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THE l^1 GLOBAL DECAY TO DISCRETE SHOCKS FOR SCALAR MONOTONE SCHEMES

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ABSTRACT. Given a family of discrete shocks ϕ of a monotone scheme, we prove that the discrete shock profile with rational shock speed η is asymptotically stable with respect to the l^1 topology $\|\cdot\|_1$: if $u^0-\phi\in l^1$, then $\|u^n-\phi_{\cdot-n\eta}\|_1\to 0$ as $n\to\infty$ under no restriction conditions of the initial data to the interval $[\inf\phi,\sup\phi]$. The asymptotic wave profile is uniquely identified from the above family by a mass parameter.

1. INTRODUCTION

We consider the monotone conservative scheme of the form

$$u_j^{n+1}=G(u_{j-k}^n,\cdots,u_{j+k'}^n),\qquad j\in\mathbb{Z},\tag{1.1}$$

where $G(u_{j-k}^n,\cdots,u_{j+k'}^n):\prod_{k+k'+1}\mathbb{R}\to\mathbb{R}$ is a monotone increasing function of each of its arguments. Equation (1.1) is said to be in conservation form if it can be written in the following way:

$$u_j^{n+1}=u_j^n-\lambda\Big[g(u_{j-k+1}^n,\cdots,u_{j+k'}^n)-g(u_{j-k}^n,\cdots,u_{j+k'-1}^n)\Big].$$

If $\lambda=\Delta t/\Delta x$, $g(u,\cdots,u)=f(u)$ and u_j^n is equated with $u(n\Delta t,j\Delta x)$, (1.1) is a first-order accurate approximation (see [3]), to the nonlinear conservation laws

$$u_t+f(u)_x=0.\tag{1.2}$$

A *discrete shock profile* connecting u_- to u_+ is a special solution of the difference equation of the form

$$u_{x-\eta}=G(u_{x-k},\cdots,u_{x+k'}),\qquad x\in\mathcal{L}_\eta,\tag{1.3}$$

with $\lim_{x\to\pm\infty}u_x=u_\pm$, the domain $\mathcal{L}_\eta$ on which (1.3) makes sense is defined as the closure of $\mathbb{Z}+\eta\mathbb{Z}$. The conservation form of the difference scheme indicates that $u_\pm$ and s satisfy the Rankine-Hugoniot condition

$$f(u_+)-f(u_-)=s(u_+-u_-),$$

where the discrete shock speed η is related to s by the relation $\eta=\lambda s$. As a sufficient condition for the existence of the discrete shocks we impose the Oleinik

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entropy condition

$$\frac{f(u)-f(u_+)}{u-u_+} < \frac{f(u_-)-f(u_+)}{u_- - u_+} \tag{1.4}$$

for u lying strictly between u_+ and u_-. We require that this discrete shock can be observed on the original grid; therefore, we assume that $\eta = p/q$ with $q > 0$ be a rational number.

Throughout this paper, we shall use the operator T introduced by Jennings [5]. Its definition is

$$(Tu)_{j-\eta} = G(u_{j-k}, \cdots, u_{j+k'}), \quad j \in \mathcal{L}_\eta.$$

It is easy to see, if u_j^n is a solution of (1.1) for $j \in \mathcal{L}_\eta$, then

$$u_j^n = (T^n u^0)_{j-n\eta}.$$

Obviously the discrete shocks with speed η are the fixed points of T. We take this operator with data in $l^\infty(\mathcal{L}_\eta)$, and write $\|\cdot\|_p$ for the discrete l^p norm for $p \in [1,\infty]$ with the usual measure. The main goal of this paper is to establish the following stability result for the discrete shock profiles of (1.1).

Theorem 1.1. *Suppose η is a rational number with $-k < -\eta < k'$, $f'(u_+) < s < f'(u_-)$ and the mesh ratio λ is suitably small.* Let $\phi : \mathcal{L}_\eta \to \mathcal{L}_\eta$ be a bounded discrete shock profile of* (1.1) *connecting u_+ to u_- and $g(u_{-k}, \cdots, u_{k'-1}) \in C^2(\mathbb{R}^{k+k'})$. Then for any function $u^0(x) \in \phi + l^1(\mathcal{L}_\eta)$, the solution $T^n u^0$ of* (1.1) *with initial data u^0 satisfies*

$$\lim_{n\to\infty} \|T^n u^0 - \phi(m(u^0))\|_1 = 0, \tag{1.5}$$

where the existence of $\phi(m(u^0))$ is stated in Lemma 2.1 *and $m(u^0)$ is a mass parameter identifying a discrete shock by the relation $m(u^0) = m(\phi)$, $m(\phi) := \sum_{x<0}(\phi_x - u_-) + \sum_{x\geq 0}(\phi_x - u_+)$.*

Remark 1.2. For the rational discrete shock speed η, the discrete shock profiles that we study are really solutions of the discrete equation of the form

$$u_{j-p} = G^{(q)}(u_{j-qk}, \cdots, u_{j+qk'}), \qquad j \in \mathbb{Z}. \tag{1.6}$$

$G^{(q)}$ is the composition of monotone increasing functions and is therefore monotone increasing. Without loss of generality we will present all proofs for the case that η is an integer and thus $L_\eta = \mathbb{Z}$, though the main results are stated in a more general setting. For definiteness we shall take $u_+ < u_-$ throughout the paper.

The study of existence and stability of discrete shocks was pioneered by Jennings [5] and has drawn much attention in subsequent years. This kind of investigation has been proved important in understanding the convergence behavior of numerical shock computations and is essential for the error analysis around shocks. For scalar conservative schemes many stability results have been obtained by various authors, see e.g., Jennings [5], Ying and Zhou [24], Jiang and Yu [6], Fan [1], Liu and Wang [7, 8, 9], and Smyrlis [21]. The existence of discrete shocks for system began with Majda and Ralston [14]; see also Michelson [13]. For stability results in system case we refer to Liu and Xin [10, 11], Szepessy [20], and Liu-Yu [12] and references cited therein. It is well known that the l^1 metric is a natural norm for the conservation

*The technical restriction on λ will be detailed in Section 6 as a *Courant-Friedrichs-Lewy* (CFL) condition, see (6.3).

schemes. The l^1 stability result for scalar monotone schemes was established by Jennings [5]; see also Osher and Ralston [15]. Jenning's result in a compact form reads as follows:

Theorem 1.3. *Let ϕ be as in Theorem 1.1 and $u^0 \in \phi + l^1(\mathcal{L}_\eta)$ such that*

$$u^0(x) \in [\inf \phi, \sup \phi] \quad \text{for all } x \in \mathcal{L}_\eta. \tag{1.7}$$

Then (1.5) *holds for $u^n = T^n u^0$.*

Theorem 1.1 improves upon Theorem 1.3 because larger perturbations are allowed in Theorem 1.1 than those stated in (1.7). In the paper [2] Freistühler and Serre have discussed the l^1 attraction of smooth viscous profiles for scalar viscous conservation laws; see also [19] for a new proof of the global stability result obtained in [2]. Although the results we present here are of a different flavor than those in the above papers, because of the technical nature of the discrete schemes, the main idea of the paper parallels the above work. The proof of Theorem 1.1 is carried out by a direct use of Theorem 1.3 and the contraction property of T. This allows us to reduce the problem to the following related l^1 stability of the constant state solutions.

Theorem 1.4. *Under the assumptions made in Theorem 1.1, assume for every $c \in \mathbb{R}$ and any function $v^0 \in c + l^1(\mathcal{L}_\eta)$*

$$\sum_{x \in \mathcal{L}_\eta} (v^0(x) - c) = 0. \tag{1.8}$$

Then the solution $v^n = T^n v^0$ of (1.1) *with data v^0 satisfies*

$$\lim_{n \to \infty} \|v^n - c\|_1 = 0. \tag{1.9}$$

The principal effort is devoted to proving this theorem by using the decay property of the discrete fundamental solution, l^2 energy estimate as well as the l^1 contraction. The energy inequality we establish in this paper is intimately related to the discrete cell entropy inequality; however, we refrain here from giving any details on the corresponding extensions, and instead we refer to [23, 16] for a detailed discussion on this link.

This paper is organized as follows. Some basic properties of discrete shocks and their parameterizations were discussed in Section 2. The parameter we use here is determined through a mass map, which is indicated by the conservation form of the numerical scheme. In Section 3 we first discuss several necessary criteria for stability for the general schemes, which include the stability condition, the dissipative condition as well as the non-resonant condition. These are all implied by the monotonicity assumption. Under some weaker settings we will derive various bounds for the discrete Green function of the linearized scheme, paving the way for our decay estimate for nonlinear one. We would like to point out that in the system case, the discrete Green functions also play an important role in the error estimate [4] as well as the stability analysis of discrete shocks [12]. Hence the bounds of Green function established in this work may be used in more general set up. In Section 4, based on a key energy inequality, we first prove the l^∞ decay as well as the l^2 decay, then using the point-wise bounds for Green function we establish the l^1 decay for small data. The proof of the further l^1 global decay becomes quite simple and was motivated by a geometrical observation in [2]. Some technical assumptions for guaranteeing the desired energy inequality are clarified in Section 5. We would like to point out that the CFL-like condition (6.3) is sufficient to guarantee the energy inequality (5.1) in Lemma 5.1, which is essentially used to ensure the l^2 decay. Unfortunately condition (6.3) is supported basically by the three point schemes. We are still unable, however, to find a simple connection between inequality (5.1) and the monotonicity of the schemes with arbitrary stencils. Finally a further property for monotone schemes is proved in the appendix.

2. Discrete shocks and mass map

Let us recall some basic facts about the discrete shocks and their parameterizations. Since f satisfies the Rankine-Hugoniot relation and the Oleinik entropy condition (1.4) across $]u_+, u_-[$, (1.1) is a strictly monotone and consistent scheme. Arguments of Jennings [5] yield

Lemma 2.1 (Jennings)**.** *Suppose $-k < -\eta < k'$ and that $G(\cdot, \cdots, \cdot)$ is a strictly monotone increasing function of each of its arguments. Then for each $u_0 \in]u_+, u_-[$, there exists a unique function ϕ, defined on $x \in \mathcal{L}_\eta$, which takes on the value u_0 at $x = 0$ and satisfies*

$$\phi_{x-\eta} = G(\phi_{x-k}, \cdots, \phi_{j+k'}), \quad x \in \mathcal{L}_\eta,$$

and

$$\phi(x) \to u_\pm, \text{ as } x \to \pm\infty, \ \ x \in \mathcal{L}_\eta.$$

The function $\phi(x)$ is a monotone function of x and depends continuously at each value of x on u_0.

One very important feature of these discrete shocks is their parameterizations. The parameter is taken as $\phi(x)|_{x=0} = u_0$, $u_0 \in]u_+, u_-[$, in Lemma 2.1. Due to the conservative nature of the scheme, the parameter can also be taken as the amount of the mass m from the Riemann data defined as

$$m(\phi) := \sum_{x \le 0} (\phi(x) - u_-) + \sum_{x > 0} (\phi(x) - u_+). \tag{2.1}$$

Thus the function ϕ is a one-parameter family of the discrete shock profile with parameter $m(\phi)$ or $\phi(0)$, respectively. The equivalence between these two parameters was shown in [9] and is summarized in the following

Lemma 2.2. *If the assumptions made in Lemma* 2.1 *hold, then for all possible discrete shocks ϕ connecting u_- to u_+ with $\phi(0) = u_0$, there exists a* 1–1 *continuous mapping from $u_0 \in]u_+, u_-[$ to $m(\phi) \in \mathbb{R}$ defined in* (2.1)*. Thus ϕ also depends continuously on any given $m(\phi) \in \mathbb{R}$.*

Let us denote by X the metric space $\phi + l^1(\mathcal{L}_\eta)$ with ϕ being a discrete shock connecting the states $u_\pm$. Obviously the mass map $m : X \to \mathbb{R}$ given in (2.1) is well defined on the metric space X. We also define the notations

$$a \vee b = \max(a, b), \quad a \wedge b = \min(a, b), \quad a^+ = a \vee 0.$$

For later reference we collect two basic facts associated with the mass map:

$\mathbf{M}_1$: For $p(x), q(x) \in X$, $m(p) = m(q) \Leftrightarrow \sum_x (p(x) - q(x)) = 0$;

M$_2$: (Mass of cut-off function) For $p \in X$ and any two bounded functions $p_1(x) \le p_2(x)$, a.e. we define $q(x) = (p(x) \vee p_1(x)) \wedge p_2(x)$. Then

$$m(q) = m(p) - \sum_x (p - p_2)^+ + \sum_x (p_1 - p)^+.$$

3. Proof of Theorem 1.1

We consider the problem (1.1) with initial data $u^0(x) \in l^\infty$. As remarked earlier we may, without loss of generality, restrict ourselves to the case that η is an integer such that $\mathcal{L}_\eta = \mathbb{Z}$. To stress ideas and constructions needed for proving the convergence in l^1 norm when the initial perturbation is relaxed beyond the strip $[\inf \phi, \sup \phi]$, we cite without proof the following well-known properties of the solution operator $\{T^n : \quad l^\infty \to l^\infty, \quad n \in \mathbb{N}\}$:

- (conservative) If $a - b \in l^1(\mathbb{Z})$, then for all $n > 0$, $T^n a - T^n b \in l^1(\mathbb{Z})$ and $\sum(T^n a - T^n b) = \sum(a - b)$.
- (l^1 contracting) If $a - b \in l^1(\mathbb{Z})$, then $T^n a - T^n b \in l^1(\mathbb{Z})$ and $\|T^n a - T^n b\|_1$ is non-increasing of $n > 0$.
- (monotonicity) If $a_j \ge b_j$ for all $j \in \mathbb{Z}$, then $T^n a_j \ge T^n b_j$ for all $n > 0$.

The above nice properties are easily seen from the following relation for given scalar discrete conservation laws:

$$\text{monotone} \Longrightarrow \ l^1 \text{ contracting} \Longrightarrow TVD \Longrightarrow \ \text{monotonicity}.$$

Let us now establish the following

Lemma 3.1. *Theorem 1.4 implies Theorem 1.1.*

Proof. Our proof mimics the one given in [2] where a similar result for viscous conservation laws was proved. For simplicity we take the zero discrete speed η in the proof to follow. The case $\eta \ne 0$ can be proven without essential difference. We consider the case $u_+ < u_-$ and introduce the notations

$$I(n) := \sum_{j \in \mathbb{Z}} (u_j^n - u_-)^+ \quad \text{and} \quad J(n) := \sum_{j \in \mathbb{Z}} (u_+ - u_j^n)^+, \tag{3.1}$$

which are well-defined finite numbers depending on n as

$$0 \le I(n), \ J(n) \le \|u^n - \phi\|_1 \le \|u^0 - \phi\|_1.$$

Moreover, we claim that

$$\lim_{n \to \infty} I(n) = 0, \quad \lim_{n \to \infty} J(n) = 0. \tag{3.2}$$

In order to continue the argument, we postpone the proof of this claim later on. Fix an arbitrary $\epsilon > 0$, there exists a $n(\epsilon) \in \mathbb{N}$ such that

$$I(n_\epsilon) \le \epsilon, \quad J(n_\epsilon) \le \epsilon$$

by claim (3.2).

Let $v_j^0 = (u_j^{n_\epsilon} \wedge u_-) \vee u_+$, and v^n be the solution of (1.1) with initial data v^0. From the mass relation in (M_2) for cut-off function, it follows that

$$m(v^0) = m(u^{n_\epsilon}) - I(n_\epsilon) + J(n_\epsilon), \quad n_\epsilon \in \mathbb{N}, \tag{3.3}$$

we then have

$$\|v^0 - u^{n_\epsilon}\|_1 \le |J(n_\epsilon) - I(n_\epsilon)| \le \epsilon. \tag{3.4}$$

Theorem 1.3 shows that

$$\lim_{n\to\infty} \|v^n - \phi(m(v^0))\|_1 = 0 \tag{3.5}$$

with the mass $m(v^0)$ determined in (3.3). By l^1 contraction property

$$\begin{aligned} \|u^{n+n_\epsilon} - \phi(m(v^0))\|_1 &\le \|u^{n+n_\epsilon} - v^n\|_1 + \|v^n - \phi(m(v^0))\|_1 \\ &\le \|u^{n_\epsilon} - v^0\|_1 + \|v^n - \phi(m(v^0))\|_1. \end{aligned} \tag{3.6}$$

Again, by the l^1 contraction, both (3.4) and (3.5) imply that

$$\lim_{n\to\infty} \sup \|u^n - \phi(m(v^0))\|_1 \le \epsilon.$$

By the conservative property the mass $m(u^n) = m(u^0)$ does not depend on ϵ, so $m(v^0)$ converges, as $\epsilon \to 0$, to the limit $m(u^0)$. Thus

$$\begin{aligned} \lim_{n\to\infty} \sup \|u^n - \phi(m(u^0))\|_1 &\le \|u^{n_\epsilon} - \phi(m(v^0))\|_1 + \|\phi(m(v^0)) - \phi(m(u^{n_\epsilon}))\|_1 \\ &\le 2\epsilon. \end{aligned}$$

Passing to the limit $\epsilon \to 0$ gives the desired result. Thus the proof of Lemma 3.1 is complete. □

Proof of the claim (3.2). Since $\phi(\pm\infty) = u_\pm$, $u_+ < u_-$ and $u^0 - \phi \in l^1$, the set $\{j : u_j^0 \ge u_-\}$ is bounded from above by $j_0 < \infty$. Thus there exists an N such that

$$u_j^0 \le u_- - \frac{I(0)}{N} \quad \text{for} \quad j > j_0.$$

Define

$$v_j^0 = \begin{cases} u_- + (u_j^0 - u_-)^+ & j \le j_0, \\ u_- - \frac{I(0)}{N} & j_0 < j < j_0 + N + 1, \\ u_- & j \ge j_0 + N + 1. \end{cases}$$

This construction implies

$$u_j^0 \le v_j^0, \quad j \in \mathbb{Z},$$

and gives

$$\begin{aligned} \sum_{j\in\mathbb{Z}} (v_j^0 - u_-) &= \sum_{j\ge j_0} (u_j^0 - u_-)^+ + \sum_{j_0<j<j_0+N+1} (-I(0)/N) + \sum_{j\ge j_0+N+1} (0) \\ &= I(0) + N \cdot (-I(0)/N) = 0 \end{aligned}$$

as well as $v_j^0 - u_- \in l^1$. Letting $T^n v_j^0$ denote the solution of (1.1) with initial data v_j^0, we see that Theorem 1.4 implies

$$\lim_{n\to\infty} \|(T^n v^0 - u_-)^+\|_1 = 0.$$

Thus for $u^n := T^n u^0$ we immediately have $\lim_{n\to\infty} I(n) = 0$ for $u_j^n \le T^n v_j^0$. Similarly, for $J(n)$ one also has $\lim_{n\to\infty} J(n) = 0$. Hence the proof of the claim (3.2) is complete.

4. Discrete Green functions

To clarify the conditions necessary for the main result in this work, we linearize the difference scheme (1.1) at the constant initial state $u_j = u$ for all j by substituting $u_j = u + \epsilon v_j$ and collecting terms of order ϵ to get the linearized difference scheme

$$v_j^{n+1} = \left(\sum_{l=-k}^{k'} G_l E^l \right) v_j^n, \tag{4.1}$$

where $G_l = \frac{\partial G}{\partial u_l}(u, \cdots, u)$, and E denotes a shift operator satisfying $E^l v_j = v_{j+l}$ and $E^0 = 1$. We may rewrite the scheme into an abstract form

$$v^{n+1} = P v^n$$

with symbol $P := (\sum_{l=-k}^{k'} G_l E^l)$.

Definition 4.1. The fundamental solution for (4.1) is the sequence H_j^n defined by

$$H_j^0 = \begin{cases} 1 & j = 0, \\ 0 & j \neq 0, \end{cases}$$

and

$$H^n = P^n H^0.$$

Also, given sequences u_j and v_j, one of which has finite support, we define the discrete convolution by

$$(u * v)_j = \sum_{l \in \mathbb{Z}} u_{j-l} v_l.$$

The usual properties of convolutions still hold

$$\|u * v\|_p \leq \|u\|_p \|v\|_1 \quad \text{and} \quad \|u * v\|_\infty \leq \|u\|_1 \|v\|_\infty.$$

Throughout this paper the l^2 norm is shortened to $\| \cdot \|$.

In the following theorem we state discrete version of Duhamel's theorem for a finite difference scheme with Pu^n as its homogeneous part.

Proposition 4.1. *The solution v_j^n of the nonhomogeneous difference equation*

$$v_j^{n+1} = P v_j^n + h_j^n$$

is given by

$$v^n = H^n * v^0 + \sum_{j=1}^{n} H^{n-j} * h^{j-1}. \tag{4.2}$$

Proof. Computing $v^{n+1} - v^n$ by using the given expression (4.2), one gets that v^n satisfies

$$v^{n+1} - v^n = (P - I) v^n + h^n.$$

□

Introducing the following notations

$$A(u) := \sum_{l=-k}^{k'} lG_l(u), \quad D(u) := \sum_{l=-k}^{k'} l^2 G_l - A(u)^2,$$

$$B(u,z) := \sum_{l=-k}^{k'} G_l z^l, \qquad z \in \mathbb{C}, \tag{4.3}$$

we then summarize the properties for the monotone scheme in the following

Lemma 4.2. *If the scheme* (1.1) *is strictly monotone, then*

(1) $D(u) > 0$*;*
(2) *the scheme is non-resonant at state* u*, i.e.,*

$$\{z \in \mathbb{C}, \quad |B(u,z)| = 1\} \cap \{|z| = 1\} = \{1\};$$

(3) *the scheme is dissipative in the sense that there exists a constant* $\delta > 0$ *such that*

$$|B(u, e^{i\theta})| \le e^{-\delta\theta^2}, \quad 0 < |\theta| \le \pi.$$

Remarks. i) The quantity $D(u)$ plays the role as a stability index. If $D(u) > 0$, then the scheme is linearly stable at u; if $D(u) < 0$, then the scheme is strongly unstable at u. $D(u) > 0$ also implies that the monotone scheme is at most the first-order accurate approximation to the conservation law (1.2).

ii) It is not hard to show that the stability condition $D(u) > 0$ when combined with the nonresonant condition is equivalent to the dissipative condition (3) of Lemma 4.2. The dissipative condition we present here amounts to the one in the sense of Kreiss i.e., $|B(u, e^{i\theta})| \le 1 - \delta|\theta|^2$ for $0 < |\theta| < \pi$, [18, p. 109].

Proof of Lemma 4.2. (1) Using the monotonicity assumption $G_l > 0$, one may apply the Cauchy-Schwartz inequality to obtain

$$D(u) = \sum l^2 G_l - \left(\sum l\sqrt{G_l} \cdot \sqrt{G_l}\right)^2 > \sum l^2 G_l - \sum l^2 G_l \sum G_l = 0$$

for $\sum G_l = 1$ ensured by the conservation form of the scheme; see [3].

(2) Assume that the scheme is resonant at state u. Then there must exist a $\theta \neq 0$ such that

$$|B(u, e^{i\theta})| = 1.$$

On the other hand, for $G_l > 0$, we have

$$|B(u, e^{i\theta})| = |\sum_l G_l e^{il\theta}| \le \sum_l |G_l| = \sum_l G_l = 1. \tag{4.4}$$

Note that the sign of equality holds if and only if the ratio of any two nonzero terms is positive, then (4.4) holds if and only if $\theta = 0$. This contradiction completes the proof of (2).

(3) By the definition of $B(\theta) := B(u, e^{i\theta})$, one has

$$B(\theta) = \exp\left\{iA\theta - \frac{1}{2}D\theta^2 + O(|\theta|^3)\right\}. \tag{4.5}$$

In fact for θ satisfying $|e^{i\theta}-1|<1$,

$$\begin{aligned}\log B(\theta) &= \log\left(\sum_{l=-k}^{k'} G_l e^{il\theta}\right)\\ &= \log\left(1+\sum_{l=-k}^{k'} lG_l i\theta - \sum_{l=-k}^{k'} G_l \frac{l^2\theta^2}{2} + O(|\theta|^3)\right)\\ &= Ai\theta - \frac{1}{2}D\theta^2 + O(|\theta|^3).\end{aligned}$$

Since $D>0$, we may take $\theta_0<D$ suitably small such that

$$|B(\theta)| \le e^{-1/4D\theta^2}, \quad 0<|\theta|\le\theta_0.$$

Fix this θ_0, then the nonresonant condition implies that

$$a_0 := \sup_{\theta_0\le\theta\le\pi} |B(\theta)| < 1.$$

Now we choose $\delta_1 = \pi^{-2}\log(\frac{1}{a_0})$. Then for θ satisfying $\theta_0 \le |\theta| \le \pi$, the estimate

$$|B(\theta)| \le e^{-\delta_1\theta^2}$$

holds. Thus taking $\delta = \min\{\delta_1, \frac{1}{4D}\}$ we finish the proof of Lemma 4.2. □

Next we collect various facts about the fundamental solution H_j^n which will be required in the subsequent analysis.

Proposition 4.3. *Assume that the scheme* (1.1) *is dissipative at* $u=0$. *Then there exists* $\epsilon(n)\to 0$ *such that the sequence* H_j^n *satisfies*

(i) $\sum_j H_j^n = 1$;

(ii) $H_j^n = 0$ *for* $-j<-kn$ *or* $-j>k'n$ *and for* $-nk\le -j\le nk'$

$$0 \le H_j^n = \frac{1}{\sqrt{2\pi nD}}e^{-\alpha^2/2} + \frac{\epsilon(n)}{\sqrt{nD}(1+\alpha^2)} \quad \text{with} \quad \alpha = \frac{j+nA}{\sqrt{nD}};$$

(iii) $\|\Delta_+^1 H_j^n\|_1 \le C/\sqrt{n}$, *where* $\Delta_+ H_j^n = H_{j+1}^n - H_j^n$,

In the proof of Proposition 4.3, we shall use an asymptotic expansion result in local limit theorems in the context of probability. We summarize a version in the following.

Asymptotic expansion in local limit theorems. Let $\{X_n\}$ be a sequence of integer-valued random variables having a common distribution. Suppose the variance $\mathrm{Var}(X_1)=\sigma^2$ and that the maximal span of the distribution of X_1 is equal to 1. Then

$$(1+|x|^2)\left(\sigma\sqrt{n}P_n(N) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right) = \epsilon(n) \tag{4.6}$$

uniformly in x. Here $x=\frac{N-nEX_1}{\sigma\sqrt{n}}$ and $\epsilon\to 0$ as $n\to\infty$.

This is a simple version of the asymptotic expansion in [17, Theorem 16, p. 207].

Proof of Proposition 4.3. (i) Part (i) holds for $n=0$ and also for $n>0$ by induction since

$$\sum_j (H_j^n - H_j^{n-1}) = (P-I)\sum_j H_j^{n-1} = (P-I)\cdot 1 = 0.$$

(ii) Similarly, $H_j^n \geq 0$ for $n = 0$, and the general case follows by induction, since u_j^n is a convex combination of u_{j+l}^{n-1} for $-k \leq l \leq k'$.

To prove the estimate in (ii), we shall use the asymptotic expansion result (4.6) for local limit theorems. We consider a sequence $\{X_n\}$ of random i.i.d. variables such that X_1 takes on the values

$$-k, -k+1, \cdots, 1, \cdots, k'.$$

Since $\sum_l G_l = 1$ and $G_l > 0$ for $-k \leq l \leq k'$, we can consider G_l as the probability distribution for the discrete random variable X_1, with $P(X_1 = -l) = G_l$ for $-k \leq l \leq k'$. Then H_j^n becomes the probability distribution for $S_n = \sum_{i=1}^n X_i$, i.e.,

$$H_j^n = P_n(j) = P(S_n = j),$$

which when combined with $S_0 = 0$ is consistent with the definition of H_j^0. We also find that

$$E(X_1) = \sum -lP(X_1 = -l) = -\sum_{l=-k}^{k'} lG_l =: -A$$

and the variance

$$\begin{aligned}\mathrm{Var}(X_1) = E(X_1 - EX_1)^2 &= \sum_{l=-k}^{k'} [-l - \sum_{j=-k}^{k'} -jG_j]^2 G_l \\ &= \sum l^2 G_l - (\sum lG_l)^2 = D = \sigma^2.\end{aligned}$$

Let $x = \alpha = \frac{j+nA}{\sqrt{nD}}$. Then the asymptotic expansion (4.6) with $N = j$ and $\sigma = \sqrt{D}$ gives

$$(1 + |\alpha|^2)\left(\sqrt{nD}H_j^n - \frac{1}{\sqrt{2\pi}}e^{-\alpha^2/2}\right) = \epsilon(n)$$

uniformly in α. Hence

$$H_j^n = \frac{1}{\sqrt{2\pi nD}}e^{-\alpha^2/2} + \frac{\epsilon(n)}{\sqrt{nD}(1+\alpha^2)}.$$

(iii) Let U_n, V_n be two mutually independent random walks starting at 0 and -1, respectively. It is a property of random walks that both move together with the same probability after the first time $U_n = V_n$. Then this first hitting time is finite, i.e., defining

$$\tau = \min\{n \geq 1, \quad U_n = V_n\}, \quad P(\tau < \infty) = 1$$

and

$$P(\tau > n) \leq C/\sqrt{n} \quad \text{with} \quad C \geq \sqrt{\frac{2}{\pi}}\sigma.$$

The above known facts in probability [22, P3 in Section 32, p. 381] lead to

$$\begin{aligned}
\sum_k |H^n_{k+1} - H^n_k| &= \sum_k |P(S_n = k+1) - P(S_n = k)| \\
&= \sum_k |P(V_n = k+j) - P(U_n = k+j)| \\
&= \sum_k |P(U_n = k \quad | \quad \tau \le n) + P(U_n = k \quad | \quad \tau > n) \\
&\qquad - P(V_n = k \quad | \quad \tau \le n) - P(V_n = k \quad | \quad \tau > n)| \\
&= \sum_k |P(U_n = k \quad | \quad \tau > n) - P(V_n = k \quad | \quad \tau > n)| \\
&\le 2P(\tau > n) \\
&\le C/\sqrt{n}.
\end{aligned}$$

Taking $\sigma = \sqrt{D}$ finishes the proof of (iii). □

Remark 4.4. A direct estimate on H^n_j without using the known results for probability is possible. The idea is to use the discrete version of the Fourier transform via defining

$$\hat{H}^n(\theta) = \left(\sum_{l=-k}^{k'} G_l e^{il\theta}\right)^n = B(u, e^{i\theta})^n =: B(\theta)^n.$$

The inverse Fourier transformation gives

$$H^n_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta}\hat{H}^n(\theta)d\theta.$$

The main task is to carefully estimate the above integral.

5. l^1 decay to constant state

The goal of this section is to prove Theorem 1.4. To this end we need to establish the following basic energy estimate.

Lemma 5.1. *Let* (1.1) *be a monotone conservative difference scheme. Then there exists a positive constant* c_0 *such that for initial data* $u^0 \in l^1$ *and all* $n \in \mathbb{N}$,

$$\|u^{n+1}\|^2 - \|u^n\|^2 + c_0\|\Delta u^n\|^2 \le 0 \tag{5.1}$$

provided λ *is suitably small.*

The restriction on λ to ensure the energy estimate (5.1) will be clarified later in Section 6. Equipped with Lemma 5.1, we proceed to investigate the decay properties of the numerical solutions.

5.1. l^∞ decay.

Lemma 5.2. *If* (5.1) *is satisfied, then*

$$\lim_{n\to\infty} \|u^n\|_\infty = 0. \tag{5.2}$$

Proof. Summing up over n in (5.1), one gets

$$\|u^n\|^2 + c_0 \sum_{i<n} \|\Delta u^i\|^2 \le \|u^0\|^2. \tag{5.3}$$

On the other hand we have

$$\|u^n\|_\infty^2 \le \sum_{j\in\mathbb{Z}} |(u_j^n)^2 - (u_{j-1}^n)^2| \le \sum_{j\in\mathbb{Z}} |u_j^n + u_{j-1}^n||\Delta u_j^n|.$$

Using the Cauchy-Schwartz inequality, one obtains

$$\|u^n\|_\infty^2 \le 2\|u^n\|\|\Delta u^n\|.$$

This, when combined with the basic estimate (5.3), gives

$$\sum_{i<n} \|u^i\|_\infty^4 \le 4\|u^0\|^2 \sum_{i<n} \|\Delta u^i\|^2 \le 4c_0^{-1}\|u^0\|^4.$$

Since $\{\|u^i\|_\infty^4\}$ is summable, we have

$$\lim_{n\to\infty} \|u^n\|_\infty = 0.$$

□

The decay property above allows us to assume $\|u^0\|_\infty \le 1$, and hence by monotonicity $\|u^n\|_\infty \le 1$.

5.2. l^2 decay.

Lemma 5.3. *Given the energy inequality* (5.1)*, there exists a* $C_0 > 0$ *such that*

$$\|u^n\| \le C_0\|u^0\|_1 n^{-1/4}, \quad n \in \mathbb{N}. \tag{5.4}$$

Proof. Since $\|u^n\|^2 \le \|u^n\|_\infty\|u^n\|_1$, the l^1 contracting combined with

$$\|u^n\|_\infty \le \sqrt{2}\|u^n\|^{1/2}\|\Delta u^n\|^{1/2}$$

gives the inequality

$$\|u^n\|^3 \le 2\|u^0\|_1^2\|\Delta u^n\|. \tag{5.5}$$

Rewriting the basic relation (5.1) as

$$(\|u^{n+1}\| - \|u^n\|)^2 + 2\|u^n\|(\|u^{n+1}\| - \|u^n\|) + c_0\|\Delta u^n\|^2 \le 0$$

and multiplying by $\|u^n\|^5$, one gets

$$2\|u^n\|^6(\|u^{n+1}\| - \|u^n\|) + c_0\|\Delta u^n\|^2\|u^n\|^5 \le 0.$$

Inserting the inequality (5.5) into the above leads to

$$\|u^{n+1}\| - \|u^n\| + M\|u^n\|^5 \le 0$$

with $M = c_0/(8\|u^0\|_1^4)$. From this inequality and the fact that $\|u^{n+1}\| \le \|u^n\|$, a direct calculation yields

$$\|u^{n+1}\|^{-4} - \|u^n\|^{-4} \ge 4M.$$

This by summation over index n gives

$$\|u^n\|^{-4} \ge 4Mn + \|u^0\|^{-4}.$$

Thus we obtain

$$\|u^n\| \le (4Mn)^{-1/4} = (2/c_0)^{1/4}\|u^0\|_1 n^{-1/4}, \quad n \in \mathbb{N}, \tag{5.6}$$

due to $M = c_0/(8\|u_0\|_1^4)$.

□

5.3. l^1 **decay for small data.**

Lemma 5.4. *Let $u^0 \in l^1$ with $\sum_j u_j^0 = 0$ and*

$$\|u^0\|_1 \le C_1^{-1}$$

for some constant $C_1 > 0$. Assuming $g(\bar{u}) \in C^2(\mathbb{R}^{k+k'})$, then

$$\lim_{n\to\infty} \|u^n\|_1 = 0.$$

Proof. We still assume that $\|u^0\|_\infty \le 1$. First we rewrite the scheme as

$$u_j^n = Pu_j^n + G(u_{j-k}^n, \cdots, u_{j+k'}^n) - Pu_j^n.$$

The nonlinear part is in conservation form and can be written as

$$G(u_{j-k}^n, \cdots, u_{j+k'}^n) - Pu_j^n = -\lambda(F_{j+1}^n - F_j^n) \tag{5.7}$$

with

$$|F_j^n| \le M \sum_{l=-k}^{k'} (u_{j+l}^n)^2. \tag{5.8}$$

In fact, let $\bar{u} = (u_{-k}, \cdots, u_{k'-1})$ and $E\bar{u} = (u_{-k+1}, \cdots, u_{k'})$. Differentiating the relation of G with respect to u_l, we get

$$G_l = \delta_{0,l} - \lambda[g_{l-1}(E\bar{u}) - g_l(\bar{u})], \quad -k \le l \le k', \tag{5.9}$$

where a subscript l denotes partial differentiation with respect to the lth components and $g_l \equiv 0$ for $l \ge k'$ or $l \le -k-1$. Define

$$F_j^n = g(u_{j-k}^n, \cdots, u_{j+k'-1}^n) - \sum_{l=-k}^{k'-1} g_l(\bar{0}) u_{j+l}^n,$$

which gives (5.7). The assumption $g(\bar{u}) \in C^2(\mathbb{R}^{k+k'})$ implies that second partial derivatives of g are bounded for $|u| \le 1$. Then the estimate for F_j^n in (5.8) is immediate.

Using the discrete Duhamel's principle (4.2), one has

$$\begin{aligned} u_j^n &= H^n * u^0 + \sum_{m=1}^{n} H^{n-m} * [-\lambda \Delta_+ F^{m-1}] \\ &= H^n * u^0 + \lambda \sum_{m=1}^{n} \Delta_- H^{n-m} * F^{m-1}. \end{aligned} \tag{5.10}$$

By the Young inequality,

$$\|u^n\|_1 \le \|H^n * u^0\|_1 + \lambda \sum_{m=1}^{n} \|\Delta_- H^{n-m} * F^{m-1}\|_1. \tag{5.11}$$

Now the $m = 1$ term in (5.11) is bounded by

$$\lambda \|\Delta_- H^{n-1} * F^0\|_1 \le \lambda \|\Delta_- H^{n-1}\|_1 \|F^0\|_1 \le C\lambda/\sqrt{n-1},$$

by (iii) in Proposition 4.3, and the $m = n$ term is bounded by

$$\begin{aligned} \lambda \|\Delta_- H^0 * F^{n-1}\|_1 &\le \lambda \|\Delta_- H^0\|_1 \|F^{n-1}\|_1 \\ &\le 2\lambda \|H^0\|_1 \cdot (k + k' + 1) M \|u^{n-1}\|_2^2 \\ &\le C\lambda/\sqrt{n-1} \end{aligned}$$

by (5.8) and the l^2 decay estimate (5.4). Therefore (5.11) gives

$$\begin{aligned}\|u^n\|_1 &\le \|H^n * u^0\|_1 + \lambda \sum_{m=2}^{n-1} \|F^{m-1}\|_1 \|\Delta_- H^{n-m}\|_1 + \frac{C\lambda}{\sqrt{n-1}} \\ &\le \|H^n * u^0\|_1 + \lambda \sum_{m=2}^{n-1} \frac{C_0^2}{\sqrt{m-1}} \frac{C}{\sqrt{n-m}} \|u^0\|_1^2 + \frac{C\lambda}{\sqrt{n-1}} \\ &\le \|H^n * u^0\|_1 + C\|u^0\|_1^2 + \frac{C\lambda}{\sqrt{n-1}},\end{aligned}$$

where the inequality $\|u^{m-1}\|^2 \le C_0^2 \|u^0\|_1^2 (m-1)^{-1/2}$ in (5.4) has been used. The fact that the sum in the second term on the right side is bounded independent of n is supported by the inequality

$$\sum_{m=2}^{n-1} \frac{1}{\sqrt{n-m}\sqrt{m-1}} \le \int_0^{n-1} \frac{1}{\sqrt{n-1-x}\sqrt{x}} dx \le \int_0^1 \frac{1}{\sqrt{1-x}\sqrt{x}} dx < \infty.$$

From the fact that the mass for u^0 is zero, i.e., $\sum_j u_j^0 = 0$, it follows that $\|H^n * u^0\|_1$ tends to zero as $n \to \infty$ (see Appendix). Hence, letting $n \to \infty$ in the above inequality, we have

$$l := \operatorname{limsup}_{n\to\infty} \|u^n\|_1 \le C_1 \|u^0\|_1^2.$$

Replacing $n = 0$ by $n = m > 0$, one also has

$$l \le C_1 \|u^m\|_1^2.$$

Passing to the limit as $m \to \infty$ yields the inequality $l \le C_1 l^2$. Recalling that $l \le \|u^0\|_1 < C_1^{-1}$ one has $l = 0$ and thus the proof of Lemma 5.4 is complete. □

End of the proof of Theorem 1.4. Let us define the space

$$Y = \{a_j, a \in l^1, \quad \sum_j a_j = 0, \|a\|_\infty \le 1\} \tag{5.12}$$

endowed with the l^1-distance. It is a convex set and $TY \subset Y$ due to the stated semigroup properties. Let

$$Y_0 = \{a \in Y, \quad \lim_{n\to\infty} \|T^n a\|_1 = 0\}$$

be an attraction basin of zero state. Obviously $0 \in Y_0$, and Y_0 is closed due to the l^1 contracting property. From Lemma 5.4 it follows that

$$\{a \in Y, \|a\|_1 \le C_0\} \subset Y_0$$

for $C_0 < C_1^{-1}$ is an attraction basin. For any $u^0 \in Y$, we shall prove that $u^0 \in Y_0$. In fact since $u^0 \in l^1$, there exists an $m \in \mathbb{N}$ such that $u^0/m \in Y_0$ with $\|u^0/m\|_1 < C_0$. Noting that

$$\|T^n u^0\|_1 \le \|T^n u^0 - T^n(u^0/m)\|_1 + \|T^n(u^0/m)\|_1,$$

we find that the first term on the right side is bounded by $\|\frac{m-1}{m} u^0\|_1 < (m-1)C_0$ because of the l^1 contracting. The second term tends to zero as n becomes large. Then for large n_1 we have $\|T^n u^0\|_1 < (m-1)C_0$ for $n > n_1$. An induction on m from $m = 2$ implies that $u^0 \in Y_0$. Thus we have $Y = Y_0$.

Now, we recall that the solution enters Y provided $u^0 \in l^\infty$. Thus the theorem is valid with the condition that $u^0 \in l^1$. Theorem 1.4 is proved.

6. Energy dissipation

Before going further, let us recall that there is a close relation between monotonicity and the presence of the viscosity terms; see [3]. Let u be a smooth solution of (1.2). The truncation error is

$$u(x, t+\Delta t) - G(\bar{u}(x,t)) = -(\Delta t)^2[\beta(u,\lambda)u_x]_x + O((\Delta t)^3),$$

where

$$\beta(u,\lambda) = \frac{1}{2\lambda^2}D(u)$$

which is positive as shown in Lemma 4.2.

Thus the monotone scheme (1.1) is a second-order accurate approximation to smooth solutions of the viscous conservation law

$$u_t + f(u)_x = \frac{\Delta x}{2\lambda}[D(u)u_x]_x.$$

These facts motivate us to establish the energy decay estimate as stated in Lemma 5.1.

Let $Q^n_{j+1/2}$ be defined as

$$Q^n_{j+1/2} = \lambda[f(u^n_{j+1}) + f(u^n_j) - 2g(u^n_{j-k+1}, \cdots, u^n_{j+k'})](\delta_+ u^n_j)^{-1} \tag{6.1}$$

with $\delta_+ u^n_j := u^n_{j+1} - u^n_j \neq 0$. Thus one may rewrite the scheme (1.1) into Q-form

$$\begin{aligned} u^{n+1}_j = u^n_j &- \frac{\lambda}{2}[f(u^n_{j+1}) - f(u^n_{j-1}] \\ &+ \frac{1}{2}\Big[Q^n_{j+1/2}(u^n_{j+1} - u^n_j) - Q^n_{j-1/2}(u^n_j - u^n_{j-1})\Big]. \end{aligned} \tag{6.2}$$

Here $Q^n_{j+1/2}$ serves as the numerical viscosity counterbalancing the effect from the numerical flux in the first bracket; see [23].

Based on this Q-form, this lemma tells us how small λ is in the following

Lemma 6.1. *The solution to the monotone scheme* (1.1) *satisfies the energy inequality* (5.1) *if*

$$\lambda \sup_{(s-u^{n+1}_j)(s-u^n_{j+1})\le 0} \left| \frac{f(u^n_{j+1}) - f(s)}{u^n_{j+1} - s} \right| \le Q^n_{j+1/2} < 1. \tag{6.3}$$

Proof. Note that for any u^n_j we have the identity

$$(u^{n+1}_j)^2 - (u^n_j)^2 + (u^{n+1}_j - u^n_j)^2 = 2u^{n+1}_j(u^{n+1}_j - u^n_j). \tag{6.4}$$

It follows from (6.2) that

$$\begin{aligned} 2u^{n+1}_j(u^{n+1}_j - u^n_j) = &-\lambda u^{n+1}_j[f(u^n_{j+1}) - f(u^n_{j-1})] + Q^n_{j+1/2}u^{n+1}_j(u^n_{j+1} - u^n_j) \\ &- Q^n_{j-1/2}u^{n+1}_j(u^n_j - u^n_{j-1}), \end{aligned}$$

where the first term on the right equals

$$\begin{aligned} -\lambda u^{n+1}_j[f(u^n_{j+1}) - f(u^n_{j-1})] = &-\lambda\int_{u^n_{j-1}}^{u^n_{j+1}} sf'(s)ds + \lambda\int_{u^{n+1}_j}^{u^n_{j+1}}[f(u^n_{j+1}) - f(s)]ds \\ &- \lambda\int_{u^{n+1}_j}^{u^n_{j-1}}[f(u^n_{j-1}) - f(s)]ds. \end{aligned}$$

To estimate the remaining two terms, we recall the identity for any $l \in \mathbb{Z}$

$$2u_j^{n+1}(u_{j+l+1}^n - u_{j+l}^n) = (u_{j+l+1}^n)^2 - (u_{j+l}^n)^2 + (u_j^{n+1} - u_{j+l}^n)^2 - (u_j^{n+1} - u_{j+l+1}^n)^2.$$

Define $F(u) = \int_0^u f'(s)sds$ and $F_j^n = F(u_j^n)$. Then a combination of these identities leads to

$$\begin{aligned}
2u_j^{n+1}(u_j^{n+1} - u_j^n) = -\lambda[\tilde{F}_{j+1/2}^n - \tilde{F}_{j-1/2}^n] + \frac{1}{2}\big[Q_{j+1/2}^n + Q_{j-1/2}^n\big](u_j^{n+1} - u_j^n)^2 \\
+ a_j^+(u_j^{n+1} - u_{j+1}^n)^2 + a_j^-(u_j^{n+1} - u_{j-1}^n)^2,
\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}
\tilde{F}_{j+1/2}^n &= F_{j+1}^n + F_j^n - \frac{1}{2}\lambda^{-1}Q_{j+1/2}^n[(u_{j+1}^n)^2 - (u_j^n)^2], \\
a_j^+ &= -\frac{1}{2}Q_{j+1/2}^n + \lambda(u_j^{n+1} - u_{j+1}^n)^{-2}\int_{u_j^{n+1}}^{u_{j+1}^n}(f(u_{j+1}^n) - f(s))ds, \\
a_j^- &= -\frac{1}{2}Q_{j-1/2}^n - \lambda(u_j^{n+1} - u_{j-1}^n)^{-2}\int_{u_j^{n+1}}^{u_{j-1}^n}(f(u_{j-1}^n) - f(s))ds.
\end{aligned}$$

To obtain the desired energy estimate, we assume a priori $u_j^n \in l^2$. Hence summation of (6.4) over $j \in \mathbb{Z}$ when combined with (6.5) gives

$$\begin{aligned}
||u^{n+1}\|^2 - \|u^n\|^2 = -\sum_j\left(1 - \frac{1}{2}[Q_{j+1/2}^n + Q_{j-1/2}^n]\right)(u_j^{n+1} - u_j^n)^2 \\
+ \sum_j a_j^+(u_j^{n+1} - u_{j+1}^n)^2 + \sum_j a_j^-(u_j^{n+1} - u_{j-1}^n)^2.
\end{aligned}$$

Note that

$$\begin{aligned}
a_j^\pm &= -(u_j^{n+1} - u_{j\pm1}^n)^{-2}\int_{u_j^{n+1}}^{u_{j\pm1}^n}[Q_{j\pm1/2}^n \mp \lambda\frac{f(u_{j\pm1}^n) - f(s)}{u_{j\pm1}^n - s}](u_{j\pm1}^n - s)ds \\
&\leq -\left[Q_{j\pm1/2}^n + \lambda\inf_{(s-u_j^{n+1})(s-u_{j\pm1}^n)\leq 0}(\mp\frac{f(u_{j\pm1}^n) - f(s)}{u_{j\pm1}^n - s})\right].
\end{aligned}$$

Therefore, by virtue of the constraint condition (6.3), there exists a positive constant β such that

$$a_j^\pm \leq 0, \qquad 1 - \frac{1}{2}[Q_{j+1/2}^n + Q_{j-1/2}^n] \geq \beta.$$

These enable us to obtain

$$||u^{n+1}\|^2 - \|u^n\|^2 + \beta\sum_{j\in\mathbb{Z}}(u_j^{n+1} - u_j^n)^2 \leq 0,$$

which when combined with the scheme itself leads to the desired estimate. $\square$

As an immediate application we show that the above condition (6.3) is just a version of the CFL condition. Let us consider the generalized Lax-Friderichs scheme of the form

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}[f(u_{j+1}^n) - f(u_{j-1}^n)] + \frac{\alpha}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{6.6}$$

Note that the original Lax-Fridrichs scheme with $\alpha = 1$ is not a strictly monotone scheme. This scheme (6.6) is well posed under the CFL condition

$$\lambda \sup |f'| \leq \alpha < 1. \tag{6.7}$$

Note that in this example $Q^n_{j+1/2} = \alpha < 1$ and the above CFL condition ensures the condition (6.3). Therefore, the energy inequality (5.1) holds for scheme (6.6), and Theorem 1.1 applies to the generalized Lax-Friderichs scheme (6.6) under the CFL condition (6.7). ☐

Remark 6.2. The role played by the *upper bound* on numerical viscosity in (6.3) is related to the hyperbolic nature of the approximated conservation laws, as it amounts to the CFL-like condition

$$\lambda |f(u_j) - h_{j+1/2} + f(u_{j+1}) - h_{j+1/2}| \leq |u_{j+1} - u_j|.$$

Unfortunately this restriction is supported by 3-point schemes.

Appendix

This appendix is devoted to proving a l^1 decay property of solutions to linearized scheme (4.1), subject to the initial data with zero mass.

Lemma A.1. *Given a discrete function H^n as in Proposition 4.1 and the data $u^0 \in l^1$, then*

$$\lim_{n\to\infty} \|H^n * u^0\|_1 = 0$$

if and only if

$$\sum_{i\in\mathbb{Z}} u^0_i = 0.$$

Proof. The necessity follows from the conservation property of the total mass. Next we shall prove the sufficiency. By the contraction property it suffices to prove the lemma for the data u^0_j satisfying $\sum_j u^0_j = 0$ and $u^0_j = 0$, $|j| \geq J$ for some finite number $J > 0$. First note that $\|H^n\|_1 = 1$ and

$$\|H^n\|_2 \leq Cn^{-1/4}$$

for any $n > 0$. In fact the l^1 estimate above follows from the positivity of H^n and $\sum_j H^n_j = 1$; and the l^2 estimate can be obtained by using the point-wise bound for H^n. Actually, using $H^n_j = \frac{1}{\sqrt{2\pi nD}} e^{-\alpha^2/2} + \frac{\epsilon(n)}{\sqrt{nD}(1+\alpha^2)}$ with $\alpha = \frac{j+nA}{\sqrt{nD}}$, and $H^n_j = 0$ for $-j > k'n$ or $-j < -kn$, we have

$$\begin{aligned}\|H^n\|_2^2 &\leq \sum_{j=-kn}^{k'n} \left[\frac{1}{\pi nD} e^{-\alpha^2} + \frac{2\epsilon(n)^2}{nD(1+\alpha^2)^2}\right] \\ &\leq \frac{C}{\sqrt{n}}\end{aligned}$$

for $\alpha = \frac{j+nA}{\sqrt{nD}}$. Equipped with these estimates, one obtains

$$\|u^n\|_2 \leq \|H^n\|_2 \|u^0\|_1 \leq Cn^{-1/4}.$$

Setting $v^n_j = \sum_{i=-\infty}^{j} u^n_i$, then $u^n_j = v^n_j - v^n_{j-1}$ and

$$v^n = H^n * v^0.$$

Clearly one can control the l^2 norm of v^n by $\|v^0\|_2$ since

$$\|v^n\|_2 \le \|H^n\|_1\|v^0\|_2 \le \|v^0\|_2.$$

This, when combined with the inequality

$$\|v^n\|_\infty \le \sqrt{2}\|v^n\|_2^{1/2}\|\Delta v^n\|_2^{1/2} = \sqrt{2}\|v^n\|_2^{1/2}\|u^n\|_2^{1/2},$$

yields the decay of $\|v^n\|_\infty$, i.e. $\lim_{n\to\infty}\|v^n\|_\infty = 0$. From the monotonicity assumption it follows that the solution u_j^n changes sign with index j still finite times, say N. Therefore we assume that u_j^n does not change sign for $J_i \le j \le J_{i+1}$. Thus

$$\sum_{i\in\mathbb{Z}}|u_i^n| = \sum_{i=1}^{N}\sum_{j=J_i}^{J_{i+1}}|u_j^n| = \sum_{i=1}^{N}|\sum_{j=J_i}^{J_{i+1}}u_j^n| = \sum_{i=1}^{N}|v_{J_{i+1}}^n - v_{J_i}^n| \le 2N\|v^n\|_\infty \to 0$$

as $n \to \infty$ since here N is a finite number. □

Remark A.2. We would like to point out that usually one can only get the decay rate for the general l^1 initial data as we considered in the above proof. The extra decay rate $n^{-1/2}$ is possible only by further specifying the initial data. We take, for instance, u^0 such that $u_j^0 = v_j^0 - v_{j-1}^0$ with $v^0 \in l^1$. Then we have

$$u^n = H^n * u^0 = \Delta H^n * v^0.$$

This, when combined with (iii) of Proposition 4.3, gives

$$\|u^n\|_1 \le \|\Delta H^n\|_1\|v^0\|_1 \le C\|v^0\|_1/\sqrt{n} \to 0.$$

For general l^1 initial data our proof is sharp.

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