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The L^p Stability of Relaxation Rarefaction Profiles

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We consider the large time behavior of solutions for a hyperbolic relaxation system. For a certain class of initial data the solution is shown to converge to relaxation rarefaction profiles at a determined asymptotic rate. The result is established without the smallness conditions of the wave strength and the initial disturbances. © 2001 Academic Press

Key Words: decay rate; rarefaction wave; relaxation.

1. INTRODUCTION

Our main point of interest is the large time behavior of solutions developed by relaxation dynamics starting with a certain class of initial data.

We consider a hyperbolic relaxation system of the form

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + au_x &= f(u) - v, \end{aligned} \tag{1.1}$$

where u, v are scalars and the constant $a > 0$ is a given constant. This system is an example of a class of relaxation systems proposed by Jin and Xin [3]. We assume that f is strictly convex, i.e.,

$$f''(u) \geqslant \alpha > 0, \qquad \text{for} \quad u \in \mathbb{R}.$$

Now, let us define a set of the equilibrium states of (1.1) as

$$\Gamma(u) := \{(u, v),\ v = f(u)\}.$$

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The initial data are asymptotically approaching the equilibrium state

$$(u, v)(x, 0) = (u_0, v_0)(x) \to \Gamma(u_\pm) \qquad \text{as} \quad x \to \pm\infty, \tag{1.2}$$

where $u_\pm$ are constants satisfying $u_- < u_+$. We assume that a is large enough so that it dominates a priori the velocities $f'(u)$; that is, the well known sub-characteristic condition

$$-\sqrt{a} < f'(u) < \sqrt{a} \tag{1.3}$$

holds for u in question. This requirement ensures the global existence of the solution (u, v), as well as their L^1 contraction property; see Natalini [12].

The purpose of this paper is to show the large time behavior of solutions and measure the decay rate to the large time wave profiles. The asymptotic behavior as $t \to \infty$ of the solutions to (1.1)–(1.2) is clearly related to that of the Riemann problem for the equilibrium conservation law,

$$\begin{aligned} &r_t + f(r)_x = 0, && (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\ &r(x, 0) = u_\pm, && \text{for} \quad \pm x > 0. \end{aligned} \tag{1.4}$$

Its entropy solution $r(x, t)$ is called the rarefaction wave, which is given explicitly as

$$r(x, t) = \begin{cases} u_-, & x < f'(u_-)\, t \\ (f')^{-1}\left(\dfrac{x}{t}\right), & f'(u_-)\, t \leqslant x \leqslant f'(u_+)\, t. \\ u_+, & x > f'(u_+)\, t. \end{cases} \tag{1.5}$$

For any $t > 0$, $r(x, t)$ is Lipschitz continuous, monotonous nondecreasing and flattens out at a linear rate as t increases. For any given constant $\gamma > 0$, one can always choose t_0, say $(\alpha\gamma)^{-1}$, to assure

$$0 \leqslant r(x, t_0)_x \leqslant \gamma.$$

Then we solve Eq. (1.1) with the initial data $\Gamma(r(x, t_0))$, the corresponding solution $(\bar{u}, \bar{v})$ connecting $\Gamma(u_\pm)$ is shown to be monotonic nondecreasing in space and flatten out in a way as the rarefaction wave (1.5). This solution can be regarded as an approximation of rarefaction wave $r(x, t)$. In this sense we call such solution $\bar{u}(x, t)$ as *the relaxation rarefaction profile*, with $\bar{v}$ being its associated flux. This relaxation rarefaction profile respects the following properties:

THEOREM 1.1. *Assume the sub-characteristic condition* (1.3) *for initial data* $\Gamma(r(x, t_0))$. *Let* $\bar{u}$ *be the relaxation rarefaction profile defined above, then there exists a* $\delta > 0$ *such that for* $t_0 > \delta$

$$0 \leqslant \bar{u}_x \leqslant \frac{K}{t + t_0} \tag{1.6}$$

for a positive constant K.

For $u_- < u_+$, we define the function space $\mathscr{L}(u_-, u_+) \subset L^\infty(\mathbb{R})^2$ as

$$\mathscr{L}(u_-, u_+) = \left\{ (u, v) \in L^\infty(\mathbb{R})^2,\ \pm \int_0^{\pm\infty} |(u, v)(x) - \Gamma(u_\pm)| < \infty \right\}.$$

Taking the general initial data $(u_0, v_0)(x) \in \mathscr{L}(u_-, u_+)$, we state the main result as follows.

THEOREM 1.2. *Let* $(u_0, v_0) \in \Gamma(r(x, t_0)) + L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ *be the initial data, and* $(\bar{u}, \bar{v})$ *be the relaxation rarefaction profile as described in Theorem* 1.1. *Suppose that the stability condition* (1.3) *holds. Then there exists* $C > 0$ *such that*

$$\|u(t) - \bar{u}(t)\|_{L^p} + \|v(t) - \bar{v}(t)\|_{L^p} \leqslant C(1+t)^{-(1/2)+(1/2p)}, \qquad \forall t \geqslant 0.$$

An immediate consequence of the above theorem is the following

COROLLARY 1.3. *Assume the assumptions made in Theorem* 1.2 *with* $\Gamma(r(x, t_0))$ *replaced by an equilibrium state* $\Gamma(u_-)$ $(u_- = u_+)$, *then*

$$\|(u, v)(t) - \Gamma(u_-)\|_{L^p} \leqslant C(1+t)^{-(1/2)+(1/2p)}, \qquad \forall t \geqslant 0.$$

The complementary situation, when $u_- > u_+$, and (1.1), (1.2) admit traveling wave solutions was treated in Mascia and Natalini [11]. It was shown there that these traveling wave solutions attract in L^1 a large class of initial data. For L^2 stability of relaxation shock profiles by energy methods see [7, 8] and by obtaining decay rates see [9]. The L^p stability of diffusion waves for a class of 2×2 hyperbolic relaxation systems proposed by Liu [6] was studied by Chern [1] using the energy method combined with the Fourier analysis. Recently Liu and Natalini [4] employed a parabolic scaling argument to establish the L^p global stability of diffusion waves of the relaxation system (1.1).

The first results on nonlinear stability of rarefaction waves, as well as the traveling waves, for hyperbolic relaxation problem were obtained by Liu [6]. The convergence towards the planar rarefaction wave for 2-D

Jin-Xin's model was shown by Luo [10]. All by use of the elegant energy method but no decay rates. Consult [13] for a survey on the stability of elementary waves for various relaxation models. In the case of viscous conservation laws, the convergence towards the rerefaction waves has been studied by many authors; see, e.g., [2, 14, 16, 18, 19] and references therein. The decay rate stated in Theorem 1.1 is close to the decay rate for viscous case [2], but our initial data here are not necessarily restricted within a box framed by $\Gamma(u_{\pm})$. This seems remarkable as relaxation mechanism is known to possess less smoothing effect than viscosity.

Our work uses the L^1-contraction and the time-weighted L^2 energy approach. Such approach allows for decay rate estimates, and has been useful in the viscous conservation laws; see, e.g., [14, 19]. Compared to the traveling shock wave, the rarefaction wave is time-varying. One does not know what the exact large time wave profile is when trying to prove it stable. The rarefaction stability results depend strongly on how well you define the approximate profile; see, e.g., [16]. In this note we introduce a so called relaxation rarefaction profile. Such profile is shown to behave like the usual rarefaction wave for the equilibrium conservation laws, and is a $L^p(p>1)$ attractor for a large class of initial data in $L^\infty+L^1\cap H^1$.

This paper is organized as follows. In Section 2, we show the properties of relaxation rarefaction profile stated in Theorem 1.1. In Section 3, we first reformulate the problem, then we establish the L^p estimates for the reformulated problem. By using the known L^1 estimate, we prove the L^p convergence rate of the solution to the relaxation rarefaction profile. In final section, we discuss the L^p convergence rate of relaxation rarefaction profile towards the rarefaction wave (1.5).

2. RELAXATION RAREFACTION PROFILES

Let $r(x, t)$ be the relaxation wave given in (1.5). Then $r(x, t)$ is Lipschitz continuous and satisfies

$$0 \leqslant r_x \leqslant 1/(\alpha t), \qquad t>0. \tag{2.1}$$

The interval in which the sub-characteristic condition is satisfied can be determined by the given initial data. More precisely for initial data (u_0, v_0), we define the range of the data as $[c, d]$ with

$$c := (\inf_x (u_0 - v_0/\sqrt{a}) + \inf_x (u_0 + v_0/\sqrt{a}))/2,$$

$$d := (\sup_x (u_0 - v_0/\sqrt{a}) + \sup_x (u_0 + v_0/\sqrt{a}))/2.$$

Then the condition (1.3) becomes

$$\theta = \sup |f'(\xi)|/\sqrt{a} < 1 \tag{2.2}$$

for $\xi \in [c, d]$; see Serre [5]

Let $\bar{\alpha} = \sup_{\xi \in [c, d]} |f''(\xi)|$ and

$$\delta := \frac{2\bar{\alpha}^2 K}{(1-\theta)\,\alpha} \qquad \text{with} \quad K = \frac{1+\theta}{\alpha(1-\theta)}. \tag{2.3}$$

Then the relaxation rarefaction wave respects the property stated in Theorem 1.1.

Now we proceed to prove Theorem 1.1 as follows.

Proof. First we regularize the initial data $\Gamma(r(x, t_0))$ by a mollifier ω_ε to obtain

$$(\bar{u}^\varepsilon, \bar{v}^\varepsilon)(x, 0) = \Gamma(r(\cdot, t_0) * \omega_\varepsilon).$$

By the regularity theory for the semilinear hyperbolic systems the solution $\{\bar{u}^\varepsilon, \bar{v}^\varepsilon\}$ is smooth. We establish the estimate (1.6) for $\bar{u}^\varepsilon$ and then psss to the limit. To simplify the presentation, we skip over this standard regularization procedure.

Setting $R_\pm = \bar{u}_x \pm \bar{v}_x/\sqrt{a}$, then one gets from differentiating Eq. (1.1) with respect to x

$$(\partial_t \pm \sqrt{a}\,\partial_x)\, R_\pm = \pm b^+ R_- \mp b^- R_+ \tag{2.4}$$

with $b^\pm(x, t) = \frac{1}{2}(1 \pm \frac{f'(\bar{u})}{\sqrt{a}})$. By the sub-characteristic condition (1.3) we have

$$\frac{1-\theta}{2} \leqslant b^\pm(x, t) \leqslant \frac{1+\theta}{2} \qquad \text{and} \qquad b^+ + b^- = 1.$$

Initially

$$R_\pm(x, 0) = \left(1 \pm \frac{f'(r)}{\sqrt{a}}\right) r_x(x, t_0) \geqslant 0.$$

By the maximum principle for (2.4), see [17], we have

$$R_\pm(x, t) \geqslant 0, \qquad t \in \mathbb{R}^+$$

which leads to

$$\bar{u}_x(x, t) = \tfrac{1}{2}(R_-(x, t) + R_+(x, t)) \geqslant 0, \qquad t \in \mathbb{R}^+.$$

To estimate the upper bound, motivated by [17] we introduce

$$R_{\pm} = q_{\pm}(x, t) + \frac{2K}{t+t_0}\, b^{\pm}(x, t).$$

Since $(\bar{u}, \bar{v})(x, 0) = \Gamma(r(x, t_0))$ we have using (2.1)

$$q_{\pm}(x, 0) = R_{\pm}(x, 0) - \frac{2K}{t_0}\, b^{\pm}(x, 0) = 2b^{\pm}(x, 0)\left[r_x(x, t_0) - \frac{K}{t_0}\right] < 0$$

for $K > 1/\alpha$. Note that

$$\bar{u}_x = \frac{1}{2}(R_+ + R_-) = \frac{1}{2}(q_+ + q_-)(x, t) + \frac{K}{t+t_0},$$

then our remaining task is to show

$$q_{\pm}(x, t) \leqslant 0, \qquad \text{for} \quad t > 0. \tag{2.5}$$

Upon substitution for R_- one obtains

$$(\partial_t - \sqrt{a}\,\partial_x)\, q_- = -b_+ q_- + b_- q_+ + \frac{2K}{t+t_0}\, b^-(x, t)$$
$$- \frac{2K}{t+t_0}\, (\partial_t - \sqrt{a}\,\partial_x)\, b^-.$$

Using the equation $\bar{u}_t + \bar{v}_x = 0$ one gets

$$(\partial_t - \sqrt{a}\,\partial_x)\, b^- = \frac{1}{2} f''(\bar{u})\, R_+ = \frac{1}{2} f''(\bar{u}) \left[q_+ + \frac{2Kb^+}{t+t_0}\right].$$

Thus

$$(\partial_t - \sqrt{a}\,\partial_x)\, q_- = -b^+ q_- + \left(b^- - \frac{Kf''(\bar{u})}{t+t_0}\right) q_+$$
$$- \frac{2K}{(t+t_0)^2}\, [Kf''(\bar{u})\, b^+ - b^-]. \tag{2.6}$$

Similarly

$$(\partial_t + \sqrt{a}\,\partial_x)\, q_+ = \left(b^+ - \frac{Kf''(\bar{u})}{t+t_0}\right) q_- - b^- q_+ - \frac{2K}{(t+t_0)^2}\, [Kf'' b^- - b^+]. \tag{2.7}$$

Noting that

$$b^{\pm} - \frac{Kf''(\bar{u})}{t+t_0} \geqslant \frac{1}{2}(1-\theta)\left[1 - \frac{2\bar{\alpha}K}{t_0(1-\theta)}\right] > 0$$

and

$$Kf''b^{\pm} - b^{\mp} \geqslant \tfrac{1}{2}K\alpha(1-\theta) - \tfrac{1}{2}(1+\theta) = 0,$$

an application of the maximum principle for the weakly coupled hyperbolic system (2.6)–(2.7) yields (2.5) since the initial data satisfy $q_{\pm}(x, 0) \leqslant 0$. This completes the proof of Theorem 1.1. ∎

Remark. From the above proof one can see that

$$\bar{u}_x \leqslant \frac{1}{\beta t}$$

for $\beta = K^{-1} < \alpha$. This is in agreement with the usual Oleinik's entropy condition for nonlinear conservation law, see, e.g., [15].

3. REFORMULATION OF THE PROBLEM

Now we reformulate the problem as follows. Letting $(\bar{u}, \bar{v})$ be a relaxation rarefaction profile defined above, we put

$$(u, v)(x, t) = (\bar{u}, \bar{v})(x, t) + (\phi, \psi)(x, t), \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.1}$$

Then the system (1.1) is reduced to

$$\begin{aligned} &\phi_t + v\psi_x = 0, \\ &\psi_t + a\phi_x = [f] - \psi \end{aligned} \tag{3.2}$$

with $[f] = f(\phi + \bar{u}) - f(\bar{u})$.

Under the assumptions made in Theorem 1.2, the initial data (ϕ_0, ψ_0) satisfy

$$(\phi_0, \psi_0) \in B := L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \tag{3.3}$$

with norm

$$\|(\phi_0, \psi_0)\|_B := \|(\phi_0, \psi_0)\|_{L^1} + \|(\phi_0, \psi_0)\|_{H^1} < \infty.$$

Hereafter by C_B we denote a constant depending on the norm $\|(\phi_0, \psi_0)\|_B$.

We define the solution space of (3.2), (3.3) as

$$X(0, T) := C^0(0, T; B), \qquad \text{with} \quad 0 < T \leqslant \infty.$$

Theorem 1.2 is the consequence of the following result.

THEOREM 3.1. *Under the assumptions made in Theorem* 1.2, *then the problem* (3.2), (3.3) *admits a unique global solution* $(\phi, \psi) \in X(0, \infty)$ *satisfying*

$$\|(\phi, \psi)(t)\|_{L^p} \leqslant C_B(1+t)^{-(1/2)+(1/2p)}, \qquad t \geqslant 0. \tag{3.4}$$

3.1. L^1 *Estimate*

To obtain the desired estimate (3.4) for large data we will use the L^1 estimate. Recalling from [12] that, under the subcharacteristic condition (1.3), one has the following uniform L^1 stability estimate

$$\|\phi(t)\|_{L^1} + \|\psi(t)\|_{L^1} \leqslant (1+\sqrt{a}) \left(\|\phi_0\|_{L^1} + \frac{1}{\sqrt{a}} \|\psi_0\|_{L^1} \right).$$

This combined with the assumption $(\phi_0, \psi_0) \in B$ in (3.3) gives

$$\|\phi(t)\|_{L^1} + \|\psi(t)\|_{L^1} \leqslant C_B. \tag{3.5}$$

Equipped with this estimate we proceed to obtain the L^2 estimate.

3.2. L^2 *Estimate*

In order to obtain the decay rate in L^2, we proceed to establish the basic energy estimate.

Put

$$E(t) = \int_{\mathbb{R}} \left[\frac{\phi^2}{2} - \phi\psi_x + \psi_x^2 + a\phi_x^2 \right] dx.$$

Then $E(0)$ is bounded since

$$E(0) \leqslant C \|(\phi_0, \psi_0)\|_{H^1}^2 \leqslant C_B.$$

LEMMA 3.2. *If* $(\phi_0, \psi_0) \in B$ *and* $(\bar{u}, \bar{v})$ *is the solution given in Theorem* 1.1, *then it holds*

$$\frac{d}{dt} E(t) + c_0 \int_{\mathbb{R}} [\phi_x^2 + \psi_x^2] \, dx \leqslant 0, \qquad t \in \mathbb{R}^+ \tag{3.6}$$

for a constant c_0 *depending on* a.

Proof. Using the relation $\phi_t = -\psi_x$ and $\psi_t = [f] - a\phi_x - \psi$ and integration by parts, we obtain

$$\frac{d}{dt} E(t) = \int_{\mathbb{R}} [\phi\phi_t - \phi_t\psi_x - \phi(\psi_t)_x + 2\psi_x(\psi_t)_x + 2a\phi_x(\phi_t)_x] \, dx$$

$$= \int_{\mathbb{R}} \{ -\phi\psi_x + \psi_x^2 + (2\psi_x - \phi)([f]_x - a\phi_{xx} - \psi_x) - 2a\phi_x\psi_{xx}\} \, dx$$

$$= -\int_{\mathbb{R}} \{a\phi_x^2 + \psi_x^2 + \phi[f]_x - 2\psi_x[f]_x\} \, dx. \tag{3.7}$$

Since f is convex with $f''(u) \geqslant \alpha$ and $\bar{u}_x \geqslant 0$, for any $t \in \mathbb{R}^+$,

$$-\int_{\mathbb{R}} \phi[f]_x \, dx = \int_{\mathbb{R}} \phi_x[f] \, dx$$

$$= \int_{\mathbb{R}} \left\{ \frac{\partial}{\partial x} \int_{\bar{u}}^{\phi + \bar{u}} f(\xi) \, d\xi - \int_0^1 \int_0^1 \eta f''(r + \xi\eta\phi) \, d\xi \, d\eta \, \phi^2 \bar{u}_x \right\} dx$$

$$\leqslant -\frac{\alpha}{2} \int_{\mathbb{R}} \phi^2 \bar{u}_x \, dx.$$

And now, we treat the last term in (3.7) from below

$$2 \int_{\mathbb{R}} \psi_x[f]_x \, dx = 2 \int_{\mathbb{R}} \psi_x\phi_x f'(u) \, dx + 2 \int_{\mathbb{R}} \psi_x[f'] \, \bar{u}_x \, dx$$

with $[f'] = f'(\phi + \bar{u}) - f'(\bar{u}) = (\int_0^1 f''(\bar{u} + \eta\phi) \, d\eta) \, \phi =: \beta(x, t) \, \phi$.

Thus

$$\frac{d}{dt} E(t) \leqslant -\int_{\mathbb{R}} \left[a\phi_x^2 + 2f'(u) \, \psi_x\phi_x + \psi_x^2 + \frac{\alpha}{2} \bar{u}_x\phi^2 + 2\beta(x, t) \, \phi\psi_x\bar{u}_x \right] dx$$

$$= -\int_{\mathbb{R}} G(x, t) \, dx.$$

Rewrite $G(x, t)$ as

$$G(x, t) = (\psi_x, \phi_x, \phi) \, M(x, t)(\psi_x, \phi_x, \phi)^T$$

with the matrix

$$M(x, t) = \begin{pmatrix} 1 & f' & \beta\bar{u}_x \\ f' & a & 0 \\ \beta\bar{u}_x & 0 & \frac{\alpha}{2} \bar{u}_x \end{pmatrix}.$$

Since $\bar{u}_x \geqslant 0$, $a-(f')^2 \geqslant (1-\theta^2)\, a > 0$ and $\frac{\alpha}{2}\bar{u}_x - \beta^2\bar{u}_x^2 = \bar{u}_x(\frac{\alpha}{2} - \beta^2\bar{u}_x) > 0$ for $\beta^2\bar{u}_x \leqslant \bar{\alpha}^2 (K/t_0) < \frac{\alpha}{2}$.

Further calculation shows that for $\bar{\alpha} \geqslant \beta \geqslant \alpha$ one has

$$D = \frac{\alpha}{2}\,\bar{u}_x \left\{ a - f'^2 - \frac{2a\beta^2}{\alpha}\,\bar{u}_x \right\} \geqslant \frac{a\alpha(1-\theta^2)}{4}\,\bar{u}_x \geqslant 0,$$

which is ensured by the fact that $\bar{u}_x \leqslant \alpha(1-\theta)/2\bar{\alpha}^2$ and $|f'(\bar{u})| \leqslant \theta\sqrt{a}$.

Therefore the matrix $M(x, t)$ is non-negative definite; more precisely, there exists $c_0 := c_0(a, \theta) > 0$ such that

$$G(x, t) \geqslant c_0[\phi_x^2 + \psi_x^2]$$

for $t \geqslant t_0 > \delta$. In fact for $t_0 > \delta$, say $t_0 = 2\delta$, a simple computation shows that one can choose $c_0 = \min\{a(1-\theta)/3(1+\theta),\ (1-\theta^2)/2(1+2\theta)\}$. Thus the proof is complete. ∎

Based on Lemma 3.2, we easily get

$$E(t) \leqslant E(0) \leqslant C_B, \tag{3.8}$$

$$\int_0^t \int_{\mathbb{R}} [\phi_x^2 + \psi_x^2]\, dx\, d\tau \leqslant c_0^{-1} E(0) \leqslant C_B. \tag{3.9}$$

Further we establish the following decay estimates:

PROPOSITION 3.3. *Let $(\bar{u}, \bar{v})$ be the relaxation rarefaction profile corresponding to an initial expansive profile $\Gamma(r(x, t_0))$, and let $(u_0, v_0) = \Gamma(r(x, t_0)) + (\phi_0, \psi_0)$ with $(\phi_0, \psi_0) \in B := H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the solution (u, v) satisfies*

$$(1+t)^\gamma E(t) + \int_0^t (1+t)^\gamma \|(\phi_x, \psi_x)(\tau)\|_{L^2(\mathbb{R})}^2\, d\tau \leqslant C(1+t)^{\gamma - 1/2}$$

for some $C > 0$ depending on the norm $\|(\phi_0, \psi_0)\|_B$.

Proof. Multiplying (3.6) by $(1+t)^\gamma$, where $\gamma > 1/2$, and integrating the resulting inequality over $[0, t]$, we obtain

$$(1+t)^\gamma E(t) + c_0 \int_0^t (1+\tau)^\gamma \|(\phi_x, \psi_x)(\tau)\|_{L^2}^2\, d\tau \leqslant E(0) + \gamma \int_0^t (1+\tau)^{\gamma-1} E(\tau)\, d\tau. \tag{3.10}$$

Setting

$$A^{\gamma}(t)=(1+t)^{\gamma}\, E(t)+\int_0^t (1+\tau)^{\gamma}\, \|(\phi_x, \psi_x)(\tau)\|_{L^2}^2\, d\tau. \tag{3.11}$$

then by (3.10) and (3.8),

$$A^{\gamma}(t)\leqslant C_B\left(1+\gamma\int_0^t (1+\tau)^{\gamma-1}\, E(\tau)\, d\tau\right).$$

It remains to estimate the last term

$$\begin{aligned}\gamma\int_0^t (1+\tau)^{\gamma-1}\, E(\tau)\, d\tau &\leqslant \gamma\max\{2, a\}\int_0^t (1+\tau)^{\gamma-1}\, \|(\phi, \phi_x, \psi_x)(\tau)\|_{L^2}^2\, d\tau\\ &\leqslant C\gamma\left\{\int_0^t (1+\tau)^{\gamma-1}\, \|(\phi_x, \psi_x)(\tau)\|_{L^2}^2\, d\tau+\int_0^t (1+\tau)^{\gamma-1}\, \|\phi(\tau)\|_{L^2}^2\, d\tau\right\}.\end{aligned}$$

The first term on the right side is bounded by $C\gamma A^{\gamma-1}(t)$ and the second is treated by using the Gagliardo–Nirenberg inequality

$$\|\phi(t)\|_{L^2}\leqslant 2^{1/3}\, \|\phi(t)\|_{L^1}^{2/3}\, \|\phi_x\|_{L^2}^{1/3}.$$

This inequality combined with the L^1-estimate $\|\phi\|_{L^1}\leqslant C_B$ yields

$$\|\phi(t)\|_{L^2}^2\leqslant C_B\, \|\phi_x(t)\|_{L^2}^{2/3}, \qquad \forall t\geqslant 0.$$

Therefore using the Hölder inequality

$$\begin{aligned}\int_0^t (1+\tau)^{\gamma-1}\, \|\phi(\tau)\|_{L^2}^2\, d\tau &\leqslant C_B\int_0^t (1+\tau)^{\gamma-1}\, \|\phi_x(\tau)\|_{L^2}^{2/3}\, d\tau\\ &= C_B\int_0^t (1+\tau)^{\gamma-1-\gamma/3}\, [\, \|\phi_x(\tau)\|_{L^2}^2\, (1+\tau)^{\gamma}]^{1/3}\, d\tau\\ &\leqslant C_B(1+t)^{(2\gamma-1)/3}\left(\int_0^t (1+\tau)^{\gamma}\, \|\phi_x(\tau)\|_{L^2}^2\, d\tau\right)^{1/3}.\end{aligned}$$

We then obtain

$$A^{\gamma}(t)\leqslant C_B(1+\gamma A^{\gamma-1}(t)+\gamma(1+t)^{(2\gamma-1)/3}\, [A^{\gamma}(t)]^{1/3}), \qquad \forall t\geqslant 0.$$

Now we will show that

$$A^{\gamma}(t) \leqslant C_B(1+t)^{\gamma-1/2}. \tag{3.12}$$

It suffices to consider $\gamma = n \in \mathbb{N}$, since, with $\gamma = n+\delta$, for some $0<\delta<1$

$$A^{n+\delta}(t) = (1+t)^{n+\delta} E(t) + \int_0^t (1+\tau)^{n+\delta} \|(\phi_x, \psi_x)(\tau)\|^2 \, d\tau \leqslant (1+t)^{\delta} A^n(t).$$

In fact (3.12) is true for $\gamma = 1$ since by (3.8), (3.9), and (3.11)

$$A^0(t) = E(t) + \int_0^t \|(\phi_x, \psi_x)(\tau)\|^2 \, d\tau \leqslant \max\{c_0^{-1}, 1\} \, E(0), \qquad \forall t > 0$$

and

$$A^1(t) \leqslant C_B(1 + \gamma \max\{c_0^{-1}, 1\} \, E(0) + \gamma(1+t)^{1/3} \, [A^1(t)]^{1/3})$$

which immediately gives

$$A^1(t) \leqslant C_B(1+t)^{1/2}.$$

Assume that $n = k$, $A^k(t) \leqslant C_B(1+t)^{k-1/2}$. We have just seen that this is true for $k = 1$,

$$\begin{aligned} A^{k+1}(t) &\leqslant C_B[1 + \gamma A^k(t) + \gamma(1+t)^{(2k+1)/3} \, (A^{k+1}(t))^{1/3}] \\ &\leqslant C_B[1 + (1+t)^{k-1/2} + (1+t)^{k+1-1/2}] \leqslant C(1+t)^{k+1-1/2}, \end{aligned}$$

and the induction step is complete.

Recalling the definition of $A^{\gamma}(t)$ given in (3.11), we then have

$$(1+t)^{\gamma} \, E(t) + \int_0^t (1+t)^{\gamma} \, \|(\phi_x, \psi_x)(\tau)\|_{L^2}^2 \, d\tau \leqslant C_B(1+t)^{\gamma-1/2},$$

which finishes the proof. ∎

3.3. L^p *Decay*

Proof of Theorem 3.1. Proposition 3.3 gives

$$\int_{\mathbb{R}} (\phi^2 + \phi_x^2 + \psi_x^2) \, dx \leqslant C(1+t)^{-1/2}. \tag{3.13}$$

Then using the Sobolev inequality and (3.13) one has

$$\|\phi\|_{L^\infty} \leqslant \sqrt{2} \, \|\phi\|_{L^2} \, \|\phi_x\|_{L^2} \leqslant C(1+t)^{-1/2}.$$

The use of this estimate and the L^1 estimate in (3.5) shows that, in L^p,

$$\|\phi\|_{L^p} \leqslant \|\phi\|_{L^\infty}^{1-1/p} \ \|\phi\|_{L^1}^{1/p} \leqslant C(1+t)^{-(1/2)+(1/2p)}.$$

Now we estimate ψ. Since

$$\|\psi\|_{L^\infty} \leqslant \sqrt{2}\ \|\psi\|_{L^2}\ \|\psi_x\|_{L^2} \leqslant \sqrt{2}\ \|\psi\|_\infty^{1/2}\ \|\psi\|_{L^1}^{1/2}\ \|\psi_x\|_{L^2},$$

by the L^1 estimate in (3.5) for ψ, we have

$$\|\psi\|_{L^\infty} \leqslant 2\ \|\psi\|_{L^1}\ \|\psi_x\|_{L^2}^2 \leqslant C_B(1+t)^{-1/2}.$$

Therefore,

$$\|\psi\|_{L^p} \leqslant C(1+t)^{-(1/2)+(1/2p)}.$$

4. CONCLUDING REMARK

Now we discuss the L^p derivation of the relaxation rarefaction profile away from the rarefaction wave given in (1.5).

Let $r(x, t)$ be the rarefaction wave given in (1.5) and $(\bar{u}, \bar{v})$ be the relaxation rarefaction profile with initial data $\Gamma(r(x, t_0))$ for $t_0 > \delta$.

Put

$$N(t) := \|\bar{u}(\cdot, t) - r(x, t+t_0)\|_{L^1}, \qquad t \geqslant 0.$$

The corresponding L^p estimate is immediate.

LEMMA 4.1 (L^p Estimate). *Let $(\bar{u}, \bar{v})$ be the solution with initial data $\Gamma(r(x, t_0))$. Then*

$$\|\bar{u}(\cdot, t) - r(\cdot, t+t_0)\|_{L^p} \leqslant CN(t)^{(1/2)+(1/2p)}\ (t+t_0)^{(1/2)-(1/2p)}.$$

Proof. Thanks to Theorem 1.1, $\bar{u}$ is monotonous nondecreasing in x and $\bar{u}_x \leqslant K/(t+t_0)$. Let $t > 0$ be fixed and set

$$w(x, t) := \bar{u}((t+t_0)\,x, t) - r(x, 1).$$

By selfsimilarity of r one has

$$\|\bar{u}(x, t) - r(x, t+t_0)\|_{L^\infty} = \|\bar{u}((t+t_0)\,x, t) - r(x, 1)\|_{L^\infty} = \|w(\cdot, t)\|_{L^\infty}.$$

Using the defined L^1 bound $N(t)$ we have

$$\|w(\cdot, t)\|_{L^1} = \frac{1}{t+t_0}\ \|\bar{u}(\cdot, t) - r(\cdot, t+t_0)\|_{L^1} = \frac{N(t)}{t+t_0}.$$

Thus using $\bar{u}_x \leqslant K/(t+t_0)$ and $r_x(x, 1) \leqslant 1/\alpha$ one obtains

$$\|w\|_{L^\infty} \leqslant \sqrt{2}\, \|w_x\|_{L^\infty}^{1/2} \|w\|_{L^1}^{1/2} \leqslant \sqrt{2} \sup_x \left[(t+t_0)\, \bar{u}_x + r_x(x, 1)\right]^{1/2} \|w\|_{L^1}^{1/2}$$

$$\leqslant C \left(\frac{N(t)}{t+t_0}\right)^{1/2}.$$

Finally, in L^p, $p \in (1, \infty)$

$$\begin{aligned} \|\bar{u}(\cdot, t) - r(\cdot, t+t_0)\|_{L^p} &\leqslant \|\bar{u}(\cdot, t) - r(\cdot, t+t_0)\|_{L^\infty}^{1-1/p} \\ &\quad \times \|\bar{u}(\cdot, t) - r(\cdot, t+t_0)\|_{L^1}^{1/p} \\ &\leqslant CN(t)^{(1/2)+(1/2p)}\, (t+t_0)^{-(1/2)+(1/2p)}. \end{aligned}$$

The proof is complete.

A standard L^1 stability estimate for Riemann invariants, see, e.g., [12], leads to the L^1 estimate

$$\begin{aligned} \|(\bar{u}, \bar{v}) - \Gamma(r(\cdot, t+t_0))\|_{L^1} &\leqslant \int_0^t \int_{\mathbb{R}} |(a - f'^2)\, r_x|\, dx\, d\tau \\ &\leqslant C(f'(u_-) - f'(u_+))\, t, \end{aligned}$$

where (1.5) has been used. However, I conjecture actually the estimate

$$N(t) \sim 1 + \ln(t+t_0). \tag{4.1}$$

These estimates enable us to conclude

$$\|u(\cdot, t) - r(\cdot, t+t_0)\|_{L^p} \leqslant C[1 + \ln(t+t_0)]^{(1/2)+(1/2p)}\, (t+t_0)^{-(1/2)+(1/2p)}, \tag{4.2}$$

which is consistent with the result for viscous case obtained in [2]. My attempts at obtaining the estimate (4.1) have not been successful.

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