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On Discreteness of the Hopf Equation

Hai-liang Liu

Iowa State University, USA (E-mail: hliu@iastate.edu)

Abstract The principle aim of this essay is to illustrate how different phenomena is captured by different discretizations of the Hopf equation and general hyperbolic conservation laws. This includes dispersive schemes, shock capturing schemes as well as schemes for computing multi-valued solutions of the underlying equation. We introduce some model equations which describe the behavior of the discrete equation more accurate than the original equation. These model equations can either be conveniently discretized for producing novel numerical schemes or further analyzed to enrich the theory of nonlinear partial differential equations.

Keywords Hopf equation, dispersive scheme, shock capturing schemes, multi-valued solutions, level set equation

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1 Introduction

We report here some results of our ongoing studies of the numerical approximation to hyperbolic problems in different applications. Hyperbolic equations are of great practical importance since they model a variety of physical phenomena that appear in fluid mechanics, astrophysics, groundwater flow, traffic flow, geometric optics, among many others.

It is well known that solutions of quasi-linear hyperbolic equations breakdown after a finite elapse of time. After appearance of singularities (collision of characteristics), one has to explore a proper notion of solutions to address the physical reality. This has been a challenging task from both theoretical and numerical points of view. In this essay I shall take a numerical point of view, and will illustrate the main ideas through the simplest model: the one-dimensional initial value problem for the Hopf equation

$$u_t + uu_x = 0, \quad u(x,0) = u_0(x). \tag{1.1}$$

There have been different ways to discretize the equation and correspondingly different phenomena are observed. Here I illustrate features of three canonical numerical discretizations: dispersive schemes, shock capturing schemes and the schemes for computing multi-valued solutions of the underlying equation; and introduce some model equations which describe the behavior of the discrete equation more accurate than the original equation.

A symmetric finite difference approximation to (1.1) with $f(u) = u^2/2$ leads to a semi-discrete central scheme

$$\dot{u}_k + \frac{f(u_{k+1}) - f(u_{k-1})}{2h} = 0, \quad u_k(0) = u_0(kh). \tag{1.2}$$

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For smooth solutions the expansion of $u_{k\pm1}$ in a Taylor series centered at $x_k = kh$ leads to

$$u_t + f(u)_x + \frac{h^2}{6}\big(f(u)\big)_{xxx} = O(h^4).$$

It is anticipated that the continuum limit ($h \to 0$) presents similarities with the well-understood zero dispersion limit[42] ($\varepsilon \to 0$) for the Korteweg-de Vries equation

$$u_t + uu_x + \varepsilon u_{xxx} = 0, \qquad \varepsilon \sim h^2. \tag{1.3}$$

From wave propagation point of view, finite differences should be selected biased in the direction determined by the sign of the characteristic speed $f'(u)$. Considering this one is led to upwind differencing schemes which have been proven to be most suitable for solving first order PDEs such as the Hopf equation. In the case $f'(u) > 0$, the upwind scheme takes the form

$$\dot{u}_k + \frac{f(u_k) - f(u_{k-1})}{h} = 0, \quad u_k(0) = u_0(kh). \tag{1.4}$$

The stable mechanism in upwind schemes lies in the numerical viscosity, which is illustrated by the viscous approximation

$$u_t + uu_x = \varepsilon u_{xx}, \qquad \varepsilon \sim h. \tag{1.5}$$

The classical characteristics method suggests to solve (1.1) by a family of ODEs

$$\begin{cases} \dfrac{dx}{dt} = f'(p), & x(0) = \alpha, \\ \dfrac{dp}{dt} = 0, & p(0) = u_0(\alpha), \end{cases} \tag{1.6}$$

with an initial position parameter $\alpha \in \mathbb{R}$. This family of ODEs suggests multi-valued solutions for the Hopf equation beyond the collision of characteristics curves. Actually the multi-valued solution is physically meaningful in the context of high frequency wave propagation. Let multi-valued solutions be expressed through a global invariant of the above ODE system, $u \in \{p, \quad \phi(t,x,p) = 0\}$, then in phase space (x,p) the level set function $\phi(t,x,p)$ solves a linear transport equation

$$\phi_t + f'(p)\phi_x = 0. \tag{1.7}$$

Level set methods can thus be used to effectively capture multi-valued solutions evolved by the Hopf equation (see [8,37,51]).

When the target solution is smooth, numerical solutions from the above three sample approximations converge to the same function. Beyond singularities, they are different in an essential manner. In Section 2 we record three canonical dispersive schemes for the Hopf equation, and correspondingly introduce several regularized model equations. Shock capturing schemes are discussed in Section 3, in which two model equations are introduced: flux refinement and the alternating evolution approximation. In Section 4 we first illustrate why multi-valued solutions are relevant, and then introduce the level set equation for describing the dynamics of the multi-valued solution. Emphasis will be placed on how physics is incorporated into the discrete models. Relevant references will be given as is necessary in each section. In the end some concluding remarks are provided.

2 Dispersive Schemes

In 1943, Von Neumann[81] employed a centered difference scheme to compute compressible flows with strong shocks in one space dimension and discovered the oscillatory nature after the shock formation. These oscillations arose due to the unbiased finite differences in the numerical scheme he chose and are characteristic of all such schemes. The continuous analogue of this phenomenon has been well studied in the context of the Korteweg-de Vries equation[42,44,79]. There have also been detailed studies of dispersive numerical schemes[27,33,45,60,74,76,80].

In this section we shall discuss three canonical dispersive schemes for the Hopf equation, and speculate variations of their model equations.

Dispersive schemes are closely related to some integrable systems, two of which have been carefully studied. One of them is the Toda lattice[74]

$$\dot{a}_k = \frac{1}{h} a_k (b_{k+1} - b_k), \quad \dot{b}_k = \frac{2}{h}(a_k^2 - a_{k-1}^2). \tag{2.1}$$

The other, introduced by Kac and Van Moerbeke in [38], is

$$\dot{p}_k + \frac{1}{2h} p_k (p_{k+1}^2 - p_{k-1}^2) = 0. \tag{2.2}$$

The equivalence of the two schemes was shown by Moser[62].

2.1 The Goodman-Lax Scheme

One dispersive scheme in this class is the semi-discrete scheme studied by Goodman and Lax[27]:

$$\dot{u}_k + u_k \frac{u_{k+1} - u_{k-1}}{2h} = 0, \quad u_k(0) = u_0(kh), \tag{2.3}$$

which is integrable, as for $u_k > 0$ the transformation $u_k = p_k^2$ reduces it to (2.2). The discrete equation (2.3) is positivity preserving and has a unique solution. Two conservation laws,

$$\sum_k u_k(t) = \sum_k u_k(0)$$

and

$$\prod_k u_k(t) = \prod_k u_k(0),$$

are crucially used in [27] to bound solutions of the scheme. In fact, all solutions are thus bounded from above by $\sum_k u_k(0)$. The uniform boundedness of $u_k(t)$ guarantees that every solution of the discrete solution which has positive data exists for all time.

From equation (2.3) we see that h measures how far the system is from being described by its continuum analog (1.1), obtained by replacing $u_k(t)$ with $u(kh, t)$ and taking the formal limit $h \to 0$. However, Goodman and Lax proved that the solution of (2.3) oscillates on mesh scale $O(h)$ and its weak limit does not satisfy the conservation law

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{2.4}$$

The dispersive feature of the scheme can be seen from the first order model equation

$$u_t + u u_x + \frac{h^2}{6} u u_{xxx} = 0, \tag{2.5}$$

which is obtained by Goodman and Lax from using Taylor's expansion on $u(kh,t)$.

It is clear that the model equation (2.5) has lost some nice features of the discrete equation. It is natural to ask

- Whether u remains positive for all time if it is so initially?
- Is the model well-posed in a neighborhood of points where $u=0$?

These are non-trivial questions to answer since the model (2.5) is fully nonlinear. Our numerical simulation shows that the solution is rather sensitive for some initial data.

We now speculate a slightly regularized model. Define $\partial_x v = [u(x+h)-u(x-h)]/(2h)$, so that $u=\mathcal{M}(h)v$, where

$$\mathcal{M}(h)=\frac{2h\partial_x}{e^{h\partial_x}-e^{-h\partial_x}}=1-\frac{h^2}{6}\partial_x^2+O(h^4). \tag{2.6}$$

Expanding $\mathcal{M}$ as above yields the regularized lowest-order approximation

$$u_t+uv_x=0,\quad u=\Big(1-\frac{h^2}{6}\partial_x^2\Big)v.$$

Upon a scaling $(x,t)\to\sqrt{\frac{6}{h}}(x,t)$ we arrive at

$$\begin{cases} u_t+uv_x=0\\ u=(I-\partial_x^2)v. \end{cases} \tag{2.7}$$

This can be regarded as a regularized version of the model equation (2.5).

2.2 The 1/3 Scheme

Another discretization, which dates back to Zabusky and Kruskal[84] and has been studied by Levermore and Liu[47], is the so called one third scheme

$$\dot{u}_k+\frac{u_{k-1}+u_k+u_{k+1}}{3}\cdot\frac{u_{k+1}-u_{k-1}}{2h}=0,\quad u_k(0)=u_0(kh). \tag{2.8}$$

This scheme is non-integrable, and large oscillations are observed after shock formation[47]. The main appeal of this scheme is that it possesses two semi-discrete local conservation laws:

$$\dot{u}_k+\frac{1}{h}[f_{k+1/2}-f_{k-1/2}]=0,\quad f_{j+1/2}=\frac{1}{6}(u_k^2+u_ku_{k+1}+u_{k+1}^2),$$
$$\dot{u}_k^2+\frac{1}{h}[g_{k+1/2}-g_{k-1/2}]=0,\quad g_{j+1/2}=\frac{1}{3}(u_k^2u_{k+1}+u_ku_{k+1}^2),$$

which reflects the local conservation of u and u^2 by the classical solution of the Hopf equation.

A model equation from using Taylor's expansion on $u(kh,t)$ is

$$u_t+uu_x+\frac{h^2}{6}(2u_xu_{xx}+uu_{xxx})=O(h^4). \tag{2.9}$$

A normalized version of its leading approximation (ignoring $O(h^4)$ term) becomes

$$u_t+uu_x+2u_xu_{xx}+uu_{xxx}=0,$$

which can be rewritten in conservative form

$$u_t + (1+\partial_x^2)\Big(\frac{u^2}{2}\Big)_x = \Big(\frac{u_x^2}{2}\Big)_x. \tag{2.10}$$

Like the semi-discrete scheme (2.8) this continuous model also has two conservation laws with

$$\Big(\frac{u^2}{2}\Big)_t + \Big(\frac{u^2}{3}(u+3u_{xx})\Big)_x = 0.$$

2.3 The Centered Scheme

A straightforward central discretization of the flux $(u^2/2)_x$ gives the third dispersive scheme:

$$\dot{u}_k + \frac{u_{k+1}+u_{k-1}}{2}\cdot\frac{u_{k+1}-u_{k-1}}{2h} = 0, \quad u_k(0) = u_0(kh), \tag{2.11}$$

which has both $\sum_k u_k$ and $\sum_k u_k^3$ as conserved quantities, but none of these quantities is enough to bound the growth of u_k, which in fact may blow up in finite time[76]. In contrast, the 1/3 scheme (2.8) conserves both $\sum_k u_k$ and $\sum_k u_k^2$, which indeed bounds the growth of the solutions.

Using Taylor expansion on $u^2(kh,t)$ we obtain

$$u_t + \Big(I + \frac{h^2}{6}\partial_x^2\Big)\Big(\frac{u^2}{2}\Big)_x = O(h^4).$$

Ignoring $O(h^4)$ and scaling $(x,t) \to \sqrt{\frac{6}{h}}(x,t)$ we have

$$u_t + (I+\partial_x^2)\Big(\frac{u^2}{2}\Big)_x = 0.$$

This model equation conserves both $\int u$ and $\int u^3$. The regularization procedure (2.6) when applied to this model gives

$$\begin{cases} u_t + v_x = 0, \\ \dfrac{u^2}{2} = (I-\partial_x^2)v. \end{cases} \tag{2.12}$$

This system also preserves $\int u^3$ for

$$(u^3)_t + 3(v^2 - v_x^2)_x = 0.$$

Finally we point out that these model equations such as (2.7), (2.10) and (2.12) are similar to another class of dispersive models

$$(I-\partial_x^2)u_t + (I-\theta\partial_x^2)\Big(\frac{u^2}{2}\Big)_x = (1-4\theta)\Big(\frac{u_x^2}{2}\Big)_x, \tag{2.13}$$

among which only two are integrable. One is the Camassa-Holm equation with $\theta = 1/3$ (see [4,24]), and the other is the Degasperis-Procesi equation with $\theta = 1/4$ (see [16]). These two equations are two recently most-studied equations in the literature. The former allows for the peakon solution, and the later even admits discontinuous solutions (see, e.g. [11,83]).

Using a renormalization technique and examining number of conservation laws, the author in [49] derived a class of dispersive nonlocal equations of the form

$$u_t + uu_x + (I - \partial_x^2)^{-1}(B(u, u_x))_x = 0, \tag{2.14}$$

where u denotes the wave motion in x direction, and B is quadratic in its arguments, including the Camasa-Holm equation $B(u,p) = u^2 + p^2/2$; and the Degasperis-Procesi equation $B(u,p) = 3u^2/2$.

Theorem 2.1[49]**.** *Suppose that $u_0 \in H_x^{3/2+}$, then there exists a time T and a unique solution u of (2.14) in the space $C_{[0,T)}H_x^{3/2+} \cap C^1_{[0,T)}H_x^{1/2+}$ such that $\lim\limits_{t\downarrow 0} u(t,\cdot) = u_0(\cdot)$. If $T_0 < \infty$ is the maximal existence time, then*

$$\lim_{t\to T_0-} \sup_{0\le\tau\le t} \|u_x(,\tau)\|_{L^\infty(\Omega)} = \infty,$$

where $\Omega = \mathbb{R}$ for initial data decaying at far fields, or $\Omega = [0,\pi]$ for periodic data.

Wave breaking criteria are identified separately for each particular model in the class (2.14), using their special features, see [49] for further details. In terms of the wave breaking phenomena the dispersive nonlocal models are essentially different from dissipative models such as those studied in [23,53].

It is hoped the above mentioned model equations from dispersive schemes be further analyzed on rigorous grounds.

3 Shock Capturing Schemes

In the context of gas dynamics, entropy satisfying shock waves are physically relevant. The notorious difficulty encountered for the satisfactory approximation of the exact solutions of hyperbolic conservation laws lies in the presence of discontinuities (shocks) in the solution.

The idea of adding numerical viscosity to an existing oscillatory centered scheme such as the scheme (2.11) to generate a shock capturing method dates back to [43] or even earlier. A well-known and beloved fully discrete scheme in this class is the celebrated Lax-Friedrich (LxF) scheme[43]:

$$u_k^{n+1} = \frac{u_{k-1}^n + u_{k+1}^n}{2} - \Delta t\Big[\frac{f(u_{k+1}) - f(u_{k-1})}{2h}\Big], \tag{3.1}$$

where f is the numerical flux, $f = u^2/2$ for the Hopf equation. The leading approximation from using the Taylor expansion gives the following model equation

$$u_t + f(u)_x = \frac{h}{2\lambda}u_{xx}, \quad \lambda := \Delta t/h.$$

The term on the right measures the amount of numerical viscosity carried in the LxF scheme. We note that the Lax-Friedrich scheme is also a powerful tool for attacking existence problems for conservation laws; consult works by Ding et.al.[5,17,18] for the isentropic gas dynamic system.

Numerically it is expected that a shock capturing scheme reflect the wave propagation information on the grids. One of the first attempts at upwinding for the equations of gas dynamics was due to Courant, Isaacson and Rees[14] in 1952. They proposed to capture information along characteristics, and evaluated the characteristic variables by interpolation from the two nearest grid values. However, the scheme is not good for nonlinear problems with shocks since it is not

in conservative form. In 1959, Godnuov proposed to evolve the cell average of the solution over $(x_{j-1/2}, x_{j+1/2})$ and solve Riemann problems at cell interfaces $x_{j\pm1/2}$ (see [26]). The Godunov scheme is thus conservative and reveals rich characteristic structures:

$$u_k^{n+1} = u_k^n - \frac{\Delta t}{h}[f(u^*(u_k, u_{k+1})) - f(u^*(u_{k-1}, u_k))], \tag{3.2}$$

where $u^*(u_k, u_{k+1})$ is the Riemann solution of $\frac{x-x_{k+1/2}}{t-t_n}$ evaluated at $x = x_{k+1/2}$. In this method, evolution of cell averages are crucial for capturing the right shock speed (ensured by the Lax-Wendroff theorem[46] for conservative schemes), and physics is thus incorporated into the scheme through the local Riemann solution u^*. A Riemann problem is a one-dimensional Cauchy problem with initial data that is constant everywhere except for a single jump discontinuity. The solution evolves into nonlinear waves that propagate coherently in time. The constant states used for the Riemann problem are the cell averages on either side of the cell interface.

Nowadays the Godunov method is regarded as the most crude and basic tool in the arsenal of shock capturing methods for hyperbolic conservation laws. A well known recipe to achieve both high-order accuracy and convergence to the entropy solution is the so called high-resolution schemes. The success has been due to two factors:

- the local enforcement of nonlinear conservation laws;
- the non-oscillatory piecewise polynomial reconstruction from evolved cell averages.

Various higher-order extensions of the Godunov scheme have been rapidly developed since 1970's, employing higher-order reconstruction of piece-wise polynomials from the cell averages, including MUSCL, TVD, PPM, ENO and WENO schemes[12,30,31,57,71,72,77,78]. A key ingredient in this procedure is the revealing of numerical solutions on cell interfaces through so called numerical fluxes — different schemes arise when different realization is adopted.

In contrast to upwind schemes, Godunov-type central schemes evaluate staggered cell averages crossing cell interfaces $x_{j+\frac{1}{2}}$, costly Riemann-solvers are thus bypassed. In the one-dimensional case, examples of such schemes for conservation laws are the second-order Nessyahu-Tadmor scheme[63], as well as numerous further extensions (see e.g. [2,34,40,41,48,58,59]).

The above two categories of schemes are somehow interlaced during their independent developments; the upwind scheme becomes Riemann solver-free when a local numerical flux is used to replace the exact Riemann solver, see Shu and Osher[71,72], and the central scheme becomes less diffusive when variable control volumes are used in deriving the scheme, see Kurganov and Tadmor[41]. The upwind feature can be further enforced[32,39].

Investigation of high resolution schemes with compact stencils using ideas such as numerical flux and limiters coined in the development of high resolution shock capturing schemes is still an active area of research (see, e.g., [10,82]). In what follows we introduce two model equations aiming to incorporate physics (entropy condition) into the solution process before discretization.

3.1 Flux Refinement

In general, all conservative shock capturing schemes can be symbolically modeled by

$$u_t + F(u^-, u^+)_x = 0, \tag{3.3}$$

where $F(u^-, u^+)$ is a refinement of the flux function $f(u)$. Here $u^- = u(x-0, t)$ and $u^+ = u(x+0, t)$ denote the left and right limits of u when discontinuity appears. Without discontinuity

$u^- = u^+$, and $F(u,u)$ is required to be consistent with the original flux $f(u)$

$$F(u,u) = f(u). \tag{3.4}$$

The weak solution of (3.3) must be understood in the integral sense. For example, if we use piecewise constant (cell averages) function to approximate the solution, integration of (3.3) over a control volume $[x_{k-1/2}, x_{k+1/2}] \times [t^n, t^{n+1}]$ leads to

$$u_k^{n+1} = u_k^n - \frac{\Delta t}{h}[F(u_k^n, u_{k+1}^n) - F(u_{k-1}^n, u_k^n)]. \tag{3.5}$$

Clearly the refined flux F plays an essential role in incorporating physics into the numerical method. Once it is adopted, the approximation of the model equation (3.3) is solely of a numerical nature and can be modified without sacrificing the physical input. Most of high resolution schemes as well as compact DG schemes can be implemented by directly discretizing the model equation (3.3).

To characterize the model equation (3.3), we shall adopt a concept of upwind-type flux:

Definition: *A refined flux F is said to be an upwind-type flux if:*
(i)(Consistency) $F(u,u) = f(u)$.
(ii)(Monotonicity) $F(u^-, u^+)$ is nondecreasing in u^- and non-increasing in u^+.
(iii)(Upwind) $F(u^-, u^+) = f(u^+)$ when $f' < 0$ for all u between u^+ and u^-; and $F(u^+, u^-) = f(u^-)$ when all $f' > 0$.
The monotonicity is required to ensure the entropy stability, but may lead to excessive dissipation. The characteristics imposed by the upwind requirement is expected to help sharpen things up again.

It is straightforward to verify that under a proper CFL restriction on the time step, the first order scheme (3.5) is a monotone scheme if (i) and (ii) are satisfied; and upwind scheme if (iii) is also ensured.

A wide variety of numerical methods with different numerical fluxes have appeared in the literature, which serve as good examples of refined flux. For scalar one-dimensional conservation laws, a well-known flux is the so called Godunov flux[64]:

$$F^{God}(u^+, u^-) = \begin{cases} \min\limits_{u^- \le u \le u^+} f(u) & \text{if } u^- \le u^+, \\ \max\limits_{u^+ \le u \le u^-} f(u) & \text{if } u^- > u^+, \end{cases} \tag{3.6}$$

which can be shown to satisfy all above three requirements.

An alternative upwind-type flux is as follows:

$$F(u^+, u^-) = \max\{f(\min(u^+, u^*)), f(\max(u^-, u^*))\}, \tag{3.7}$$

where u^* satisfies $f'(u)(u - u^*) \ge 0$, that is, f' changes sign at u^*. This flux is an improved version of that proposed in [15]

$$F(u^+, u^-) = f(\min(u^+, u^*)) + f(\max(u^-, u^*)) - f(u^*). \tag{3.8}$$

Indeed (3.7) gives the same approximation as the Godunov flux for many convex fluxes including $f = u^2/2$ for the Hopf equation.

The ENO-Engquist flux[20]

$$F^{EO}(u^+, u^-) = \frac{1}{2}\Big(f(u^+) + f(u^-) - \int_{u^-}^{u^+} |f'(s)| ds\Big) \tag{3.9}$$

also satisfies the above three requirements. However, when dealing with system of conservation laws, the local LxF flux becomes a simpler choice:

$$F(u^+,u^-)=f\Big(\frac{u^++u^-}{2}\Big)-\sigma\Big(\frac{u^+-u^-}{2}\Big), \tag{3.10}$$

where σ is taken as local maximum of eigenvalues of $f'(u)$ (see [71,72]).

The equation (3.3) provides a great deal of advantage for a direct numerical discretization. It would be interesting to develop a rigorous theory for this model equation (3.3).

3.2 Alternating Evolution

We now turn to another approximate equation for hyperbolic conservation laws

$$\partial_t U+\partial_x f(U)=0,\quad U(x,0)=U_0(x), \tag{3.11}$$

using an idea of alternative samples: a continuous equation once realized on grids, the true solution is replaced by a cartoon of samples on grids.

Let U be denoted by two samples $\{u,v\}$, an alternating evolution approximation to systems of hyperbolic conservation laws was proposed in [50] as follows:

$$\partial_t u+\partial_x f(v)=\frac{1}{\varepsilon}(v-u), \tag{3.12}$$

$$\partial_t v+\partial_x f(u)=\frac{1}{\varepsilon}(u-v), \tag{3.13}$$

with the initial condition

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\qquad x\in\mathbb{R}. \tag{3.14}$$

Here, $\varepsilon>0$ is a scale parameter of user's choice. In what follows, we consider (u_0,v_0) to be in hyperbolic region of (3.11) and (3.12) only, otherwise, it can be made so by a change of reference frame in (3.11) such that the flux $f(U)$ is replaced by $f(U)+CU$. To see the consistency of this approximation with the conservation law (3.11), we introduce two new variables

$$\phi:=\frac{u+v}{2},\quad \psi:=\frac{u-v}{2}.$$

Hence (ϕ,ψ) satisfy the following system of balance laws

$$\partial_t\phi+\partial_x\Big(\frac{f(\phi+\psi)+f(\phi-\psi)}{2}\Big)=0, \tag{3.15}$$

$$\partial_t\psi+\partial_x\Big(\frac{f(\phi-\psi)-f(\phi+\psi)}{2}\Big)=-\frac{4}{\varepsilon}\psi. \tag{3.16}$$

The system (3.15) is in conservative form, uniformly in ε, and the stiff source term now appears only in (3.16). The ψ is thus forced to approach zero exponentially as $\varepsilon\downarrow 0$, and as a consequence, (3.11) is found to be the limit of (3.15). Convergence of the approximate solution to the entropy solution of the scalar conservation laws (3.11) in multi-dimensional setting is established in [50].

Theorem 3.1[50]. *For any $(u_0,v_0)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and each fixed ε, (3.12), (3.13) admits a unique weak solution $(u^\varepsilon,v^\varepsilon)$ on $\mathbb{R}^d\times\mathbb{R}^+$ such that $(u,v)\in C([0,\infty);L^1(\mathbb{R}^d))$. Moreover, there exists a bounded measurable function $U(x,t)$ on $\mathbb{R}^d\times\mathbb{R}^+$ such that as $\varepsilon\downarrow 0$*

$$u^\varepsilon\to U(x,t),\quad u^\varepsilon-v^\varepsilon\to 0\quad (x,t)\in\mathbb{R}^d\times\mathbb{R}^+.$$

Also U is the entropy solution of (3.11) with initial data $U_0(x) = \frac{1}{2}(u_0(x) + v_0(x))$ for $x \in \mathbb{R}^d$.

Among nice features of the AE approximation is its accuracy of no method error once initial data is properly prepared.

Corollary 3.1. *Let U be the entropy solution of the scalar conservation laws (3.11) with initial data $U_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $(u^\varepsilon, v^\varepsilon)$ be the weak solution subject to initial data with $(u_0, v_0) \in L^\infty(\mathbb{R}^d)$. Then it holds*

$$\|u^\varepsilon - v^\varepsilon\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} e^{-4t/\varepsilon}.$$

Furthermore,
(i) if $u_0 + v_0 = 2U_0$, then

$$\lim_{t/\varepsilon \to \infty} \|u^\varepsilon(\cdot, t) - U(\cdot, t)\|_{L^1(\mathbb{R}^d)} = \lim_{t/\varepsilon \to \infty} \|v^\varepsilon(\cdot, t) - U(\cdot, t)\|_{L^1(\mathbb{R}^d)} = 0.$$

(ii) if $u_0 = v_0 = U_0$, then

$$u^\varepsilon(x, t) = v^\varepsilon(x, t) = U(x, t)$$

almost everywhere in $\mathbb{R}^d \times \mathbb{R}^+$.

This feature may be vividly illustrated for the linear scalar conservation laws with constant speed a:

$$\partial_t U + a\partial_x U = 0, \quad U(x, 0) = U_0(x), \tag{3.17}$$

with an explicit solution $U(x, t) = U_0(x - at)$. In such a case the AE approximation system becomes

$$\partial_t u + a\partial_x v = \frac{1}{\varepsilon}(v - u), \quad \partial_t v + a\partial_x u = \frac{1}{\varepsilon}(u - v).$$

Its exact solution can be written as

$$u = \frac{1}{2}u_0(x - at)(1 + e^{-4t/\varepsilon}) + \frac{1}{2}v_0(x - at)(1 - e^{-4t/\varepsilon}),$$
$$v = \frac{1}{2}u_0(x - at)(1 - e^{-4t/\varepsilon}) + \frac{1}{2}v_0(x - at)(1 + e^{-4t/\varepsilon}).$$

From this we see that

- if we choose $u_0 = v_0 = U_0$ initially, then $u = v = U_0(x - at)$ gives the exact solution for the linear advection equation (3.17);
- both u and v converge exponentially to the average of $u_0(x - at)$ and $v_0(x - at)$ as ε/t becomes small. In particular, the quantity $u + v$ is $(u_0 + v_0)(x - at)$ independent of ε. This shows that the exact solution for conservation laws can still be precisely captured if $u_0 + v_0 = 2U_0$ is enforced initially.

This lays the foundation for using the AE approximation as a tool in constructing high resolution numerical schemes for systems of hyperbolic conservation laws. The discretization procedure in [50] is summarized as follows:

Let $\{x_j\}$ be a uniform partition of the domain in $\mathbb{R}$ with $\Delta x = x_{j+1} - x_j$ and $x_{j\pm1} = x_j \pm \Delta x$.

Lax-Friedrichs Averaging: Let $\overline{u}(x)$ denote the sliding average of u over

$$I_x = \{\xi | \quad x - \Delta x \le \xi \le x + \Delta x\}.$$

Integration of the 1-D AE system (3.12), (3.13) over I_x gives

$$\frac{d}{dt}\overline{u}(x,t) = -\frac{\overline{u}(x,t)}{\varepsilon} + \frac{1}{\varepsilon}L[v](x,t), \tag{3.18}$$

$$\frac{d}{dt}\overline{v}(x,t) = -\frac{\overline{v}(x,t)}{\varepsilon} + \frac{1}{\varepsilon}L[u](x,t), \tag{3.19}$$

where x serves as a moving parameter and

$$L[U](x,t) := \overline{U}(x,t) - \frac{\varepsilon}{2\Delta x}\big[f(U(x+\Delta x,t)) - f(U(x-\Delta x,t))\big]. \tag{3.20}$$

Alternate sampling: The numerical scheme can be constructed by sampling both formulations (3.18) and (3.19) over alternating grids for u and v, respectively, with proper approximation of $L[U]$ defined in (3.20). For example, if we sample (3.18) at x_{2j} and (3.19) at x_{2j+1}, respectively, then we have

$$\frac{d}{dt}u_{2j}(t) = \frac{1}{\varepsilon}[-u_{2j}(t) + L_{2j}[v](t)], \tag{3.21}$$

$$\frac{d}{dt}v_{2j+1}(t) = \frac{1}{\varepsilon}[-v_{2j+1}(t) + L_{2j+1}[u](t)]. \tag{3.22}$$

The high accuracy of the scheme is realized via two steps: (i) high-order reconstruction U from averages $U_k(t)$ and evaluation of $L[U](t)$ accordingly; (ii) higher-order approximation of the above ODE system.

Higher order reconstruction: To initiate the algorithm, at $t = 0$, we employ the initial data:

$$u_0(x) = v_0(x) = U_0(x), \qquad x \in \mathbb{R}.$$

The grid values are defined through the Lax-Friedrichs averaging

$$U_k^0 = \frac{1}{2\Delta x}\int_{I_k} U_0(\xi)d\xi, \qquad k \in \mathbb{Z},$$

where $I_k := [x_{k-1}, x_{k+1}]$. Given grid values U_k^n, say $n = 0$, we assign them to u and v alternatively,

$$U_k^n = \begin{cases} u_k^n, & k = 2j, \\ v_k^n, & k = 2j+1, \end{cases}$$

and then reconstruct piecewise polynomials $p[U^n](x)$ over intervals $\{I_k\}$ for k even and odd, respectively, such that

$$U_k^n = \frac{1}{2\Delta x}\int_{I_k} p[U^n](x)dx, \qquad k \in \mathbb{Z}. \tag{3.23}$$

This piecewise polynomial reconstruction is conservative, (3.23), and should also be accurate of desired order, and non-oscillatory[30,31,63,78].

With such a reconstructed $p[U^n]$, we proceed to evaluate $L[U^n]$ as follows:

(i) At any even node $k = 2j$, using piecewise polynomial $p[v^n]$ constructed over $[x_{2j}, x_{2j+2}]$ we obtain

$$L_{2j}[v^n] = \frac{1}{2\Delta x}\int_{I_{2j}} p[v^n](x)dx - \frac{\varepsilon}{2\Delta x}\big[f(p[v^n]_{2j+1}) - f(p[v^n]_{2j-1})\big]. \tag{3.24}$$

(ii) At any odd node $k = 2j + 1$, using piecewise polynomials $p[u^n](x)$ constructed over $[x_{2j-1}, x_{2j+1}]$ we obtain $L_{2j+1}[u^n]$ as

$$L_{2j+1}[u^n] = \frac{1}{2\Delta x}\int_{I_{2j+1}} p[u^n](x)dx - \frac{\varepsilon}{2\Delta x}\big[f(p[u^n]_{2j+2}) - f(p[u^n]_{2j})\big]. \tag{3.25}$$

These procedures together with the Runge-Kutta time discretization[29] of the system (3.21), (3.22) enable us to design a new class of high-resolution schemes for hyperbolic conservation laws, called AE schemes. We refer the reader interested in AE schemes to [50], and [1], in which a class of local AE schemes are proved to be TVD, and extensively tested.

4 Multi-valued Solutions

The Hopf equation when arising in the context of high frequency wave propagation, the classical entropy solutions are not adequate in describing the wave behavior beyond the singularity, where multi-valued solutions in physical space are needed. Therefore, capturing multi-valued solutions by efficient algorithms is an important issue.

4.1 Asymptotic Wave Dynamics

We now illustrate the necessity for considering multi-valued solutions via a scaled one-dimensional Schrödinger equation

$$i\varepsilon\psi_t^\varepsilon = -\frac{\varepsilon^2}{2}\psi_{xx}^\varepsilon, \tag{4.1}$$

subject to a highly oscillatory initial wave function

$$\psi^\varepsilon(x,0) = A_0(x)\exp(iS_0(x)/\varepsilon), \tag{4.2}$$

where ε is the scaled Planck constant. In the *semiclassical regime* which corresponds to a small value $\varepsilon \ll 1$, the wave field ψ^ε and its associated physical observables become highly oscillatory within the wave length $O(\varepsilon)$, and thus direct numerical simulation of the wave field becomes prohibitively costly. A natural way to remedy this problem is to use some approximate models which can resolve the small-scale in the wave field. The classical approach is the WKB (Wentzel-Kramers-Brillouin) method, which are asymptotic approximations obtained as the small scale goes to zero. The WKB ansatz consists of representing the wave field function ψ^ε in the form

$$\psi^\varepsilon(x,t) = A^\varepsilon(x,t)\exp(iS(x,t)/\varepsilon), \quad A^\varepsilon(x,t) = \sum_{j=0}^{\infty}\varepsilon^j A_j(x,t). \tag{4.3}$$

With this decomposition, the most singular part of the wave field is characterized by two quantities, the phase function S which satisfies a nonlinear eikonal equation and the amplitude function A which, to leading order, satisfies a transport equation, i.e.,

$$S_t + \frac{1}{2}|S_x|^2 = 0, \quad (A_0^2)_t + (A_0^2 S_x)_x = 0.$$

Correspondingly, the phase gradient $u = S_x$ and position density $\rho = |A_0|^2$ solves the pressureless Euler equation

$$\begin{cases} u_t + uu_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{4.4}$$

As is known, using the *entropy or viscosity* solution notion in solving the transport equation for ρ may lead to a measure-valued solution. This means that the intensity $|A|^2$ may develop a Dirac delta function supported along shock curves of the phase variable S. (See e.g. [3,6,7,19,28,70]). These singularities are called caustics and the energy of the wave becomes infinite there. This clearly contradicts to the a priori estimates for the linear wave equation (4.1).

A natural way to avoid such difficulties is to seek multi-valued u corresponding to crossing waves. This means that in general for every non-caustic location in space and time, (x, t), a set of functions $u_i,\ i = 1, \cdots$ is constructed. Each of these functions is a solution of the Hopf equation in a neighborhood of (x, t) with suitable boundary conditions that couple them together. This set of solutions is referred to as the multi-valued solution of the Hopf equation. Each u_i is called a branch of the multi-valued solution. In the context of wave propagation, each branch corresponds to an arrival of certain phases at that location while the viscosity solution picks out the phase corresponding to the first arrival wave.

4.2 Level Set Equation

It is clear both dispersive schemes and the shock capturing schemes documented in previous sections can not be used as numerical devices when multi-valuedness does occur in the solution. Computation of multi-valued solutions for the WKB system has been a challenging task. There have been several methods suggested in literature, including the ray tracing method, Hamilton-Jacobi based method, kinetic moments method as well as the wave front methods. We make no attempt to comment on these methods, but refer to [21] for a seminar survey on computational high-frequency wave propagation. More recently, a geometric view point has been adopted in place of the kinetic one in phase space; here we shall outline the corresponding level set methods developed in [8,35–37,51,52]. Traditionally the level set method has been a highly successful computational technique for capturing the evolution of curves and surfaces[67,68] with applications in diverse areas such as multi-phase fluids, computer vision, imaging processing, optimal shape design, etc.

For the Hopf equation $u_t + uu_x = 0$ with initial data $u(x, 0) = g(x)$, the solution develops singularities when characteristic collide at a physical location. The existence of the characteristic ODE solutions in configuration space suggests multi-valued solutions beyond the collision of characteristic curves. In fact, the multi-valued nature of this problem can be seen from the following implicit formula for the solution

$$u(x, t) = g(x - u(x, t)t); \tag{4.5}$$

when $g' < 0$, it is possible to have more than one solution at each (x, t).

To track a multi-valued solution, we may simply evolve the graph of $u(x, t)$ as a surface in the (x, p) space by a velocity field prescribed by the characteristics. In this way, the above implicit function is equivalent to the level set formulation

$$\phi(t, x, p) = 0, \quad \phi := p - g(x - pt). \tag{4.6}$$

This implicit representation is still valid beyond the crossing of characteristics. We can easily derive the evolution equation for ϕ in the phase space (x, p):

$$\phi_t + p\phi_x = 0, \tag{4.7}$$

subject to the initial data

$$\phi(0,x,p) = p - g(x).$$

We can thus capture the multi-valued solution of the above quasi-linear equation by solving the level set equation (4.7) followed by a post-projection procedure

$$u_i \in \{p| \quad \phi(t,x,p) = 0\}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{4.8}$$

It was suggested in [56] that the multi-valued density in (4.4) can also be evaluated within the same level set framework:

$$\rho_i \in \Big\{ \frac{f(t,x,p)}{|\phi_p|} \Big| \quad \phi(t,x,p) = 0 \Big\}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{4.9}$$

where f is obtained by solving

$$f_t + pf_x = 0, \quad f(0,x,p) = \rho_0(x).$$

Such a level set formulation based on the graph evolution for transport equation $u_t + uu_x = 0$ can be traced back to Jacobi (who did not consider multi-valued solutions), and has been extensively used in various contexts (see e.g. [13,22,25,65,75]). These earlier works focused either on the solution before the formation of multiple values or on preventing multi-valued solutions.

In [8,37], the level set formulation was first used as a numerical device to capture multi-valued solutions for general quasi-linear hyperbolic equations

$$u_t + F(u,x) \cdot \nabla_x u = B(u,x).$$

The corresponding level set equation is

$$\phi_t + F(p,x) \cdot \nabla_x \phi + B(p,x)\phi_p = 0. \tag{4.10}$$

In addition to the semiclassical approximation in Schrödinger equations, geometric optics applied to high frequency wave propagation problems often leads to a weakly coupled system of a Hamilton-Jacobi equation for phase S and a transport equation for position density

$$\begin{cases} S_t + H(x, \nabla_x S) = 0, \\ \rho_t + \nabla_x \big(\rho \nabla_p H(x, \nabla_x S) \big) = 0. \end{cases} \tag{4.11}$$

A key idea to compute this system in [8,37,51] is to represent the $n-$dimensional bi-characteristic manifold of the Hamiltonian-Jacobi equation in phase space by an implicit vector level set function $\phi(t,x,p)$, whose components solves the same Liouville equation

$$\phi_t + \nabla_p H \cdot \nabla_x \phi - \nabla_x H \cdot \nabla_p \phi = 0. \tag{4.12}$$

The multi-valued phase gradient, $u = \nabla_x S$, is realized by the zero level set, $u \in \{p, \phi(t,x,p) = 0\}$. Based on the level set framework in the phase space, the amplitude is evaluated by

$$\rho(t,x) = \int f(t,x,p)\delta(\phi)dp, \tag{4.13}$$

where the quantity f also solves the same Liouville equation (4.12) but with $f(0,x,p) = \rho_0(x)$ as initial data. The multi-valued higher moments can be also resolved by integrating f along

the bi-characteristic manifold in the phase directions (see [35,36]). We refer the reader to the review article [52] for further details.

4.3 Algorithm

The level set methods for computing multi-valued solutions are hybrid numerical schemes – splitting the solution process into three parts:

- Construct the level set function to imbed the given initial data;
- Evolve the level set equation in configuration space by un upwind solution algorithm;
- Realize the multi-valued solutions by projection onto the common zeros of level set functions such as (4.8) and (4.9).

In computing the level set equation in phase space, since the area of interest is close to the zero level set, it is possible to use fast local level set techniques in the same manner as in, e.g., [8,9,66,69], which will reduce the computational cost and also reduce the storage requirements [61].

In more complicated situations deriving the level set equation for modeling the intrinsic motion law remains a challenging task [54,55]. For example, for systems of hyperbolic equations with interaction of different characteristic wave fields, further techniques need to be developed for deriving an efficient level set equation.

5 Concluding Remarks

We have presented three different ways of discretization for the Hopf equation and general hyperbolic equations, highlighting different phenomenon captured by the chosen discretization. The dispersive schemes are presented mainly for the Hopf equation, the suggested model equations are related to some integrable equations for shallow water problems. For shock capturing methods both the flux refinement and the alternating evolution approximation are valid for general system of hyperbolic conservation laws, as well as other convection-dominated equations. The level set method is a powerful numerical device for computing multi-valued solutions for both the Hopf equation as well as more general WKB systems. In the future, we shall further analyze these model equations on rigorous grounds, as well as explore them to construct more efficient numerical methods.

Finally we remark that discrete schemes on lattices often have richer dynamics than its idealized continuous analogue. It is also important to understand the impact of the discreteness on the overall dynamics by studying each scheme as a discrete dynamic system on lattices.

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