

GLOBAL REGULARITY, AND WAVE BREAKING PHENOMENA IN A CLASS OF NONLOCAL DISPERSIVE EQUATIONS

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ABSTRACT. This paper is concerned with a class of nonlocal dispersive models – the θ -equation proposed by H. Liu [On discreteness of the Hopf equation, *Acta Math. Appl. Sin. Engl. Ser.* **24**(3)(2008)423–440]:

$$(1 - \partial_x^2)u_t + (1 - \theta\partial_x^2)\left(\frac{u^2}{2}\right)_x = (1 - 4\theta)\left(\frac{u_x^2}{2}\right)_x,$$

including integrable equations such as the Camassa-Holm equation, $\theta = 1/3$, and the Degasperis-Procesi equation, $\theta = 1/4$, as special models. We investigate both global regularity of solutions and wave breaking phenomena for $\theta \in \mathbb{R}$. It is shown that as θ increases regularity of solutions improves: (i) $0 < \theta < 1/4$, the solution will blow up when the momentum of initial data satisfies certain sign conditions; (ii) $1/4 \leq \theta < 1/2$, the solution will blow up when the slope of initial data is negative at one point; (iii) $\frac{1}{2} \leq \theta \leq 1$ and $\theta = \frac{2n}{2n-1}$, $n \in \mathbb{N}$, global existence of strong solutions is ensured. Moreover, if the momentum of initial data has a definite sign, then for any $\theta \in \mathbb{R}$ global smoothness of the corresponding solution is proved. Proofs are either based on the use of some global invariants or based on exploration of favorable sign conditions of quantities involving solution derivatives. Existence and uniqueness results of global weak solutions for any $\theta \in \mathbb{R}$ are also presented. For some restricted range of parameters results here are equivalent to those known for the b -equations [e.g. J. Escher and Z. Yin, Well-posedness, blow-up phenomena, and global solutions for the b -equation, *J. reine angew. Math.*, **624** (2008)51–80.]

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1. INTRODUCTION

In recent years nonlocal dispersive models have been investigated intensively at different levels of treatments: modeling, analysis as well as numerical simulation. The model derives in several ways, for instance, (i) the asymptotic modeling of shallow water waves [43, 20, 21]; (ii) renormalization of dispersive operators [43, 37]; and (iii) model equations of some dispersive schemes [38]. The peculiar feature of nonlocal dispersive models is their ability to capture both global smoothness of solutions and the wave breaking phenomena.

In this work we focus on a class of nonlocal dispersive models – the θ -equation of the form

$$(1 - \partial_x^2)u_t + (1 - \theta\partial_x^2) \left(\frac{u^2}{2} \right)_x = (1 - 4\theta) \left(\frac{u_x^2}{2} \right)_x, \quad (1.1)$$

subject to the initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}. \quad (1.2)$$

The equation can be formally rewritten as

$$u_t - u_{txx} + uu_x = \theta uu_{xxx} + (1 - \theta)u_x u_{xx}, \quad (1.3)$$

which when $0 < \theta < 1$ involves a convex combination of nonlinear terms uu_{xxx} and $u_x u_{xx}$. This class was identified by H. Liu [38] in his study of model equations for some dispersive schemes to approximate the Hopf equation

$$u_t + uu_x = 0.$$

The model (1.1) under a transformation links to the so called b -model,

$$u_t - \alpha^2 u_{txx} + c_0 u_x + (b + 1)uu_x + \Gamma u_{xxx} = \alpha^2 (bu_x u_{xx} + uu_{xxx}).$$

which has been extensively studied in recent years [17, 18, 24, 25, 28, 29]. Both classes of equations are contained in the more general class derived in [37] using renormalization of dispersive operators and number of conservation laws.

In (1.1), two equations are worth of special attention: $\theta = \frac{1}{3}$ and $\theta = \frac{1}{4}$. The θ -equation when $\theta = \frac{1}{3}$ reduces to the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, in which $u(t, x)$ denotes the fluid velocity at time t in the spatial x direction [3, 19, 30]. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [13, 15]. It has a bi-Hamiltonian structure [26, 33] and is completely integrable [3, 7]. Its solitary waves are smooth if $c_0 > 0$ and peaked in the limiting case $c_0 = 0$, cf. [4]. The orbital stability of the peaked solitons is proved in [12], and that of the smooth solitons in [14]. The explicit interaction of the peaked solitons is given in [1].

The Cauchy problem for the Camassa-Holm equation has been studied extensively. It has been shown that this problem is locally well-posed [8, 41] for initial data $u_0 \in H^{3/2+}(\mathbb{R})$. Moreover, it has global strong solutions [6, 8] and also admits finite time blow-up solutions [6, 8, 9]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [2, 10, 11, 44]. The advantage of the Camassa-Holm equation in comparison with the KdV equation,

$$u_t + uu_x + \Gamma u_{xxx} = 0,$$

lies in the fact that the Camassa-Holm equation has peaked solitons and models the peculiar wave breaking phenomena [4, 9].

Taking $\theta = \frac{1}{4}$ in (1.1) we find the Degasperis-Procesi equation [18]. The Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as that for the Camassa-Holm shallow water

equation [20, 21]. An inverse scattering approach for computing n -peakon solutions to the Degasperis-Procesi equation was presented in [36]. Its traveling wave solutions were investigated in [32, 42]. The formal integrability of the Degasperis-Procesi equation was obtained in [16] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to the Camassa-Holm peakons [16].

The study of the Cauchy problem for the Degasperis-Procesi equation is more recent. Local well-posedness of this equation is established in [46] for initial data $u_0 \in H^{3/2+}(\mathbb{R})$. Global strong solutions are proved in [22, 34, 47] and finite time blow-up solutions in [22, 34, 46, 47]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$, see e.g. [22, 47] and global entropy weak solutions belonging to the class $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and to the class $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, cf. [5].

Though both the Degasperis-Procesi and the Camassa-Holm equation share some nice properties, they differ in that the DP equation has not only peakon solutions [16] and periodic peakon solutions [48], but also shock peakons [35] and the periodic shock waves [23].

The main quest of this paper is to see how regularity of solutions changes in terms of the parameter θ . With this in mind we present a relative complete picture of solutions of problem (1.1)-(1.2) for different choices of θ .

Theorem 1.1. [Global regularity] *Let $u_0 \in H^{3/2+}(\mathbb{R})$ and $m_0 := (1 - \partial_x^2)u_0$.*

- i) *For any $\theta \neq 0$, if in addition $u_0 \in L^1(\mathbb{R})$, and m_0 has a definite sign ($m_0 \leq 0$ or $m_0 \geq 0$ for all $x \in \mathbb{R}$), then the solution remains smooth for all time. Moreover, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, we have*

$$(1) \quad m(t, x)u(t, x) \geq 0 \text{ and}$$

$$\|m_0\|_{L^1(\mathbb{R})} = \|m(t, \cdot)\|_{L^1(\mathbb{R})} = \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}.$$

$$(2) \quad \|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} \text{ and}$$

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u(t, \cdot)\|_1 \leq \frac{\sqrt{2}}{2} e^{\frac{1-\theta-3|t|}{2}} \|u_0\|_{L^1(\mathbb{R})} \|u_0\|_1.$$

- ii) *For $\frac{1}{2} \leq \theta \leq 1$, if in addition $u_0 \in W^{2, \frac{\theta}{1-\theta}}(\mathbb{R})$, then the solution remains smooth for all time.*
- iii) *For $\theta = \frac{2n}{2n-1} \in (1, 2)$, $n \in \mathbb{N}$, if in addition $u_0 \in W^{3, \frac{\theta}{\theta-1}}(\mathbb{R})$, then the solution remains smooth for all time.*

Remark 1.1. The result stated in i) recovers the global existence result of strong solutions to the Camassa-Holm equation in [8] and the Degasperis-Procesi equation in [47].

Theorem 1.2. [Blow up criterion] *Let $u_0 \in H^{3/2+}(\mathbb{R})$ and $m_0 := (1 - \partial_x^2)u_0$.*

- i) *For $0 < \theta \leq \frac{1}{4}$ and a fixed x^* , if $u_0(x^* + x) = -u_0(x^* - x)$ and $(x - x^*)m_0(x) \leq 0$ for any $x \in \mathbb{R}$, then the solution must blow up in finite time strictly before $T^* = -\frac{1}{u_x(0, x^*)}$ provided $u_x(0, x^*) < 0$.*
- ii) *For $\frac{1}{4} \leq \theta < \frac{1}{2}$, if $u_0(x^* + x) = -u_0(x^* - x)$ for any $x \in \mathbb{R}$ and $u_x(0, x^*) < 0$, then the solution must blow up in finite time strictly before $T^* = \frac{2\theta}{(2\theta-1)u_x(0, x^*)}$.*

Remark 1.2. The result stated in Theorem 1.2 shows that strong solutions to the θ -equation (1.1)-(1.2) for $0 < \theta < \frac{1}{2}$ may blow up in finite time, while Theorem 1.1 shows that in the case $\frac{1}{2} \leq \theta \leq 1$ every strong solution to the θ -equation (1.1) exists globally in time. This presents a clear picture for global regularity and blow-up phenomena of solutions to the θ -equation for all $0 < \theta \leq 1$.

We shall present main ideas for proofs of the above results, a further refined analysis could be done following those presented in [24] for the b -equation. Note that the case $\theta = 0$ is a borderline case and not covered by the above results, we shall present a detailed account for this case. For completeness, we also present global existence results for weak solutions to characterize peakon solutions to (1.1) for any $\theta \in \mathbb{R}$.

The rest of this paper is organized as follows. In §2, we present some preliminaries including how the θ -equation relates to other class of dispersive equations, the local well-posedness and some key quantities to be used in subsequent analysis. In §3, we show how global existence of smooth solutions is established. The ideas for deriving the precise blow-up scenario is given in §4. A detailed account for the case of $\theta = 0$ is presented in §5. Two existence and uniqueness results on global weak solutions and one example for peakon solutions to (1.1) for any $\theta \in \mathbb{R}$ are given in §6.

2. PRELIMINARIES

2.1. The θ -equation and its variants. The θ -equation of the form

$$(1 - \partial_x^2)u_t + (1 - \theta \partial_x^2) \left(\frac{u^2}{2} \right)_x = (1 - 4\theta) \left(\frac{u_x^2}{2} \right)_x, \quad (2.1)$$

up to a scaling of $t \rightarrow \frac{t}{\theta}$ for $\theta \neq 0$, can be rewritten into a class of B-equations

$$u_t + uu_x + [Q * B(u, u_x)]_x = 0, \quad (2.2)$$

where $Q = \frac{1}{2}e^{-|x|}$ and

$$B = \left(\frac{1}{\theta} - 1 \right) \frac{u^2}{2} + \left(4 - \frac{1}{\theta} \right) \frac{u_x^2}{2}.$$

The B -class with B being quadratic in u and u_x was derived in [37] by using a renormalization technique and examining number of conservation laws. In this B -class the Camasa-Holm equation corresponds to $B(u, u_x) = u^2 + u_x^2/2$; and the Degasperis-Procesi equation corresponds to $B(u, u_x) = 3u^2/2$. The local well-posedness for (2.2) with initial data $u_0(x)$ was established in [37].

Theorem 2.1. [37] *Suppose that $u_0 \in H_x^{3/2+}$ and $B(u, p)$ are quadratic functions in its arguments, then there exists a time T and a unique solution u of (2.2) in the space $C([0, T]; H^{3/2+}(\mathbb{R})) \cap C^1([0, T]; H^{1/2+}(\mathbb{R}))$ such that $\lim_{t \downarrow 0} u(t, \cdot) = u_0(\cdot)$. If $T < \infty$ is the maximal existence time, then*

$$\lim_{t \rightarrow T} \sup_{0 \leq \tau \leq t} \|u_x(\cdot, \tau)\|_{L^\infty(\Omega)} = \infty,$$

where $\Omega = \mathbb{R}$ for initial data decaying at far fields, or $\Omega = [0, \pi]$ for periodic data.

Wave breaking criteria are identified separately for several particular models in class (2.2), using their special features, see [37] for further details.

For $\theta \neq 0$, the class of θ -equations can also be transformed into the b -equation of the form

$$u_t - \alpha^2 u_{txx} + c_0 u_x + (b+1)uu_x + \Gamma u_{xxx} = \alpha^2 (bu_x u_{xx} + uu_{xxx}). \quad (2.3)$$

In fact, if we set

$$\begin{aligned} u(t, x) &= c_0 \theta + \tilde{u}(\tau, z), \\ z &= \alpha \left(x - \theta \left(1 + \frac{\Gamma}{\alpha^2} \right) t \right), \\ \tau &= \alpha \theta t, \end{aligned}$$

then a straightforward calculation leads to

$$(1 - \alpha^2 \partial_z^2) \tilde{u}_\tau + c_0 \tilde{u}_z + \frac{1}{\theta} \tilde{u} \tilde{u}_z + \Gamma \tilde{u}_{zzz} = \alpha^2 \left(\left(\frac{1}{\theta} - 1 \right) \tilde{u}_z \tilde{u}_{zz} + \tilde{u} \tilde{u}_{zzz} \right).$$

Setting

$$\theta = \frac{1}{b+1}$$

and changing variables (\tilde{u}, τ, z) back to (u, t, x) we thus obtain the so-called b -equation (2.3).

Note that the θ -equation does not include the case $b = -1$, which has been known un-physical. Also $\theta = 0$ case is not in the class of B -equation (2.2) either.

2.2. Local well-posedness and a priori estimates. In order to prove our main results for different cases, we need to establish the following local existence result.

Theorem 2.2. [Local existence] *Let $u_0 \in H^{3/2+}(\mathbb{R})$, then exists a $T = T(\theta, \|u_0\|_{3/2+}) > 0$ and a unique solution in*

$$C([0, T]; H^{3/2+}(\mathbb{R})) \cap C^1([0, T]; H^{1/2+}(\mathbb{R})).$$

The solution depends continuously on the initial data, i.e. the mapping

$$u_0 \rightarrow u(\cdot, u_0) : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R})), \quad s > 3/2$$

is continuous. Moreover, if $T < \infty$ then $\lim_{t \rightarrow T} \|u(t, \cdot)\|_s = \infty$.

The proof for $\theta \neq 0$ follows from that for the b -equation in [24] or for the B -equation in [38].

Furthermore we have the following result.

Theorem 2.3. *Let $u_0 \in H^{3/2+}(\mathbb{R})$ be given and assume that T is the maximal existence time of the corresponding solution to (1.1) with the initial data u_0 . If there exists an $M > 0$ such that*

$$\|u_x(t, x)\|_{L^\infty(\mathbb{R})} \leq M, \quad t \in [0, T],$$

then the $H^s(\mathbb{R})$ -norm of $u(t, \cdot)$ does not blow up for $t \in [0, T]$.

Let u be the solution in $C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$, it suffices to verify how $\|u(t, \cdot)\|_s$ depends on $\|u_x(t, \cdot)\|_\infty$. Here we could carry out a careful energy estimate to obtain a differential inequality of the form

$$\frac{d}{dt} \|u(t, \cdot)\|_s \leq C \|u_x(t, \cdot)\|_\infty \|u(t, \cdot)\|_s.$$

The claim then follows from the Gronwall inequality. A detailed illustration of such a procedure for the case $\theta = 0$ will be given in §5.

Remark 2.1. This result is fundamental for us to prove or disprove the global existence of strong solutions. More precisely, global existence follows from a priori estimate on $\|u_x(t, \cdot)\|_\infty$, and the finite time blow up of $\|u_x(t, \cdot)\|_\infty$ under certain initial conditions reveals the wave breaking phenomena.

3. GLOBAL REGULARITY

3.1. Key invariants and favorable sign conditions. Let T be the life span of the strong solution $u \in C([0, T]; H^{3/2+}(\mathbb{R})) \cap C^1([0, T]; H^{1/2+}(\mathbb{R}))$. We now look at some key estimates valid for $t \in [0, T]$. First since the θ -equation is in conservative form, so

$$\int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_0 dx. \quad (3.1)$$

Let $m = (1 - \partial_x^2)u$, then

$$u = (1 - \partial_x^2)^{-1}m = Q * m, \quad (3.2)$$

which implies

$$\int_{\mathbb{R}} m dx = \int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_0 dx = \int_{\mathbb{R}} m_0 dx. \quad (3.3)$$

Moreover the equation (1.1) can be reformulated as

$$m_t + \theta u m_x + (1 - \theta) m u_x = 0. \quad (3.4)$$

For any $\alpha \in \mathbb{R}$, let $x = x(t, \alpha)$ be the curve determined by

$$\frac{d}{dt}x = \theta u(t, x), \quad x(0, \alpha) = \alpha$$

for $t \in [0, T)$. Then $F = \frac{\partial x}{\partial \alpha}$ solves

$$\frac{d}{dt}F = \theta u_x F$$

as long as u remains a strong solution. Along the curve $x = x(t, \alpha)$ we also have

$$\frac{d}{dt}m = (\theta - 1)u_x m.$$

These together when canceling the common factor u_x leads to the following global invariant:

$$m(t, x(t, \alpha))F^{\frac{1}{\theta}-1} = m_0(\alpha), \quad \forall \alpha \in \mathbb{R}. \quad (3.5)$$

From this Lagrangian identity we see that m has a definite sign once m_0 has. Correspondingly it follows from (3.2) that u has a definite sign

$$\text{sign}(m) = \text{sign}(m_0) = \text{sign}(u) \quad (3.6)$$

provided that m_0 has a definite sign on \mathbb{R} .

From (3.5) it follows

$$\int_{\mathbb{R}} |m|^{\frac{\theta}{1-\theta}}(t, x(t, \alpha))F d\alpha = \int_{\mathbb{R}} |m|^{\frac{\theta}{1-\theta}} dx = \int_{\mathbb{R}} |m_0|^{\frac{\theta}{1-\theta}} dx,$$

which yields the following estimate:

$$\frac{d}{dt} \int_{\mathbb{R}} |m|^{\frac{\theta}{1-\theta}} dx = 0. \quad (3.7)$$

Inspired by [17] we identify another conservation laws as follows

$$\frac{d}{dt} \int_{\mathbb{R}} \left((1 - \theta)^2 m^{\frac{2-\theta}{\theta-1}} m_x^2 + \theta^2 m^{\frac{\theta}{\theta-1}} \right) dx = 0. \quad (3.8)$$

This conserved quantity will be used for some cases in the range $\theta > 1$.

Multiplying (3.4) by $m = u - u_{xx}$, and integrating by parts, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} m^2 dx = (3\theta - 2) \int_{\mathbb{R}} u_x m^2 dx, \quad (3.9)$$

which suggests that $\theta = \frac{2}{3}$ is a critical point for the blow-up scenario. Note that

$$\|u(t, \cdot)\|_2 \leq \|m(t, \cdot)\|_{L^2} \leq \sqrt{2} \|u(t, \cdot)\|_2. \quad (3.10)$$

Both (3.9) and (3.10) together enable us to conclude the following

Theorem 3.1. *Assume $u_0 \in H^{3/2+}(\mathbb{R})$. If $\theta = \frac{2}{3}$, then every solution to (1.1)-(1.2) remains regular globally in time. If $\theta < \frac{2}{3}$, then the solution will blow up in finite time if and only if the slope of the solution becomes unbounded from below in finite time. If $\theta > \frac{2}{3}$, then the solution will blow up in finite time if and only if the slope of the solution becomes unbounded from above in finite time.*

Remark 3.1. This result not only covers the corresponding results for the Camassa-Holm equation in [6, 45] and the Degasperis-Procesi equation in [46], but also presents another different possible blow-up mechanism, i.e., if $\theta > \frac{2}{3}$, then the solution to (1.1) blows up in finite time if and only if the slope of the solution becomes unbounded from above in finite time.

3.2. Global existence: proof of Theorem 1.1. Let T be the maximum existence time of the solution u with initial data $u_0 \in H^s$. Using a simple density argument we can just consider the case $s = 3$. Based on Theorem 2.2 and Theorem 2.3 it suffices to show the uniform bound of $\|u_x(t, \cdot)\|_\infty$ for all cases presented in Theorem 1.1.

The proof of the first assertion i) is based on the global invariant (3.5), which implies (3.6), i.e., m has a definite sign for $t > 0$ as long as m_0 has a definite sign. Then for any $(t, x) \in [0, T) \times \mathbb{R}$,

$$\begin{aligned} |u_x(t, x)| &= |Q_x * m| \leq \|Q_x\|_\infty \|m\|_{L^1} = \frac{1}{2} \left| \int_{\mathbb{R}} m dx \right| \\ &= \frac{1}{2} \left| \int_{\mathbb{R}} u dx \right| = \frac{1}{2} \left| \int_{\mathbb{R}} u_0 dx \right| \leq \frac{1}{2} \|u_0\|_{L^1}. \end{aligned}$$

Then $T = \infty$.

The second assertion ii) follows from the use of (3.7), i.e.,

$$\int_{\mathbb{R}} |m|^{\frac{\theta}{1-\theta}} dx = \int_{\mathbb{R}} |m_0|^{\frac{\theta}{1-\theta}} dx \leq \|u_0\|_{W^{2,p}}, \quad p = \frac{\theta}{1-\theta} \in [1, \infty].$$

From $m \in L^p(\mathbb{R})$ and $u - u_{xx} = m$ it follows that $u \in W^{2,p}(\mathbb{R})$. By the Sobolev imbedding theorem, we see that $W^{2,p}(\mathbb{R}) \subset C^1(\mathbb{R})$. Thus $T = \infty$.

The last assertion iii) follows from the use of (3.8) with $\theta = \frac{2n}{2n-1}$, which upon integration leads to

$$\int_{\mathbb{R}} (m^{2n-2} m_x^2 + 4n^2 m^{2n}) dx = \int_{\mathbb{R}} (m_0^{2n-2} m_{0x}^2 + 4n^2 m_0^{2n}) dx.$$

From this we see that $m \in L^\infty$, for

$$m^{2n} = \int_{-\infty}^x 2nm^{2n-1} m_x dx \leq \frac{1}{2} \int_{\mathbb{R}} (m^{2n-2} m_x^2 + 4n^2 m^{2n}) dx.$$

Using $u = Q * m$ we obtain that $u \in W^{2,\infty}$; that is $|u_x|$ is uniformly bounded. Thus $T = \infty$.

4. BLOW UP PHENOMENA: PROOF OF THEOREM 1.2

For the blow up analysis, one needs to find a way to show that $d = u_x$ will become unbounded in finite time. Rewriting (2.1) as

$$u_t + \theta u u_x = \frac{Q_x}{2} * [(1 - 4\theta)u_x^2 + (\theta - 1)u^2].$$

Notice that $Q_{xx} = Q - \delta(x)$; a direct differentiation in x of the above equation leads to

$$d_t + \theta u d_x + \left(\frac{1}{2} - \theta\right) d^2 = \frac{1-\theta}{2} u^2 + \frac{Q}{2} * [(1 - 4\theta)d^2 + (\theta - 1)u^2].$$

For $\theta < \frac{1}{2}$ there is no control on u^2 term while we track dynamics of d . The idea here, motivated by that used in [24], is to focus on a curve $x = h(t)$ such that $u(t, h(t)) = 0$ and $h(0) = x^*$. On this curve

$$\dot{d} + \left(\frac{1}{2} - \theta\right) d^2 = \frac{Q}{2} * [(1 - 4\theta)u_x^2 + (\theta - 1)u^2](t, h(t)). \quad (4.1)$$

Two cases are distinguished:

(i) $\frac{1}{4} \leq \theta < \frac{1}{2}$. In this range of θ , the right-hand side of (4.1) is non-positive. We thus have

$$\dot{d} + \left(\frac{1}{2} - \theta\right) d^2 \leq 0,$$

for which d will become unbounded from below in finite time as long as $d(0, h(0)) = u_x(0, x^*) < 0$.

(ii) $0 < \theta < \frac{1}{4}$. In this range of θ we also need control the nonlocal term. If we can identify some initial data such that

$$Q * [(1 - 4\theta)u_x^2 + (\theta - 1)u^2](t, h(t)) \leq (1 - 4\theta) [u_x^2 - u^2](t, h(t)) = (1 - 4\theta)d(t)^2. \quad (4.2)$$

Then we have

$$\dot{d} + \left(\frac{1}{2} - \theta\right) d^2 \leq \left(\frac{1}{2} - 2\theta\right) d^2.$$

That is

$$\dot{d} + \theta d^2 \leq 0.$$

Again in this case d will become unbounded from below in finite time once $d(0, h(0)) = u_x(0, x^*) < 0$.

Now we verify that the assumptions in Theorem 1.2 are sufficient for claim (4.2) to hold. From $u_0(x^* + x) = -u_0(x^* - x)$ for any $x \in \mathbb{R}$, it follows that $u_0(x^*) = 0$, and $u(t, x^* + x) = -u(t, x^* - x)$ due to symmetry of the equation. We then have $u(t, x^*) = 0$, leading to the case $h(t) = x^*$.

We further assume that

$$(x - x^*)m_0(x) \leq 0,$$

which combined with (3.5) yields

$$(x - x^*)m(t, x) \leq 0.$$

This relation enables one to use a similar argument as (5.3)-(5.10) in [24] to obtain

$$Q * [u_x^2 - u^2](t, x^*) \leq (u_x^2 - u^2)(t, x^*).$$

Hence

$$\begin{aligned} Q * [(1 - 4\theta)u_x^2 + (\theta - 1)u^2](t, x^*) &= (1 - 4\theta)Q * [u_x^2 - u^2](t, x^*) - 3\theta Q * [u^2](t, x^*) \\ &\leq (1 - 4\theta)Q * [u_x^2 - u^2](t, x^*) \\ &\leq (1 - 4\theta)(u_x^2 - u^2)(t, x^*), \end{aligned}$$

which leads to (4.2) as desired.

5. A DETAILED ACCOUNT OF THE CASE $\theta = 0$

In this section, we establish the local well-posedness and present the precise blow-up scenario and global existence results for the θ -equation with $\theta = 0$, i.e.,

$$u_t - u_{txx} = u_x u_{xx} - uu_x. \quad (5.1)$$

Note that $(1 - \partial_x^2)^{-1}f = Q * f$ for all $f \in L^2(\mathbb{R})$ and $Q * m = u$ for $m = u - u_{xx}$. Using this relation, we can rewrite (5.1) as follows:

$$\begin{cases} u_t = \partial_x Q * \left(\frac{1}{2}u_x^2 - \frac{1}{2}u^2\right), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (5.2)$$

or in the equivalent form:

$$\begin{cases} u_t = \partial_x (1 - \partial_x^2)^{-1} \left(\frac{1}{2}u_x^2 - \frac{1}{2}u^2\right), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (5.3)$$

Theorem 5.1. *Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, there exists a $T = T(\|u_0\|_s) > 0$, and a unique solution u to (5.1) such that*

$$u = u(\cdot, u_0) \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^s(\mathbb{R})).$$

The solution depends continuously on the initial data, i.e. the mapping

$$u_0 \rightarrow u(\cdot, u_0) : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^s(\mathbb{R}))$$

is continuous. Moreover, if $T < \infty$ then $\lim_{t \rightarrow T} \|u(t, \cdot)\|_s = \infty$.

Proof. Set $f(u) = \partial_x Q * (\frac{1}{2}u_x^2 - \frac{1}{2}u^2) = \partial_x(1 - \partial_x^2)^{-1}(\frac{1}{2}u_x^2 - \frac{1}{2}u^2)$. Let $u, v \in H^s$, $s > \frac{3}{2}$. Note that H^{s-1} is a Banach algebra. Then, we have

$$\begin{aligned} & \| f(u) - f(v) \|_s \\ &= \| \partial_x(1 - \partial_x^2)^{-1} \left(\frac{1}{2}(u^2 - v^2) + \frac{1}{2}(u_x^2 - v_x^2) \right) \|_s \\ &\leq \frac{1}{2} \| (u - v)(u + v) \|_{s-1} + \frac{1}{2} \| (u_x - v_x)(u_x + v_x) \|_{s-1} \quad (5.4) \\ &\leq \frac{1}{2} \| u - v \|_s \| u + v \|_s + \frac{1}{2} \| \partial_x(u - v) \|_{s-1} \| u_x + v_x \|_{s-1} \\ &\leq (\| u \|_s + \| v \|_s) \| u - v \|_s. \end{aligned}$$

This implies that $f(u)$ satisfies a local Lipschitz condition in u , uniformly in t on $[0, \infty)$.

Next we show that for every $t_0 \geq 0$, $u(t_0) \in H^s(\mathbb{R})$, the Cauchy problem (5.1) has a unique mild solution u on an interval $[t_0, t_1]$ whose length is bounded below by

$$\delta(\|u(t_0)\|_s) = \frac{\|u(t_0)\|_s}{r^2(t_0)} = \frac{1}{2r(t_0)},$$

where $r(t_0) = 2\|u(t_0)\|_s$. Set $t_1 = t_0 + \delta(\|u(t_0)\|_s)$. Let us define by

$$\|u\|_{C([t_0, t_1]; H^s(\mathbb{R}))} := \sup_{t \in [t_0, t_1]} \|u\|_s$$

the norm of u as an element of $C([t_0, t_1]; H^s(\mathbb{R}))$. For a given $u(t_0) \in H^s$ we define a mapping $F : C([t_0, t_1]; H^s(\mathbb{R})) \rightarrow C([t_0, t_1]; H^s(\mathbb{R}))$ by

$$(Fu)(t) = u(t_0) + \int_{t_0}^t f(u(s)) ds, \quad t_0 \leq t \leq t_1. \quad (5.5)$$

The mapping F defined by (5.5) maps the ball of radius $r(t_0)$ centered at 0 of $C([t_0, t_1]; H^s(\mathbb{R}))$ into itself. This follows from the following estimate

$$\begin{aligned} \|(Fu)(t)\|_s &\leq \|u(t_0)\|_s + \int_{t_0}^t \|f(u(s)) - f(0)\|_s ds \\ &\leq \|u(t_0)\|_s + r^2(t_0)(t - t_0) \\ &\leq 2\|u(t_0)\|_s = r(t_0), \end{aligned} \quad (5.6)$$

where we have used the relations (5.4)-(5.5), $f(0) = 0$ and the definition of t_1 .

By (5.4) and (5.5), we have

$$\|(Fu)(t) - (Fv)(t)\|_s \leq 2r(t_0)(t - t_0)\|u - v\|_{C([t_0, t_1]; H^s(\mathbb{R}))}. \quad (5.7)$$

Using (5.5) and (5.7) and induction on n , we obtain

$$\begin{aligned} \|(F^n u)(t) - (F^n v)(t)\|_s &\leq \frac{(2r(t_0)(t-t_0))^n}{n!} \|u-v\|_{C([t_0, t_1]; H^s(\mathbb{R}))} \\ &\leq \frac{(2r(t_0)\delta(\|u(t_0)\|_s))^n}{n!} \|u-v\|_{C([t_0, t_1]; H^s(\mathbb{R}))} \\ &\leq \frac{1}{n!} \|u-v\|_{C([t_0, t_1]; H^s(\mathbb{R}))}. \end{aligned} \quad (5.8)$$

For $n \geq 2$ we have $\frac{1}{n!} < 1$. Thus, by a well known extension of the Banach contraction principle, we know that F has a unique fixed point u in the ball of $C([t_0, t_1]; H^s(\mathbb{R}))$. This fixed point is the mild solution of the following integral equation associated with Eq.(5.1):

$$u(t, x) = u(t_0, x) + \int_{t_0}^t \partial_x Q * \left(\frac{1}{2}u_x^2 - \frac{1}{2}u^2\right)(\tau, x) d\tau. \quad (5.9)$$

Next, we prove the uniqueness of u and the Lipschitz continuity of the map $u(t_0) \rightarrow u$. Let v be a mild solution to (5.1) on $[t_0, t_1)$ with initial data $v(t_0)$. Note that $\|u\|_s \leq 2\|u(t_0)\|_s$ and $\|v\|_s \leq 2\|v(t_0)\|_s$. Then

$$\begin{aligned} &\|u(t) - v(t)\|_s \\ &\leq \|u(t_0) - v(t_0)\|_s + \int_{t_0}^t \|f(u) - f(v)\|_s d\tau \\ &\leq \|u(t_0) - v(t_0)\|_s + (\|u(t_0)\|_s + \|v(t_0)\|_s) \int_{t_0}^t \|u(t) - v(t)\|_s d\tau. \end{aligned} \quad (5.10)$$

An application of Gronwall's inequality yields

$$\|u(t) - v(t)\|_s \leq e^{(\|u(t_0)\|_s + \|v(t_0)\|_s)(t_1 - t_0)} \|u(t_0) - v(t_0)\|_s.$$

Therefore

$$\|u - v\|_{C([t_0, t_1]; H^s(\mathbb{R}))} \leq e^{(\|u(t_0)\|_s + \|v(t_0)\|_s)(t_1 - t_0)} \|u(t_0) - v(t_0)\|_s, \quad (5.11)$$

which implies both the uniqueness of u and the Lipschitz continuity of the map $u(t_0) \rightarrow u$.

From the above we know that if u is a mild solution of (5.1) on the interval $[0, \tau]$, then it can be extended to the interval $[0, \tau + \delta]$ with $\delta > 0$ by defining on $[\tau, \tau + \delta]$, $u(t, x) = v(t, x)$ where $v(t, x)$ is the solution of the following integral equation

$$v(t, x) = u(\tau) + \int_{\tau}^t f(v(s, x)) ds, \quad \tau \leq t \leq \tau + \delta,$$

where δ depends only on $\|u(\tau, \cdot)\|_s$. Let T be the maximal existence time of the mild solution u of (5.1). If $T < \infty$ then $\lim_{t \rightarrow T} \|u(t, \cdot)\|_s = \infty$. Otherwise there is a sequence $t_n \rightarrow T$ such that $\|u(t_n, \cdot)\|_s \leq C$ for all n . This would yield that for each t_n , near enough to T , u defined on $[0, t_n]$ can be extended to $[0, t_n + \delta]$ where $\delta > 0$ is independent of t_n . Thus u can be extended beyond T . This contradicts the definition of T .

Note that $u \in C([0, T]; H^s(\mathbb{R}))$ and $f(u)$ satisfies locally Lipschitz conditions in u , uniformly in t on $[0, T)$. Then we have that $f(u(t, x))$ is continuous in t . Thus it follows from (5.7)-(5.9) that

$$u(t, x) \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^s(\mathbb{R}))$$

is the solution to (5.1). This completes the proof of the theorem. \square

Next, we present the precise blow-up scenario for solutions to Eq.(5.1).

We first recall the following two useful lemmas.

Lemma 5.1. [31] *If $r > 0$, then $H^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is an algebra. Moreover*

$$\|fg\|_r \leq c(\|f\|_{L^\infty(\mathbb{R})}\|g\|_r + \|f\|_r\|g\|_{L^\infty(\mathbb{R})}),$$

where c is a constant depending only on r .

Lemma 5.2. [31] *For $\Lambda = (1 - \partial_x^2)^{1/2}$. If $r > 0$, then*

$$\|[\Lambda^r, f]g\|_{L^2(\mathbb{R})} \leq c(\|\partial_x f\|_{L^\infty(\mathbb{R})}\|\Lambda^{r-1}g\|_{L^2(\mathbb{R})} + \|\Lambda^r f\|_{L^2(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}),$$

where c is a constant depending only on r .

Then we prove the following useful result.

Theorem 5.2. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ be given and assume that T is the existence time of the corresponding solution to Eq.(5.1) with the initial data u_0 . If there exists $M > 0$ such that*

$$\|u_x(t, x)\|_{L^\infty(\mathbb{R})} \leq M, \quad t \in [0, T],$$

then the $H^s(\mathbb{R})$ -norm of $u(t, \cdot)$ does not blow up on $[0, T]$.

Proof. Let u be the solution to Eq.(5.1) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and let T be the maximal existence time of the solution u , which is guaranteed by Theorem 5.1. Throughout this proof, $c > 0$ stands for a generic constant depending only on s .

Applying the operator Λ^s to Eq.(5.2), multiplying by $\Lambda^s u$, and integrating over \mathbb{R} , we obtain

$$\frac{d}{dt}\|u\|_s^2 = 2(u, f_1(u))_s + 2(u, f_2(u))_s, \quad (5.12)$$

where

$$f_1(u) = \partial_x(1 - \partial_x^2)^{-1}\left(-\frac{1}{2}u^2\right) = -(1 - \partial_x^2)^{-1}(uu_x)$$

and $f_2(u) = \partial_x(1 - \partial_x^2)^{-1}\left(\frac{1}{2}u_x^2\right)$.

Let us estimate the first term of the right-hand side of Eq.(5.12).

$$\begin{aligned} |(f_1(u), u)_s| &= |(\Lambda^s(1 - \partial_x^2)^{-1}(u\partial_x u), \Lambda^s u)_0| \\ &\leq |(\Lambda^{s-1}(u\partial_x u), \Lambda^{s-1}u)_0| \\ &\leq |([\Lambda^{s-1}, u]\partial_x u, \Lambda^{s-1}u)_0 + (u\Lambda^{s-1}\partial_x u, \Lambda^{s-1}u)_0| \\ &\leq \|[\Lambda^{s-1}, u]\partial_x u\|_0 \|\Lambda^{s-1}u\|_0 + \frac{1}{2} |(u_x \Lambda^{s-1}u, \Lambda^{s-1}u)_0| \\ &\leq (c\|u_x\|_{L^\infty(\mathbb{R})} + \frac{1}{2}\|u_x\|_{L^\infty(\mathbb{R})})\|u\|_{s-1}^2 \\ &\leq c\|u_x\|_{L^\infty(\mathbb{R})}\|u\|_s^2. \end{aligned} \quad (5.13)$$

Here, we applied Lemma 5.2 with $r = s - 1$. Then, let us estimate the second term of the right-hand side of (5.12).

$$\begin{aligned} |(f_2(u), u)_s| &\leq \|f_2(u)\|_s \|u\|_s \leq \frac{1}{2}\|u_x^2\|_{s-1}\|u\|_s \\ &\leq c(\|u_x\|_{L^\infty(\mathbb{R})}\|u\|_{s-1})\|u\|_s \\ &\leq c\|u_x\|_{L^\infty(\mathbb{R})}\|u\|_s^2, \end{aligned} \quad (5.14)$$

where we used Lemma 5.1 with $r = s$. Combining inequalities (5.13)-(5.14) with (5.12), we obtain

$$\frac{d}{dt}\|u\|_s^2 \leq cM\|u\|_s^2.$$

An application of Gronwall's inequality yields

$$\|u(t)\|_s^2 \leq \exp(cMt)\|u(0)\|_s^2. \quad (5.15)$$

This completes the proof of the theorem. \square

We now present the precise blow-up scenario for Eq.(5.1).

Theorem 5.3. *Assume that $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then the solution to Eq.(5.1) blows up in finite time if and only if the slope of the solution becomes unbounded from below in finite time.*

Proof. Applying Theorem 5.1 and a simple density argument, it suffices to consider the case $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(5.1) with initial data $u_0 \in H^3(\mathbb{R})$. From Theorem 5.1 we know that $u \in C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^3(\mathbb{R}))$.

Multiplying Eq.(5.1) by u and integrating by parts, we get

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) dx = 2 \int_{\mathbb{R}} uu_x u_{xx} dx - 2 \int_{\mathbb{R}} u^2 u_x dx = - \int_{\mathbb{R}} u_x u_x^2 dx. \quad (5.16)$$

Differentiating Eq.(5.1) with respect to x , then multiplying the obtained equation by u_x and integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (u_x^2 + u_{xx}^2) dx &= -2 \int_{\mathbb{R}} u_x u_{xx}^2 dx + 2 \int_{\mathbb{R}} uu_x u_{xx} dx \\ &= -2 \int_{\mathbb{R}} u_x u_x^2 dx - \int_{\mathbb{R}} u_x^3 dx. \end{aligned} \quad (5.17)$$

Summing up (5.16) and (5.17), we have

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) dx = - \int_{\mathbb{R}} u_x (u_x^2 + u_{xx}^2) dx. \quad (5.18)$$

If the slope of the solution is bounded from below on $[0, T) \times \mathbb{R}$, then there exists $M > 0$ such that

$$\frac{d}{dt} \|u\|_2 \leq M \|u\|_2.$$

By means of Gronwall's inequality, we have

$$\|u(t, \cdot)\|_2 \leq \|u(0, \cdot)\|_2 \exp\{Mt\}, \quad \forall t \in [0, T).$$

By Theorem 5.2, we see that the solution does not blow up in finite time.

On the other hand, by Theorem 5.1 and Sobolev's imbedding theorem, we see that if the slope of the solution becomes unbounded from below in finite time, then the solution will blow up in finite time. This completes the proof of the theorem. \square

Remark 5.1. Theorem 5.3 shows that (5.1) has the same blow-up scenario as the Camassa-Holm equation [6, 45] and the Degasperis-Procesi equation [46] do.

Finally, we show that there exist global strong solutions to Eq.(5.1) provided the initial data u_0 satisfies certain sign conditions.

Lemma 5.3. *Assume that $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Let $T > 0$ be the existence time of the corresponding solution u to (5.1). Then we have*

$$m(t, x) = m_0(x) \exp^{-\int_0^t u_x(\tau, x) d\tau}, \quad (5.19)$$

where $(t, x) \in [0, T) \times \mathbb{R}$ and $m = u - u_{xx}$. Moreover, for every $(t, x) \in [0, T) \times \mathbb{R}$, $m(t, x)$ has the same sign as $m_0(x)$ does.

Proof. Let $T > 0$ be the maximal existence time of the solution u with initial data $u_0 \in H^s(\mathbb{R})$.

Due to $u(t, x) \in C^1([0, T]; H^s(\mathbb{R}))$ and $H^s(\mathbb{R}) \subset C(\mathbb{R})$, we see that the function $u_x(t, x)$ are bounded, Lipschitz in the space variable x , and of class C^1 in time. For arbitrarily fixed $T' \in (0, T)$, Sobolev's imbedding theorem implies that

$$\sup_{(s, x) \in [0, T'] \times \mathbb{R}} |u_x(s, x)| < \infty.$$

Thus, we infer from the above inequality that there exists a constant $K > 0$ such that

$$e^{-\int_0^t u_x(\tau, x) d\tau} \geq e^{-tK} > 0 \quad \text{for } (t, x) \in [0, T'] \times \mathbb{R}. \quad (5.20)$$

By Eq.(5.1) and $m = u - u_{xx}$, we have

$$m_t(t, x) = -u_x(t, x)m(t, x). \quad (5.21)$$

This implies that

$$m(t, x) = m_0(x) \exp^{-\int_0^t u_x(\tau, x) d\tau}.$$

By (5.20), we see that for every $(t, x) \in [0, T] \times \mathbb{R}$, $m(t, x)$ has the same sign as $m_0(x)$ does. This completes the proof of the lemma. \square

Lemma 5.4. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ be given. If $m_0 := (u_0 - u_{0,xx}) \in L^1(\mathbb{R})$, then, as long as the solution $u(t, \cdot)$ to Eq.(5.1) with initial data u_0 given by Theorem 5.1 exists, we have*

$$\int_{\mathbb{R}} u(t, x) dx = \int_{\mathbb{R}} u_0 dx = \int_{\mathbb{R}} m_0 dx = \int_{\mathbb{R}} m(t, x) dx.$$

Proof. Again it suffices to consider the case $s = 3$. Let T be the maximal time of existence of the solution u to Eq.(5.1) with initial data $u_0 \in H^3(\mathbb{R})$.

Note that $u_0 = Q * m_0$ and $m_0 = (u_0 - u_{0,xx}) \in L^1(\mathbb{R})$. By Young's inequality, we get

$$\|u_0\|_{L^1(\mathbb{R})} = \|Q * m_0\|_{L^1(\mathbb{R})} \leq \|Q\|_{L^1(\mathbb{R})} \|m_0\|_{L^1(\mathbb{R})} \leq \|m_0\|_{L^1(\mathbb{R})}.$$

Integrating Eq.(3.2) by parts, we get

$$\frac{d}{dt} \int_{\mathbb{R}} u dx = \int_{\mathbb{R}} \partial_x Q * \left(\frac{1}{2} u_x^2 - \frac{1}{2} u^2 \right) dx = 0.$$

It then follows that

$$\int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_0 dx.$$

Due to $m = u - u_{xx}$, we have

$$\begin{aligned} \int_{\mathbb{R}} m dx &= \int_{\mathbb{R}} u dx - \int_{\mathbb{R}} u_{xx} dx = \int_{\mathbb{R}} u dx \\ &= \int_{\mathbb{R}} u_0 dx = \int_{\mathbb{R}} u_0 dx - \int_{\mathbb{R}} u_{0,xx} dx = \int_{\mathbb{R}} m_0 dx. \end{aligned}$$

This completes the proof of the lemma. \square

We now present the first global existence result.

Theorem 5.4. *Let $u_0 \in H^s(\mathbb{R})$ $s > \frac{3}{2}$ be given. If $m_0 := u_0 - \partial_x^2 u_0 \in L^1(\mathbb{R})$ is nonnegative, then the corresponding solution to Eq.(5.2) is defined globally in time. Moreover, $I(u) = \int_{\mathbb{R}} u dx$ is a conservation law, and that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, we have*

(i) $m(t, x) \geq 0$, $u(t, x) \geq 0$ and

$$\|m_0\|_{L^1(\mathbb{R})} = \|m(t)\|_{L^1(\mathbb{R})} = \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}.$$

(ii) $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}$ and

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u(t, \cdot)\|_1 \leq \frac{\sqrt{2}}{2} e^{\frac{t}{2} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1.$$

Proof. As we mentioned before that we only need to prove the above theorem for $s = 3$. Let $T > 0$ be the maximal existence time of the solution u with initial data $u_0 \in H^3(\mathbb{R})$.

If $m_0(x) \geq 0$, then Lemma 5.3 ensures that $m(t, x) \geq 0$ for all $t \in [0, T)$. Noticing that $u = Q * m$ and the positivity of Q , we infer that $u(t, x) \geq 0$ for all $t \in [0, T)$. By Lemma 5.4, we obtain

$$\begin{aligned} -u_x(t, x) + \int_{-\infty}^x u(t, x) dx &= \int_{-\infty}^x (u - u_{xx}) dx \\ &= \int_{-\infty}^x m dx \leq \int_{-\infty}^{\infty} m dx = \int_{\mathbb{R}} m_0 dx = \int_{\mathbb{R}} u_0 dx. \end{aligned} \quad (5.22)$$

Therefore, from (5.22) we find that

$$u_x(t, x) \geq - \int_{\mathbb{R}} u_0 dx = -\|u_0\|_{L^1(\mathbb{R})}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \quad (5.23)$$

On the other hand, by $m(t, x) \geq 0$ for all $t \in [0, T)$, we obtain

$$u_x(t, x) - \int_{-\infty}^x u dx = - \int_{-\infty}^x (u - u_{xx}) dx = - \int_{-\infty}^x m dx \leq 0.$$

By the above inequality and $u(t, x) \geq 0$ for all $t \in [0, T)$, we get

$$u_x(t, x) \leq \int_{-\infty}^x u dx \leq \int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_0 dx = \|u_0\|_{L^1(\mathbb{R})}. \quad (5.24)$$

Thus, (5.23) and (5.24) imply that

$$|u_x(t, x)| \leq \|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \quad (5.25)$$

By Theorem 5.2 and the above inequality, we deduce that $T = \infty$. Recalling finally Lemma 5.4, we get assertion (i).

Multiplying (5.1) by u and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2(t, x) + u_x^2(t, x)) dx &= \int_{\mathbb{R}} (uu_x u_{xx} - u^2 u_x) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}} u_x^3 dx \leq \frac{1}{2} \|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} u_x^2 dx. \end{aligned} \quad (5.26)$$

An application of Gronwall's inequality leads to

$$\int_{\mathbb{R}} (u^2(t, x) + u_x^2(t, x)) dx \leq e^{t\|u_0\|_{L^1(\mathbb{R})}} \int_{\mathbb{R}} (u_0^2 + u_{0,x}^2) dx. \quad (5.27)$$

Consequently,

$$\|u(t, \cdot)\|_1 \leq e^{\frac{t}{2}\|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1. \quad (5.28)$$

On the other hand,

$$u^2(t, x) = \int_{-\infty}^x uu_x dx - \int_x^{\infty} uu_x dx \leq \frac{1}{2} \int_{\mathbb{R}} (u^2 + u_x^2) dx = \frac{1}{2} \|u(t, \cdot)\|_1^2. \quad (5.29)$$

Combining (5.28) with (5.29), we obtain assertion (ii). This completes the proof of the theorem. \square

In a similar way to the proof of Theorem 5.4, we can get the following global existence result.

Theorem 5.5. *Let $u_0 \in H^s(\mathbb{R})$ $s > \frac{3}{2}$ be given. If $m_0 := u_0 - \partial_x^2 u_0 \in L^1(\mathbb{R})$ is non-positive, then the corresponding solution to Eq.(5.1) is defined globally in time. Moreover, $I(u) = \int_{\mathbb{R}} u dx$ is invariant in time, and that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, we have*

(i) $m(t, x) \leq 0$, $u(t, x) \leq 0$ and

$$\|m_0\|_{L^1(\mathbb{R})} = \|m(t)\|_{L^1(\mathbb{R})} = \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}.$$

(ii) $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}$ and

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{\sqrt{2}}{2} \|u(t, \cdot)\|_1 \leq \frac{\sqrt{2}}{2} e^{\frac{t}{2} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1.$$

6. GLOBAL WEAK SOLUTIONS

In this section, we present some results on global weak solutions to characterize peakon solutions to (1.1) for any $\theta \in \mathbb{R}$ provided initial data satisfy certain sign conditions.

Let us first introduce some notations to be used in the sequel. We let $M(\mathbb{R})$ denote the space of Radon measures on \mathbb{R} with bounded total variation. The cone of positive measures is denoted by $M^+(\mathbb{R})$. Let $BV(\mathbb{R})$ stand for the space of functions with bounded variation and write $\mathbb{V}(f)$ for the total variation of $f \in BV(\mathbb{R})$. Finally, let $\{\rho_n\}_{n \geq 1}$ denote the mollifiers

$$\rho_n(x) := \left(\int_{\mathbb{R}} \rho(\xi) d\xi \right)^{-1} n \rho(nx), \quad x \in \mathbb{R}, \quad n \geq 1,$$

where $\rho \in C_c^\infty(\mathbb{R})$ is defined by

$$\rho(x) := \begin{cases} e^{\frac{1}{x^2-1}}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| \geq 1. \end{cases}$$

Note that the b -equation for any $b \in \mathbb{R}$ has peakon solutions with corners at their peaks, cf. [17, 28, 29]. Thus, the θ -equation for any $\theta \in \mathbb{R}$ has also peakon solutions, see Example 6.1 below. Obviously, such solutions are not strong solutions to (1.1) for any $\theta \in \mathbb{R}$. In order to provide a mathematical framework for the study of peakon solutions, we shall first give the notion of weak solutions to (1.1).

Equation (1.1) can be written as

$$u_t + \theta u u_x + \partial_x (1 - \partial_x^2)^{-1} B(u, u_x) = 0, \quad B = (1 - \theta) \frac{u^2}{2} + (4\theta - 1) \frac{u_x^2}{2}.$$

If we set

$$F(u) := \frac{\theta u^2}{2} + Q * \left[(1 - \theta) \frac{u^2}{2} + (4\theta - 1) \frac{u_x^2}{2} \right],$$

then the above equation takes the conservative form

$$u_t + F(u)_x = 0, \quad u(0, x) = u_0, \quad t > 0, \quad x \in \mathbb{R}. \quad (6.1)$$

In order to introduce the notion of weak solutions to (6.1), let $\psi \in C_0^\infty([0, T] \times \mathbb{R})$ denote the set of all the restrictions to $[0, T] \times \mathbb{R}$ of smooth functions on \mathbb{R}^2 with compact support contained in $(-T, T) \times \mathbb{R}$.

Definition 6.1. Let $u_0 \in H^1(\mathbb{R})$. If u belongs to $L_{loc}^\infty([0, T]; H^1(\mathbb{R}))$ and satisfies the following identity

$$\int_0^T \int_{\mathbb{R}} (u \psi_t + F(u) \psi_x) dx dt + \int_{\mathbb{R}} u_0(x) \psi(0, x) dx = 0$$

for all $\psi \in C_0^\infty([0, T] \times \mathbb{R})$, then u is called a weak solution to (6.1). If u is a weak solution on $[0, T]$ for every $T > 0$, then it is called a global weak solution to (6.1).

The following proposition is standard.

Proposition 6.1. (i) Every strong solution is a weak solution.
(ii) If u is a weak solution and $u \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$, $s > \frac{3}{2}$, then it is a strong solution.

Referring to an approximation procedure used first for the solutions to the Camassa-Holm equation [11], a partial integration result in Bochner spaces [39] and Helly's theorem [40] together with the obtained global existence results and two useful a priori estimates for strong solutions, e.g., Theorem 1.1 and Theorems 5.4-5.5, we may obtain the following uniqueness and existence results for the global weak solution to (6.1) for any $\theta \in \mathbb{R}$ provided the initial data satisfy certain sign conditions.

Theorem 6.1. Let $u_0 \in H^1(\mathbb{R})$ be given. Assume that

$$(u_0 - u_{0,xx}) \in M^+(\mathbb{R}).$$

Then (6.1) for any $\theta \in \mathbb{R}$ has a unique weak solution

$$u \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{loc}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

with initial data $u(0) = u_0$ and

$$(u(t, \cdot) - u_{xx}(t, \cdot)) \in M^+(\mathbb{R})$$

is uniformly bounded for all $t \in \mathbb{R}_+$.

In the following, we only present main steps of the proof of the theorem, a refined analysis could be done following those given in [24] for the b -equation.

A sketch of existence proof of weak solutions

Step 1. Given $u_0 \in H^1(\mathbb{R})$ and $m_0 := u_0 - u_{0,xx} \in M^+(\mathbb{R})$. Then one can show that

$$\|u_0\|_{L^1(\mathbb{R})} \leq \|m_0\|_{M^+(\mathbb{R})}.$$

Let us define $u_0^n := \rho_n * u_0 \in H^\infty(\mathbb{R})$ for $n \geq 1$. Note that for all $n \geq 1$,

$$m_0^n := u_0^n - u_{0,xx}^n = \rho_n * (m_0) \geq 0.$$

By Theorem 1.1 and Theorem 5.4, we obtain that there exists a unique strong solution to (6.1),

$$u^n = u^n(\cdot, u_0^n) \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R})), \quad \forall s \geq 3.$$

Step 2. By a priori estimates in Theorem 1.1 and Theorem 5.4, Young's inequality and energy estimate for (6.1), we may get

$$\int_0^T \int_{\mathbb{R}} ([u^n(t, x)]^2 + [u_x^n(t, x)]^2 + [u_t^n(t, x)]^2) dx dt \leq M, \quad (6.2)$$

where M is a positive constant depending only on θ , T , $\|Q_x\|_{L^2(\mathbb{R})}$, and $\|u_0\|_1$. It then follows from (6.2) that the sequence $\{u^n\}_{n \geq 1}$ is uniformly bounded in the space $H^1((0, T) \times \mathbb{R})$. Thus, we can extract a subsequence such that

$$u^{n_k} \rightharpoonup u \quad \text{weakly in } H^1((0, T) \times \mathbb{R}) \text{ for } n_k \rightarrow \infty \quad (6.3)$$

and

$$u^{n_k} \rightarrow u \quad \text{a.e. on } (0, T) \times \mathbb{R} \text{ for } n_k \rightarrow \infty, \quad (6.4)$$

for some $u \in H^1((0, T) \times \mathbb{R})$. From Theorem 1.1 (i) and the fact $\|u_0^n\|_1 \leq \|u_0\|_1$ we see that for any fixed $t \in (0, T)$, the sequence $u_x^{n_k}(t, \cdot) \in BV(\mathbb{R})$ satisfies

$$\mathbb{V}[u_x^{n_k}(t, \cdot)] \leq 2\|m_0\|_{M(\mathbb{R})}.$$

Step 3. This, when applying Helly's theorem, cf. [40], enables us to conclude that there exists a subsequence, denoted still by $\{u_x^{n_k}(t, \cdot)\}$, which converges to the function $u_x(t, \cdot)$ for a.e. $t \in (0, T)$. A key energy estimate is of the form

$$\|B(u^n, u_x^n)\|_{L^2(\mathbb{R})} \leq C(\|u_0\|_1),$$

which ensures B admits a weak limit. This when combined with the fact that (u^n, u_x^n) converges to (u, u_x) as well as $Q_x \in L^2$ leads to the assertion that u satisfies (6.1) in distributional sense.

Step 4. From equation (6.1) we see that $u_t^{n_k}(t, \cdot)$ is uniformly bounded in $L^2(\mathbb{R})$, and $\|u^{n_k}(t, \cdot)\|_1$ is uniformly bounded for all $t \in (0, T)$. This implies that the map $t \mapsto u_t^{n_k}(t, \cdot) \in H^1(\mathbb{R})$ is weakly equi-continuous on $[0, T]$. Recalling the Arzela-Ascoli theorem and a priori estimates in Theorem 1.1 and Theorem 5.4, we may prove

$$u \in L_{loc}^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap L_{loc}^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \quad \text{and} \quad u_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R}).$$

Step 5. Since u solves (6.1) in distributional sense, we have

$$\rho_n * u_t + \rho_n * (\theta u u_x) + \rho_n * \partial_x Q * \left((1 - \theta) \frac{u^2}{2} + (4\theta - 1) \frac{u_x^2}{2} \right) = 0,$$

for a.e. $t \in \mathbb{R}_+$. Integrating the above equation with respect to x on \mathbb{R} and then integrating by parts, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_n * u \, dx = 0.$$

By a partial integration result in Bochner spaces [39] and Young's inequality, we may prove that

$$\int_{\mathbb{R}} u(t, \cdot) \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \rho_n * u(t, \cdot) \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \rho_n * u_0 \, dx = \int_{\mathbb{R}} u_0 \, dx.$$

Using the above conservation law, we get

$$\begin{aligned} \|u(t, \cdot) - u_{xx}(t, \cdot)\|_{M(\mathbb{R})} &\leq \|u(t, \cdot)\|_{L^1(\mathbb{R})} + \|u_{xx}(t, \cdot)\|_{M(\mathbb{R})} \\ &\leq \|u_0\|_{L^1(\mathbb{R})} + 2 \|m_0\|_{M(\mathbb{R})} \leq 3 \|m_0\|_{M(\mathbb{R})}, \end{aligned}$$

for a.e. $t \in \mathbb{R}_+$. Note that $u^{n_k}(t, x) - u_{xx}^{n_k}(t, x) \geq 0$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then the above inequality implies that $(u(t, \cdot) - u_{xx}(t, \cdot)) \in M^+(\mathbb{R})$ for a.e. $t \in \mathbb{R}_+$.

Since $u(t, x) = Q * (u(t, x) - u_{xx}(t, x))$, it follows that

$$\begin{aligned} |u(t, x)| &= |Q * (u(t, x) - u_{xx}(t, x))| \\ &\leq \|Q\|_{L^\infty(\mathbb{R})} \|u(t, \cdot) - u_{xx}(t, \cdot)\|_{M(\mathbb{R})} \leq \frac{3}{2} \|m_0\|_{M(\mathbb{R})}. \end{aligned}$$

This shows that $u(t, x) \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R})$ in view of Step 4. This proves the existence of global weak solutions to (6.1).

Uniqueness of the weak solution

Let

$$u, v \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{loc}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

be two global weak solutions of (6.1) with initial data u_0 . Set

$$N := \sup_{t \in \mathbb{R}_+} \{ \|u(t, \cdot) - u_{xx}(t, \cdot)\|_{M(\mathbb{R})} + \|v(t, \cdot) - v_{xx}(t, \cdot)\|_{M(\mathbb{R})} \}.$$

From Step 5, we know that $N < \infty$. Let us set

$$w(t, \cdot) = u(t, \cdot) - v(t, \cdot), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

and fix $T > 0$. Convoluting Eq.(6.1) for u and v with ρ_n and with $\rho_{n,x}$ respectively, using Young's inequality and following the procedure described on page 56-59 in [11], we may deduce that

$$\frac{d}{dt} \int_{\mathbb{R}} |\rho_n * w| dx = C \int_{\mathbb{R}} |\rho_n * w| dx + C \int_{\mathbb{R}} |\rho_n * w_x| dx + R_n(t), \quad (6.5)$$

and

$$\frac{d}{dt} \int_{\mathbb{R}} |\rho_n * w_x| dx = C \int_{\mathbb{R}} |\rho_n * w| dx + C \int_{\mathbb{R}} |\rho_n * w_x| dx + R_n(t), \quad (6.6)$$

for a.e. $t \in [0, T]$ and all $n \geq 1$, where C is a generic constant depending on θ and N , and that $R_n(t)$ satisfies

$$\begin{cases} \lim_{n \rightarrow \infty} R_n(t) = 0 \\ |R_n(t)| \leq K(T), \quad n \geq 1, \quad t \in [0, T]. \end{cases}$$

Here $K(T)$ is a positive constant depending on θ , T , N and the $H^1(\mathbb{R})$ -norms of $u(0)$ and $v(0)$.

Summing (6.5) and (6.6) and then using Gronwall's inequality, we infer that

$$\begin{aligned} \int_{\mathbb{R}} (|\rho_n * w| + |\rho_n * w_x|)(t, x) dx &\leq \int_0^t e^{2C(t-s)} R_n(s) ds + \\ &e^{2Ct} \int_{\mathbb{R}} (|\rho_n * w| + |\rho_n * w_x|)(0, x) dx, \end{aligned}$$

for all $t \in [0, T]$ and $n \geq 1$. Note that $w = u - v \in W^{1,1}(\mathbb{R})$. Using Lebesgue's dominated convergence theorem, we may deduce that for all $t \in [0, T]$

$$\int_{\mathbb{R}} (|w| + |w_x|)(t, x) dx \leq e^{2Ct} \int_{\mathbb{R}} (|w| + |w_x|)(0, x) dx.$$

Since $w(0) = w_x(0) = 0$, it follows from the above inequality that $u(t, x) = v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$. This proves the uniqueness of the global weak solution to (6.1).

In a similar way to the proof of Theorem 6.1, we can get the following result.

Theorem 6.2. *Let $u_0 \in H^1(\mathbb{R})$ be given. Assume that*

$$(u_{0,xx} - u_0) \in M^+(\mathbb{R}).$$

Then (6.1) for any $\theta \in \mathbb{R}$ has a unique weak solution

$$u \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{loc}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

with initial data $u(0) = u_0$ and

$$(u_{xx}(t, \cdot) - u(t, \cdot)) \in M^+(\mathbb{R})$$

is uniformly bounded for all $t \in \mathbb{R}_+$.

Remark 6.1. Theorems 6.1-6.2 cover the recent results for global weak solutions of the Camassa-Holm equation in [11] and the Degasperis-Procesi equation in [47].

Example 6.1. (Peakon solutions) Consider (1.1) for any $\theta \in \mathbb{R}$. Given the initial datum $u_0(x) = ce^{-|x|}$, $c \in \mathbb{R}$. A straightforward computation shows that

$$u_0 - u_{0,xx} = 2c \delta(x) \in M_+(\mathbb{R}) \quad \text{if } c \geq 0$$

and

$$u_{0,xx} - u_0 = -2c \delta(x) \in M_+(\mathbb{R}) \quad \text{if } c < 0.$$

One can also check that

$$u(t, x) = ce^{-|x-ct|}$$

satisfies (1.1) for any $\theta \in \mathbb{R}$ in distributional sense. Theorems 6.1-6.2 show that $u(t, x)$ is the unique global weak solution to (1.1) for any $\theta \in \mathbb{R}$ with the initial data $u_0(x)$. This weak solution is a peaked solitary wave which is analogous to that of the b -equation, cf. [24].

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