BOUNDARY CONDITIONS FOR THE MICROSCOPIC FENE MODELS

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Abstract. We consider the microscopic equation of FENE (finite extensible nonlinear elasticity) models for polymeric fluids under a steady flow field. This paper will show for the underlying Fokker-Planck type of equations, any pre-assigned distribution on boundary will become redundant once the non-dimensional number $Li := \frac{Hb}{k_B T} \geq 2$, where H is the elasticity constant, $\sqrt{b}$ is the maximum dumbbell extension, T is the temperature and k_B is the usual Boltzmann constant. Moreover, if the probability density function is regular enough for its trace to be defined on the sphere $|m| = \sqrt{b}$, then the trace is necessarily zero when $Li > 2$. These results are consistent with our numerical simulations as well as some recent well-posedness results by pre-assuming a zero boundary distribution.

1. Introduction

The two-scale macro-micro models have been proven successful in describing the dynamics of many polymeric fluids. The systems usually consist of a macroscopic momentum equation (the force balance equation) and a microscopic evolution of the probability distribution functions (PDF) through Fokker-Planck equations [4, 7]. The coupling of the micro-macro interaction is through the transport of the PDF in the microscopic equations and the induced elastic stress in the macroscopic equations. It is the through this interaction, the competition between the kinetic energy and the (multi-scale) elastic energies give arise to all different hydrodynamical and rheological properties of these materials [7, 15].

Let y be the macroscopic Eulerian (observer's) coordinate, m the microscopic molecule configurational variable. The distinguished representation of the variables represents the nature of the scale separation of these models. Assume $u = u(y,t)$ be the macroscopic velocity field of the flow and $y(X,t)$ be the induced flow map (trajectory) with macroscopic material coordinate X. $f = f(t, m, y)$, $(m, y) \in \mathcal{R}^{2d}$ is the probability distribution function of the molecule separation. The dependence of the macroscopic coordinate of f attributes to the macroscopic anisotropy of the materials. Moreover, the models assume that the microscopic deformation is the same as the macroscopic deformation through the macroscopic covariant (or anti-covariant) deformation of m. In the case of $m = F\tilde{m}$, with $F = \frac{\partial y}{\partial X}$ the (macroscopic) deformation tensor induced by the flow map $y(X, t)$, and $\tilde{m}$ the undeformed configuration, we have the following microscopic evolution equation [17]:

$$(1) \quad \partial_t f + u \cdot \nabla_y f + \nabla_m \cdot (\kappa m f) = \frac{2}{\gamma}\left[(\nabla_m \cdot (\nabla_m U f) + k_B T \Delta_m f)\right],$$

where $\kappa = \nabla_y u$ is the strain rate tensor; U denotes the spring potential; γ is the friction coefficient, T is the absolute temperature, and k_B is the Boltzmann constant.

Equation (1) models both the convection and stretching of the polymers by the macroscopic flow, and the microscopic convection diffusion evolution. The later mechanism can be interpreted with its corresponding stochastic differential equation [11, 16]

$$dm = \left(-\frac{2}{\gamma}\nabla_m U\right) dt + \sqrt{\frac{4k_B T}{\gamma}} dW_t,$$

where W_t is a standard Brownian motion, which in turn gives the Fokker-Planck dynamic to the PDF.

Much of the material properties can be attributed to different microscopic energies. The simplest spring potential is given by the Hookean law $U(m) = H|m|^2/2$, where H is the elasticity constant. This potential has the distinguished feature that the system is closed under 2nd moment closure, which yields the usual Oldroyd models [2, 7, 17].

A more commonly used model is with the following FENE (Finite Extendible Nonlinear Elasticity) potential:

$$U(m) = -\frac{Hb}{2} log(1 - |m|^2/b),$$

which takes into account a finite-extensibility constraint by assigning infinite energy when the molecule length approaches $\sqrt{b}$, the maximum dumbbell extension [2]. In this case, the convective spring force becomes

$$\nabla_m U = \frac{Hm}{1 - |m|^2/b}, \tag{2}$$

which also becomes infinity on the boundary of $B_{\sqrt{b}}$. Intuitively, this should mean that the Fokker-Planck equation (1) is only defined on the open ball $\Omega = \{m \in \mathcal{R}^d : \quad |m|^2 < b\}$, where $d = 2, 3$, disregard of the boundary condition pre-imposed on the solution of (1). On the other hand, the diffusion due to the thermo-fluctuation does have infinite propagation speed. Also, the Brownian motion is unbounded in L^∞ norm.

The main complexity with the FENE potential lies mainly with the singularity of the equation at the boundary. In this paper, we will discuss boundary conditions for the Fokker-Planck equation alone, with the fluid velocity being steady and homogeneous. The velocity gradient will be treated as a constant matrix.

Observe that since equation (1) experiences singularity on the sphere $|m| = \sqrt{b}$, the data may not be necessarily well defined. The main issues of our interest in this work are:

- if the boundary conditions are necessary or redundant;
- if the PDF solution is regular enough to have a trace on the boundary, what is its trace, no matter if the data is pre-imposed or not.

These issues for the underlying FENE models are fundamental and attracted much attention in the study of the wellposedness in certain weighted Sobolev spaces, see e.g. [14], as well as in the 2D SDE framework [13]. However the how issue remained to be mystic.

Our key observation in this paper is that the answer to above questions hinges on whether the non-dimensional quantity

$$Li := \frac{Hb}{k_B T} \tag{3}$$

crosses a critical value 2. The main goals of this paper are to show two new results, namely (i) for the underlying Fokker-Planck equation (1), any boundary condition will become redundant once the non-dimensional number $Li \geq 2$; (ii) if the probability density function f is regular enough for its trace to be defined on the sphere $|m| = \sqrt{b}$, then the trace is necessarily zero when $Li > 2$. Physically speaking, no boundary condition is necessary for the case with given spring at low temperature or the case at fixed temperature with large product of H and b.

To put our work in a proper perspective we recall that well-posedness of the coupled micro-macro system or Fokker-Planck equation alone have attracted much attention recently [5, 6, 8, 12]. In particular, the local existence results of [22], in the weighted Sobolev space will also force the zero boundary condition. Up to now, most works are with prescribed boundary conditions. On the other hand, numerical simulations seemed to indicate that the solutions are not sensible to such boundary conditions. We also refer to [14] for the study of large-time behavior of the coupled micro-macro system and a rigorous formulation of the no-flux boundary condition.

We now proceed to identify the key quantity Li defined in (3) by making the following scaling

$$(y, u, t, m, b) \to \left(\frac{y}{L_0}, \frac{u}{U_0}, \frac{t}{T_c}, \frac{m}{l}, \frac{b}{l^2} \right)$$

where $T_c := L_0/U_0$ is the macroscopic convective time scale and $l := \sqrt{\frac{k_B T}{H}}$ serves as the mesoscopic length scale of the spring. We further introduce the non-dimensional parameter $De = \frac{T_r}{T_c}$, where $T_r = \frac{\gamma}{4H}$ characterizes the mesoscopic relaxation time scale of the spring, and De is often called the Deborah number, which is a unique parameter in non-Newtonian fluids.

Putting all above together and still using (y, u, t, m, b) for the scaled quantities, equation (1) thus reduces to

$$\partial_t f(t, m) + \nabla_m \cdot (\kappa m f) = \frac{1}{2De} (\nabla_m \cdot (\nabla_m U f) + \Delta_m f), \tag{4}$$

where $\kappa = \nabla_y u$ is the steady homogeneous velocity gradient. $Tr(\kappa) = \nabla_y \cdot u = 0$ for the incompressible flow. Note that the corresponding square of radius for the non-dimensional configuration variable m is exactly the key parameter $Li = b/l^2 = \frac{Hb}{k_B T}$ as given in (3) (though still denoted by b in what follows). The potential in non-dimensional form thus reads

$$\nabla_m U = \frac{m}{1 - |m|^2/b}. \tag{5}$$

In order to simplify the notation, we simply take $De = 1$ and obtain

$$\partial_t f(t,m) + \nabla_m \cdot (\kappa m f) = \frac{1}{2}(\nabla_m \cdot (\nabla_m U f) + \Delta_m f). \tag{6}$$

For equation (6) with scaled potential (5) our first result is that boundary conditions are unnecessary when $b \geq 2$.

Theorem 1.1. *Let $T^* > 0$ be any fixed number. Consider the equation (6) in the domain $\Omega(T^*) := \{(t,m),\ 0 \leq t \leq T^*, |m| < \sqrt{b}\}$. When $b < 2$ the PDF on boundary must be imposed on $|m|^2 = b$ and when $b \geq 2$, any pre-assigned distribution on boundary $|m|^2 = b$ will become redundant, and in this case the original problem can be formulated as follows:*

to find a distribution function $f(t,m)$ such that in $\Omega(T^*)$ the equation (6) holds and $f(0,m) = f_0(m)$, where $f_0 \geq 0$ is a given bounded measurable function that is supported in $\Omega(0)$.

Here we want to make several remarks:
(1) The statement in this theorem is justified based on the use of Fichera's criterion [3], which does not apply to other than Dirichlet type of boundary conditions.
(2)The appropriate function space for weak solutions may be chosen as a subspace of the usual Hilbert space, restricted with a proper weight to take care of the boundary singularity. The interior regularity is ensured by the classical parabolic theory. Here we choose to leave the solution space and regularity of the PDF unspecified since our focus is on whether boundary conditions are necessary for whatever appropriate spaces to be adopted. Of course based on the above a priori statement one may further investigate the global well-posedness, which is beyond the scope of this paper.
(3)The corresponding analogue of this statement in the SDE framework is known [13], where for the 2D case the authors show that if $b \geq 2$, then the trajectories of the stochastic process representing the evolution of the end-to-end vector does not touch the boundary of radius $\sqrt{b}$, which means the polymer does not reach its maximal extensibility.

Our second result determines the trace of the probability density function on the sphere $|m|^2 = b$.

Theorem 1.2. *Let $f_0(m)$ be a bounded measurable function in $|m| \leq \sqrt{b}$ with* $\mathrm{supp}(f_0(m)) \subset \{m, |m| \leq \sqrt{b^*}\}$*, $b^* < b$. Then for $b > 2$ the solution $f(t, m)$ of equation (6) remains bounded and satisfies*

$$|f| \leq |f_0| \left(\frac{b - |m|^2}{b - b^*} \right)^{b/2 - \alpha} e^{Kt},$$

where α and K satisfy

$$0 < \alpha < \frac{b}{2} - 1, \quad K > K^* := \frac{\beta^2}{16b\alpha(b - 2 - 2\alpha)} - \rho(b - 2\alpha)$$

with $\beta = \rho(b - \alpha)r^2 + 2\alpha(d + b - 2 - 2\alpha)$ and $\rho = \sqrt{Tr(\kappa^\top \kappa)}$.

In comparison we mention that the solution to the stochastic differential equation associated with (6) is shown to exist and have trajectorial uniqueness if and only if $b \geq 2$ [12]. We also refer [5] for an existence result with prescribed zero boundary data.

Our results also show that for $b \geq 2$ well-posedness requires no prescribed boundary value on $|m|^2 = b$ and for $b > 2$ the distribution function, if regular enough to have a trace, must have zero trace

$$f(t, m)|_{|m| = \sqrt{b}} = 0. \tag{7}$$

In other words, one is not allowed to prescribe boundary data on the sphere $|m| = \sqrt{b}$ other than (unnecessary) $f = 0$ or a natural no-flux boundary condition.

The difficulty of the problem lies in the singularity of the equation occurring at boundary. The key to our approach is to rewrite the equation into a second order equation having standard nonnegative characteristic form, for which we apply the Fichera function criterion to check when boundary conditions are unnecessary [3, 18, 20]. We further investigate the trace of the PDF on the sphere $|m| = \sqrt{b}$ where no data is pre-imposed. Our approach is to convert the equation by a delicate transformation in such a way that the resulting equation supports a maximum principle.

This paper is organized as follows: in Section 2, we use the Fichera function criterion to prove Theorem 1.1. Section 3 is devoted to the trace analysis of PDF on the sphere $|m| = \sqrt{b}$. The presentation is split into two parts, without and with homogeneous flow involved.

2. The Fichera Function and Boundary Conditions

We first introduce the following transformation

$$f(t,m) = g(t,m)\exp(-U(m)), \tag{8}$$

which gives

$$\partial_t g + \nabla_m \cdot (\kappa m g) - \nabla_m U \cdot (\kappa m g) = \frac{1}{2}[\Delta_m g - \nabla_m U \cdot \nabla_m g]. \tag{9}$$

The right hand side of the equation becomes the dual form of the original Fokker-Planck equation [9, 19]. We note a different transformation in [6, 10, 21], $f(t,m) = g(t,m)\exp(-U(m)/2)$, which was used to remove the singularity at boundary in the resulting equation.

Applying further re-scaling:

$$x = \sqrt{2}m, \quad r^2 = 2b, \quad v(t,x) = g(t,m), \tag{10}$$

we obtain

$$\partial_t v + \nabla_x \cdot (\kappa x v) + a(x) \cdot (\nabla_x v - \kappa x v) = \Delta_x v, \tag{11}$$

where

$$a(x) := \frac{bx}{r^2 - |x|^2}.$$

Note that $\nabla_x \cdot (\kappa x) = Tr(\kappa) = 0$, the above equation reduces to

$$\partial_t v + (a(x) + \kappa x) \cdot \nabla_x v - a(x) \cdot \kappa x v = \Delta_x v. \tag{12}$$

Once v is determined, the PDF f can be recovered through

$$f(t,m) = v(t, \sqrt{2}m)(1 - |m|^2/b)^{b/2}. \tag{13}$$

Rewrite the equation (12) as

$$L(v) = 0, \quad x \in \mathcal{R}^d, \tag{14}$$

where

$$\begin{aligned} L(v) \;&:=\; (r^2 - |x|^2)\Delta_x v - (r^2 - |x|^2)v_t \\ &\quad -(bx + (r^2 - |x|^2)\kappa x) \cdot \nabla_x v + b x^\top \kappa x v \end{aligned}$$

has a standard form

$$L(v) :\equiv a^{kj}(\xi) v_{\xi_k \xi_j} + b^k(\xi) v_{\xi_k} + c(\xi) v = 0, \quad k, j = 0 \cdots d.$$

Here the repeated indices are summed from 1 to d, $\xi = (t, x)$,

$$a^{00} = 0, \quad b^0 = -(r^2 - |x|^2), \quad c(\xi) = b x^\top \kappa x$$

and

$$a^{kk}(\xi) = (r^2 - |x|^2), \quad b^k = -[bx_k + (r^2 - |x|^2)\kappa_{kj}x_j], \quad k = 1 \cdots d.$$

Note that the new equation is degenerate at boundary $|x| = r$.

This second order equation has nonnegative characteristic form in domain $\Omega(T^*) = \{(t,x), \quad 0 < t < T^*, \quad |x| < r\}$ for any $T^* > 0$, since

$$a^{kj}(\xi) y_k y_j \geq 0$$

for any real vector y and any point $\xi \in \Omega(T^*)$. Hence there are no negative characteristic points on the boundary.

We will employ the method of the Fichera function [1, 3, 18, 20] to study the corresponding relevant boundary value points on the boundary [18].

The Fichera function can be defined as follows [3, 18]

$$\Gamma(t,x) = (b^k - a^{kj}_{\xi_j}) n_k, \tag{15}$$

where $N = (n_0, n_1, \cdots, n_d)$ denotes the inward normal vector to the boundary $\partial\Omega(T^*)$. Γ will not change sign at the points where $a^{kj} n_k n_j = 0$ of the boundary $\partial\Omega(T^*)$, under smooth non-degenerate changes of independent variables ξ. More important in our case, no boundary condition shall be imposed on the portion of the boundary such that $\partial\Omega(T^*)$ where $\Gamma > 0$ and $\Gamma = 0$. The boundary conditions are only needed for the negative Γ points and the negative characteristic points on the boundary.

Next we check the points on $\partial\Omega(T^*)$. At boundary $|x| = r, \quad 0 < t < T^*$, one has $n_0 = 0, \quad n_k = -x_k/r, \quad k = 1 \cdots d$, thus

$$\begin{aligned}
\Gamma(t,x) &= (b^k - a^{kj}_{\xi_j}) n_k = \sum_{k=1}^{d} (b^k - a^{kk}_{x_k}) n_k \\
&= \sum_{k=1}^{d} (-bx_k + 2x_k) \cdot (-x_k/r) \\
&= (b-2)r.
\end{aligned}$$

If $\Gamma \geq 0$, that is $b \geq 2$, no boundary condition is needed. Otherwise, in the case $b < 2$, an appropriate boundary condition has to be imposed.

We now examine the boundary $t = T^*$ and $|x| < r$, on which one has $n^0 = -1, \quad n^k = 0$, thus

$$\begin{aligned}\Gamma(t,x) &= (b^k - a^{kj}_{\xi_j})n_k = b^0 n_0 \\ &= r^2 - |x|^2 > 0.\end{aligned}$$

No condition needs to be imposed at $t = T^*$ either. Similarly at $t = 0, \quad |x| < r$, one obtains $\Gamma(t,x) = |x|^2 - r^2 < 0$, thereby a condition at $t = 0$, the initial condition, has to be imposed. These finish the proof of Theorem 1.1.

3. The Trace of the Distribution Function on the Sphere $|m| = \sqrt{b}$ for $b > 2$

We now restrict ourselves to the case of $b > 2$. The proof in the above section shows that the presence of the fluid velocity does not affect the relevancy of the boundary points with respect to the equation. In this section, we will show, if the solution exists and assumes a trace on the boundary, the trace of the resulting probability density function has to be zero.

3.1. **No flow case.** We will start from the case $\kappa = 0$. The equation (11) becomes:

$$\partial_t v + a(x) \cdot \nabla_x v = \Delta_x v, \quad a(x) := \frac{bx}{r^2 - |x|^2}, \tag{16}$$

with the initial condition:

$$v(0,x) = v_0(x) = f_0(\frac{x}{\sqrt{2}})(1 - |x|^2/r^2)^{-b/2}, \quad \text{supp}(v_0) \subset [-r, r],$$

we are going to show that there exists an α satisfying $0 < \alpha < b/2$ and a $K > 0$ such that

$$|v(t,x)| \le M e^{Kt}(r^2 - x^2)^{-\alpha}, \quad \forall t > 0.$$

Combinning with the original transformation (8) and (10), which yield

$$f(t,m) = v(t,x)(1 - |x|^2/r^2)^{b/2},$$

and we arrive at:

$$|f(t,m)| \le C(r^2 - x^2)^{b/2-\alpha} e^{Kt}. \tag{17}$$

This leads to the zero trace for PDF:

$$f(t,m)|_{|m|^2=b} = 0, \quad \forall t > 0.$$

The main difficulty of Theorem 1.2 lies in the singularity at boundary.

The equation (14) for v solves $L(v) = 0$ with

$$L(v) := (r^2 - |x|^2)\Delta_x v - bx \cdot \nabla_x v - (r^2 - |x|^2)\partial_t v.$$

We now introduce the transformation

$$v(t,x) := w(t,x)(r^2 - |x|^2)^{-\alpha} e^{Kt},$$

with α and K to be determined. A simple calculation gives

$$\begin{aligned}
\partial_t v &= (w_t + Kw)(r^2 - |x|^2)^{-\alpha} e^{Kt}, \\
\nabla_x v &= [\nabla_x w(r^2 - |x|^2)^{-\alpha} + 2\alpha w x(r^2 - |x|^2)^{-\alpha-1}]e^{Kt}, \\
\Delta_x v &= [\Delta_x w(r^2 - |x|^2)^{-\alpha} + 4\alpha x \cdot \nabla_x w(r^2 - |x|^2)^{-\alpha-1} \\
&\quad +4\alpha(\alpha+1)w|x|^2(r^2 - |x|^2)^{-\alpha-2}]e^{Kt} \\
&\quad +2\alpha d w(r^2 - |x|^2)^{-(\alpha+1)} e^{Kt}.
\end{aligned}$$

Substitution of these terms into the equation $L(v) = 0$ multiplied by $(r^2 - |x|^2)^{\alpha+1} e^{-Kt}$ gives

$$A(w) = 0,$$

where the operator $A(w)$ is defined as

$$\begin{aligned}
A(w) &:= (r^2 - |x|^2)^2 \Delta_x w + (4\alpha - b)(r^2 - |x|^2)x \cdot \nabla_x w \\
&\quad -(r^2 - |x|^2)^2 \partial_t w + c(x)w,
\end{aligned}$$

in which the coefficient

$$c(x) = -K(r^2 - |x|^2)^2 + 2\alpha[dr^2 + (2\alpha + 2 - d - b)|x|^2].$$

In order to apply a maximum principle to $A(w) = 0$, we need to choose α and K such that $c < 0$ in $\Omega(T^*)$. Setting $|x|^2 = \theta r^2$, we have

$$\begin{aligned}
c &= -Kr^4(1-\theta)^2 + 2\alpha d r^2 + 2\alpha(2\alpha + 2 - d - b)\theta r^2 \\
&= -r^2 \left\{ Kr^2\theta^2 - 2((2\alpha + 2 - d - b)\alpha + Kr^2)\theta + Kr^2 - 2d\alpha \right\}.
\end{aligned}$$

Thus as a function of θ, c achieves its maximum

$$c = K^{-1}\alpha \left\{ 2Kr^2(2\alpha + 2 - b) + \alpha(2\alpha + 2 - d - b)^2 \right\},$$

at

$$\theta^* = 1 + \frac{\alpha}{Kr^2}(2 - d - b + 2\alpha).$$

The coefficient c can be made negative if its maximum value is negative, which is true provided that

$$\alpha < \frac{b}{2} - 1$$

and

$$Kr^2 > \frac{\alpha(d+b-2-2\alpha)^2}{2(b-2-2\alpha)} > 0.$$

With these choices we apply the maximum principle [18] to the equation $A(w) = 0$, and find that w achieves a positive maximum only at initial time, i.e. in the region $\{(0,x), \quad |x| < r^2\}$. Therefore we have

$$0 \le w(t,x) \le \|w(0,\cdot)\|_{L^\infty}.$$

Note that

$$w_0(x) = v_0(x)(r^2 - |x|^2)^\alpha = f_0(m) r^h (r^2 - |x|^2)^{\alpha - b/2}.$$

Assume that $f_0(m) \neq 0$ for $|m|^2 \le b^* < b$. Then

$$\|w_0\|_\infty \le \|f_0\|_\infty r^b (r^2 - 2b^*)^{\alpha - b/2}.$$

Thus from

$$f(t,m) = v(t,x)(1 - |x|^2/r^2)^{b/2} = w(t,x) r^{-b} (r^2 - |x|^2)^{b/2-\alpha} e^{Kt},$$

it follows that

$$|f(t,m) \le \|f_0\|_\infty \le \left(\frac{r^2 - |x|^2}{r^2 - 2b^*}\right)^{b/2-\alpha} e^{Kt}.$$

Replacing $r^2 = 2b$ and $|x|^2 = 2|m|^2$ we have obtained the desired estimate stated in Theorem 1.2 . Therefore the trace of f on the sphere $|m| = \sqrt{b}$ must be null.

3.2. **Coupled with flow $\kappa \neq 0$.** We now show the null trace when flow is involved.

In this case the equation (14) has the form $L(v) = 0$ with

$$\begin{aligned} L(v) \quad := \quad & (r^2 - |x|^2)\Delta_x v - (bx + (r^2 - |x|^2)\kappa x) \cdot \nabla_x v \\ & -(r^2 - |x|^2)\partial_t v + (bx^\top \kappa x) v. \end{aligned}$$

We again apply the transformation

$$v(t,x) := w(t,x)(r^2 - |x|^2)^{-\alpha} e^{Kt},$$

with α and K to be determined. This transformation applied to $L(v)(r^2 - |x|^2)^{\alpha+1}e^{-Kt}$ leads to the following equation

$$B(w) = 0,$$

with the operator $B(w)$ being

$$\begin{aligned} B(w) &= (r^2 - |x|^2)^2 \Delta_x w + (r^2 - |x|^2)[(4\alpha - b)x \\ &\quad -(r^2 - |x|^2)\kappa x] \cdot \nabla_x w - (r^2 - |x|^2)^2 \partial_t w + c(x)w, \end{aligned}$$

and the coefficient of the last term being

$$\begin{aligned} c(x) &= -K(r^2 - |x|^2)^2 + 2\alpha[dr^2 + (2\alpha + 2 - d - b)|x|^2] \\ &\quad +(b - 2\alpha)x^\top \kappa x(r^2 - |x|^2). \end{aligned}$$

As the similar argument as in the no-flow case, we proceed to determine α and K so that c stays negative in $\Omega(T)$. Let ρ be the largest eigenvalues of the deformation tensor

$$S = (\kappa + \kappa^\top)/2,$$

one has

$$x^\top \kappa x \le \rho |x|^2.$$

We first choose α in such a way that $\alpha < b/2 - 1$, which implies $h - 2\alpha > 0$. Thus we obtain

$$\begin{aligned} c(x) \le \bar{c} &= -K(r^2 - |x|^2)^2 + 2\alpha[dr^2 + (2\alpha + 2 - d - b)|x|^2] \\ &\quad +\rho(b - 2\alpha)|x|^2(r^2 - |x|^2) \\ &= -[K + \rho(b - 2\alpha)]|x|^4 + [2Kr^2 + \rho(h - 2\alpha)r^2 \\ &\quad +2\alpha(2\alpha + 2 - d - b)]|x|^2 + 2\alpha dr^2 - Kr^4 \\ &\le 2\alpha dr^2 - Kr^4 \\ &\quad + \frac{[2Kr^2 + \rho(b - 2\alpha)r^2 + 2\alpha(2\alpha + 2 - d - b)]^2}{4[K + \rho(h - 2\alpha)]} \\ &\le 2\alpha r^2(2\alpha + 2 - b) + \frac{\beta^2}{4[K + \rho(b - 2\alpha)]}, \end{aligned}$$

where

$$\beta := \rho(b - 2\alpha)r^2 - 2\alpha(2\alpha + 2 - d - b).$$

Therefore we can choose K such that

$$K > \frac{\beta^2}{8\alpha r^2(b - 2 - 2\alpha)} - \rho(b - \alpha),$$

and the following is always true

$$c(x) \le \bar{c} < 0, \quad |x| \le r^2.$$

We thus can apply the maximum principle [18] to the equation

$$B(w) = 0,$$

and obtain that u achieves its positive maximum only at initial time, i.e. in the region $\{(0,x), \quad |x| < r^2\}$. This will give the result that

$$0 \le w(t,x) \le \|w(0,\cdot)\|_{L^\infty}.$$

Converting back to f and we prove the results stated in Theorem 1.2.

Acknowledgments. We acknowledge the discussion and suggestion with Gary Lieberman. We also thank referees for valuable suggestions and pointing out two relevant references [13, 14]. H Liu's research was partially supported by the NSF grant DMS-0505975 and the Ames Laboratory (DOE facility). C. Liu's research was partially supported by the NSF grants DMS-0405850 and DMS-0509094. We also want to thank the IMA for the hospitality and support during our staying for the *Thematic Year on Mathematics of Materials and Macromolecules: Multiple Scales, Disorder, and Singularities.*

References

[1] M. Bardi and S. Bottacin,*Characteristic and irrelevant boundary points for viscosity solutions of nonlinear degenerate elliptic equations*, Dip. di Mathematica P. E. A, University di Padova, 25, 1998.

[2] R. B. Bird, C. F. Curtiss, R. C. Armstrong and P. Hassager (1987) *Dynamics of Polymeric Liquids, Volume 2: Kinetic Theory*, Weiley Interscience, New York.

[3] G. Fichera (1956) *Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine*, Atti. Accad. Naz. Lincei. Mem. CI. Sci. Fis. Mat. Nat. Sez. I (8) **5**, pp. 1–30.

[4] R.B. Bird, O. Hassager, R.C. Armstrong and C.F. Curtiss (1977) *Dynamics of Polymeric Fluids*, **2**, Kinetic Theory. John Wiley & Sons, New York.

[5] J.W. Barrett, C. Schwab and E. Suli (2005) *Existence of global weak solutions for some polymeric flow models*, Math. Models Methods Appl. Sci. **15**(6), pp. 939–983.

[6] Q. Du, C. Liu and P. Yu (2005) *FENE dumbbell model and its several linear and nonlinear closure approximations*, SIAM Journal on Multiscale Modeling and Simulation Volume 4, Number 3 pp. 709—731.

[7] M. Doi and S. F. Edwards (1986) *The Theory of Polymer Dynamics*, Oxford University Press.

[8] W.-N. E, T. Li and P. Zhang (2004) *Well-posedness for the dumbell model of polymeric fluids*, Comm. Math. Phys. **248**(2), pp. 409–427.

[9] C. W. Gardiner (1997) *Handbook of Stochastic Methods*, Springer.

[10] Y. Guo (2003) *The Vlasov-Maxwell-Boltzmann system near Maxwellians*, J Invent. Math., **153**, pp. 593–630.

[11] P. Halin, G. Lielens, R. Keunings and V. Legat (1998) *The Lagrangian particle method for macroscopic and Micro-Macro viscoelastic flow computations*, J. Non-Newtonian Fluid Mech. **79**, pp. 387–403.

[12] B. Jourdain, T. Lilièvre and C. Le Bris (2004) *Existence of solutions for a micro-macro modelling of polymeric fluid: the FENE model*, Journal of Functional Analysis, **209**, pp. 162–193.

[13] B. Jourdain, T. Lilièvre (2002), *Mathematical analysis of a stochastic differential equation arising in the micro-macro modelling of polymeric fluids*, p.205–223, in Probabilistic Methods in Fluids Processings of the Swansea 2002 Workshop, Eds: I M Davies, N. Jacob, A. Truman, O Hassan, K. Morgan and N.P. Weatherill.

[14] C. Le Bris, B. Jourdain, T. Lilièvre and F. Otto (2005), *Long-time asymptotic of a multi-scale model for polymeric fluid flows*, preprint CRM-3188.

[15] R. G. Larson (1995) *The Structure and Rheology of Complex Fluids*, Oxford.

[16] C. Le Bris and P.-L. Lions (2004) *Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications*, Ann. Mat. Pura Appl. (4) 183, no. 1, pp. 97–130.

[17] F. H. Lin, C. Liu and P. Zhang (2005) *On a Micro-Macro Model for Polymeric Fluids near Equilibrium*, submitted for publication in Comm. Pure Appl. Math.

[18] O. A. Oleinik and E. V. Radkevič (1973) *Second Order Equations with Nonnegative Characteristic Form*, translated from Russian by Paul Fife, AMS, Providence Rhode Island and Plenum Press, New York.

[19] F. Otto and C. Villani (2000) *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173**(2), pp. 361–400.

[20] K. Taira (1988) *Diffusion Processes and Partial Differential Equations*, Academic Press.

[21] P. Yu, Q. Du and C. Liu (2005) *¿From Micro to Macro dynamics via a new moment closure approximation to FENE model of polymeric fluids*, Journal on Multiscale Modeling and Simulation , Vol 3, No 4, pp. 895 — 917.

[22] H. Zhang and P. Zhang (2005) *Local existence for the FENE-Dumbbell model of polymeric fluids*, Arch. Rat. Mech. Anal. to appear.

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