

wherever a Dirichlet boundary condition is imposed. In practice, one should take β_0 about between the minimum of $\Gamma_1(\beta_1)$ over β_1 and $\Gamma_1(0)$, that is

$$\beta_0 \in \left[\frac{k^2}{4}, k^2\right]. \tag{67}$$

On the boundary face with a Dirichlet boundary data, one would have to take larger β_0, say $2k^2$. The choice of β_1 can be flexible, the guiding principle is to use β_1 as a leverage so β_0 can be determined from a fixed range. Therefore the range of β_1 should be

$$0 \leq \beta_1 < \frac{3}{k^2 - 1}, \tag{68}$$

for which $\Gamma(\beta_1) \in \left[\frac{k^2}{4}, k^2\right]$. One particular good choice is

$$\beta_1 = \frac{1}{2k(k+1)} \tag{69}$$

with which superconvergence has been proven and numerically observed at both nodes and other special points [19] for linear convection-diffusion equations. For $k = 2$ it means $\beta_1 = 1/12$, which is clearly in the range of $(0, 1)$, as given in (68).

Also it has been proven in [39] third order maximum-principle-preserving DDG schemes are possible if

$$\beta_0 \geq 1, \quad \beta_1 \in \left[\frac{1}{8}, \frac{1}{4}\right]. \tag{70}$$

Multi-dimensional rectangular grids: we assume that the domain Ω can be partitioned into rectangular meshes

$$\Omega = \cup_{\alpha=1}^{N} K_\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)$, $N = (N_1, \cdots, N_d)$, and $K_\alpha = I^1_{\alpha_1} \times \cdots \times I^d_{\alpha_d}$, where $I^i_{\alpha_i} = [x^i_{\alpha_i - 1/2}, x^i_{\alpha_i+1/2}]$ for $\alpha_i = 1, \cdots, N_i$. The lengths of cells are denoted by $h^{x^i} = \max_{1\leq \alpha_i \leq N_i} |I^i_{\alpha_i}|$, with $h = \max_{1\leq i\leq d} h^{x^i}$ being the maximum mesh size. For uniform meshes with size h we proceed to estimate $\Gamma_d(\beta_1)$. Using this we obtain

$$\begin{aligned}
\int_{\partial K} (\partial_n v - \beta_1 h \partial_n^2 v)^2 ds &= \sum_{i=1}^d \int_{K_\alpha / I^i_{\alpha_i}} |\partial_{x^i} v - \beta_1 h \partial^2_{x_i} v|^2_{x^i = x^i_{\alpha_i \pm 1/2}} d\hat{x}^i \\
&\leq \sum_{i=1}^d \frac{2\Gamma_1(\beta_1)}{h} \int_K |\partial_{x^i} v|^2 dx \\
&= \frac{2\Gamma_1(\beta_1)}{h} \int_K |\nabla v|^2 dx,
\end{aligned}$$

where $d\hat{x}^i = \prod_{j\neq i} dx^j$. Therefore,

$$\Gamma_d(\beta_1) = \Gamma_1(\beta_1).$$

Hence we have the following

Theorem 6.1. *For multi-dimensional rectangular meshes, using either P^k or Q^k elements, the bilinear operator defined in (53) is coercive if the pair (β_0, β_1) is chosen to satisfy*

(71a) $$\beta_0 > \Gamma_1(\beta_1) \quad \text{for } e \in \Gamma^0_h,$$

(71b) $$\beta_0 > 2k^2 \quad \text{for } e \in \partial\Omega_D.$$

Remark 6.1. Based on this result the admissible parameter when using structured grids can be chosen as follows: when $k = 2$ the range for β_0 at interior faces should be $2 \sim 4$, at the domain boundary where the Dirichlet data are imposed, β_0 needs to be bigger than 8; β_1 can be chosen as either $\frac{1}{12}$ or any other number in $\left[\frac{1}{8}, \frac{1}{4}\right]$. For $k \geq 3$, take β_1 as given in (69), and β_0 as in (67) about $\Gamma(\beta_1)$ —for example, one may take $(\beta_0, \beta_1) \sim (6, 1/24)$ in numerical simulations when using polynomials of degree 3.

Remark 6.2. In multi-dimensional unstructured grids, the size of K, the choice of characteristic length h_e, and the length of ∂K are all involved in the implicit expression $\Gamma_d(\beta_1)$. A detailed characterization of $\Gamma_d(\beta_1)$ on 2D triangular meshes is possible, but the obtained estimates are rather complicated. In practice, one should use the choice for the uniform grids as a guide, and adjust in terms of the ratio h_e/h_K to suit the specific meshes under consideration.

7. Summary and concluding remarks

In this paper we have provided a framework for analyzing the DDG schemes for multi-dimensional convection-diffusion equations and the Poisson equation. We presented an admissible range for DDG flux parameters to ensure the optimal L^2 error estimates for smooth solutions. The main technique used in this paper is the energy analysis, which works for arbitrary triangulation in multiple dimensions. Our analysis and results are presented for homogeneous Dirichlet boundary conditions, with extensions to non-homogeneous boundary conditions, including the Dirichlet boundary condition mixed with the Neumann or the Robin boundary condition, on different parts of the domain boundary. As in finite element analysis, the use of a proper projection and its approximation property is crucial. In this work we use a global projection defined by the DDG formulation for an elliptic problem, instead of the explicit global projection introduced in [33]. An advantage of using such projection is that the projection error once obtained already gives the error estimate for the corresponding elliptic problem.

The DDG methods for multidimensional systems, like the compressible Navier-Stokes equations can be defined by simply applying the procedure described for the multi-dimensional scalar equation to each component of the unknown. This convergence theory of the DDG method explains the success of its practical implementation in viscous fluid problems; see [25].

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