KdV Dynamics in
the Plasma-Sheath Transition

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Abstract—A mathematical model is formulated to catch the dynamics hidden in the plasma-sheath transition layer and the inner sheath layer for planar motion of a plasma. It is shown that the rescaled potential in the plasma-sheath transition layer and inner layer is governed by a perturbed KdV equation, through which some of the complex interactions and couplings among physical mechanisms acting in the plasma-sheath formation process are elucidated. This model is analyzed and compared with the model used by Fokas in his study of nonlinear dispersive initial boundary value problems [1]. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Many problems in science involve structures on several distinct length scales. Usually the relevant length scales are not known *a priori*, but emerge from an attempt of the system to reach its equilibrium state. In plasma physics, for example, the typical length scale can be predicated by dimensional analysis, but the sheath transition and inner layer are determined by a complex interplay of the internal dynamics. The plasma-sheath transition is a fundamental problem in plasma physics and a good discussion is found in the book by Lieberman and Lichtenberg [2]. Mathematically it provides a challenge to the applied analyst in that there are multiple scales which must be resolved to obtain an adequate description of the physical process.

  Matched asymptotic expansions provide a powerful method to predict limiting behavior. Using this method, several authors [3–11] beginning with Franklin and Ockendon [3] have described the plasma-sheath transition by analysis of the balance laws on three relevant space scales for

  (A) the bulk quasi-neutral plasma,

(B) the plasma-sheath transition layer, and

(C) the sheath layer.

Region (B) is governed by the Painlevé equation, governing the electrical potential in the transition layer joining quasi-neutral plasma to space charge sheath in a weakly ionized plasma.

The intent of the present work is to provide a derivation of a new model, describing both the plasma-sheath transition layer and the sheath inner layer.

For an ionized plasma consisting of electrons and ions, one -imensional motion is described by a normalized Euler-Poisson system of the form

$$\partial_t n + \partial_x(nu) = ze^{-\phi}, \tag{1.1}$$

$$\partial_t u + u\partial_x u = \partial_x \phi - f(u) - ze^{-\phi}un^{-1}, \tag{1.2}$$

$$\epsilon^2 \partial_x^2 \phi = n - e^{-\phi}, \tag{1.3}$$

where u represents ion velocity, n the ion density, and ϕ is the electric potential (both suitably scaled), x is a dimensionless space variable, $-L < x < x_w$, where $x_w = -\epsilon^{4/5}b(t\epsilon^{-2/5})$ is the location of a possibly moving wall. Here the prescription of the moving boundary has taken the two fundamental length scales (the sheath scale and the intermediate scale) into consideration, and the regular fixed boundary is just a special case of $b = 0$. The electron density is given by Boltzmann's relation and has been set equal to $e^{-\phi}$. At $x = x_w$, we prescribe boundary conditions $\phi = \phi_w(t, \epsilon)$ and $u = u_w(t, \epsilon)$. In fact, the main goal of this paper is to derive a KdV model for the plasma sheath transition and compare with the recent result of [1] for a similar yet subtly different problem. A formal derivation of the current model in Section 2 provides us the necessary background information, and the discussion in Section 3 is devoted to a quantum formulation of the current Euler-Poisson system, which further elucidates the dispersive nature of the underlying force in the system. It would be of interest to derive the same boundary layer separation directly using the quantum formulation in Section 3.

2. SIMPLIFICATION OF THE BASIC EQUATIONS

For completeness, we provide the relevant balance laws for hydrodynamic models for plasma [2].

Let m_i denote the ion mass, n_i the ion density, u_i the ion velocity, m_e the electron mass, n_e the electron density, u_e the electron velocity, Φ the electric potential, and Z denotes the rate of ionization. The balance laws of mass and momentum for ions are

$$\partial_\tau n_i + \partial_X(n_i u_i) = Zn_e,$$

$$\partial_\tau(n_i u_i) + \partial_X\left(n_i u_i^2\right) = -\frac{en_i}{m_i}\partial_X\Phi - n_i\frac{\tilde{f}(u_i)}{\lambda},$$

where $\tilde{f}/\lambda$ denotes the ion friction and $\lambda > 0$ the constant ion collision mean free path, and the balance laws of mass and momentum for electrons are

$$\partial_\tau n_e + \partial_X(n_e u_e) = Zn_e,$$

$$\partial_\tau(n_e u_e) + \partial_X\left(n_e u_e^2 + \frac{p_e}{m_e}\right) = \frac{en_e}{m_e}\partial_X\Phi,$$

where the pressure is given by $p_e = kn_e T_e$, here T_e denotes the electron temperature, the ion temperature is zero. In addition, Φ satisfies Poisson's equation

$$-\frac{\epsilon_0}{e}\partial_X^2\Phi = n_i - n_e,$$

where ϵ_0 is the permittivity of free space. Usually, the ions are heavy compared to the electrons, i.e., $m_i \gg m_e$. Passing to the limit $m_e \to 0$ in the momentum equation for electron one can formally obtain

$$\partial_X(kT_e n_e) = en_e\partial_X\Phi.$$

Integration in terms of X gives

$$n_e = n_{ch} \exp\left(\frac{e\Phi}{kT_e}\right).$$

This is the well-known Boltzmann relation for electrons, in which n_{ch} denotes the characteristic charged particle density, e the electron charge, and k Boltzmann's constant.

The above systems may be further simplified if we introduce quantities

$$c_s = \sqrt{\frac{kT_e}{m_i}}, \qquad \lambda_D = \sqrt{\frac{\epsilon_0 kT_e}{n_{ch}e^2}}$$

representing the ion sound speed and the electron Debye length. Indeed introducing the following dimensionless variables:

$$t = \frac{\tau c_s}{\lambda}, \qquad x = \frac{X}{\lambda}, \qquad \epsilon = \frac{\lambda_D}{\lambda},$$

$$\frac{n_i}{n_{ch}} \to n_i, \qquad \frac{u_i}{c_s} \to u_i \qquad \phi = \frac{-e\Phi}{kT_e},$$

$$\frac{n_e}{n_{ch}} \to n_e, \qquad \frac{u_e}{c_s} \to u_e, \qquad m = \frac{m_e}{m_i}, \qquad z = Z\lambda/c_s,$$

the above coupled Euler-Poisson system may be rewritten as

$$\partial_t n_i + \partial_x (n_i u_i) = z n_e, \tag{2.1}$$

$$\partial_t (n_i u_i) + \partial_x (n_i u_i^2) = n_i \partial_x \phi - n_i f(u_i), \qquad f(u_i) := \frac{\tilde{f}(c_s u_i)}{c_s^2}, \tag{2.2}$$

and

$$\partial_t n_e + \partial_x (n_e u_e) = z n_e, \tag{2.3}$$

$$\partial_t (n_e u_e) + \partial_x (n_e u_e^2) + \frac{1}{m} \partial_x n_e = -\frac{n_e}{m} \partial_x \phi, \tag{2.4}$$

coupled with Poisson's equation

$$\epsilon^2 \partial_x^2 \phi = n_i - n_e. \tag{2.5}$$

Recall that the limit $m \to 0$ in the above momentum equation (2.4) yields Boltzmann's relation

$$n_e = \exp(-\phi).$$

Hence, the limit system $(m \to 0)$ may be rewritten in nondimensional form

$$\partial_t n + \partial_x (nu) = z e^{-\phi},$$
$$\partial_t u + u \partial_x u = \partial_x \phi - f(u) - z e^{-\phi} u n^{-1},$$
$$\epsilon^2 \partial_x^2 \phi = n - e^{-\phi},$$

which is exactly the system (1.1)–(1.3) stated in Section 1.

3. QUANTUM FORMULATION

In this section, we will derive the system of Schrödinger equations such that their semiclassical limit coincides with the Euler-Poisson system with linear damping $f(u_i) = \alpha u_i$ in the momentum equation for ions. This partially justifies the dispersive nature of the force imposed by the Possion equation.

Let the desired Schrödinger equation take the form

$$i\hbar\partial_t\psi^\hbar = -\frac{\hbar^2}{2}\Delta_x\psi^\hbar + \left[V + \frac{i\hbar}{2}Q\right]\psi^\hbar, \tag{3.1}$$

with the potential V and nonhomogeneous term Q to be determined. We remark in passing that the connection between Schrödinger equations and the classical hydrodynamical equations was already noted in 1927 by Madelung, in the context of semiclassical limit of the nonlinear Schrödinger equation. To this end, one identifies two physical relevant observable quantities—the fluid density $|\psi|^2$, and the fluid velocity $u^\hbar := \hbar\nabla_x\arg\psi^\hbar$. For the semiclassical regime, it is customary to consider the following WKB (after Wentzel, Kramers, and Brillouin) ansatz

$$\psi^\hbar = A^\hbar(x,t)\exp\left(i\frac{S(x,t)}{\hbar}\right),$$

with $A^\hbar \geq 0$ assuming that the phase and the amplitude are sufficiently smooth, and we expand the amplitude: in powers of $\hbar$:

$$A^\hbar = A_0 + \hbar A_1 + \hbar^2 A_2 + \cdots.$$

Insertion of this expression into (3.1) leads to the following relation between the wave phase and its amplitude

$$-A^\hbar\left(\partial_t S + \frac{1}{2}|\nabla S|^2 + V(x)\right) + \frac{i\hbar}{2}\left(2\partial_t A^\hbar + A^\hbar\Delta_x S + 2\nabla_x A^\hbar\cdot\nabla_x S - QA^\hbar\right) + \frac{\hbar^2}{2}\Delta_x A^\hbar = 0.$$

Nullifying the expressions related to the first two powers of $\hbar$, we derive the WKB system with corrector term QA_0^2

$$\partial_t S + \frac{1}{2}|\nabla S|^2 + V = 0,$$

and the leading order of the amplitude A_0 solves the forced transport equation

$$\partial_t A_0^2 + \nabla\cdot\left(A_0^2\nabla S\right) = QA_0^2.$$

When $\hbar$ is small the leading term A_0 becomes significant. Set $(\rho, U) := (A_0^2, \nabla S)$, one then has the following system:

$$\partial_t\rho + \nabla\cdot(\rho U) = Q\rho,$$
$$\partial_t U + U\cdot\nabla U = -\nabla V.$$

In order to recover (2.1),(2.2), it suffices to take $(\rho, U) := (n_i, u_i)$, $Q := Q_i = zn_en_i^{-1}$, and $V := V_i$ such that $-\nabla_x V_i = \nabla\phi - \alpha\nabla_x S_i - Q\nabla_x S_i$, i.e.,

$$V_i = \alpha S_i + z\nabla_x^{-1}\left[\frac{n_e}{n_i}\nabla_x S_i\right] - \phi.$$

Assume the wave function for ions is ψ_i and for electron is ψ_e, therefore,

$$n_i = |\psi_i|^2, \qquad n_e = |\psi_e|^2.$$

Hence, the Poisson equation (2.5) becomes

$$\epsilon^2\Delta_x\phi = |\psi_i|^2 - |\psi_e|^2,$$

or $\phi = \epsilon^{-2}\Delta^{-1}(|\psi_i|^2 - |\psi_e|^2)$. Note that the phase $S_i = \hbar \arg(\psi_i)$. A combination of the above facts gives

$$Q_i = z\,|\psi_e|^2\,|\psi_i|^{-2},$$
$$V_i = -\epsilon^{-2}\Delta^{-1}\left(|\psi_i|^2 - |\psi_e|^2\right) + \alpha\hbar \arg(\psi_i) + \nabla_x^{-1}\left[Q_i\hbar\nabla_x\left(\arg\psi_i\right)\right].$$

To close the system, we need to derive the equation for ψ_e. To this end, we take $(\rho, U) := (n_e, u_e)$, $Q := Q_e = z$, and

$$V := V_e = \frac{1}{m}\left[\phi + ln\,|\psi_e|^2\right] + z\hbar\arg(\psi_e),$$

such that $-\nabla V_e = -(1/m)\nabla_x[\phi + ln\,n_e] - zu_e$.

From the above analysis, we see that the scaled Euler-Poisson system (2.1)–(2.4) with (2.5) can be formally realized as a semiclassical limit of the following coupled Schrödinger Poisson system for $\Psi = (\psi_i, \psi_e)$:

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2}\Delta\Psi + \left[V + \frac{i\hbar}{2}Q\right]\Psi, \qquad (3.2)$$

with $V = (V_i, V_e)^\top$ and $Q = (Q_i, Q_e)^\top$ defined above.

Passing to the limit $m \to 0$ in the second equation of (3.2), one has

$$\phi + ln\,|\psi_e|^2 = 0, \qquad \text{i.e.,} \quad |\psi_e|^2 = e^{-\phi}.$$

The limiting Schrödinger-Poisson equation for $\psi := \psi_i$ becomes

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi + \left[\alpha\hbar\arg(\psi) - \phi + \nabla_x^{-1}\left(z\hbar e^{-\phi}\,|\psi|^{-2}\,\nabla_x\left(\arg(\psi)\right)\right)\right]\psi, \qquad (3.3)$$

$$\epsilon^2\Delta\phi = |\psi|^2 - e^{-\phi}, \qquad (3.4)$$

which is the desired quantum description of the Euler-Poisson equation (1.1)–(1.3) for $f(u) = \alpha u$.

4. BOUNDARY LAYERS AND TRANSITION LAYER

4.1. The Initial Boundary Value Problem

We wish to show the initial boundary value problem for (1.1)–(1.3) with initial conditions

$$n = 1, \qquad u = 1, \qquad \phi = 0, \qquad -\infty < x < 0, \qquad t = 0, \qquad (4.1)$$

and boundary conditions

$$u = u_w(t, \epsilon), \quad \phi = \phi_w(t, \epsilon), \qquad \text{at } x = x_w(t, \epsilon), \quad t > 0. \qquad (4.2)$$

Consistency requires that $\phi_w(0, \epsilon) = 0$ and $u_w(0, \epsilon) = 1$.

In our formulation, there will be two fundamental length scales:

$$\xi = \frac{x}{\epsilon} + \epsilon^{-1/5}b(\tau), \qquad \text{the sheath scale (inner),}$$
$$y = \frac{x}{\epsilon^{4/5}} + b(\tau), \qquad \text{the intermediate scale (outer),}$$

and the time scale

$$\tau = \frac{t}{\epsilon^{2/5}}.$$

For convenience, set $\delta = \epsilon^{2/5}$. Let us rewrite the Euler-Poisson system (1.1)–(1.3) at the intermediate scale by introducing the asymptotic expansions

$$u = 1 + \sum_{i=1}^{\infty} \delta^i u_i,$$

$$n = 1 + \sum_{i=1}^{\infty} \delta^i n_i,$$

$$\phi = \sum_{i=1}^{\infty} \delta^i \phi_i.$$

Substitution into the conservation of mass equation (1.1) with $D_\tau := \partial_y + b'(\tau)\partial_y$ gives

$$\delta^{-1} D_\tau \left[\delta n_1 + \delta^2 n_2 \right] + \delta^{-2} \partial_y \left[\delta(n_1 + u_1) + \delta^2(n_1 u_1 + n_2 + u_2) \right] = z(1 - \delta\phi_1) + O\left(\delta^2\right),$$

and $O(\delta^{-1})$ terms are balanced if we take $n_1 = -u_1 + c(\tau)$, where $c(\tau)$ denotes an arbitrary function of τ, so that

$$n = 1 - \delta \left(u_1 - c(\tau) \right) + \delta^2 n_2 + \cdots, \tag{4.3}$$

and $O(1)$ balance yields

$$D_\tau n_1 + \partial_y(n_1 u_1) = -\partial_y(n_2 + u_2) + z. \tag{4.4}$$

Substitution of the expansion into the momentum equation (1.2) yields

$$\delta^{-1} D_\tau \left[\delta u_1 + \delta^2 u_2 \right] + \delta^{-2} \left(1 + \delta u_1 + \delta^2 u_2 \right) \partial_y \left(\delta u_1 + \delta^2 u_2 \right)$$
$$= \delta^{-1} \partial_y \left[\phi_1 + \delta\phi_2 \right] - f(1) - f'(1)(\delta u_1) - z \frac{(1 + \delta u_1 + \cdots)(1 - \delta\phi_1)}{1 + \delta n_1 + \delta_2 n_2 + \cdots} + O\left(\delta^2\right),$$

where Taylor's expansion for $f(1 + \delta u_1 + \cdots) = f(1) + f'(1)(\delta u_1 + \cdots)$ has been used. Balance at levels of $O(\delta^{-1})$ and $O(1)$ gives

$$\begin{aligned} O\left(\delta^{-1}\right): & \qquad \partial_y(u_1 - \phi_1) = 0, \\ O(1): & \qquad D_\tau u_1 + u_1 \partial_y u_1 + \partial_y u_2 = \partial_y \phi_2 - f(1) - z, \end{aligned} \tag{4.5}$$

respectively. Also, Poisson's equation (1.3) yields

$$\epsilon^2 \delta^{-4} \partial_y^2 \left[\delta\phi_1 + \delta^2\phi_2 \right] = \delta(\phi_1 + n_1) + \delta^2 \left(n_2 + \phi_2 - \frac{\phi_1^2}{2} \right) + O\left(\delta^3\right).$$

Since for $\delta = \epsilon^{2/5}$, $\epsilon^2\delta^{-4} = \delta$, the relations from balance of terms $O(\delta)$ and $O(\delta^2)$ give

$$O(\delta): \qquad n_1 + \phi_1 = 0, \tag{4.6}$$

$$O(\delta^2): \qquad \partial_y^2 \phi_1 = n_2 + \phi_2 - \frac{\phi_1^2}{2}. \tag{4.7}$$

Substitution of (4.6),(4.7) into (4.5) gives

$$D_\tau \phi_1 + c'(\tau) + c(\tau)\partial_y \phi_1 + \phi_1 \partial_y \phi_1 = \partial_y(\phi_2 - u_2) - f(1) - z. \tag{4.8}$$

Note that (4.4) with $n_1 = -\phi_1$ and $u_1 = -n_1 + c(\tau)$ leads to

$$\partial_y(n_2 + u_2) = D_\tau \phi_1 + \partial_y \phi_1^2 + c(\tau)\partial_y \phi_1,$$

and (4.9) gives

$$\partial_y \left(n_2 + \phi_2 \right) = \partial_y^3 \phi_1 + \phi_1 \partial_y \phi_1.$$

Both, when inserted into (4.8a), yield the following equation

$$2 D_\tau \phi_1 + \partial_y \phi_1^2 + 2c(\tau)\partial_y \phi_1 - \partial_y^3 \phi_1 = -f(1) - z - c'(\tau).$$

In order to determine $c(\tau)$, we need to impose an additional condition. Assume that the momentum nu approaches the state $nu = 1$ as $y \to -\infty$ for all τ. Thus, one has $(1 + \delta u_1 + \cdots)(1 + \delta u_1 + \cdots) \to 1$ as $y \to -\infty$ and $n_1 + u_1 \to 0$ as $y \to -\infty$. This combined with the fact that $n_1 + u_1 = c(\tau)$ implies $c(\tau) \equiv 0$ and so the scaled potential ϕ_1 is governed by the "KdV $= -f(1) - z$" equation

$$2 D_\tau \phi_1 + \partial_y \phi_1^2 - \partial_y^3 \phi_1 = -f(1) - z. \tag{4.9}$$

4.2. Sheath Inner Layer Solution

Set $\xi = x/\epsilon + b(\tau)/\epsilon^{1/5}$ and $\tau = t/\epsilon^{2/5}$, we then have

$$\epsilon^{3/5} \left[\partial_\tau + \epsilon^{-1/5} b'(\tau)\partial_\xi \right] n + \partial_\xi (nu) = \epsilon z e^{-\phi}, \tag{4.10}$$

$$\epsilon^{3/5} \left[\partial_\tau + \epsilon^{-1/5} b'(\tau)\partial_\xi \right] u + u\partial_\xi u = \partial_\xi \phi - \epsilon f(u) - \epsilon z u n^{-1} e^{-\phi}, \tag{4.11}$$

$$\partial_\xi^2 \phi = n - e^{-\phi}. \tag{4.12}$$

Thus, to leading order in ϵ, the solution is a profile satisfying

$$\partial_\xi (nu) = 0, \tag{4.13}$$

$$u\partial_\xi u = \partial_\xi \phi, \tag{4.14}$$

$$\partial_\xi^2 \phi = n - e^{-\phi}. \tag{4.15}$$

The original expansions for the transition layer give

$$\lim_{\substack{\epsilon \to 0 \\ y \to 0}} u(y, \tau) = 1 + \epsilon^{2/5} u_1(y, \tau) + \cdots = 1,$$

$$\lim_{\substack{\epsilon \to 0 \\ y \to 0}} n(y, \tau) = 1 + \epsilon^{2/5} n_1(y, \tau) + \cdots = 1,$$

$$\lim_{\substack{\epsilon \to 0 \\ y \to 0}} \phi(y, \tau) = \epsilon^{2/5} \phi_1(y, \tau) + \cdots = 0,$$

and hence, the matching condition for the steady inner sheath solution is

$$\lim_{\xi \to -\infty} u(\xi, \tau) = \lim_{\xi \to -\infty} n(\xi, \tau) = 1, \qquad \lim_{\xi \to -\infty} \phi(\xi, \tau) = 0. \tag{4.16}$$

Thus, integration of (4.11) subject to the above boundary conditions yields

$$u^2 = 1 + 2\phi. \tag{4.17}$$

Substitution of (4.17) into the Poisson equation (4.15) gives the classical sheath equation

$$\partial_\xi^2 \phi = (1 + 2\phi)^{-1/2} - e^{-\phi}, \tag{4.18}$$

where $nu = 1$ derived from (4.13) has been used. Its energy integral is

$$\frac{1}{2} \left(\partial_\xi \phi \right)^2 = \sqrt{1 + 2\phi} + e^{-\phi} - 2, \tag{4.19}$$

where again (4.16) is used. Consistency of the wall boundary implies that we must restrict ourselves to the case when u_w, ϕ_w satisfy (4.17)–(4.19).

5. JUSTIFICATION OF THE KdV DYNAMICS

We are now in a position to modify the KdV $= -f(1) - z$ model to match the dynamics hidden in the inner layer solution.

Recall that the scaled quantities

$$y = \epsilon^{-4/5}x + b(\tau), \qquad \xi = \epsilon^{-1}x + \epsilon^{-1/5}b(\tau)$$

give

$$y = \epsilon^{1/5}\xi.$$

If we write equation (4.9) in the independent and dependent variables ξ and $\phi = \epsilon^{2/5}\phi_1$, we obtain

$$2\epsilon^{3/5}D_\tau\phi = \partial_\xi\left[\partial_\xi^2\phi - \phi^2\right] - \epsilon(f(1) + z).$$

Note that the steady inner solution reads

$$\partial_\xi^2\phi - \phi^2 = F(\phi),$$

where

$$F(\phi) := (1 + 2\phi)^{-1/2} - e^{-\phi} - \phi^2 \sim O\left(\phi^3\right), \qquad \text{for } |\phi| < \frac{1}{2}.$$

We thus introduce the model

$$2\epsilon^{3/5}D_\tau\phi = \partial_\xi\left[\partial_\xi^2\phi - \phi^2 - F(\phi)\right] - \epsilon(f(1) + z).$$

We now change back to the variables y and ϕ_1 to obtain

$$2D_\tau\phi_1 = \partial_y^3\phi_1 - 2\phi_1\partial_y\phi_1 - \epsilon^{-2/5}F'\left(\epsilon^{2/5}\phi_1\right)\partial_y\phi_1 - (f(1) + z),$$

which is a perturbed KdV $= -f(1) - z$ equation. In order to normalize the above equation, we introduce

$$\psi = \frac{\sqrt[3]{2}}{6}\left(\phi_1 + \frac{f(1) + z}{2}\tau\right),$$

$$\eta = \sqrt[3]{2}\left(y + \frac{f(1) + z}{4}\tau^2\right).$$

Then consider $\phi_1 = (6/\sqrt[3]{2})\psi - (f(1) + z/2)\tau$:

$$2D_\tau\phi_1 = \frac{12}{\sqrt[3]{2}}\left[D_\tau\psi + \frac{\sqrt[3]{2}}{2}(f(1) + z)\tau\partial_\eta\psi\right] - (f(1) + z),$$

$$\left\{\partial_y, \partial_y^2, \partial_y^3\right\}\phi_1 = \frac{6}{\sqrt[3]{2}}\left\{\sqrt[3]{2}\partial_\eta, \sqrt[3]{4}\partial_\eta^2, 2\partial_\eta^3\right\}\psi,$$

and so

$$D_\tau\psi + [6\psi + g(\epsilon, \psi, \tau)]\partial_\eta\psi - \partial_\eta^3\psi = 0, \tag{5.1}$$

where

$$g := 3\epsilon^{-2/5}G\left(\epsilon^{2/5}\left(\frac{6}{\sqrt[3]{2}}\psi - \frac{f(1) + z}{2}\tau\right)\right),$$

with G determined by

$$G(\phi) := F'(\phi) = e^{-\phi} - 2\phi - (1 + 2\phi)^{-3/2}. \tag{5.2}$$

As before, we take $\phi = \phi_w(\epsilon^{2/5}\tau, \epsilon)$ at the wall $y = 0$ and from the intermediate sheath layer, we see $\partial_\xi^2\phi = (1 + 2\phi_w)^{-1/2} - e^{-\phi_w}$ at $y = 0$ as well.

6. FROM PERTURBED KdV TO KdV

As is well known, KdV equation of the form

$$\partial_\tau q + \partial_\eta q + 6q\partial_\eta q - \partial_\eta^3 q = 0, \tag{6.1}$$

with periodic or decaying data on $(-\infty, \infty)$, is a completely integrable system and the solution of its corresponding initial value problem can be explicitly solved via the celebrated inverse scattering approach [12]. It is also believed that any perturbation imposed on the original KdV would easily render the failure of approach due to the loss of the integrability. In this section, we wish to bridge between our proposed model (5.1) and the exact KdV equation (6.1).

Clearly, 'KdV $= -f(1) - z$' equation is fundamentally different from the exact equation because the presence of the dissipation imposed by the damping and ionization. For the damping and ionization free case $f(1) = 0$, $z = 0$, we will show the perturbed equation can be linked to the exact KdV equation by a nontrivial transformation, see [13,14].

First, we replace the perturbed equation (5.1) by keeping only the leading perturbation term in $g(\epsilon, \psi)$. It follows from (5.2) that

$$G(\phi) = \left[1 - \phi + \frac{\phi^2}{2} + \cdots\right] - 2\phi - \left[1 - 3\phi + \frac{15}{2}\phi^2 + \cdots\right] = -7\phi^2 + O\left(\phi^3\right), \qquad \text{as } |\phi| \to 0,$$

which upon substitution of its leading term into (5.1) gives a simplified perturbed equation

$$\partial_\tau \psi + b'(\tau)\partial_\eta \psi + \left(6\psi - \frac{1176}{\sqrt[3]{4}}\epsilon^{2/5}\psi^2\right)\partial_\eta \psi - \partial_\eta^3 \psi = 0. \tag{6.2}$$

Let ψ be the solution of the equation (6.2), and introduce a transformation

$$q := -\psi - \frac{14}{\sqrt[3]{2}}\epsilon^{1/5}\partial_\eta \psi + \frac{196}{\sqrt[3]{4}}\epsilon^{2/5}\psi^2. \tag{6.3}$$

A simple calculation gives

$$- \left[\partial_\tau q + b'(\tau)\partial_\eta q + 6q\partial_\eta q - \partial_\eta^3 q\right]$$
$$= \left(1 + \frac{14}{\sqrt[3]{2}}\partial_\eta + \frac{392}{\sqrt[3]{4}}\epsilon^{2/5}\psi\right)\left[\partial_\tau \psi + b'(\tau)\partial_\eta \psi + \left(6\psi - \frac{1176}{\sqrt[3]{4}}\epsilon^{2/5}\psi^2\right)\partial_\eta \psi - \partial_\eta^3 \psi\right],$$

from which we can conclude that the new unknown q satisfies a KdV-type equation

$$\partial_\tau q + b'(\tau)\partial_\eta q + 6q\partial_\eta q - \partial_\eta^3 q = 0, \tag{6.4}$$

if ψ satisfies the perturbed equation (6.2). Note that given q, ψ is not uniquely determined from the transformation (6.3). Nevertheless, such transformation does lead us from perturbed KdV equation to the exact KdV equation with possibly time dependent linear convection (6.4). But now, we are in a extremely interesting situation. If $b'(\tau) = 1$, i.e., $b(\tau) = \tau$, then the initial boundary value problem for (6.1) is exactly the equation considered by Fokas [1] in the study of the boundary value problem for (6.1) *on the right half-line*. But the case here is subtly different. In Fokas's problem, waves move away from the boundary $\eta = 0$ while in our case on the negative half-line waves move into the wall. In fact, this can be seen as the source of the sheath formation. It thus seems very interesting to know if a result such as Fokas's can be obtained for the negative half-line problem.

7. CONCLUSIONS

The intent of this investigation is to formulate a unified model to describe the dynamics hidden in the plasma-sheath transition layer and inner layer for weakly ionized plasma. The main observation in this work is that above-mentioned dynamics is governed by a KdV equation, which reflects the dispersive mechanism hidden in the physical process.

The solution methodology is to use asymptotic methods to simplify the governing equations. The asymptotic expansions take advantage of the many different length and time scales in the problem, and the varying magnitudes of material parameters. In particular, the discrepancy in length scales allows us to isolate the sheath transition region from both presheath region and the inner sheath region. Again, this discrepancy allows us to combine the sheath transition and the inner layer into one model equation—a modified KdV equation.

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