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THE LIP^+-STABILITY AND ERROR ESTIMATES FOR A RELAXATION SCHEME*

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Abstract. We show the discrete lip^+-stability for a relaxation scheme proposed by Jin and Xin [*Comm. Pure Appl. Math.,* 48 (1995), pp. 235–277] to approximate convex conservation laws. Equipped with the lip^+-stability we obtain global error estimates in the spaces $W^{s,p}$ for $-1 \leq s \leq 1/p$, $1 \leq p \leq \infty$ and pointwise error estimates for the approximate solution obtained by the relaxation scheme. The proof uses the framework introduced by Nessyahu and Tadmor [*SIAM J. Numer. Anal.,* 29 (1992), pp. 1505–1519]. We also show a maximum principle for the relaxation scheme when the initial data are in an equilibrium state.

Key words. relaxation scheme, convex conservation laws, lip^+-stability, maximum principle

AMS subject classifications. 35L65, 65M10, 65M15

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1. Introduction.

Relaxation schemes are a class of nonoscillatory numerical schemes for systems of conservation laws proposed by Jin and Xin [3]. They are motivated by relaxation models for flows which are not in thermodynamic equilibrium, i.e., they constitute more general and more accurate models of certain physical phenomena. The relaxation schemes provide a new way of perturbing, even regularizing, systems of conservation laws and approximating their solutions. In this sense they are to be seen as an interesting tool of analysis. The computational results that are available, see, e.g., [3], as well as Aregba-Driollet and Natalini [2], indicate that the relaxed schemes obtained in the limit $\epsilon \to 0$ provide a quite promising class of new schemes. We point out that the main assets of these schemes are that they neither require the computation of the Jacobians of fluxes for the conservation laws nor the use of Riemann-solvers. This is needed for flows in which a real gas law has to be used in place of the frequently used assumption of an ideal gas, e.g., the two phase flow for cryogenic gases. In such cases it may be too expensive or even impossible to calculate Riemann solutions or even flux Jacobians. This important property is shared by other schemes, such as the high resolution central schemes introduced by Nessyahu and Tadmor [15]; see also Kurganov and Tadmor [5] for references on recent developments.

To make things more precise we want to consider a scalar conservation law. We take a convex flux function $f \in C^3(\mathbb{R})$ and initial data $u_0 \in L^\infty(\mathbb{R})$ and consider the Cauchy problem

$$u_t + f(u)_x = 0 \tag{1.1}$$

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with initial data

$$u(x,0) = u_0(x). \tag{1.2}$$

For this problem we want to approximate the global weak entropy solution by a relaxation scheme.

We choose a time step Δt, a spatial mesh size Δx, a parameter a which will be related to the characteristic speed of the conservation law, and a small relaxation parameter $\epsilon > 0$. From these we define the mesh ratio $\lambda = \frac{\Delta t}{\Delta x}$, the CFL parameter $\mu = \sqrt{a}\lambda \in]0,1[$ and the scale parameter $k = \frac{\Delta t}{\epsilon}$. The mesh is given by the points $(j\Delta x, n\Delta t)$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The approximate solution takes the discrete values u_j^n at the mesh points. Further, relaxation schemes involve the discrete relaxation fluxes v_j^n. We want to consider the following *semi-implicit relaxation scheme:*

$$\begin{aligned} &u_j^{n+1} - u_j^n + \frac{\lambda}{2}(v_{j+1}^n - v_{j-1}^n) - \frac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0, \qquad j \in \mathbb{Z}, \quad n \in \mathbb{N}, \\ &v_j^{n+1} - v_j^n + \frac{a\lambda}{2}(u_{j+1}^n - u_{j-1}^n) - \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) = -k[v_j^{n+1} - f(u_j^{n+1})]. \end{aligned} \tag{1.3}$$

The discrete initial data are given by averaging the initial data u_0 over mesh cells $I_j = \big](j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x\big[$, i.e., taking

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx \quad \text{and setting} \;\; v_j^0 = f(u_j^0). \tag{1.4}$$

The difference $v - f(u)$ measures a deviation from an equilibrium in relaxation models. If we have $v = f(u)$, we say that our variables are *in an equilibrium state.* We are interested in the relaxation limit where k is large, i.e., ϵ is small and the conservation law is being approximated. In this case the source term becomes stiff. Moreover, we specifically will have to require in our analysis that for some positive constant c the *scale condition*

$$0 < c \le k \tag{1.5}$$

holds. This means that $\Delta t/\epsilon$ is bounded from below away from zero. When making Δx small, i.e., considering convergence of the scheme, we assume that λ is fixed and therefore Δt is automatically made smaller. By the scale condition we have to make ϵ smaller appropriately.

This scheme has been studied in various preceding papers. Note that in the limit $\epsilon \to 0$ it reduces to the generalized Lax–Friedrichs scheme, i.e., with the numerical viscosity $Q = 1$ replaced by CFL parameter $Q = \mu$. For the original Lax–Friedrichs scheme the lip^+ stability and error estimates can be found in Nessyahu and Tadmor [14], as well as in [18]. Our present results when taking the limit case $\epsilon = 0$ provide the lip^+ stability and error estimates for the generalized Lax–Friedrichs scheme. The convergence theory for the relaxation scheme (1.3) can be found in Aregba-Driollet and Natalini [1], Wang and Warnecke [23], Yong [24], and Tang and Wu [21]. Based on proper total variation bounds, independent of ϵ and Δx, for the approximate solutions the convergence of a subsequence of $(u_j^n, v_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ to the unique entropy solution of the initial value problem (1.1) was established by standard compactness arguments. The L^1-convergence rate for the relaxation scheme (1.3) was obtained by Liu and Warnecke [10]. The effect of initial layers was also studied there.

As already mentioned above, the presence of relaxation mechanisms is widespread in the context of both continuum mechanics as well as kinetic theories; see, e.g., [22] for physical examples. These mechanisms motivate the class of nonoscillatory numerical schemes for conservation laws introduced by Jin and Xin [3] to which the scheme (1.3) belongs. The development of relaxation approximations to hyperbolic conservation laws has caught considerable attention in recent years; see Natalini [12] as well as Katsoulakis and Tzavaras [6]. The corresponding relaxation schemes were also introduced based on established relaxation approximations; see Aregba-Driollet and Natalini [1], [2], as well as Katsoulakis, Kossioris, and Makridakis [7]. Concerning the asymptotic convergence of relaxation systems to the corresponding equilibrium conservation laws as the rate of relaxation, i.e., the relaxation parameter, tends to zero there are already many papers (consult Natalini [13]) for an overview for recent developments for hyperbolic relaxation problems.

Based on extensions of Kruzhkov and Kuznetzov-type error estimates, convergence rates for various relaxation approximations have been established; see, e.g., [20], [6], [10], and [7]. The Lip' theory was introduced by Tadmor [16] and explored with various coauthors in [14], [4], and [17]. Using it the convergence rates for relaxation systems approximating convex conservation laws were investigated. The heart of the matter is to establish the Lip^+ stability, which combined with the Lip' consistency to give sharp estimates. To establish the Lip^+ stability for a hyperbolic relaxation model, Tadmor and Tang [17] introduced a transformation so that they could use the maximum principle for the reduced equations. For piecewise smooth solutions with finitely many discontinuities Teng [19] proved a first order convergence rate.

The main goals of this paper are to show three new results, namely the discrete maximum principle in Theorem 2.1, the discrete lip^+-stability in Theorem 3.1, and the error estimates in Theorem 4.2. Most of the effort goes into proving the discrete lip^+-stability of the relaxation scheme (1.3). In order to obtain the lip^+-stability, two ingredients play an important role. The first is the subcharacteristic condition $-\sqrt{a} < f'(u) < \sqrt{a}$; see, e.g., Liu [8] or Whitham [22], which is necessary for the convergence of relaxation approximations to conservation laws [13]. The second is the convexity of the flux function f since the entropy condition enforced by the discrete lip^+-stability holds only for conservation laws with convex flux functions. Under the subcharacteristic condition we establish the maximum principle for the relaxation scheme when the initial data are in an equilibrium state. We point out that if the initial data are not in an equilibrium state this kind of maximum principle does not hold. However, an L^∞-bound for approximate solutions can still be obtained in terms of an L^∞-bound of the initial data, provided one assumes a more strict subcharacteristic condition; see [11], [23], and [1]. In the previous papers, e.g., [1], [2], [10], [23], a (strict) maximum principle was proved for the continuous case only. Here we prove it for the discrete approximations of the method used in our paper. Equipped with the discrete lip^+-stability we obtain global error estimates in the spaces $\mathrm{W}^{s,p}$ for $-1 \le s \le 1/p$, $1 \le p \le \infty$ and a pointwise error estimate for the scheme (1.3), i.e., Theorem 4.2, by using the Lip' theory following Tadmor and his coauthors, e.g., [14], [16].

The main difficulty is to obtain the discrete lip^+-stability. First we rewrite the semi-implicit relaxation scheme (1.3) in a well-known manner as an explicit scheme in terms of Riemann invariants $\mathbf{R}_j^n$. The new tool devised in this paper is the use of a bounded discrete function $\mathbf{A}_j^n$ such that the estimate

$$(\mathbf{R}_j^n - \mathbf{R}_{j-1}^n)/\Delta x \le \mathbf{A}_j^n \|u_j^0\|_{lip^+}$$

holds for all $j \in \mathbb{Z}$ and all $n \in \mathbb{N}$. This bound implies the desired lip^+-stability for the discrete solution $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$. To this end we have to carefully choose the $\mathbf{A}_j^n$ in such a way that they can be used to deal with the relaxation terms and take care of the upwind scheme for the convection parts of the relaxation scheme at the same time. Our choice of the $\mathbf{A}_j^n$ was inspired by a transformation given by Tadmor and Tang [17] for continuous models, though a direct analogue of their approach was not feasible in the discrete case studied here. Such a technique was also used by Liu and Natalini [9] to study the long time diffusive behavior of the relaxation system leading to (1.3) for $\epsilon = 1$. An interesting open problem is the extension of this work to second order scheme. At the moment this does not seem straight forward and will have to be considered in future work.

In this paper we shall use the following seminorms introduced and used, e.g., in Nessyahu and Tadmor [14]:

$$\|w\|_{Lip^+(\mathbb{R})} := \operatorname{esssup}_{x,y\in\mathbb{R},x\neq y}\left(\frac{w(x)-w(y)}{x-y}\right)^+, \qquad \text{with } (\cdot)^+ := \max(\cdot,0),$$

which reduces to the usual $Lip(\mathbb{R})$ norm with $(\cdot)^+$ replaced by $|\cdot|$. We let $\|w\|_{Lip'(\mathbb{R})}$ denote the Lip-dual seminorm defined as

$$\sup_{\psi}\frac{(\phi-\hat{\phi}_0,\psi)}{\|\psi\|_{Lip(\mathbb{R})}}, \quad \text{where} \quad \hat{\phi}_0 = \int_{\mathrm{supp}\phi}\phi dx.$$

A discrete lip^+-seminorm is defined for discrete functions w as

$$\|w\|_{lip^+} := \max_{j\in\mathbb{Z}}\left(\frac{w_{j+1}-w_j}{\Delta x}\right)^+.$$

2. The discrete maximum principle. This section is devoted to establishing a maximum principle for the relaxation scheme. We take the *Riemann invariants*

$$\mathbf{R}_j^n := \begin{pmatrix} R_{1,j}^n \\ R_{2,j}^n \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\big(u_j^n - \frac{v_j^n}{\sqrt{a}}\big) \\ \frac{1}{2}\big(u_j^n + \frac{v_j^n}{\sqrt{a}}\big) \end{pmatrix} \tag{2.1}$$

and define as usual the *Maxwellians*

$$\mathbf{M}(u_j^n) := \begin{pmatrix} M_1(u_j^n) \\ M_2(u_j^n) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\big(u_j^n - \frac{f(u_j^n)}{\sqrt{a}}\big) \\ \frac{1}{2}\big(u_j^n + \frac{f(u_j^n)}{\sqrt{a}}\big) \end{pmatrix}. \tag{2.2}$$

Then the relaxation scheme (1.3) can be rewritten as

$$\mathbf{R}_j^{n+1} = \mathbf{R}_j^{n+\frac{1}{2}} + k\big(\mathbf{M}(u_j^{n+1}) - \mathbf{R}_j^{n+1}\big) \tag{2.3}$$

with

$$\mathbf{R}_j^{n+\frac{1}{2}} := \begin{pmatrix} R_{1,j}^{n+\frac{1}{2}} \\ R_{2,j}^{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} (1-\mu)R_{1,j}^n + \mu R_{1,j+1}^n \\ (1-\mu)R_{2,j}^n + \mu R_{2,j-1}^n \end{pmatrix}. \tag{2.4}$$

Throughout this paper we assume that μ is fixed and satisfies the CFL condition

$$0 < \mu = \sqrt{a}\lambda < 1. \tag{2.5}$$

We let $\mathbf{E} = (1,1)$, then $u_j^n = \sum_{i=1}^2 R_{i,j}^n = \mathbf{E} \cdot \mathbf{R}_j^n$. Multiplying the scheme (2.3) by $\mathbf{E}$ we have

$$u_j^{n+1} = \mathbf{E} \cdot \mathbf{R}_j^{n+\frac{1}{2}} =: u_j^{n+\frac{1}{2}} \tag{2.6}$$

due to $\mathbf{E} \cdot (\mathbf{M}(u_j^{n+1}) - \mathbf{R}_j^{n+1}) = 0$. Using (2.6) we may rewrite the semi-implicit scheme (2.3) as an *explicit scheme*

$$\mathbf{R}_j^{n+1} = \frac{1}{1+k}\mathbf{R}_j^{n+\frac{1}{2}} + \frac{k}{1+k}\mathbf{M}(\mathbf{E} \cdot \mathbf{R}_j^{n+\frac{1}{2}}) \tag{2.7}$$

with $\mathbf{R}_j^{n+\frac{1}{2}}$ defined as in (2.4).

THEOREM 2.1 (maximum principle). *Assume that the following bounds are given,*

$$b_1 \leq u_0(x) \leq b_2, \tag{2.8}$$

and that the subcharacteristic condition

$$-\sqrt{a} \leq f'(u) \leq \sqrt{a} \quad \text{for} \quad b_1 \leq u \leq b_2 \tag{2.9}$$

holds. Then any solution $(u_j^n, v_j^n)_{j\in\mathbb{Z}, n\in\mathbb{N}}$ *of the scheme* (1.3) *with initial data given by* (1.4) *satisfies the bounds*

$$b_1 \leq u_j^n \leq b_2 \quad \text{for} \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{2.10}$$

Proof. We prove the theorem by induction. It is obvious that the averaging (1.4) maintains the bounds $b_1 \leq u_j^0 \leq b_2$ for the discrete data.

The Maxwellians $M_i(u)$ are nondecreasing functions for $b_1 \leq u \leq b_2$ under the subcharacteristic condition (2.9). This is easily seen by differentiating (2.2). Then with $v_j^0 = f(u_j^0)$ we have for $i = 1, 2$, $j \in \mathbb{Z}$ that

$$R_{i,j}^0 = \frac{1}{2}\left(u_j^0 + (-1)^i \frac{v_j^0}{\sqrt{a}}\right) = M_i(u_j^0) \leq M_i(b_2).$$

Similarly we can show that

$$M_i(b_1) \leq R_{i,j}^0, \quad i = 1, 2.$$

For the induction we assume that the bounds

$$M_i(b_1) \leq R_{i,j}^n \leq M_i(b_2) \quad \text{for} \quad i = 1, 2, \quad j \in \mathbb{Z} \tag{2.11}$$

are given. We have just seen that this is true for $n = 0$. Now we prove these estimates for $n + 1$. By (2.7) and (2.4) we have

$$\begin{aligned} R_{1,j}^{n+1} &= \frac{1}{1+k}\left[(1-\mu)R_{1,j}^n + \mu R_{1,j+1}^n\right] \\ &+ \frac{k}{1+k}M_1\left[(1-\mu)R_{1,j}^n + \mu R_{1,j+1}^n + (1-\mu)R_{2,j}^n + \mu R_{2,j-1}^n\right]. \end{aligned} \tag{2.12}$$

Note that for any $u \in \mathbb{R}$ we have $M_1(u) + M_2(u) = u$. By this fact and the induction hypothesis (2.11), which we use for $j-1$, j and $j+1$, we obtain the estimates

$$b_1 \leq \mathbf{E} \cdot \mathbf{R}_j^{n+\frac{1}{2}} = (1-\mu)(R_{1,j}^n + R_{2,j}^n) + \mu R_{1,j+1}^n + \mu R_{2,j-1}^n \leq b_2.$$

For the second term in (2.12) we take the right-hand inequality just derived and use the fact that $\mathbf{M}(u)$ is a nondecreasing function for $b_1 \leq u \leq b_2$. We apply the right-hand estimate in (2.11) to the first term. Thereby we obtain the estimate

$$R_{1,j}^{n+1} \leq \frac{1}{1+k} M_1(b_2) + \frac{k}{1+k} M_1(b_2) = M_1(b_2) \quad \text{for} \quad j \in \mathbb{Z}.$$

Similarly we obtain the remaining bounds

$$M_1(b_1) \leq R_{1,j}^{n+1}, \qquad M_2(b_1) \leq R_{2,j}^{n+1} \leq M_2(b_2) \quad \text{for} \quad j \in \mathbb{Z},$$

i.e., (2.11) is true for any $n \in \mathbb{N}$. The addition of the inequalities (2.11) for $i = 1,\ 2$ yields the estimates for the u_j^n as asserted in (2.10). □

3. The *lip*$^+$-stability.

In this section we need the assumption that f is convex, i.e.,

$$f''(u) > 0 \quad \text{for} \quad u \in \mathbb{R}. \tag{3.1}$$

Further we assume that the initial data satisfy the uniform bound $|u_0(x)| \leq b < \infty$ for $x \in \mathbb{R}$. Therefore, by (1.4) the discrete initial data inherit this bound, i.e.,

$$|u_j^0| \leq b \quad \text{for} \quad j \in \mathbb{Z}. \tag{3.2}$$

We choose $a > 0$ satisfying the subcharacteristic condition

$$\sup_{|u|\leq b} |f'(u)| < \sqrt{a}. \tag{3.3}$$

It follows from Theorem 2.1 that the discrete solution $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ given by the scheme (1.3) satisfies the same L^∞-bound as initial data, i.e.,

$$|u_j^n| \leq b \quad \text{for} \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{3.4}$$

Since $f \in C^3$ and convex, there exist positive constants γ, α_1, α_2, K such that

$$\sup_{|u|\leq b} |f'(u)| = \gamma < \sqrt{a}, \tag{3.5}$$

$$\alpha_1 \leq f''(u) \leq \alpha_2 \quad \text{for} \quad -b \leq u \leq b, \tag{3.6}$$

$$\sup_{|u|\leq b} |f'''(u)| = K. \tag{3.7}$$

THEOREM 3.1 (lip^+-stability). *Assume that*

$$\|u_0\|_{Lip^+(\mathbb{R})} =: L < \infty, \tag{3.8}$$

the parameter $a > 0$ is suitably large, Δx is suitably small, and the scale parameter k satisfies the scale condition (1.5).

Then the approximate solution $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ given by the relaxation scheme (1.3) *with initial data* (1.4) *satisfies the lip$^+$-stability. More precisely, the following estimate holds:*

$$u_j^n - u_{j-1}^n \leq 2L\Delta x \qquad \text{for} \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{3.9}$$

Note that from the estimate (3.15) below we can actually obtain the estimate

$$u_i^n - u_{j-1}^n \le \left(1 - \frac{\alpha_2 L\Delta x}{2\sqrt{a}}\right)^{-1} L\Delta x,$$

which, as the mesh size Δx becomes finer, recovers the optimal estimate for the continuous case in [17].

Proof. We define the difference

$$\overline{\mathbf{R}}_j^n := \mathbf{R}_j^n - \mathbf{R}_{j-1}^n. \tag{3.10}$$

By the mean value theorem we find for any $j \in \mathbb{Z}$, $n \in \mathbb{N}$ a value ξ_j^n between $u_j^{n+\frac{1}{2}}$ and $u_{j-1}^{n+\frac{1}{2}}$ such that

$$\left(u_j^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}\right) f'(\xi_j^n) = f(u_j^{n+\frac{1}{2}}) - f(u_{j-1}^{n+\frac{1}{2}}). \tag{3.11}$$

We set for any $u \in \mathbb{R}$

$$\mathbf{M}'(u) := \begin{pmatrix} M_1'(u) \\ M_2'(u) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1 - \frac{f'(u)}{\sqrt{a}}) \\ \frac{1}{2}(1 + \frac{f'(u)}{\sqrt{a}}) \end{pmatrix}. \tag{3.12}$$

The explicit form of the scheme (2.7) and using (3.11) together with (2.6) then gives

$$\overline{\mathbf{R}}_j^{n+1} = \frac{1}{1+k}\overline{\mathbf{R}}_j^{n+\frac{1}{2}} + \frac{k}{1+k}\mathbf{M}'(\xi_j^n)\left(\mathbf{E}\cdot\overline{\mathbf{R}}_j^{n+\frac{1}{2}}\right). \tag{3.13}$$

We choose a discrete vector function $\mathbf{A}_j^n$ as

$$\mathbf{A}_j^n := \begin{pmatrix} A_{1,j}^n \\ A_{2,j}^n \end{pmatrix} = \begin{pmatrix} M_1'(u_{j-1}^n) \\ M_2'(u_j^n) \end{pmatrix}. \tag{3.14}$$

The heart of the matter is to prove the inequality

$$\overline{\mathbf{R}}_j^n \le \mathbf{A}_j^n L\Delta x, \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{3.15}$$

This estimate combined with (3.5) and (3.14) yields (3.9) as follows:

$$\begin{aligned} u_j^n - u_{j-1}^n = \mathbf{E}\cdot\overline{\mathbf{R}}_j^n &\le \left(1 - \frac{f'(u_{j-1}^n) - f'(u_j^n)}{2\sqrt{a}}\right) L\Delta x \\ &\le \left(1 + \frac{\gamma}{\sqrt{a}}\right) L\Delta x \le 2L\Delta x. \end{aligned}$$

It remains to prove (3.15). For this purpose we define

$$\mathbf{P}_j^n := \overline{\mathbf{R}}_j^n - \mathbf{A}_j^n L\Delta x. \tag{3.16}$$

Then (3.15) becomes

$$\mathbf{P}_j^n \le 0, \quad \text{i.e.,} \quad P_{i,j}^n \le 0, \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}, \quad i = 1, 2. \tag{3.17}$$

We shall prove (3.17) by induction. First let us consider the case $n = 0$. We have by inserting (3.10) together with (2.1) and (3.14) into (3.16) and using the definition of the discrete initial data (1.4) as well as the mean value theorem

$$\begin{aligned}
P_{1,j}^0 &= \frac{1}{2}\left(u_j^0 - \frac{v_j^0}{\sqrt{a}}\right) - \frac{1}{2}\left(u_{j-1}^0 - \frac{v_{j-1}^0}{\sqrt{a}}\right) - \frac{1}{2}\left(1 - \frac{f'(u_{j-1}^0)}{\sqrt{a}}\right) L\Delta x \\
&= \frac{1}{2}\left[(u_j^0 - u_{j-1}^0) - \frac{f(u_j^0) - f(u_{j-1}^0)}{\sqrt{a}}\right] - \frac{1}{2}\left(1 - \frac{f'(u_{j-1}^0)}{\sqrt{a}}\right) L\Delta x \\
&= \frac{1}{2}\Big[(u_j^0 - u_{j-1}^0) - L\Delta x\Big]\left(1 - \frac{f'(u_{j-1}^0)}{\sqrt{a}}\right) \\
&\quad - \frac{1}{2\sqrt{a}}\Big(f(u_j^0) - f(u_{j-1}^0) - f'(u_{j-1}^0)(u_j^0 - u_{j-1}^0)\Big) \\
&= \frac{1}{2}\Big[(u_j^0 - u_{j-1}^0) - L\Delta x\Big]\left(1 - \frac{f'(u_{j-1}^0)}{\sqrt{a}}\right) - \frac{1}{4\sqrt{a}} f''(\xi_j^0)(u_j^0 - u_{j-1}^0)^2
\end{aligned}$$

and

$$\begin{aligned}
P_{2,j}^0 &= \frac{1}{2}\left(u_j^0 + \frac{v_j^0}{\sqrt{a}}\right) - \frac{1}{2}\left(u_{j-1}^0 + \frac{v_{j-1}^0}{\sqrt{a}}\right) - \frac{1}{2}\left(1 + \frac{f'(u_j^0)}{\sqrt{a}}\right) L\Delta x \\
&= \frac{1}{2}\Big[(u_j^0 - u_{j-1}^0) - L\Delta x\Big]\left(1 + \frac{f'(u_j^0)}{\sqrt{a}}\right) \\
&\quad - \frac{1}{2\sqrt{a}}\Big(f(u_{j-1}^0) - f(u_j^0) + f'(u_j^0)(u_j^0 - u_{j-1}^0)\Big) \\
&= \frac{1}{2}\Big[(u_j^0 - u_{j-1}^0) - L\Delta x\Big]\left(1 + \frac{f'(u_j^0)}{\sqrt{a}}\right) - \frac{1}{4\sqrt{a}} f''(\tilde{\xi}_j^0)(u_j^0 - u_{j-1}^0)^2,
\end{aligned}$$

where ξ_j^0 and $\tilde{\xi}_j^0$ are intermediate values between u_{j-1}^0 and u_j^0. Thus (3.17) with $n = 0$ follows from the assumption (3.8), the convexity of the flux function, i.e., $f'' > 0$ and the subcharacteristic condition (3.3).

We now assume that (3.17) is true for n. It remains to prove (3.17) for $n+1$, i.e.,

$$P_{i,j}^{n+1} \le 0 \qquad \text{for} \quad i = 1, 2, \quad j \in \mathbb{Z}. \tag{3.18}$$

To this end we insert the relation (3.16) into (3.13) and get

$$\begin{aligned}
\mathbf{P}_j^{n+1} = \frac{1}{1+k} & \begin{pmatrix} (1-\mu)P_{1,j}^n + \mu P_{1,j+1}^n \\ (1-\mu)P_{2,j}^n + \mu P_{2,j-1}^n \end{pmatrix} \\
+ \frac{k}{1+k}\mathbf{M}'(\xi_j^n)\mathbf{E} \cdot & \begin{pmatrix} (1-\mu)P_{1,j}^n + \mu P_{1,j+1}^n \\ (1-\mu)P_{2,j}^n + \mu P_{2,j-1}^n \end{pmatrix} + (\mathbf{Q}_1^n + \mathbf{Q}_2^n)L\Delta x.
\end{aligned} \tag{3.19}$$

The vectors $\mathbf{Q}_1^n$ and $\mathbf{Q}_2^n$ are given as follows:

$$\begin{aligned}
\mathbf{Q}_1 = \begin{pmatrix} Q_{1,1}^n \\ Q_{2,1}^n \end{pmatrix} &:= -\mathbf{A}_j^{n+1} + \begin{pmatrix} (1-\mu)A_{1,j}^n + \mu A_{1,j+1}^n \\ (1-\mu)A_{2,j}^n + \mu A_{2,j-1}^n \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2\sqrt{a}}\Big(f'(u_{j-1}^{n+1}) - (1-\mu)f'(u_{j-1}^n) - \mu f'(u_j^n)\Big) \\ -\frac{1}{2\sqrt{a}}\Big(f'(u_j^{n+1}) - (1-\mu)f'(u_j^n) - \mu f'(u_{j-1}^n)\Big) \end{pmatrix}
\end{aligned} \tag{3.20}$$

and, using the fact that by (3.12) we have $M_2'(u) = 1 - M_1'(u)$,

$$\mathbf{Q}_2 = \begin{pmatrix} Q_{1,2}^n \\ Q_{2,2}^n \end{pmatrix} := \frac{k}{1+k}\Big(\mathbf{M}'(\xi_j^n)\mathbf{E} - \mathbf{Id}\Big) \cdot \begin{pmatrix} (1-\mu)A_{1,j}^n + \mu A_{1,j+1}^n \\ (1-\mu)A_{2,j}^n + \mu A_{2,j-1}^n \end{pmatrix}$$

$$= \frac{k}{1+k} \begin{pmatrix} -M_2'(\xi_j^n) & M_1'(\xi_j^n) \\ M_2'(\xi_j^n) & -M_1'(\xi_j^n) \end{pmatrix} \begin{pmatrix} \frac{1-\mu}{2}\Big(1 - \frac{f'(u_{j-1}^n)}{\sqrt{a}}\Big) + \frac{\mu}{2}\Big(1 - \frac{f'(u_j^n)}{\sqrt{a}}\Big) \\ \frac{1-\mu}{2}\Big(1 + \frac{f'(u_j^n)}{\sqrt{a}}\Big) + \frac{\mu}{2}\Big(1 + \frac{f'(u_{j-1}^n)}{\sqrt{a}}\Big) \end{pmatrix}. \tag{3.21}$$

For the last term in (3.19) we have the following estimate.

Lemma 3.2 (key lemma). *If the assumptions of Theorem 3.1 and the induction assumption (3.17) hold, then there exists a constant $c(\mu) > 0$ such that with α_2 given as in (3.6) we have the following estimate:*

$$\mathbf{Q}_1 + \mathbf{Q}_2 \leq -\frac{c(\mu)\alpha_2}{\sqrt{a}}\Big(P_{1,j}^n + P_{1,j+1}^n + P_{2,j-1}^n + P_{2,j}^n\Big)\mathbf{E}^\tau. \tag{3.22}$$

We continue the argument and postpone the proof of this lemma to the end of this section. It follows from (3.19) and Lemma 3.2 that

$$\begin{aligned} P_{1,j}^{n+1} \leq \; & \frac{1}{k+1}\Big[(1-\mu)P_{1,j}^n + \mu P_{1,j+1}^n\Big] \\ & + \frac{k}{k+1}\cdot\frac{1}{2}\Big(1 - \frac{f'(\xi_j^n)}{\sqrt{a}}\Big)\Big((1-\mu)(P_{1,j}^n + P_{2,j}^n) + \mu(P_{1,j+1}^n + P_{2,j-1}^n)\Big) \\ & - \frac{c(\mu)\alpha_2}{\sqrt{a}}(P_{1,j}^n + P_{1,j+1}^n + P_{2,j-1}^n + P_{2,j}^n)L\Delta x. \end{aligned}$$

The induction assumption $P_{i,j}^n \leq 0$ yields the estimate

$$\begin{aligned} P_{1,j}^{n+1} \leq \; & \frac{1}{k+1}\Big((1-\mu)P_{1,j}^n + \mu P_{1,j+1}^n\Big) \\ & + \left(\frac{k}{k+1}\cdot\frac{1}{2}\left(1 - \frac{f'(\xi_j^n)}{\sqrt{a}}\right)\cdot\min\{\mu,(1-\mu)\} - \frac{c(\mu)\alpha_2 L\Delta x}{\sqrt{a}}\right) \\ & \cdot (P_{1,j}^n + P_{1,j+1}^n + P_{2,j-1}^n + P_{2,j}^n). \end{aligned} \tag{3.23}$$

Using the scale condition (1.5), i.e., the assumption that k is bounded away from zero, we see that $\frac{k}{k+1}$ is also bounded away from zero. Due to this and the subcharacteristic condition (3.3) the coefficient of the second term on the right-hand side of (3.23) is nonnegative when Δx is small enough. Thereby $P_{1,j}^{n+1} \leq 0$ follows immediately. Analogously, we obtain

$$P_{2,j}^{n+1} \leq 0, \quad j \in \mathbb{Z}.$$

These estimates complete the proof of the theorem. $\square$

Remark. A slightly more general stability estimate of the form

$$u_j^n - u_{j-1}^n \leq \frac{2}{\beta n\Delta t + (L\Delta x)^{-1}}$$

for a positive constant $\beta \leq \alpha_1$ can be analogously obtained just by proving that

$$\mathbf{P}_j^n := \overline{\mathbf{R}}_j^n - \mathbf{A}_j^n \frac{1}{\beta n\Delta t + (L\Delta x)^{-1}} \leq 0$$

instead of (3.17) with $\mathbf{P}_j^n$ defined as in (3.16).

Proof of the key lemma 3.2. First we collect three identities deduced from the relaxation scheme (2.3) and (2.6):

$$u_j^{n+1} - u_j^n = \mathbf{E} \cdot (\mathbf{R}_j^{n+\frac{1}{2}} - \mathbf{R}_j^n) = \mu \mathbf{E} \cdot \begin{pmatrix} R_{1,j+1}^n - R_{1,j}^n \\ R_{2,j-1}^n - R_{2,j}^n \end{pmatrix} = \mu(\overline{R}_{1,j+1}^n - \overline{R}_{2,j}^n), \tag{3.24}$$

$$\begin{aligned} u_j^{n+1} - u_{j-1}^{n+1} &= \mathbf{E} \cdot (\mathbf{R}_j^{n+\frac{1}{2}} - \mathbf{R}_{j-1}^{n+\frac{1}{2}}) = \mathbf{E} \cdot \left(\overline{\mathbf{R}}_j^n + \mu \begin{pmatrix} \overline{R}_{1,j+1}^n - \overline{R}_{1,j}^n \\ \overline{R}_{2,j-1}^n - \overline{R}_{2,j}^n \end{pmatrix} \right) \\ &= (1-\mu)(\overline{R}_{1,j}^n + \overline{R}_{2,j}^n) + \mu(\overline{R}_{1,j+1}^n + \overline{R}_{2,j-1}^n), \end{aligned} \tag{3.25}$$

$$u_j^n - u_{j-1}^n = \mathbf{E} \cdot (\mathbf{R}_j^n - \mathbf{R}_{j-1}^n) = \overline{R}_{1,j}^n + \overline{R}_{2,j}^n. \tag{3.26}$$

These identities will be used repeatedly.

Using the mean value theorem we have for the first component $Q_{1,1}^n$ of $\mathbf{Q}_1^n$ in (3.20)

$$Q_{1,1}^n = \frac{1}{2\sqrt{a}} \Big(f''(\xi_{j-1}^{n+\frac{1}{2}})(u_{j-1}^{n+1} - u_{j-1}^n) - \mu f''(\xi_{j-\frac{1}{2}}^n)(u_j^n - u_{j-1}^n) \Big).$$

Taking (3.24), (3.26), and the relation $\overline{\mathbf{R}}_j^n = \mathbf{P}_j^n + \mathbf{A}_j^n L\Delta x$ one obtains

$$\begin{aligned} Q_{1,1}^n &= \frac{1}{2\sqrt{a}} \Big(\mu f''(\xi_{j-1}^{n+\frac{1}{2}})(\overline{R}_{1,j}^n - \overline{R}_{2,j-1}^n) - \mu f''(\xi_{j-\frac{1}{2}}^n)(\overline{R}_{1,j}^n + \overline{R}_{2,j}^n) \Big) \\ &= -\frac{\lambda}{2} \Big[f''(\xi_{j-1}^{n+\frac{1}{2}}) P_{2,j-1}^n + f''(\xi_{j-\frac{1}{2}}^n) P_{2,j}^n - \Big(f''(\xi_{j-1}^{n+\frac{1}{2}}) - f''(\xi_{j-\frac{1}{2}}^n) \Big) P_{1,j}^n \Big] \\ &\quad -\frac{\lambda}{2} \left[f''(\xi_{j-\frac{1}{2}}^n) + \frac{f''(\xi_{j-\frac{1}{2}}^n) f'(u_j^n) + \Big(2 f''(\xi_{j-1}^{n+\frac{1}{2}}) - f''(\xi_{j-\frac{1}{2}}^n) \Big) f'(u_{j-1}^n)}{2\sqrt{a}} \right] L\Delta x. \end{aligned}$$

Recalling the induction hypothesis (3.17), i.e., $\mathbf{P}_j^n \leq 0$, and using the bounds for f' f'' given in (3.5), (3.6) one obtains the estimate

$$Q_{1,1}^n \leq -\frac{\lambda \alpha_2}{2}(P_{2,j-1}^n + P_{2,j}^n + P_{1,j}^n) - \frac{\lambda}{2}\Big(\alpha_1 - \frac{3\alpha_2 \gamma}{2\sqrt{a}}\Big) L\Delta x. \tag{3.27}$$

Note that if $f'' = \alpha_1$, then the last term in (3.27) may alternatively be estimated sharper as

$$-\frac{\lambda}{2}\alpha_1 \Big(1 - \frac{\gamma}{\sqrt{a}}\Big) L\Delta x.$$

It is negative under the subcharacteristic condition $\gamma < \sqrt{a}$.

Now we proceed to estimate $Q_{1,2}^n$. A straightforward evaluation of $Q_{1,2}^n$ in (3.21) yields

$$Q_{1,2}^n = \frac{k}{4(1+k)} \left(\frac{f'(u_j^n) + f'(u_{j-1}^n) - 2f'(\xi_j^n)}{\sqrt{a}} + \frac{(2\mu - 1) f'(\xi_j^n)(f'(u_j^n) - f'(u_{j-1}^n))}{a} \right). \tag{3.28}$$

The estimate to be derived from this inequality, which will be (3.36) below, will be obtained in four steps. Three of these steps are needed to estimate the first quotient.

Our first two steps are estimates of the differences appearing in the first quotient in (3.28). We make repeated use of the mean value theorem in the following form:

$$f(u) - f(v) = \int_0^1 f'(\theta u + (1-\theta)v)d\theta\,(u-v).$$

Using the definition of $f'(\xi_j^n)$ in (3.11) and (2.6) we obtain

$$f'(u_{j-1}^n) - f'(\xi_j^n) = -\int_0^1\int_0^1 f_1''(\theta,\theta_1)\Big[\theta(u_j^{n+1} - u_{j-1}^n) + (1-\theta)(u_{j-1}^{n+1} - u_{j-1}^n)\Big]d\theta_1 d\theta, \tag{3.29}$$

where

$$f_1''(\theta,\theta_1) := f''(\theta_1 u_{j-1}^n + (1-\theta_1)(\theta u_j^{n+1} + (1-\theta)u_{j-1}^{n+1})).$$

By (3.25) and (3.24) we have

$$\begin{aligned} J &:= \theta(u_j^{n+1} - u_{j-1}^n) + (1-\theta)(u_{j-1}^{n+1} - u_{j-1}^n) = \theta(u_j^{n+1} - u_{j-1}^{n+1}) + (u_{j-1}^{n+1} - u_{j-1}^n) \\ &= \theta\big(\mu(\overline{R}_{1,j+1}^n + \overline{R}_{2,j-1}^n) + (1-\mu)(\overline{R}_{1,j}^n + \overline{R}_{2,j}^n)\big) + \mu(\overline{R}_{1,j}^n - \overline{R}_{2,j-1}^n). \end{aligned}$$

Inserting the relation (3.16) gives

$$\begin{aligned} J &= \theta\Bigg[\mu(P_{1,j+1}^n + P_{2,j-1}^n) + (1-\mu)(P_{1,j}^n + P_{2,j}^n) \\ &\quad + \left(\frac{\mu\big(f'(u_{j-1}^n) - f'(u_j^n)\big)}{2\sqrt{a}} + \frac{(1-\mu)\big(f'(u_j^n) - f'(u_{j-1}^n)\big)}{2\sqrt{a}} + 1\right)L\Delta x\Bigg] \\ &\quad + \mu(P_{1,j}^n - P_{2,j-1}^n) - \mu\frac{f'(u_{j-1}^n)}{\sqrt{a}}L\Delta x. \end{aligned}$$

Having obtained this expression for J we use (3.3), (3.5), (3.6), and the induction hypothesis (3.17) to obtain from (3.29) the estimate

$$\begin{aligned} f'(u_{j-1}^n) - f'(\xi_j^n) &\le -\alpha_2\left[\frac{\mu}{2}(P_{1,j+1}^n + P_{2,j-1}^n) + \frac{1-\mu}{2}(P_{1,j}^n + P_{2,j}^n) + \mu P_{1,j}^n\right] \\ &\quad + \alpha_2\frac{(|2\mu-1| + \mu)\gamma}{\sqrt{a}}L\Delta x - \int_0^1\int_0^1 f_1''(\theta,\theta_1)\theta d\theta_1 d\theta L\Delta x. \end{aligned} \tag{3.30}$$

Now we get to the second step of estimating the second difference in the first quotient in (3.28). Defining analogously as above

$$f_2''(\theta,\theta_1) := f''(\theta_1 u_j^n + (1-\theta_1)(\theta u_j^{n+1} + (1-\theta)u_{j-1}^{n+1}))$$

one has

$$\begin{aligned} f'(u_j^n) - f'(\xi_j^n) &= -\int_0^1\int_0^1 f_2''(\theta,\theta_1)\Big[\theta(u_j^{n+1} - u_j^n) + (1-\theta)(u_{j-1}^{n+1} - u_j^n)\Big]d\theta_1 d\theta \\ &= -\int_0^1\int_0^1 f_2''(\theta,\theta_1)\Big[\mu(\overline{R}_{1,j+1}^n - \overline{R}_{2,j}^n) \\ &\quad + (1-\theta)\Big(-\mu(\overline{R}_{1,j+1}^n + \overline{R}_{2,j-1}^n) - (1-\mu)(\overline{R}_{1,j}^n + \overline{R}_{2,j}^n)\Big)\Big]d\theta_1 d\theta \end{aligned}$$

$$
\begin{aligned}
= -\int_0^1 \int_0^1 f_2''(\theta,\theta_1)\Bigg[&\mu(P_{1,j+1}^n - P_{2,j}^n) - \mu\frac{f'(u_j^n)}{\sqrt{a}}L\Delta x \\
&+(1-\theta)\Bigg(- \mu(P_{1,j+1}^n + P_{2,j-1}^n) - (1-\mu)(P_{1,j}^n + P_{2,j}^n) \\
&- \left(\mu\frac{f'(u_{j-1}^n) - f'(u_j^n)}{2\sqrt{a}} + (1-\mu)\frac{f'(u_j^n) - f'(u_{j-1}^n)}{2\sqrt{a}} + 1\right) L\Delta x\Bigg)\Bigg] d\theta_1 d\theta.
\end{aligned}
$$

This gives us the following inequality:

$$
\begin{aligned}
f'(u_j^n) - f'(\xi_j^n) \leq\ & -\alpha_2\mu P_{1,j+1}^n + \alpha_2\frac{(|1-2\mu|+\mu)\gamma}{\sqrt{a}}L\Delta x \\
& + \int_0^1 \int_0^1 f_2''(\theta,\theta_1)(1-\theta)d\theta_1 d\theta L\Delta x.
\end{aligned} \tag{3.31}
$$

As the third step in deriving the estimate for $Q_{1,2}^n$ we have to take care of the integrals involving second derivatives f'' in (3.30) and (3.31). For this purpose we define

$$
I := \int_0^1 \int_0^1 \Big(f_2''(\theta,\theta_1)(1-\theta) - f_1''(\theta,\theta_1)\theta\Big)d\theta_1 d\theta
$$

which becomes zero for the case f'' =const. For general convex flux functions this term has to be treated carefully since the integral in (3.31) is positive. It is not obviously dominated by other negative terms. We estimate I as follows. Using the mean value theorem again we get

$$
\begin{aligned}
I = \ & \int_0^1 \int_0^1 \theta\Big[f''\Big(\theta_1 u_j^n + (1-\theta_1)\big((1-\theta)u_j^{n+1} + \theta u_{j-1}^{n+1}\big)\Big) \\
& - f''\Big(\theta_1 u_{j-1}^n + (1-\theta_1)\big(\theta u_j^{n+1} + (1-\theta)u_{j-1}^{n+1}\big)\Big)d\theta_1\Big]d\theta \\
= & \int_0^1 \int_0^1 \int_0^1 \theta f'''\Big(\theta_2\xi_1 + (1-\theta_2)\xi_2\Big)(\xi_1 - \xi_2)d\theta_2 d\theta_1 d\theta,
\end{aligned} \tag{3.32}
$$

where

$$
\xi_1 - \xi_2 := \theta_1(u_j^n - u_{j-1}^n) + (1-\theta_1)(1-2\theta)(u_j^{n+1} - u_{j-1}^{n+1}).
$$

Using (3.26) and $\overline{\mathbf{R}}_j^n = \mathbf{P}_j^n + \mathbf{A}_j^n L\Delta x$ we have

$$
\begin{aligned}
\xi_1 - \xi_2 = \theta_1(\overline{R}_{1,j}^n + \overline{R}_{2,j}^n) + (1-\theta_1)(1-2\theta)\Big[&(1-\mu)(\overline{R}_{1,j}^n + \overline{R}_{2,j}^n) + \mu(\overline{R}_{1,j+1}^n + \overline{R}_{2,j-1}^n)\Big] \\
= \theta_1(P_{1,j}^n + P_{2,j}^n) &+ \theta_1\Big(1 - \frac{f'(u_{j-1}^n) - f'(u_j^n)}{2\sqrt{a}}\Big)L\Delta x \\
+ (1-\theta_1)(1-2\theta)&\Big[\mu(P_{1,j+1}^n + P_{2,j-1}^n) + (1-\mu)(P_{1,j}^n + P_{2,j}^n)\Big] \\
+ (1-\theta_1)(1-2\theta)&\Big[1 - \frac{(2\mu-1)(f'(u_j^n) - f'(u_{j-1}^n))}{2\sqrt{a}}\Big]L\Delta x.
\end{aligned} \tag{3.33}
$$

Substituting K for $\sup_{|u|\le b}|f'''(u)|$ and (3.33) into (3.32) gives the desired estimate for I:

$$
\begin{aligned}
I \le K \Bigg(&-\frac{1}{4}(P_{1,j}^n + P_{2,j}^n) - \frac{1}{12}\Big[\mu(P_{1,j+1}^n + P_{2,j-1}^n) + (1-\mu)(P_{1,j}^n + P_{2,j}^n)\Big] \\
&+ \Big[\frac{1}{4}\left(1 + \frac{\gamma}{\sqrt{a}}\right) + \frac{1}{12}\Big(1 + \frac{|2\mu - 1|\gamma}{\sqrt{a}}\Big)\Big] L\Delta x \Bigg).
\end{aligned} \tag{3.34}
$$

As the fourth step in estimating $Q_{1,2}^n$ we have to consider the second quotient in (3.28). Using (3.26) and similar arguments as above we get

$$
\begin{aligned}
\frac{(2\mu-1)f'(\xi_j^n)}{a}\Big(f'(u_j^n) - f'(u_{j-1}^n)\Big) &= \frac{(2\mu-1)f'(\xi_j^n)}{a} f''(\xi_{j-\frac{1}{2}}^n)(u_j^n - u_{j-1}^n) \\
&= \frac{(2\mu-1)f'(\xi_j^n)f''(\xi_{j-\frac{1}{2}}^n)}{a} \\
&\quad \cdot \left[P_{2,j}^n + P_{1,j}^n + \left(1 - \frac{f'(u_{j-1}^n) - f'(u_j^n)}{2\sqrt{a}}\right) L\Delta x\right] \\
&\le -\alpha_2 \frac{\gamma|1-2\mu|}{a}(P_{2,j}^n + P_{1,j}^n) + \alpha_2 \frac{2|1-2\mu|\gamma}{a} L\Delta x.
\end{aligned} \tag{3.35}
$$

Now we get back to (3.28). We insert the estimates (3.30) and (3.31) together with (3.34) and also (3.35) to give

$$
\begin{aligned}
Q_{1,2}^n \le &-\frac{c(\mu)(\alpha_2 + KL\Delta x)}{\sqrt{a}}(P_{1,j}^n + P_{1,j+1}^n + P_{2,j-1}^n + P_{2,j}^n) \\
&+ \frac{c(\mu)\alpha_2\gamma}{a} L\Delta x + \frac{c(\mu)KL^2\Delta x^2}{\sqrt{a}},
\end{aligned} \tag{3.36}
$$

where $c(\mu)$ is a generic constant depending on μ. Using $\mu = \sqrt{a}\lambda$ one obtains from (3.27) and (3.36) that

$$
\begin{aligned}
Q_{1,1}^n + Q_{1,2}^n \le &-\frac{c(\mu)\alpha_2 + KL\Delta x}{\sqrt{a}}(P_{1,j}^n + P_{1,j+1}^n + P_{2,j-1}^n + P_{2,j}^n) \\
&- \frac{\lambda}{2}\Big(\alpha_1 - \frac{3\alpha_2\gamma}{2\sqrt{a}} - \frac{c(\mu)\alpha_2\gamma}{\sqrt{a}} - c(\mu)KL\Delta x\Big) L\Delta x.
\end{aligned}
$$

Choosing a suitably large in order to make the last brackets nonpositive and Δx suitably small one arrives at the estimate

$$
Q_{1,1}^n + Q_{1,2}^n \le -\frac{c(\mu)\alpha_2}{\sqrt{a}}(P_{1,j}^n + P_{1,j+1}^n + P_{2,j-1}^n + P_{2,j}^n). \tag{3.37}
$$

Analogously, we obtain such an estimate for $Q_{2,1}^n + Q_{2,2}^n$. Thus the proof of the key lemma is complete. ∎

Remark. In the proof of the key lemma we can see that there exists a positive constant $c(\mu, \alpha_1, \alpha_2, \gamma)$ such that $a > c(\mu, \alpha_1, \alpha_2, \gamma)$ is sufficient for all arguments in the proof of the theorem related to the choice of a. The smallness assumption for Δx depends on the quantities $\mu, \alpha_1, \alpha_2, \gamma, L$, and K.

4. Error estimates. In this section we will consider error estimates for the discrete solution given by the relaxation scheme (1.3) with the initial data (1.4) as an approximation to the solution u of the Cauchy problem for the conservation law (1.1) and the initial condition (1.2). We are following the Lip' theory developed by Nessyahu and Tadmor [14], [16].

First we extend our discrete solution $(u_j^n, v_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ given at the grid points to a piecewise bilinear function by setting

$$\left(u^{\Delta,\epsilon}(x,t), v^{\Delta,\epsilon}(x,t)\right) := \sum_{j\in\mathbb{Z},n\in\mathbb{N}} (u_j^n, v_j^n)\Lambda_j^n(x,t),$$

where $\Lambda_j^n(x,t) := \Lambda_j(x)\Lambda^n(t)$ with

$$\Lambda_j(x) = \frac{1}{\Delta x}\min(x - x_{j-1}, x_{j+1} - x)_+,$$
$$\Lambda^n(t) = \frac{1}{\Delta t}\min(t - t_{j-1}, t_{j+1} - t)_+.$$

By Theorem 3.1 we have

$$\|u^{\Delta,\epsilon}\|_{Lip^+(\mathbb{R})} \le 2L.$$

In order to use the results in [14, Theorem 2.1], we still have to discuss the Lip'-consistency.

LEMMA 4.1 (Lip'-consistency). *The approximation generated by the relaxation scheme* (1.3) *with the initial data* (1.4) *on a time interval* $[0,T]$ *satisfies the following truncation error estimate for* $u_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$*:*

$$\left\|u_t^{\Delta,\epsilon} + f(u^{\Delta,\epsilon})_x\right\|_{Lip'(\mathbb{R},[0,T])} \le C_T(\Delta x + \epsilon), \tag{4.1}$$

where C_T *is a positive constant depending on the final time* T.

Proof. Let N denote the number of time steps on $[0,T]$, i.e., $T = t_N = N\Delta t$. We set

$$Z_j^n = f(u_j^n) - v_j^n \quad \text{for } (j,n) \in \mathbb{Z} \times \{1,\dots,N\}.$$

Then it follows from the first equation of (1.3) that

$$\begin{aligned}\Delta x(u_j^{n+1} - u_j^n) = &-\frac{\Delta t}{2}\left[f(u_{j+1}^n) - f(u_{j-1}^n)\right] \\ &+ \frac{\Delta x\sqrt{a}}{2}\left[\lambda(u_{j+1}^n - u_j^n) - \lambda(u_j^n - u_{j-1}^n)\right] + \frac{\Delta t}{2}[Z_{j+1}^n - Z_{j-1}^n].\end{aligned} \tag{4.2}$$

We consider the test function $\phi \in C_0^\infty(\mathbb{R}^2)$, set $t_n = n\Delta t$, and define the piecewise bilinear interpolant $\hat\phi(x,t) = \sum_{j\in\mathbb{Z},n\in\mathbb{N}_0}\phi(x_j,t_n)\Lambda_j^n(x,t)$. We further set

$$H^{\Delta x} := \sum_{k=1}^{4} T_k^{\Delta x}$$

with $T_1^{\Delta x}, \dots, T_4^{\Delta x}$ as defined by Nessyahu and Tadmor [14, Equation (3.5)]. Here we additionally need

$$H^\epsilon := \sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\phi(x_j,t_n)\frac{\Delta t}{2}(Z_{j+1}^n - Z_{j-1}^n).$$

Then we have, as in [14], the relation

$$\big(\partial_t u^{\Delta,\epsilon}(x,t) + \partial_x f\big(u^{\Delta,\epsilon}(x,t)\big), \phi\big)_{x,t} = H^{\Delta x} + H^{\epsilon}. \tag{4.3}$$

For the relation between (4.2) and (4.3), see the appendix of [14]. The following estimate is shown in [14, Equation (3.7)]:

$$H^{\Delta x} \leq \mathrm{Const} \cdot \Delta x \|u^{\Delta,\epsilon}\|_{L^1([0,T],BV_x)} \|\phi\|_{Lip(\mathbb{R}\times[0,T])}. \tag{4.4}$$

Now we estimate H^{ϵ}, which comes from the relaxation term. Using summation by parts

$$\begin{aligned} |H^{\epsilon}| &= \left| \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \frac{\Delta t}{2} Z_{j+1}^n \left[\phi(x_j,t_n) - \phi(x_{j+2},t_n)\right] \right| \\ &\leq \Delta t \Delta x \big\|\phi(\cdot,t)\big\|_{Lip(\mathbb{R},[0,T])} \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} |Z_{j+1}^n|. \end{aligned}$$

Recall that $\sum_j |f(u_j^n) - v_j^n|\Delta x \leq C\epsilon$ was shown in [23, Lemma 6]. This combined with the above estimate leads to

$$|H^{\epsilon}| \leq CT\epsilon \big\|\phi(\cdot,t)\big\|_{Lip(\mathbb{R},\ [0,T])}. \tag{4.5}$$

Equipped with the above estimates (4.4), (4.5) we have

$$\left| \big(\partial_t u^{\Delta,\epsilon}(x,t) + \partial x f(u^{\Delta,\epsilon}), \phi\big)_{x,t} \right| \leq C_T(\Delta x + \epsilon) \|\phi\|_{Lip(\mathbb{R},[0,T])}$$

which implies (4.1). ◻

Furthermore, we show that the approximate solutions $u^{\Delta,\epsilon}$ are also Lip'-consistent with the initial data. We first note that the $u^{\Delta,\epsilon}$ are clearly conservative, for by (4.2) and our choice of the discrete initial data,

$$\int_{\mathbb{R}} u^{\Delta,\epsilon}(x,t_n)dx = \frac{\Delta x}{2} \sum_{j\in\mathbb{Z}} (u_j^n + u_{j+1}^n) = \frac{\Delta x}{2} \sum_{j\in\mathbb{Z}} (u_j^0 + u_{j+1}^0) = \int_{\mathbb{R}} u_0(x)dx.$$

Moreover, these initial conditions are Lip'-consistent. In fact we have

$$\begin{aligned} \big|\big(u^{\Delta,\epsilon}(x,0) - u_0(x), \phi(x)\big)\big| &= \left| \big(u^{\Delta,\epsilon}(x,0) - u_0), \phi(x) - \phi(x_{j+\frac{1}{2}})\big) \right| \\ &\leq \Delta x \|\phi\|_{Lip(\mathbb{R},\ [0,T])} \sum_j \int_{x_j}^{x_{j+1}} \left|u^{\Delta,\epsilon}(x,0) - u_0(x)\right| dx \\ &\leq C(\Delta x)^2 \big\|u_0(x)\big\|_{BV} \|\phi\|_{Lip(\mathbb{R},\ [0,T])} \end{aligned}$$

which yields

$$\big\|u^{\Delta x,\epsilon}(x,0) - u_0(x)\big\|_{Lip'(\mathbb{R})} \leq C\big\|u_0\big\|_{BV} (\Delta x)^2. \tag{4.6}$$

Now we can use the result in [14, Theorem 2.1] and get the error estimate

$$\begin{aligned} \big\|u^{\Delta,\epsilon}(\cdot,T) - u(\cdot,T)\big\|_{Lip'(\mathbb{R})} &\leq C_T \Big[\big\|u^{\Delta x,\epsilon}(\cdot,T) - u_0(x)\big\|_{Lip'(\mathbb{R})} \\ &\quad + \big\|u_t^{\Delta,\epsilon} + f(u^{\Delta,\epsilon})_x\big\|_{Lip'(\mathbb{R},[0,T])} \Big] \\ &\leq C_T(\Delta x + \epsilon) = O(\Delta x + \epsilon). \end{aligned} \tag{4.7}$$

The Lip' error estimate (4.7) may now be interpolated into the $W^{s,p}$-error estimates along the lines of Nessyahu and Tadmor [14, Corollaries 2.2, 2.4]. The resulting error estimates are summarized as follows.

THEOREM 4.2. *Consider the convex scalar conservation law* (1.1) *with* Lip^+-*bounded initial data* u_0 *and* $v_0 = f(u_0)$. *Then the relaxation scheme with discrete initial data* $(u_j^0, f(u_j^0))_{j\in\mathbb{Z}}$ *converges. The piecewise linear interpolants* $u^{\Delta,\epsilon}$ *satisfy the convergence rate estimates*

$$\|u^{\Delta,\epsilon}(\cdot,T) - u(\cdot,T)\|_{W^{s,p}} \le C_T(\Delta x+\epsilon)^{\frac{1-sp}{2p}} \quad \text{for} \quad -1\le s\le \frac{1}{p},\ 1\le p\le\infty, \tag{4.8}$$

as well as

$$|u^{\Delta,\epsilon}(x,T) - u(x,T)| \le \text{Const}_{x,T}(\Delta x+\epsilon)^{\frac{1}{3}}, \tag{4.9}$$

with

$$\text{Const}_{x,T} \sim 1 + \left|u_x(\cdot,T)\right|_{L^\infty(x-(\Delta x+\epsilon)^{1/3},\ x+(\Delta x+\epsilon)^{1/3})}.$$

Remarks.

1. When $(s,p) = (-1,1)$ the error estimate (4.8) turns into the Lip' error estimate

$$\|u^{\Delta,\epsilon}(\cdot,t) - u(\cdot,t)\|_{Lip'(\mathbb{R})} \le O(\epsilon+\Delta x).$$

2. When $(s,p) = (0,1)$ the error estimate (4.8) yields an L^1-convergence rate of order $O(\sqrt{\Delta x+\epsilon})$ which is consistent with the result obtained in [10] for conservation laws with possibly nonconvex flux functions.

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