Wave Breaking in a Class of Nonlocal Dispersive Wave Equations

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Abstract

The Korteweg de Vries (KdV) equation is well known as an approximation model for small amplitude and long waves in different physical contexts, but wave breaking phenomena related to short wavelengths are not captured in. In this work we consider a class of nonlocal dispersive wave equations which also incorporate physics of short wavelength scales. The model is identified by a renormalization of an infinite dispersive differential operator, followed by further specifications in terms of conservation laws associated with the underlying equation. Several well-known models are thus rediscovered. Wave breaking criteria are obtained for several models including the Burgers-Poisson system, the Camassa-Holm type equation and an Euler-Poisson system. The wave breaking criteria for these models are shown to depend only on the negativity of the initial velocity slope relative to other global quantities.

1   Introduction

In water wave theory, one usually takes asymptotic expansion in small parameters around a canonical wave governed by the incompressible Euler equation. As is known in general the Euler equation represents the long wavelength limit, but it often happens that the dynamics predicted by the Euler equation involves relatively short-wavelength scales. Many competing models have been suggested to capture one aspect or another of the classical water-wave problem, see e.g. [7, 8, 28, 37, 58].

Here we will consider a class of nonlocal dispersive equations of the form

$$u_t + uu_x + [Q * (B(u, u_x))]_x = 0, \tag{1.1}$$

where u denotes the wave motion in x direction, Q is a kernel generated by the inverse of a differential operator L:

$$L^{-1}f(x) = Q * f = \int_{\mathbb{R}} Q(x - y)f(y)dy$$

and B is quadratic in its arguments. A typical operator is $L = (I - \partial_x^2)$ with associated peakon kernel $Q = e^{-|x|}/2$. Such a kernel is identified using certain rational approximations to the dispersive operator and the function B is to be classified in terms of conservation laws. This way the physics of both short and long wavelengths may be captured though a specific physical example is not used here as a modelling vehicle.

In the context of water waves, one of the best known local models is probably the Korteweg de Vries (KdV)-equation

$$u_t + uu_x + \gamma^2 u_{xxx} = 0, \quad \gamma = Const, \tag{1.2}$$

where u denotes the fluid velocity. This equation possess soliton solutions — coherent structures that interact nonlinearly among themselves then reemerge, retaining their identity and showing particle-like scattering behavior. A KdV equation of this type was also found in [48] when authors were trying to model the dynamics hidden in the plasma sheath transition, where the full dynamics is governed by a normalized Euler-Poisson system of the form

$$n_t + (nu)_x = 0, \tag{1.3}$$
$$u_t + uu_x = \phi_x, \tag{1.4}$$
$$\epsilon^2 \phi_{xx} = n - e^{-\phi}, \tag{1.5}$$

where u represents ion velocity, n the ion density and ϕ electrical potential. The parameter $\epsilon > 0$ stands for a scaled Debey length (a characteristic length scale in plasma). The linearized equation of (1.3)-(1.5) takes the form

$$(I - \partial_x^2)\phi_{tt} - \phi_{xx} = 0,$$

from which one easily finds the dispersive relation

$$\omega(k) = k(1 + k^2)^{-1/2}.$$

This relation near $k = 0$ shares the similar shape to that obtained for the KdV equation (1.2), but these two dispersive relations are very different for large k. As we have seen in shallow water wave theory, the nonlinear shallow water equations which neglect dispersion altogether lead to the finite time wave breaking. On the other hand the third derivative term in the KdV equation will prevent this ever happening in its solutions. But in both cases, the long wave assumption under which the equations were derived is no longer valid. Since some waves appear to break in this way, if the depth is small enough. Therefore in [58] Whitham raised an intriguing question: what kind of mathematical equation can describe waves with both breaking and peaking? The improved model proposed by Whitham is in the form (1.1) with $B = -u$ and

$$Q(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \sqrt{\frac{tanh(k)}{k}} e^{ikx} dk,$$

yet the analysis of this model has not been done completely.

In this paper we discuss a class of nonlocal dispersive models (1.1). By applying a renormalization procedure to an infinite dispersive differential operator we identify a

typical kernel Q in the first place, then we restrict to a special class that B is quadratic in its arguments. Several well-known models are rediscovered when further identification of B is made in terms of conservation laws. We should point out that our arguments are mainly motivated by mathematical structures of the model, it is certainly possible to justify the model in other ways, say from Euler equations with free surface or from the Boussinesq system, e.g. [37].

The wave breaking criterion is established for several model equations of the form (1.1) and the system (1.3)-(1.5). The wave breaking criteria found for these models depend only on the relative size of the initial velocity slope and other conserved quantities. Our approach is to trace the dynamics of the solution gradient along the characteristics. Tracking dynamics along characteristics or particle path proves to be a powerful technique in the study of singularity formation for hyperbolic equations, see e.g. [36, 41, 43], as well as the recent study of critical threshold phenomena in Eulerian flows [26, 46, 47]. However, adaption of this elementary method to nonlocal models as such considered in this paper is rather subtle. Special features of each model have to be incorporated to yield a corresponding blow-up criterion.

The nonlocal equation (1.1) can also be written as a system

$$u_t + uu_x = \phi_x, \tag{1.6}$$
$$-L(\phi) = B(u, u_x), \tag{1.7}$$

or a higher order local equation

$$L(u_t + uu_x) + [B(u, u_x)]_x = 0,$$

which is often seen in the description of water wave models. We will take one of these three forms whenever it is more convenient for our analysis.

There are some distinguished special cases of this equation with the peakon kernel $Q = (I - \partial_x^2)^{-1}$:

- A shallow water model proposed by Whitham [58]

$$u_t + \frac{3c_0}{2h_0}uu_x + [K * u]_x = 0, \quad K = \frac{\pi}{4}\exp(-\pi|x|/2)$$

  which modulo a proper scaling is just (1.1) with $B = u$ and $L = (I - \partial_x^2)$;

- The Camassa-Holm equation [7] as a shallow water wave model,

$$u_t - u_{txx} + 3uu_x + 2ku_x = 2u_xu_{xx} + u_{xxx},$$

  corresponding to (1.1) with $B = 2ku + u^2 + \frac{1}{2}u_x^2$ and $L = (I - \partial_x^2)$;

- The Dai model for small deformation waves in thin compressible elastic rods [25]

$$u_t - u_{txx} + 3uu_x = \gamma(2u_xu_{xx} + u_{xxx}), \quad \gamma \in \mathbb{R}.$$

This model can be rewritten as

$$u_t + \gamma uu_x + (I - \partial_x^2)^{-1}\left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)_x = 0,$$

which corresponds to (1.1) with $B = \frac{3-\gamma}{2\gamma}u^2 + \frac{1}{2}u_x^2$, modulo a variable scaling $u \to \gamma u$.

- Rosenau's regularization model [52]

$$u_t + uu_x = Q * u - u, \quad Q = \frac{1}{2}\exp(-|x|),$$

which can be rewritten as (1.1) with $B = -u_x$ and $L = (I - \partial_x^2)$. This model also occurs in the one-dimensional radiating gas motion, see e.g. [31].

A more recent model is referred to as the Degaspevis-Procesi equation in [32], which corresponds to $B = 1.5u^2$ or more general $B = \frac{b}{2}u^2 + \frac{3-b}{2}u_x^2$. Other non-local models of form (1.1) with a different kernel also exist, but they are not relevant to the procedure we follow in §2; see, for example with $Q = (-\partial_x^2)^{-1}$ and $B = B(u, u_x)$, the short wave equation [33],$(u_t + uu_x)_x = u$, corresponding to (1.1) with $B = u$; the Vakhnenko equation [51, 57], $(u_t + uu_x)_x + u = 0$, which governs the propagation of waves in a relaxing medium; the Hunter-Saxton-Zheng equation, $(u_t + uu_x)_x = 0.5u_x^2$, a model for nematic liquid crystals [34, 35]. Note also that if $Q = H$ is a Hilbert operator, then the Benjamin-Ono equation [6] is found as (1.1) with $B = u_x$. Among others, both the Camassa-Holm equation and the Benjamin-Ono equation have been shown to be completely integrable equations. The intensive investigation of these two equations have produced so many elegant results, see e.g. [24, 56, 59] and references therein.

This paper is organized as follows. In §2 we formally identify a differential operator as $L = (I - \partial_x^2)^{-1}$ via using some scaling arguments and a renormalization technique. In §3 we discuss different choices of B by looking at second or more conservation laws. Through this procedure we rediscover several well known models in this class, including the Burgers-Poisson system, the Camassa-Holm equation. In §4 we discuss the local existence of the initial value problem of (1.1) subject to initial data $u_0 \in H_x^{3/2+}$ for $B = au + bu_x + cu^2 + du_x^2$. In particular we prove that the maximal existence interval is finite if and only if $\|u_x\|_{L^\infty}$ becomes unbounded at finite time. Finally §5-7 are devoted to identifying the wave breaking criterion for three different models including the Burgers-Poisson system, the Camassa-Holm equation and the Euler-Poisson system (1.3)-(1.5).

2 Renormalization

Let $u(t, x)$ denote the wave motion characterized by unidirectional propagation in the $+x$ direction, we start with the following dispersive wave equation

$$u_t + uu_x + (\tilde{L}u)_x = 0, \tag{2.1}$$

in which $\tilde{L}$ is a linear operator. This form was first suggested by Whitham [58] in order to keep intact the exact dispersion operator $\tilde{L}$ according to linearized theory, but to simulate nonlinear effects by the first-order approximation appropriate to long waves.

Appealing to Fourier's principle we may characterize the operator $\tilde{L}$ in terms of the wave numbers. Let $\check{u}(k)$ denote the Fourier transform of $u(x)$, i.e. $\check{u} = \mathcal{F}u$ and $u = \mathcal{F}^{-1}\check{u}$. If $\tilde{L}u := c(-i\partial_x)u(t, x)$, then

$$\mathcal{F}(\tilde{L}u) = c(k)\check{u}(k). \tag{2.2}$$

For most systems in question, the function $c(k)$ has a smooth maximum with nonzero curvature at $k = 0$, say $c(0) = 1$ and $c''(0) = -2\gamma^2$. Thus an approximation for sufficiently small k (i.e. sufficiently long waves) is

$$c(k) \sim 1 - \gamma^2 k^2.$$

Corresponding to which the definition of (2.2) formally gives

$$\tilde{L} = I + \gamma^2 \partial_x^2.$$

This leads to the KdV equation

$$u_t + u_x + u u_x + \gamma^2 u_{xxx} = 0.$$

Some other local models can be written down with equal justification. On the other hand since $c(k)$ is generally bounded function such that $c(k) \to 0$ as $|k| \to \infty$, $\tilde{L}$ is a smoother operator than identity, in fact obliterating small-scale features. However this equation allows no comparable moderation of nonlinear effects on such features. It is therefore to be expected that, for any initial wave form with significant short-wave components, a real solution will exist for only a short time [5]. Hence mathematical models which can incorporate both long wave and short wave effects are highly preferable.

In order to find a better model we must build the boundedness of $c(k)$ into a suitable approximation that can not be of polynomial type. To highlight the idea we make a hyperbolic scaling

$$(t, x) \to \left(\frac{t}{\epsilon}, \frac{x}{\epsilon} \right)$$

to the equation (2.1), obtaining

$$u_t + u u_x + (\tilde{L}^\epsilon u)_x = 0,$$

where $\tilde{L}^\epsilon u = c(-i\epsilon\partial_x)u$. Thus

$$\mathcal{F}(\tilde{L}^\epsilon u) = c(\epsilon k)\breve{u}(k).$$

Consider the dispersive part of the Tylor expansion of $c(\epsilon k)$

$$c(\epsilon k) \sim 1 + \sum_{j=1}^{\infty} \mu_j (-1)^j \epsilon^{2j} k^{2j}, \quad \mu_j = Const. > 0$$

Note that if we terminate the expansion at any finite order, we can only have approximation of polynomial type. To ensure the boundedness of c we make the approximate summation of the whole expansion by taking

$$c = 1 + \sum_{j=1}^{\infty} \mu_j (-1)^j (\epsilon k)^{2j} \sim \frac{1}{1 + l^2 \epsilon^2 k^2} + O(\epsilon^4), \quad l^2 = \mu_1 > 0.$$

Thus the bounded approximation can be realized by defining its rational approximation

$$c^* = \frac{1}{1 + l^2 \epsilon^2 k^2},$$

which gives the approximate operator

$$\tilde{L}^\epsilon u \sim \int_{\mathbb{R}} \frac{u(t, y)}{2\epsilon l} \exp\left(-\frac{x - y}{\epsilon l}\right) dy \equiv Q * u.$$

Making a hyperbolic scaling $(x, t) \to (\epsilon l t, \epsilon l x)$ we have

$$\tilde{L} \sim (I - \partial_x^2)^{-1}$$

with associated kernel

$$Q = \frac{1}{2} \exp(-|x|).$$

We note that the summing procedure of this nature has been explored in e.g. [52, 54, 55] in the context of eliminating truncation instability of the Chapman-Enskog expansion in Hydrodynamics. With this approximate nonlocal operator the improved version of (2.1) can be written as

$$u_t + u u_x + [Q * u]_x = 0.$$

The wave breaking phenomenon is indeed captured in this model, see §5. This model, modulo a proper scaling, was also the one considered by Whitham [58] as a model for shallow water wave motion featuring weaker dispersivity than the KdV equation. See also [49] for a study of this model.

In order to also incorporate the peaking phenomena into the model we need to include the nonlinearity into the global convolution. Hence we propose a more general class of nonlocal models by replacing u with $B(u, u_x)$, i.e.,

$$u_t + u u_x + [Q * B(u, u_x)]_x = 0,$$

where Q is a Green function of some differential operator L and B is quadratic in its arguments. The precise characterization of B will be made next section by further checking the number of conservation laws associated with the underlying model.

3   Conservation Laws

A conservation law has the form

$$P_t + J_x = 0, \tag{3.1}$$

where P is the conserved density, and $-J$ is the flux of P. Here we seek conserved quantities as functionals of $u(t, x)$, the dependent variable of the equation

$$u_t + u u_x + [Q * B(u, u_x)]_x = 0. \tag{3.2}$$

There is a close relationship between conservation laws and constants of motion. For example, if one assumes either that u is periodic in x or u and its derivatives vanish sufficiently rapidly at infinity, each conservation law yields

$$\int P dx = \int P_0 dx$$

where the integrals are taken over either the periodic domain or the infinite domain. For any given convolution flux B, the first density -flux pair is obvious

$$(P, J) = \left(u, \quad \frac{1}{2}u^2 + Q * B(u, u_x) \right),$$

which corresponds to the usual mass conservation

$$\int u(t, x) dx = \int u_0(x) dx.$$

Recall that the KdV equation (1.2) is blessed with infinite many conserved quantities; the following three are classical:

$$I_1(u) = \int u dx, \quad I_2(u) = \int \frac{1}{2}u^2 dx, \quad I_3(u) = \int \left(\frac{1}{3}u^3 - \gamma^2 u_x^2 \right) dx.$$

Motivated by these conserved quantities we proceed to seek more conservation laws for the nonlocal model (3.2). To make it precisely we restrict to a special class in which $L = (I - \partial_x^2)^{-1}$ and B is quadratic in its arguments [1]

$$B(p, q) = ap + bq + cp^2 + dq^2.$$

Thus the nonlocal model is equivalent to the following system

$$u_t + uu_x = \phi_x, \tag{3.3}$$
$$\phi_{xx} = \phi_x + au + bu_x + cu^2 + du_x^2. \tag{3.4}$$

Let the second density be $P = \frac{1}{2}u^2$. Equation (3.3) multiplied by u gives

$$\left(\frac{1}{2}u^2 \right)_t + \left(\frac{1}{3}u^3 \right)_x = u\phi_x.$$

We discuss the possible conservation or decay of $\int P dx$ with different choices of coefficients in B. Two cases are of special interest. First we consider the case $a \neq 0$, $b = c = d = 0$. We may take $B = au$ with $a = 1$. It follows from (3.4), i.e., $\phi_{xx} = \phi + u$ one has

$$u\phi_x = \phi_x(\phi_{xx} - \phi) = \frac{1}{2} \left(\phi_x^2 - \phi^2 \right)_x,$$

which gives the following conserved pair

$$(P, J) = \left(\frac{1}{2}u^2, \quad \frac{1}{3}u^3 + \frac{1}{2}\phi^2 - \frac{1}{2}\phi_x^2 \right),$$

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[1] for the stability of the model the mixed term $pq = uu_x$ is necessarily to be absent.

leading to the usual L^2 energy preservation

$$\int \frac{1}{2} u^2 dx = \int \frac{1}{2} u_0^2 dx.$$

This choice corresponds to the equation found in §2 and can be written as

$$u_t + uu_x = (\partial_x^2 - I)^{-1} u_x. \tag{3.5}$$

The corresponding system

$$u_t + uu_x = \phi_x, \quad \phi_{xx} = \phi + u,$$

also called the Burgers-Poisson system, was recently studied in [27], where local smooth existence, global weak solution as well as traveling wave solutions are discussed. For this model we shall identify a wave breaking criterion in §5.

Second, we look at the case $b < 0$ and $a = c = d = 0$. In such a case, the energy conservation is lost. Let $B = bu_x$ with $b = -1$, then from (3.4), i.e., $\phi_{xx} = \phi - u_x$ one has

$$u\phi_x = (u\phi)_x - \phi u_x = [u\phi + \phi\phi_x]_x - [\phi^2 + \phi_x^2],$$

which leads to the decay of the L^2 energy

$$\frac{d}{dt} \int \frac{u^2}{2} dx = -\|\phi\|_1^2 < 0.$$

This model, i.e., (3.3)-(3.4) with $a = c = d = 0$ and $b = -1$ appears as a 1-D model for the radiating gas motion, see [31, 39] and also Rosenau's regularization model [52, 53]. The single equation may be written as

$$u_t + uu_x = (I - \partial_x^2)^{-1} u_{xx} = Q * u - u, \quad Q = \frac{1}{2} e^{-|x|}.$$

In [46] we found a lower threshold of initial profile for the wave breaking and an upper threshold for the global smoothness, which confirms the remarkable critical threshold phenomena associated with this model.

In order to include the quadratic terms in B, we need to check other conserved quantities. Note that the equation (3.2) also takes the form

$$(I - \partial_x^2)u_t + (I - \partial_x^2)(uu_x) + [B(u, u_x)]_x = 0.$$

This when combined with $u_t + uu_x = [Q * B(u, u_x)]_x$ suggests that $u(I - \partial_x^2)u$ may serve as a potential conserved quantity. Integration by parts gives

$$\int u(u - u_{xx}) dx = \int (u^2 + u_x^2) dx.$$

We thus take the next conserved density $P = u^2 + u_x^2$. A brief calculation gives

$$P_t + \tilde{J}_x = S, \quad S := 2bu_x^2 + (2d - 1)u_x^3,$$

where the flux is

$$\tilde{J} = -\frac{2(c-1)}{3}u^3 + au^2 - uu_x^2 - 2\phi u.$$

To ensure the stability property it is necessary to take $d = 1/2$ so that the mixed term in S vanishes. With this choice the rate term S on the right becomes $2bu_x^2$, having the fixed sign $sgn(b)$. If $b = 0$, the Hamiltonian $\int Pdx$ is conserved, i.e.,

$$\int (u^2 + u_x^2)dx = Const.$$

In this case

$$B(u, u_x) = au + cu^2 + \frac{1}{2}u_x^2,$$

the corresponding system reads

$$u_t + uu_x = \phi_x \tag{3.6}$$

$$\phi_{xx} = \phi + au + cu^2 + \frac{1}{2}u_x^2, \quad a, c \in \mathbb{R}. \tag{3.7}$$

This second conserved pair is

$$(P_2, J_2) = \left(u^2 + u_x^2, -\frac{2(c-1)}{3}u^3 + au^2 - uu_x^2 - 2\phi u\right).$$

A further calculation gives the third less obvious conserved pair

$$P_3 = \frac{1+2c}{3}u^3 + au^2 + uu_x^2,$$

$$J_3 = \phi^2 + \frac{1}{4}u^4 + u^2u_x^2 - \phi u^2 - \phi_x^2.$$

Note that the so called Camassa-Holm (CH) equation corresponds to the case $a = 2\kappa$ and $c = 1$, i.e., $B = 2\kappa u + u^2 + \frac{1}{2}u_x^2$; and the evolution equation can be written as

$$u_t + uu_x + (I - \partial_x^2)^{-1}\partial_x[2\kappa u + u^2 + \frac{1}{2}u_x^2] = 0,$$

where k is a positive constant related to the critical shallow water speed, see [7]. Also in this case one recovers the constants of motion obtained in [21]. The CH equation was obtained by Fuchssteiner and Fokas [30] by the method of recursion operators, independently derived by Camassa and Holm [7] from physical principles, as an approximate model for the shallow water equation. The first asymptotically complete derivation of the CH equation was given in [29]. The alternative derivation of the CH equation as a model for shallow water waves was obtained in [37]. The CH equation is integrable, and therefore enjoys infinite number of conservative quantities, see e.g. [2, 10, 12, 8, 21, 42]. From the above analysis we record here its three local conserved quantities: the mass $I_1 = \int udx$ and the Hamiltonians

$$I_2 = \int (u^2 + u_x^2)dx, \quad I_3 = \int (u^3 + 2\kappa u^2 + uu_x^2)dx.$$

We also note two more important aspects of CH equation. The equation is a re-expression of geodesic flow on the diffeomorphism group of the circle (in the periodic case) or of the line, see e.g. [11, 40]. This geometric interpretation leads to a proof that the Least Action Principle holds also for the CH equation, see [18, 19]. The second aspect concerns the fact that the solitary waves of the CH equation are smooth if $\kappa \neq 0$ and peaked if $\kappa = 0$, see [7]. In the latter case they have to be understood as weak solutions in the sense of [20], while for $\kappa \neq 0$ they are classical solutions. In both cases they are solitons, cf. [1, 24, 38] and they are stable cf. [22].

We note that the CH equation was also found independently by Dai [25] as a model for small deformation waves in thin compressible elastic rods. Under proper scaling Dai's equation may be written as

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + u_{xxx}), \quad \gamma \in \mathbb{R}.$$

Here u measures the radial stretch relative to an equilibrium. The physical parameter γ ranges from -29.5 to 3.41. Dai's model (for $\gamma \neq 0$) can be rewritten as

$$u_t + \gamma uu_x + (I - \partial_x^2)^{-1} \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} u_x^2 \right)_x = 0.$$

This corresponds to our model (3.6)-(3.7) with $a = 0$ and $c = (3-\gamma)/2\gamma$, modulo a variable scaling $u \to \gamma u$.

For the special case $\gamma = 0$ we find the BBM equation [4], a well-known model for surface waves in a channel

$$u_t + (I - \partial_x^2)^{-1} \left(\frac{3}{2} u^2 \right)_x = 0.$$

The solutions are global, but the equation is not integrable despite it has a Hamiltonian structure.

Finally we remark on the dissipative case with $b < 0$, say b$= -1$. In this case

$$B(u, u_x) = au - u_x + cu^2 + \frac{1}{2} u_x^2$$

and the corresponding model is

$$u_t + uu_x = \phi_x \tag{3.8}$$

$$\phi_{xx} = \phi + au - u_x + cu^2 + \frac{1}{2} u_x^2, \quad a, c \in \mathbb{R}. \tag{3.9}$$

For this system the Hamiltonian is not conserved but decays according to

$$\frac{d}{dt} \int (u^2 + u_x^2) dx = -2 \int u_x^2 dx \leq 0.$$

The analysis of this model will be explored elsewhere.

4   Local Wellposedness

In order to discuss the finite time wave breaking, we must know whether solutions exist at least for short times.

In this section, we address the general question of the formation of singularities for solutions to

$$\partial_t u + u u_x = (\partial_x^2 - I)^{-1} \partial_x B(u, u_x), \tag{4.1}$$

subject to initial data $u_0 \in H^s(\Omega)$, where $\Omega = \mathbb{S}$ for periodic data or $\Omega = \mathbb{R}$ for initial data vanishing rapidly at far fields on the whole line. Here $B(u, u_x) = au + bu_x + cu^2 + du_x^2$, in which a, b, c and d are known constants. For the local wellposedness of the Camassa-Holm equation we refer to [45, 9].

Theorem 1. *Suppose that $u_0 \in H_x^{3/2+}$, then there exists a time T and a unique solution u of (4.1) in the space $C_{[0,T)} H_x^{3/2+} \cap C_{[0,T)}^1 H_x^{1/2+}$ such that $\lim_{t \downarrow 0} u(t, \cdot) = u_0(\cdot)$. If $T_0 < \infty$ is the maximal existence time, then*

$$\lim_{t \to T_0-} \sup_{0 \le \tau \le t} \| u_x(\cdot, \tau) \|_{L^\infty(\Omega)} = \infty. \tag{4.2}$$

Proof. We prove only the case $\Omega = \mathbb{R}$, the case with periodic data can be done similarly. The proof of the local existence involves a standard iteration scheme combined with a closed energy estimate, details are omitted. We now show (4.2). Set $\Lambda := (I - \partial_x^2)^{1/2}$ and use the norm

$$\| f \|_{H^q}^2 := \int_{\mathbb{R}} (1 + |\xi|^2)^q |\check{f}(\xi)|^2 d\xi < \infty.$$

We thus have

$$\| f \|_{H^q} = \| \Lambda^q f \|_{L^2}.$$

The equation (4.1) can be rewritten as

$$\Lambda^2 (u_t + u u_x) = -B(u, u_x).$$

For any $q \in (1/2, s-1]$ with $\forall s > 3/2$, applying $\Lambda^q u \Lambda^q$ to both sides of this equation and integrating with respect to x, one obtains

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2 \right) dx = \int \Lambda^q u \Lambda^q (u u_x) dx$$
$$- \int \Lambda^{q+1} u \Lambda^{q+1}(u u_x) + \int \Lambda^q u_x \Lambda^q B(u, u_x) dx,$$

where we have transformed the convection term via

$$\int \Lambda^q u \Lambda^{q+2}(u u_x) dx = 2 \int \Lambda^q u \Lambda^q (u u_x) dx + \int \Lambda^{q+1} u \Lambda^{q+1}(u u_x) dx.$$

Using the quadratic form of B we have

$$\int \Lambda^q u_x \Lambda^q B(u, u_x) dx \le C(a, b, c, d) \left(\int (\Lambda^q u)^2 + (\Lambda^q u_x)^2 dx \right.$$
$$\left. + \left| \int \Lambda^q u \Lambda^q (u u_x) dx \right| + \left| \int \Lambda^q u_x \Lambda^q (u_x^2) dx \right| \right).$$

We recall the following two estimates [45]

$$\left| \int \Lambda^q u \Lambda^q (u u_x) dx \right| \le c_q \|u_x\|_\infty \|u\|_{H^q}^2,$$
$$\left| \int \Lambda^q u \Lambda^q (u^2) dx \right| \le c_q \|u\|_\infty \|u\|_{H^q}^2$$

for any $q \ge 0$ and $u \in H_x^q \cap \{\|u_x\|_\infty < \infty\}$. A combination of the above facts leads to

$$\|u\|_{H^r}^2 \le \|u_0\|_{H^r}^2 + C \int_0^t (\|u_x\|_\infty + 1) \|u(\cdot, \tau)\|_{H^r} d\tau, \quad r \in (3/2, s].$$

It follows from the Gronwall's inequality that

$$\|u\|_{H^r}^2 \le \|u_0\|_{H^r}^2 \exp \left(C \int_0^t (\|u_x(\cdot, \tau)\|_\infty + 1) d\tau \right).$$

Therefore, if $\lim_{t \to T_0} \sup \|u_x\|_\infty < \infty$, it would lead to the boundedness of $\|u\|_{H^r}$. One may therefore extend the solution to the space $C_{[0, T_1)} H_x^r$ for some $T_1 > T_0$, which contradicts the assumption that $T_0 < \infty$ is a maximal existence interval. ∎

In subsequent sections we shall identify the wave breaking criterion for several equations of current interest. We do so by mainly checking the possible blow-up of the velocity slope u_x.

5   Wave breaking in the Burgers-Poisson system

In this section we analyze spatially periodic solutions of

$$u_t + u u_x = \phi_x, \tag{5.1}$$
$$\phi_{xx} = \phi + u, \quad u(0, x + 1) = u(0, x). \tag{5.2}$$

If the second equation is replaced by $\phi_{xx} = u$, the above system becomes

$$\partial_x (u_t + u u_x) = u,$$

which is the so-called short wave equation investigated by Hunter [33]. The system (5.1)-(5.2) when written into one equation reads

$$u_t + u u_x + (I - \partial_x^2)^{-1} \partial_x u = 0.$$

The operator $(I - \partial_x^2)^{-1}$ acting on $L^2(\mathbb{S})$ has the following representation

$$[(I - \partial_x^2)^{-1}f](x) = \int_{\mathbb{S}} Q(x - y)f(y)dy, \quad f \in L^2(\mathbb{S})$$

with the Green's function

$$Q(x) = \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh\left(\frac{1}{2}\right)}, \quad x \in \mathbb{S}.$$

As shown in §3 the system (5.1)-(5.2) yields the following conservation laws

$$u_t + (u^2/2 + Q * u)_x = 0$$

and

$$(u^2)_t + \left(\frac{2}{3}u^3 + \phi^2 - \phi_x^2\right)_x = 0. \tag{5.3}$$

Even so, solutions of the system needs not remain smooth for all time $t > 0$. The nonlocal source $-[Q * u]_x$ cannot prevent the nonlinear "breaking" of smooth solutions when their slope is sufficiently large.

Theorem 2. *Consider the system (5.1), (5.2) subject to smooth periodic initial data $u_0(x)$ satisfying $u_0(x + 1) = u_0(x)$. Set $M = \|u_0\|_\infty$ and $m = -\inf \partial_x u_0(x)$. If*

$$\frac{m^3}{4 + m(1 + \sup|Q|)} > 4M, \tag{5.4}$$

then smooth solutions of the system (5.1), (5.2) breaks down in finite time before $t^* = 2/m$.

Remark. The bound $\sup|Q|$ is given precisely by

$$\sup|Q| = \frac{\cosh(0.5)}{2\sinh(0.5)}.$$

The condition (5.4) with $\sup|Q|$ replaced by 0 becomes the breaking condition

$$\frac{m^3}{4 + m} > 4M,$$

which coincides with the condition obtained by Hunter [33] for the short wave equation of the form

$$u_t + uu_x = \int_0^x u(t, y)dy - \int_0^1 \int_0^x u(t, y)dydx.$$

Proof of Theorem 2. From (5.3) it follows that any smooth periodic solution satisfies the conservation of energy

$$\int_0^1 u^2(t, x)dx = \int_0^1 u_0^2(x)dx \leq M^2.$$

Set $v = \int_0^x u(t,y)dy$, then

$$|v(t,x)| \le \int_0^1 |u(t,x)|dx \le \left[\int_0^1 u(t,x)^2 dx\right]^{1/2} \le M.$$

Thus the source term can be written as

$$-Q * u_x = -(I - \partial_x^2)^{-1}v_{xx} = v - Q * v,$$

which implies that

$$|Q * u_x| = |v - Q * v| \le |v| + |Q|_1|v| \le 2M, \quad |Q|_1 = 1.$$

Now we rewrite the system (5.1)-(5.2) in terms of characteristic coordinates $(t, x(t, \alpha))$. This gives

$$x = x(t, \alpha), \quad u(t, x) = U(t, \alpha), \quad -Q * u_x = G(t, \alpha),$$

where

$$\dot{x} = U, \quad x(0, \alpha) = \alpha, \tag{5.5}$$
$$\dot{U} = G, \quad U(0, \alpha) = u_0(\alpha). \tag{5.6}$$

The implicit integration leads to

$$x(t, \alpha) = \alpha + \int_0^t U(s, \alpha)ds, \quad U(t, \alpha) = u_0(\alpha) + \int_0^t G(s, \alpha)ds.$$

The solution remains smooth if and only of $x_\alpha \ne 0$. Thus to prove the smooth solution break down, we shall show x_α vanishes. Differentiation of x with respect to α and using U_α shows that

$$x_\alpha = 1 + u_0'(\alpha)t + \int_0^t \int_0^s G_\alpha(\tau, \alpha)d\tau ds.$$

Note that

$$G_\alpha = x_\alpha G_x = x_\alpha(-Q * u_{xx}) = x_\alpha(U - Q * u|_{(t,x(t,\alpha))}).$$

We choose $x^* \in (0, 1)$ such that

$$u_0'(x^*) = \inf u_0'(\alpha) = -m.$$

Using the fact that $|U| \le M(1 + 2t)$ and $|Q * u| \le \sup|Q||u|_1 \le M \sup|Q|$ we obtain

$$|G_\alpha(t, x^*)| \le \left(1 + \frac{t^2}{2} \sup_{0 \le s \le t} |G_\alpha(s, x^*)|\right)(1 + \sup|Q| + 2t)M.$$

Set $b(t) := M(1 + \sup|Q| + 2t)t^2/2$. Using (5.4) we find that

$$b(t) \le b(2/m) = M(1 + \sup|Q| + 4/m)\frac{2}{m^2} < 1/2, \quad 0 \le t \le 2/m.$$

This gives

$$q(t) := \sup_{0 \le s \le t} |G_\alpha(s, x^*)| t^2/2 \le \frac{b(t)}{1 - b(t)} < 1, \qquad 0 \le t \le 2/m.$$

At $(t, x) = (\frac{2}{m}, x^*)$ we have

$$x_\alpha(\frac{2}{m}, x^*) \le 1 - m \cdot \frac{2}{m} + q(\frac{2}{m}) = q(\frac{2}{m}) - 1 < 0.$$

Since $x_\alpha = 1$ when $t = 0$, $x_\alpha(t, x^*)$ must vanish at some time $t^* \in (0, 2/m)$. Thus the velocity slope $u_x = U_\alpha/x_\alpha$, $u_x(t, x(t, x^*))$ must approach $-\infty$ as $t \uparrow t^* < 2/m$ if $U_\alpha(t^*, x^*) \ne 0$. In fact

$$U_\alpha(t^*, x^*) \le -m + \frac{2}{t^*} q(t^*) < 0.$$

For this to be true, we need to show $q(t^*) < \frac{mt^*}{2}$. Using the definition for $b(t)$, $t^* \le \frac{2}{m}$ and the breaking condition (5.4) we have

$$\begin{aligned}
b(t^*) &= M(1 + \sup|Q| + 2t^*)(t^*)^2/2 \\
&= \frac{mt^*}{2 + mt^*} \cdot \frac{M(2 + mt^*)(1 + \sup|Q| + 2t^*)t^*}{2m} \\
&\le \frac{mt^*}{2 + mt^*} \cdot \frac{4M(4 + (1 + \sup|Q|)m)}{m^3} \\
&< \frac{mt^*}{2 + mt^*}.
\end{aligned}$$

This bound leads to the desired bound

$$q(t^*) \le \frac{b(t^*)}{1 - b(t^*)} < \frac{mt^*}{2}.$$

The proof is now complete. ∎

6 Wave breaking in the Camassa-Holm-type equations

This section is devoted to analyzing spatially periodic solutions to the model

$$u_t + uu_x = \phi_x, \tag{6.1}$$

$$\phi_{xx} = \phi + au + cu^2 + \frac{1}{2}u_x^2. \tag{6.2}$$

Taking $a = 0$ and $c = 1$ in system (6.1)-(6.2) we find the well studied Camassa-Holm equation, for which the global well-posedness was proved for initial data $u_0 \in H^{3/2+}$ provided that $u_0 \in L_x^1$ and $(I - \partial_x^2)u_0$ does not change sign [13]. On the other hand, wave breaking at finite time does occur for some initial data, see e.g. [13, 14]. The Camassa-Holm equation is remarkable since it combines complete integrability with the wave breaking phenomena.

We shall investigate the wave breaking for the spatially periodic solutions to the model (6.1), (6.2) and identify a breaking criterion for it. The corresponding criteria for the Camassa-Holm equation and Dai's model follow immediately.

Let $u(t, x)$ be a smooth solution to (6.1) (6.2) for $0 \leq t < T$. By differentiating (6.1) and using (6.2), we have for $d := u_x$

$$d_t + ud_x + d^2 = au + cu^2 + \frac{d^2}{2} - Q * (au + cu^2 + \frac{d^2}{2}), \quad t \in (0, T).$$

The smoothness of u ensures that there exists a smooth characteristic curve $x = x(t, \alpha)$ satisfying

$$\frac{d}{dt}x(t, \alpha) = u(t, x(t, \alpha)), \quad x(0, \alpha) = \alpha, \quad \alpha \in \mathbb{S}.$$

Therefore the dynamics of d is governed by

$$\dot{x} = u, \quad \cdot := \partial_t + u\partial_x, \tag{6.3}$$

$$\dot{d} + \frac{d^2}{2} = G(t, x), \tag{6.4}$$

where

$$G(t, x) = au + cu^2 - Q * (au + cu^2 + u_x^2/2)\Big|_{(t, x(t, \alpha))}.$$

A priori bound of G would suffice for us to establish a breaking criterion. Recall that one important feature of the system (6.1), (6.2) is the conserved H^1 energy, i.e.,

$$\int (u^2 + u_x^2)dx = \int (u_0^2 + u_{0,x}^2)dx =: \|u_0\|_1^2$$

holds for all time before the solution breaks down. Armed with this conserved quantity we are able to obtain an a priori bound of G.

Lemma 1. *Let u be a smooth solution to (6.1), (6.2) subject to initial data $u_0 \in H^1$, for $0 \leq t \leq T$. Then*

$$G(t, x) \leq A, \quad t \in [0, T], \tag{6.5}$$

where

$$A := \left(|a|(1 + \sup |Q|) + \frac{|c|}{2} \left[(1 + \sup |Q|) + (1 - \sup |Q|)\text{sgn}(c) \right] \|u_0\|_1 \right) \|u_0\|_1.$$

Proof. The periodicity of u (or $u \in H^s$) suggests that there exists a $x^* \in (0, 1)$ such that $u_x(t, x^*) = 0$. Thus we have

$$u^2(t, x) = \int_{x^*}^x (u^2)_x dx \leq 2 \int |uu_x|dx \leq \int (u^2 + u_x^2)dx = \|u_0\|_1^2.$$

From this bound and $Q > 0$ it follows

$$
\begin{aligned}
G(t,x) &\leq |a||u| + \max\{c,0\}u^2 + \sup|Q| \int_0^1 (|a||u| + \max\{-c,0\}u^2)dx \\
&\leq |a|(1 + \sup|Q|)|u|_\infty + \max\{c,0\}|u|^2 + \sup|Q|\max\{-c,0\} \int u^2 dx \\
&\leq |a|(1 + \sup|Q|)\|u_0\|_1 + \max\{c,0\}\|u_0\|_1^2 + \sup|Q|\max\{-c,0\}\|u_0\|_1^2 \\
&= A.
\end{aligned}
$$

∎

Lemma 2. *Consider the Cauchy problem (6.1)-(6.2) subject to initial data u_0. The maximal existence time T is finite if and only if u_x becomes unbounded from below in finite time.*

Proof. Let the life span $T < \infty$ and assume that for some constant $M > 0$ we have

$$
u_x(t,x) \geq -M, \quad (t,x) \in [0,T) \times \mathbb{S}. \tag{6.6}
$$

On the other hand the above dynamics (6.4) with negative quadratic term $-d^2/2$ on the right suggests that the quantity $d(t,x)$ must have the upper bound, i.e.,

$$
d(t,x) \leq d(0,\alpha) + AT < \infty.
$$

By Theorem 1 one must have $T = \infty$. This contradiction ensures that

$$
\lim_{t \to T-} \left(\inf_{x \in \mathbb{R}} u_x(t,x)\right) = -\infty.
$$

∎

We point out that the result stated in Lemma 2 is the same as that obtained in [17] for the CH equation. We are now in a position to state the lower threshold for the wave breaking.

Theorem 3. *Consider the problem (6.1)-(6.2) subject to the periodic initial profile $u_0 \in C^1(\mathbb{S})$. Set $\beta(c,Q) := 1_{c>0} + 1_{c<0}\sup|Q|$ and*

$$
A := (|a|(1 + \sup|Q|) + |c|\beta(c,Q)\|u_0\|_1)\|u_0\|_1.
$$

If the initial velocity slope is negative with

$$
\inf_{x \in \mathbb{S}} \partial_x u_0(x) < -\sqrt{2A}.
$$

Then the maximal existence time of the smooth solution is finite before

$$
t^* = \frac{1}{-\sqrt{2A} - \inf_{x \in \mathbb{S}} \partial_x u_0(x)}.
$$

Note that for the periodic problem the quantity $\sup |Q|$ is given by

$$\sup |Q| = \frac{\cosh(0.5)}{2\sinh(0.5)}.$$

The argument used also works for initial problem on the whole domain $\Omega = \mathbb{R}$, for which $Q = \frac{1}{2}\exp(-|x|)$ with $\sup |Q| = \frac{1}{2}$. Therefore the breaking criterion becomes

$$\inf_{x \in \mathbb{R}} \partial_x u_0(x) < -\sqrt{2A},$$

where

$$A = \frac{1}{2}\left(3|a| + (3|c| + c)\|u_0\|_1/2\right)\|u_0\|_1.$$

From this result we write down the breaking condition for the Camassa-Holm equation $(a = 0, c = 1)$.

Theorem 4. *Consider the Camassa-Holm equation*

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

subject to initial profile $u_0 \in H^3(\Omega)$. The maximal existence time of the smooth solution is finite provided that the initial velocity slope is negative with

$$\inf_{x \in \Omega} \partial_x u_0(x) < -\sqrt{2}\|u_0\|_1,$$

where $\Omega = [0, 1]$ for periodic initial data; and $\Omega = \mathbb{R}$ for initial value problem on the whole line.

We note that two sufficient conditions for finite time wave breaking were obtained earlier for the Camassa-Holm equation.
(1) [13] Assume $u_0 \in H^3(\mathbb{S})$ satisfies

$$\int_0^1 u_0 dx = 0 \quad or \quad \int_0^1 u_0(u_0^2 + u_{0,x}^2)dx = 0.$$

(2) [15] Assume $u_0 \in C^\infty$ is such that

$$\min[u_0'(x)] + \max[u_0'(x)] \leq -2\sqrt{3}\|u_0\|_1.$$

Our results show that it is the negative initial velocity slope that leads to the finite time wave breaking.

Also as mentioned in §3, Dai's model can be rewritten as

$$u_t + \gamma uu_x + (I - \partial_x^2)^{-1}\partial_x \left(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right) = 0,$$

which corresponds to our model system (6.1), (6.2) with $a = 0$ and $c = (3-\gamma)/2\gamma$, modulo a variable scaling $u \to \gamma u$. Hence taking into account of the scaling $u \to \gamma u$ we may put the breaking criterion in the following form

$$\inf\{\gamma\partial_x u_0(x)\} < -\sqrt{2A},$$

where

$$A = |c|\beta(c, Q)\|\gamma u_0\|_1^2 = \frac{1}{2}\beta(c, Q)|\gamma(3 - \gamma)|\|u_0\|_1^1.$$

In summary we have

Theorem 5. *Consider Dai's model*

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + u_{xxx}), \quad \gamma \neq 0,$$

subject to smooth initial profile $u_0 \in H^3(\Omega)$. The maximal existence time of the smooth solution is finite provided that the scaled initial velocity slope $\gamma\partial_x u0(x)$ is negative with

(a) $\inf_{x\in\Omega} \gamma\partial_x u_0(x) < -\sqrt{\gamma(3 - \gamma)}\|u_0\|_1$ for $0 < \gamma < 3$;

(b) $\inf_{x\in\Omega} \gamma\partial_x u_0(x) < -\sqrt{\sup|Q|\gamma(\gamma - 3)}\|u_0\|_1$ for $\gamma < 0$ or $\gamma > 3$. In the above $\Omega = [0, 1]$ and $\sup|Q| = \frac{\cosh(0.5)}{2\sinh(0.5)}$ for periodic initial data; $\Omega = \mathbb{R}$ and $\sup|Q| = 1/2$ for initial value problem on the whole line $x \in \mathbb{R}$.

We note that the above conditions are sharper than those obtained in [22] via a different approach. The physical interpretation of Dai's model suggests that the solutions should be bounded and that blow up should occur when the rod is broken. The upper threshold obtained depends on the parameter γ in a quite interesting manner. As it states for the finite time blow up to happen the initial velocity slope must be relatively negative for $\gamma > 0$ and relatively positive for $\gamma < 0$. $\gamma = 0$ and $\gamma = 3$ are two special cases. When $\gamma = 0$, all solutions are global. When $\gamma = 3$, the solution always experiences finite-time blow up since in this case breaking criterion is always met for either periodic initial data or $u_0 \in H^s$.

Proof of Theorem 3. Evaluating the above d equation (6.4) at $x(t, \alpha)$ and using $G(t, x) \leq A$ stated in (6.5) we have

$$\dot{d} + d^2/2 = G(t, x(t, \alpha)) \leq A, \quad t \in (0, T).$$

That is

$$\dot{d} \leq -(d + \sqrt{2A})(d - \sqrt{2A})/2, \quad t \in (0, T). \tag{6.7}$$

For a fixed $\alpha \in \mathbb{S}$ if $d_0(\alpha) < -\sqrt{2A}$, then we claim that

$$d(t) < d_0(\alpha), \quad t \in (0, T). \tag{6.8}$$

If this would not be true, there is some $t_0 \in (0, T)$ with $d(t) < d_0$ on $[0, t_0)$ and $d(t_0) = d_0$ by the continuity of d in time. But in this case

$$\dot{d} \leq -(d + \sqrt{2A})(d - \sqrt{2A}) < 0, \quad t \in (0, t_0).$$

An integration over $(0, t_0)$ yields

$$d(t_0) < d_0,$$

which contradicts to our assumption that $d(t_0) = d_0$ for $t_0 < T$. This implies that (6.8) holds.

Combining (6.8) with (6.7) we obtain

$$\dot{d} \le -(d + \sqrt{2A})^2, \quad t \in (0, T),$$

which upon integration yields

$$d(t) \le -\sqrt{2A} + \left[t - \frac{1}{-\sqrt{2A} - d_0}\right]^{-1}.$$

From this we find that $d(t) \to -\infty$ before t reaches $\frac{1}{-\sqrt{2A} - d_0}$. This proves that the solution breaks down in finite time once $d_0 \ge -\sqrt{2A}$ fails. ∎

7 Wave breaking in an Euler-Poisson system

In this section we turn to the Cauchy problem of an Euler-Poisson system on the whole line

$$n_t + (nu)_x = 0, \tag{7.1}$$
$$u_t + uu_x = \phi_x, \tag{7.2}$$
$$\epsilon^2 \phi_{xx} = n - e^{-\phi}, \tag{7.3}$$

subject to smooth initial data

$$(n, u)(0, x) = (n_0, u_0)(x), \quad x \in \mathbb{R}. \tag{7.4}$$

This Euler-Poisson system is a simplified model for the one dimensional motion of ions in the weakly ionized plasma. Its relation with the KdV equation has been discussed under various conditions, see e.g. [48, 44, 3]. Here we want to show wave breaking does occur for this system, which is not shared by the KdV equation.

For smooth solution, it is easy to check that there are two obvious conserved pairs

$$(P_1, J_1) = (n, nu), \quad (P_2, J_2) = \left(nu, nu^2 + e^{-\phi} - \frac{1}{2}\phi_x^2\right),$$

which corresponds to the conservation of mass and momentum. A somewhat less obvious conserved pair is

$$P_3 = \frac{1}{2}nu^2 - n\phi - e^{-\phi} - \frac{1}{2}\phi_x^2,$$
$$J_3 = \phi_t\phi_x - nu\phi + \frac{1}{2}nu^3.$$

In contrast to the KdV equation, the wave breaking for the smooth solution of this system is unavoidable once the velocity slope is sufficient negative. Our task here is to identify a lower threshold for the wave breaking to happen at a finite time.

Let $(n, u)(t, x)$ be a smooth solution to (7.1)-(7.4) for $0 \le t < T$. By differentiating (7.2) and using the Poisson equation, we have for $d := u_x$

$$d_t + ud_x + d^2 = [n - e^{-\phi}]\epsilon^{-2}, \quad t \in (0, T).$$

The smoothness of u ensures that there exists a smooth characteristic curve $x = x(t, \alpha)$ satisfying

$$\frac{d}{dt}x(t, \alpha) = u(t, x(t, \alpha)), \quad x(0, \alpha) = \alpha, \quad \alpha \in \mathbb{R}.$$

Therefore the dynamics of (n, d) is governed by

$$\dot{x} = u, \quad \dot{\ } := \partial_t + u\partial_x, \tag{7.5}$$

$$\dot{n} = -nd \tag{7.6}$$

$$\dot{d} = -d^2 + n(1 - \gamma)\epsilon^{-2}, \quad \gamma(t) := \frac{e^{-\phi}}{n}\Big|_{(t, x(t, \alpha))}. \tag{7.7}$$

Lemma 3. *Consider the Cauchy problem (7.1)-(7.4). The maximal existence time T is finite if and only if u_x becomes unbounded from below in finite time.*

This can be proved by a similar argument as that used in the proof of Lemma 2. Details are omitted.

The lower threshold is given in the following

Theorem 6. *Consider the Cauchy problem (7.1)-(7.4) with the initial profile $(n_0, u_0) \in C^1(\mathbb{R})$. If (n_0, u_0) are bounded and the initial velocity slope fails to satisfy*

$$\partial_x u_0(x) \ge -\sqrt{2n_0(x)}/\epsilon \tag{7.8}$$

at some point x. Then the maximal existence time of the smooth solution is finite before

$$t^* = \inf_{\alpha \in \mathbb{R}} \left(\frac{2}{-d_0 - \sqrt{d_0^2 - 2n_0\epsilon^{-2}}} \right).$$

Remarks. 1) As $\epsilon \downarrow 0$, the condition (7.8) is almost always true, i.e., wave breaking may not happen for given initial smooth profile with sufficiently small ϵ. This partially explains why the KdV equation could be used as an approximate model in this scaling regime.
2) This condition is same as the critical threshold found in [26] for the Euler-Possion system (7.1)-(7.2), with Possion system (7.3) replaced by

$$\epsilon^2 \phi_{xx} = n.$$

Proof of Theorem 6. Evaluating the above dynamic system (7.6)-(7.7) at $x = x(t, \alpha)$ we have

$$\dot{n} = -nd,$$
$$\dot{d} = -d^2 + n[1 - \gamma(t)]\epsilon^{-2}.$$

From this coupled system it follows that

$$\frac{d}{dt}\left(\frac{d}{n}\right) = \frac{n\dot{d} - d\dot{n}}{n^2} = (1 - \gamma(t))\epsilon^{-2},$$

whose integration leads to

$$\frac{d}{n} = \frac{d_0}{n_0} + \left(t - \int_0^t \gamma(\tau)d\tau\right)\epsilon^{-2}.$$

This when combined with the mass equation $\dot{n} = -nd$ gives

$$\dot{n} = -n^2\left[\frac{d_0}{n_0} + \left(t - \int_0^t \gamma(\tau)d\tau\right)\epsilon^{-2}\right].$$

Integration over time once leads to

$$n = \frac{n_0}{1 + d_0 t + n_0\left(t^2/2 - \int_0^t \int_0^\tau \gamma(s)ds d\tau\right)\epsilon^{-2}}.$$

Note that $\gamma(t) > 0$. The above formula suggests

$$n \geq \frac{n_0}{1 + d_0 t + \frac{1}{2}n_0 t^2 \epsilon^{-2}}.$$

For negative $d_0 < -\sqrt{2n_0}/\epsilon^2$ the right hand side becomes unbounded at a finite time before

$$t^* = \frac{2}{-d_0 - \sqrt{d_0^2 - 2n_0\epsilon^{-2}}}.$$

The question left is whether d remains bounded if n does. This can be seen from

$$n(t) = n_0(\alpha)\exp\left(-\int_0^t d(\tau, x(\tau, \alpha))d\tau\right),$$

which becomes unbounded when $t \to T^*$ if and only if

$$\lim_{t \to T^*} \inf_{0 \leq \tau \leq t} d(\tau, x(\tau, \alpha)) = -\infty,$$

therefore $d \to -\infty$ at some point. This proves that the solution breaks down in finite time once $d_0 \geq -\sqrt{2n_0}/\epsilon^{-2}$ fails at some point. $\blacksquare$

Remark. The above wave breaking criterion suggests that if the initial velocity slope is not too negative relative to the initial density n_0, then the global solution may exist. In fact from the above analysis we see that if one could have a priori bound $\max_{0 \leq \tau \leq t} \gamma(\tau) \leq \gamma^* < 1$, then

$$n \leq \frac{n_0}{1 + d_0 t + \frac{1}{2}n_0 t^2(1 - \gamma^*)\epsilon^{-2}}.$$

The RHS remains bounded for all time if

$$d_0 > -\sqrt{2n_0(1-\gamma^*)}\epsilon^{-2}.$$

Obviously for small $\epsilon << 1$, any given initial data would satisfy this upper threshold, therefore global existence seems always ensured. One may argue that the above bound needs not to be true since it suffices if

$$\frac{2}{t^2}\int_0^t\int_0^s\gamma(\tau)d\tau ds \leq \gamma* < 1.$$

The existence of an upper threshold hinges on the existence of such a bound, which remains to be verified.

References

[1] BEALS R, SATTINGER D and SZMIGIELSKI J, Multi-peakons and a theorem of Stieltjes, *Inverse Probl.* **15** (1999), L1–L4.

[2] BEALS R, SATTINGER D and SZMIGIELSKI J, Multipeakons and the classical moment problems, *Adv. Math.* **154** (2000), 229–257.

[3] BEN YOUSSEF W and COLIN T, Rigorous derivation of KdV-type systems from a general class of nonlinear hyperbolic systems, *Math. Model. Numer. Anal.* **34** (2000), 873–911.

[4] BENJAMIN B, BONA J and MAHONY J, *Philos. Trans. R. Soc. A* **272** (1972), 47.

[5] BENJAMIN T B, Lectures on nonlinear wave motion, *Lect. Appl. Math.* **15** (1974) 3–48.

[6] BENJAMIN T B, Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.* **29** (1967), 559–582.

[7] CAMASSA R and HOLM D D, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[8] CAMASSA R, HOLM D D and HYMANN J, A new integrable shallow water equation, *Adv. Appl. Mech.* **31** (1994), 1–33.

[9] CONSTANTIN A, On the Cauchy problem for the periodic Camassa-Holm equation, *J. Diff. Equ.* **141** (1997), 218–235.

[10] CONSTANTIN A, On the inverse spectral problem for Camassa-Holm equation, *J. Funct. Anal.* **155** (1998), 352–363.

[11] CONSTANTIN A, Existence of permanent and breaking waves for a shallow water eqution: a geometric approach, *Ann. Inst. Fourier* (Grenoble) **50** (2000), 321–362.

[12] CONSTANTIN A, On the scattering problem for the Camassa-Holm equation, *Proc. R. Soc. Ser. A* **457** (2001), 953–970.

[13] CONSTANTIN A and ESCHER J, Well-posedness, global existence, and blow up phenomena for a periodic quasi-linear hyperbolic equation, *Commun. Pure Appl. Math.* **51** (1998), 475–504.

[14] CONSTANTIN A and ESCHER J, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* **181** (1998), 229–243.

[15] CONSTANTIN A and ESCHER J, On the structure of a family of quasilinear equations arising in shallow water theory, *Math. Ann.* **312** (1998), 403–416.

[16] CONSTANTIN A and ESCHER J, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa* **26** (1998), 303–328.

[17] CONSTANTIN A and ESCHER J, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.* **233** (2000), 75–91.

[18] CONSTANTIN A and KOLEV B, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A.* **35** (2002), R51–R79.

[19] CONSTANTIN A and KOLEV B, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** (2003), 787–804.

[20] CONSTANTIN A and MOLINET L, Global weak solutions for a shallow water equation, *Commun. Math. Phys.* **211** (2000), 45–61.

[21] CONSTANTIN A and MCKEAN H P, A shallow water equation on a circle, *Commun. Pure Appl. Math.* **52** (1999), 949–982.

[22] CONSTANTIN A and STRAUSS W A, Stability of a class of solitary waves in compressible elastic rods, *Phys. Lett. A* **270** (2000), 140–148.

[23] CONSTANTIN A and STRAUSS W A, Stability of peakons, *Commun. Pure Appl. Math.* **53** (2000), 603–610.

[24] CONSTANTIN A and STRAUSS W A, Stability of the Camassa-Holm solitons, *J. Nonlinear Sci.* **12** (2002), 415–422.

[25] DAI H-H, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mech.* **127** (1998), 193–207.

[26] ENGELBERG S, LIU H and TADMOR E, Critical threshold in Euler-Poisson equations, *Indiana Univ. Math. J.* **50** (2001), 109–157.

[27] FELLNER K and SCHMEISER C, Burgers-Poisson: A nonlinear dispersive model equation, *SIAM J. Appl. Math.* **64** (2004), 1509–1525.

[28] FOKAS A S, On a class of physically important integrable equations, *Physica D.* **87** (1995), 145–150.

[29] FOKAS A S and LIU Q M, Asymptotic integrability of water waves, *Phys. Rev. Lett.* **77** (1996), 2347–2351.

[30] FUCHSSTEINER B and FOKAS A S, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D* **4** (1981/82), 47–66.

[31] HAMER K, Nonlinear effects on the propagation of sound waves in a radiating gas, *Quart. J. Mech. Appl. Math.*, **24** (1971), 155–168.

[32] HOLM D D and HONE A N W, A class of equations with peakon and pulson solutions (with a appendix by BRADEN H and BYATT-SMITH J), *J. Nonlinear Math. Phys.* **12** Suppl. 1 (2005), 380–394.

[33] HUNTER J K, Numerical solutions of some nonlinear dispersive wave equations, *Lect. Appl. Math.* **26** (1990), 301–316.

[34] HUNTER J K and SAXTON R, Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991), 1498–1521.

[35] HUNTER J K and ZHENG Y X, On a completely integrable nonlinear hyperbolic variational equations, *Physica D* **79** (1994), 361–386.

[36] JOHN F, Formation of singularities in one-dimensional nonlinear wave propagation, *Commun. Pure Appl. Math.*, **27** (1974), 337-405.

[37] JOHNSON R S, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** (2002), 63–82.

[38] JOHNSON R S, On solutions of the Camassa-Holm equation, *Proc. R. Soc. Ser. A* **459** (2003), 1687–1708.

[39] KAWASHIMA S and NISHIBATA S, Shock waves for a model system of the radiating gas, *SIAM J. math. Anal.* **30** (1999), 95–117.

[40] KOURANBAEVA S, The Camasaa-Holm equation as a geodesic flow on the differmorphism group, *J. Math. Phys.* **40** (1999), 857–868.

[41] LAX P, Development of Singularities of Solutions of Nonlinear Hyperbolic Partial Differential Equations, *J. Math. Phys.*, **5** (1964), 611–613.

[42] LENELLS J, The scattering approach for the Camassa-Holm equation, *J. Nonlinear Math. Phys.* **9** (2002), 953–970.

[43] LIU T P, Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations, *J. Differ. Equ.*, **33** (1979), 92–111.

[44] LI Y and SATTINGER D H, Soliton collisions in the ion acoustic plasma equations, *J. Math. Fluid Mech.* **1** (1999), 117–130.

[45] LI Y A and OLVER P J, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive models wave equation, *J. Differ. Equ.*, **162** (2000), 27–63.

[46] LIU H L and TADMOR E, Critical thresholds in a convolution model for nonlinear conservation laws, *SIAM J. Math. Anal.* **33** (2002), 930–945.

[47] LIU H L and TADMOR E, Critical thresholds in 2-D restricted Euler-Poisson equations, *SIAM J. Appl. Math.* **63** (2003), 1889–1910.

[48] LIU H L and SLEMROD M, KdV dynamics in the plasma-sheath transition, *Appl. Math. Lett.* **17** (2004), 401–410.

[49] NAUMKIN P and SHISHMAREV I, Nonlinear nonlocal equations in the Theory of Waves, *Transl. Math. Monographs* **133**, Amererican Mathematical Society, Providence, 1994.

[50] ONO H, Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Jpn.*, **39** (1975), 1082–1091.

[51] PARKES E J and VAKHNENKO V O, The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method, *Chaos Solitons Fractals* **13** (2002), 1819–1826.

[52] ROSENAU P, Extending hydrodynamics via the regularization of the Chapman-Enskog expansion, *Phys. Rev. A* **40** (1989), 7193–7196.

[53] SCHOCHET S and TADMOR E, Regularized Chapman-Enskog expansion for scalar conservation laws, *Arch. Ration. Mech. Anal.* **119** (1992), 95–107.

[54] SLEMROD M, A renormalization method for the Chapman-Enskog expansion, *Physica D* **109** (1997), 257–273.

[55] SLEMROD M, Renormalization of the Chapman-Enskog expansion: isothermal fluid flow and Rosenau saturation, *J. Stat. Phys.* **91** (1998), 285–305.

[56] TAO T, Global well-posedness of the Benjamin-Ono equation in $H^1(R)$, *J. Hyperbolic Differ. Equ.* **1** (2004), 27–49.

[57] VAKHNENKO V O, Solitions in a nonlinear model medium, *J. Phys. A.* **25** (1992), 4181–4187.

[58] WHITHAM G B, Linear and nonlinear waves, Pure and Applied Mathematics, Wiley-Inter science, New York-London-Sydney, 1974.

[59] XIN Z and ZHANG P, On the uniqueness and large time behavior of the weak solutions to a shallow water equation, *Commun. Partial Differ. Equ.* **27** (2002), 1815–1844.