Critical Thresholds in Relaxation Systems with Resonance of Characteristic Speeds

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Abstract

In this paper, we consider hyperbolic relaxation systems arising from dynamic continuum traffic flow models including the well-known Payne and Whitham (PW) model. The equilibrium characteristic speed resonates with one characteristic speed of the full relaxation system in many physical scenarios in traffic flow, for which the usual subcharacteristic condition only marginally holds. In spite of this obstacle, we prove global in time regularity and finite time singularity formation of solutions simultaneously by showing the critical threshold phenomena associated with the underlying relaxation systems. We identify a lower threshold for finite time singularity in solutions and an upper threshold for the global existence of the smooth solution for two traffic models with different choices of traffic pressure and equilibrium velocity. The obtained thresholds are represented in terms of the initial slopes of the Riemann invariants and the initial density. The set of initial data leading to global smooth solutions is large, in particular allowing initial velocity of negative slope.

Keywords. Critical thresholds, singularity formation, quasi-linear relaxation model, global regularity, traffic flow.

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Short Running Title. Critical Thresholds in Relaxation Systems

1. Introduction

Consider the following quasi-linear hyperbolic system with relaxation

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \frac{\rho}{\tau}(v_e(\rho) - u), \end{cases} \quad x \in R, \ t > 0 \tag{1}$$

subject to the initial data

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in R, \tag{2}$$

where $\tau > 0$ is the relaxation time, $p(\rho)$ is the pressure with $p'(\rho) > 0$ and $v_e(\rho)$ is the equilibrium velocity with $v_e'(\rho) < 0$. This system arises from a continuum model of traffic flows, see [40, 43].

We are concerned with both global in time regularity and finite time singularity in solutions to such a relaxation system. As is known, the typical well-posedness result of a one dimensional system of quasi-linear hyperbolic balance laws asserts that either a solution exists for all time (global existence of the smooth solution) or else there is a finite time such that slopes of the solution become unbounded as the life span is approached (finite-time singularity), see e.g. Lax [19], John [17], Liu [34], Nishida [39], Dafermos and Hsiao [5], Luskin [37], Wang and Chen [42], Engelberg, Liu and Tadmor [7]. For the underlying relaxation system, our quest is whether there are critical thresholds for the initial data such that the global existence of the smooth solution or the finite time singularity depends only on whether the initial data cross such critical thresholds.

In the context of traffic flows, the system is often written as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x + \frac{p(\rho)_x}{\rho} = \frac{1}{\tau}(v_e(\rho) - u), \end{cases} \quad x \in R, \ t > 0, \tag{3}$$

where the first equation in (3) is a conservation law, while the second one describes drivers' acceleration behavior. The acceleration consists of a relaxation to an equilibrium speed-density relation and an anticipation factor which expresses the effect of drivers' reacting to conditions downstream. The third term in the second equation in (3) corresponds to the anticipation factor in comparison with pressure gradient in the momentum equation for gas flows. $p(\rho)$ is the traffic pressure and $v_e(\rho)$ is the desired equilibrium speed. It is often assumed that the desired equilibrium speed $v_e(\rho)$ is decreasing and satisfies $v_e(0) = v_f$ and $v_e(\rho_{\max}) = 0$ where v_f is the free flow speed and $\rho_{\max}$ is the maximum of concentration. $\tau > 0$ corresponds to drivers' responding time to the traffic. Note that system (1) and system (3) are equivalent if we consider smooth solutions.

Several well-known traffic flow models are special cases of (3). We are interested in two physical scenarios.

The first model is (3) with

$$p(\rho) = c_0^2 \rho. \tag{4}$$

This is a classical dynamic continuum model of traffic flow: the Payne [40] and Whitham [43] (PW) model. The PW model has been adopted in the study of traffic jam dynamics by Helbing [12], Jin and Zhang [15], Kerner and Konhäuser [16], and Li [23]. A global weak solution of the Cauchy problem was obtained in [36]. The data from the Lincoln Tunnel, New York obtained by Greenberg in 1959 [8] suggest

$$v_e(\rho) = c \ln \frac{\rho_{\max}}{\rho}, \quad 0 < \rho \leq \rho_{\max} \tag{5}$$

for some $c > 0$. The logarithmic formula does not give a finite value for velocity as $\rho \to 0$, but the continuum theory would be on dubious ground for very light traffic. In fact, we show in Section 3 that ρ is bounded away from zero if ρ_0 is. The equilibrium velocity we adopt is defined in (5) with $c = c_0$ and $\rho_{\max} = 1$. That is

$$v_e(\rho) = c_0 ln \frac{1}{\rho}, \quad 0 < \rho \leq 1. \tag{6}$$

The second model is Zhang's model [44]: (3) with

$$p(\rho) = \frac{v_f^2}{3} \rho^3 \tag{7}$$

and

$$v_e(\rho) = v_f (1 - \rho) \tag{8}$$

where v_f is the free flow speed. The equilibrium velocity defined in (8) is rescaled from an actual measurement done by Greenshields [10]. A global weak solution of the Cauchy problem (3) (2) with (7) and (8) for initial data of bounded total variation was obtained in [20] and the L^1 stability theory was established in [21].

The system (3) describes the relaxation process in traffic flows. In fact, relaxation phenomena arise naturally in many physical situations such as elasticity with memory, gas flow with thermal-non-equilibrium, phase transition, magneto-hydrodynamics, water waves. For hyperbolic relaxation problems, it has been shown by Whitham [43] that a sub-characteristic type condition is necessary for linear stability of the system. A remarkable development of the stability theory for various relaxation systems have appeared in past decades, see e.g., [3, 14, 18, 28, 27, 24, 35, 38, 41], relaying on some sub-characteristic type structure conditions [35]. Nonlinear stability of the traveling wave solutions of (3) with more general $p(\rho)$ and $v_e(\rho)$ is obtained again under the subcharacteristic conditions (16) by Li and Liu [24]. It is shown that the model (3) under the above two different settings supports only a marginal subcharacteristic condition (18), that is, the equilibrium characteristic speed resonates with one characteristic speed of the full relaxation system. The phenomenon also occurs in other traffic flow models, see, e.g., [1, 9, 22, 45]. Previous techniques of analysis relying upon such a subcharacteristic condition cannot be applied. Novel techniques for analyzing the underlying nonlinear dynamics are required.

In this paper we study the critical threshold phenomenon for system (3) with (4) (6) and with (7) (8), respectively. For each case we identify an upper threshold for the global existence of the smooth solution and a lower threshold for the finite time breakdown. In other words we show that only traffic with initial velocity gradient below certain threshold experiences congestion and the traffic stays smooth for initial velocity gradient above certain negative threshold. The technique is to track the nonlinear dynamics of the slopes of the Riemann invariants. For hyperbolic balance laws such as (3), the coupling of different characteristic fields makes it difficult to detect a sharp critical threshold, as observed in [31] for a 1D Euler-poisson system with pressure effects. Nevertheless, for the relaxation system (3) with (4) (6), we are able to decouple the ratio of the slope of one Riemann invariant and the half power of the density from the system and to track its dynamics, see (36). This and the *a priori* estimates of solutions in L^∞ enable us to identify the asserted thresholds. For the relaxation system (3) with (7) (8), the analysis can be performed in a similar manner.

The critical threshold phenomenon was first observed and studied in [7] for a class of Euler-poisson equations; and further extended to other problems of various types such as a convolution model for nonlinear conservation laws [29], nonlocal dispersive wave equations [25]. The study of multi-D critical threshold phenomena becomes more challenging, and a new tool of *spectral dynamics* has been first introduced in [30] to estimate the velocity gradient. Using spectral dynamics as a crucial tool the critical threshold phenomena have been justified for several interesting models [32, 33, 26].

The first result tells us the upper threshold for the global smoothness of (3) with (4) and (6).

Theorem 1.1 [Global in time regularity] Consider the relaxation system (3) with (4) and (6), subject to initial data (2) satisfying $(\rho_0, u_0) \in C^1(R) \times C^1(R)$ and

$$0 < \nu \leq \rho_0(x) \leq 1$$

for all $x \in R$ and for some $\nu > 0$. Denote

$$m = \frac{1}{\nu} e^{\frac{\|u_0\|_\infty}{c_0}}. \tag{9}$$

If both

$$-\frac{2}{\tau\sqrt{m}} \leq \frac{u_0'(x)}{\sqrt{\rho_0(x)}} + c_0 \frac{\rho_0'(x)}{\rho_0(x)^{\frac{3}{2}}} \leq 0$$

and

$$\frac{u_0'(x)}{\sqrt{\rho_0(x)}} - c_0 \frac{\rho_0'(x)}{\rho_0(x)^{\frac{3}{2}}} \geq 0$$

hold for all $x \in R$, then the Cauchy problem (3), (2) admits a global smooth solution satisfying

$$\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty + c_0|ln\nu|, \quad t \in R^+$$

and

$$m^{-1} \leq \rho(x,t) \leq m, \quad (x,t) \in R \times R^+.$$

Furthermore, we have the following estimates on the first derivatives

$$\min_{x \in R} A^+(x) \leq \frac{u_x(x,t)}{\sqrt{\rho(x,t)}} + c_0 \frac{\rho_x(x,t)}{\rho^{3/2}(x,t)} \leq 0,$$

and

$$\min_{x \in R} \frac{A^-(x)}{\sqrt{\rho_0(x)} + \sqrt{m}A^-(x)t/2} \leq \frac{u_x(x,t)}{\sqrt{\rho(x,t)}} - c_0 \frac{\rho_x(x,t)}{\rho^{3/2}(x,t)} \leq \max_{x \in R} A^-(x),$$

where

$$A^{\pm}(x) = \frac{u_0'(x)}{\sqrt{\rho_0(x)}} \pm c_0 \frac{\rho_0'(x)}{\rho_0^{3/2}(x)}. \tag{10}$$

The result on lower threshold for finite time singularity is summarized below.

Theorem 1.2 [Finite-time singularity] Consider the same problem as stated in Theorem 1.1. If

$$\frac{u_0'(x)}{\sqrt{\rho_0(x)}} + c_0 \frac{\rho_0'(x)}{\rho_0(x)^{\frac{3}{2}}} \geq -\frac{2\sqrt{m}}{\tau}$$

fails to hold at any point $x \in R$, then the solution of Cauchy problem (3) (2) must develop singularity in a finite time T^*. Moreover,

$$\lim_{t \to T^*} \left(\min_{x \in R} \left\{ \frac{u_x(x,t)}{\sqrt{\rho(x,t)}} + c_0 \frac{\rho_x(x,t)}{\rho^{3/2}(x,t)} \right\} \right) = -\infty$$

and

$$T^* < \tau \min_{x \in R} ln \left| \frac{A^+(x)}{A^+(x) + \frac{2\sqrt{m}}{\tau}} \right|,$$

where m and $A^{\pm}$ are given in (9) and (10), respectively.

For lower and upper thresholds for system (3) with (7) and (8), we state our results as below.

Theorem 1.3 [Upper and lower thresholds] Consider the relaxation system (3) with (7) and (8), subject to initial data (2) satisfying $(\rho_0, u_0) \in C^1(R) \times C^1(R)$

(i) If both

$$-\frac{1}{\tau} \leq u_0'(x) + v_f \rho_0'(x) \leq 0$$

and

$$u_0'(x) - v_f \rho_0'(x) \geq 0$$

hold for all $x \in R$, then the Cauchy problem (3), (2) with (7) and (8) admits a unique global smooth solution.

(ii) If

$$u_0'(x) + v_f \rho_0'(x) \geq -\frac{1}{\tau}$$

fails to hold at any point $x \in R$, then the solution of Cauchy problem (3) (2) with (7) and (8) must develop finite time singularity.

Concerning these theorems, several remarks are in order.
Remarks:
(i) The set of initial data leading to global regularity is rich. In particular, it allows the initial Riemann invariant of negative slope. This is in sharp contrast to the generic breakdown in homogeneous hyperbolic systems [19].
(ii) No smallness of data is assumed for the global existence of the smooth solution. The critical thresholds we identified reveal the genuine nonlinear phenomena hidden in the system.
(iii) Note that the bounds for the derivatives of the initial Riemann invariants are of order $\frac{1}{\tau}$. This implies that the smaller the relaxation time τ, the larger the set of initial data leading to global smooth solutions. This means that the shorter the drivers' reaction time, the larger the set of initial conditions leading to global smooth traffic flows. This is compared with the fact that in a class of optimal velocity models, the smaller the relaxation time τ is, the larger the linear stable region is, [2]. Similar phenomena occur in other problems. For example, in Euler-Poisson equations for plasma sheath problem, the small Debey length does delay the finite time breakdown [25]; also small Rossby number in rotational Euler equations helps to prevent breakdown from happening in $O(1)$ time [33].

We now conclude this section by outlining the rest of this paper. In Section 2 we present preliminaries about the hyperbolic relaxation system (3) and reformulation of corresponding results in terms of the Riemann invariants. Section 3 contains *a priori* estimates of solutions in L^∞ norm for (3). Section 4 is devoted to identifying the upper threshold for global existence of smooth solutions, as well as the lower threshold for the finite time singularity formation. This is done by deriving the a priori estimate of the derivatives of the solution.

2. Preliminaries and Reformulation of the Problem

For the pressure and the equilibrium velocity defined in (4), (6) and in (7) (8), it is easy to check that

$$p'(\rho) = (\rho v_e'(\rho))^2 > 0. \tag{11}$$

Thus the system (3) is a strictly hyperbolic balance law, the characteristic speeds being

$$\lambda_1(\rho, u) = u + \rho v_e'(\rho) < u - \rho v_e'(\rho) = \lambda_2(\rho, u). \tag{12}$$

The corresponding right eigenvectors of the Jacobian of the flux are

$$r_i(\rho, u) = (\rho, (-1)^{i+1} v_e'(\rho))^T, \quad i = 1,\ 2.$$

Both characteristic families are genuinely nonlinear

$$\nabla\lambda_i(\rho, v) \cdot r_i(\rho, v) = (-1)^{i+1} \frac{d^2}{d\rho^2}(\rho v_e(\rho)) \neq 0, \quad i = 1,\ 2.$$

Recall that in the usual relaxation limit, $\tau \to 0^+$, the leading order of the relaxation system (3) is the LWR (Lighthill, Whitham and Richards) model

$$\rho_t + (q(\rho))_x = 0, \tag{13}$$

where

$$q(\rho) = \rho v_e(\rho) \tag{14}$$

is the equilibrium flux which is the fundamental diagram in traffic flow. The equilibrium characteristic speed is

$$\lambda_*(\rho) = q'(\rho) = v_e(\rho) + \rho v_e'(\rho). \tag{15}$$

The so-called subcharacteristic condition is

$$\lambda_1 < \lambda_* < \lambda_2 \tag{16}$$

on the equilibrium curve $u = v_e(\rho)$. (16) was shown to be a necessary condition for linear stability, Whitham [43].

It can be derived formally [35], in the same spirit as the classical Chapman-Enskog expansion, that the relaxation process is approximated by a viscous conservation law

$$\rho_t + (q(\rho))_x = (\beta(\rho)\rho_x)_x \tag{17}$$

where

$$\beta(\rho) = -\tau(\lambda_* - \lambda_1)(\lambda_* - \lambda_2).$$

Note that (17) is dissipative, $\beta(\rho) > 0$, provided that subcharacteristic condition (16) is satisfied. Similar to the diffusion, the relaxation term has smoothing and dissipative effects for the hyperbolic conservation laws. Nonlinear stability of the traveling wave solutions of (3) with more general $p(\rho)$

and $v_e(\rho)$ is obtained under the subcharacteristic conditions (16) by Li and Liu [24].

From (12) and (15) we see that in both cases, (3) with (4) (6) and (3) with (7) (8), the subcharacteristic condition (16) is only satisfied marginally

$$\lambda_1 = \lambda_* < \lambda_2. \tag{18}$$

Thus the diffusion term in the Chapman-Enskog expansion of (3) vanishes,

$$\beta(\rho) = 0.$$

Hence previous stability analysis based on such a dissipation mechanism cannot be applied. We shall track the nonlinear dynamics along characteristic fields.

We now start reformulation of (3) with (4) and (6). Set $w = ln\rho$. Multiplying the first equation of (3) by $\frac{1}{\rho}$, we have

$$\begin{cases} w_t + uw_x + u_x = 0, \\ u_t + uu_x + c_0^2 w_x = \frac{1}{\tau}(-c_0 w - u). \end{cases} \tag{19}$$

Multiplying system (19) by the left eigenvectors of the Jacobian of the flux

$$l_i(w,u) = ((-1)^i c_0, 1), \quad i = 1,\ 2,$$

we have

$$\begin{cases} R_t^- + \lambda_1 R_x^- = -\frac{1}{\tau} R^+, \\ R_t^+ + \lambda_2 R_x^+ = -\frac{1}{\tau} R^+, \end{cases} \tag{20}$$

where

$$\lambda_1 = \frac{R^- + R^+}{2} - c_0, \quad \lambda_2 = \frac{R^- + R^+}{2} + c_0 \tag{21}$$

and the Riemann invariants

$$\begin{cases} R^-(w,u) = u - c_0 w \\ R^+(w,u) = u + c_0 w \end{cases} \tag{22}$$

define a one-to-one mapping from (ρ, u), $\rho > 0$, to (R^-, R^+) in the phase space.

The corresponding initial data is

$$(R^-, R^+)(x,0) = (R_0^-, R_0^+)(x) = (u_0 - c_0 ln\rho_0, u_0 + c_0 ln\rho_0)(x). \qquad (23)$$

In order to prove Theorem 1.1 and Theorem 1.2, it suffices to establish the following for Cauchy problem (20), (23).

Theorem 2.1 Consider the system (20) subject to C^1 bounded initial data (23). Let m be defined in (9). If

$$0 \geq \frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}} \geq -\frac{2}{\tau\sqrt{m}}, \quad x \in R$$

and

$$R_{0,x}^-(x) \geq 0, \quad x \in R,$$

then the Cauchy problem (20) (23) has a unique smooth solution for all time $t > 0$.

Moreover, we have

$$0 \geq \frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}}, \quad (x,t) \in R \times R^+,$$

and

$$\max_{x\in R} \frac{R_{0,x}^-(x)}{\sqrt{\rho_0(x)}} \geq \frac{R_x^-(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R_{0,x}^-(x)}{\sqrt{\rho_0(x)} + \frac{\sqrt{m}}{2} R_{0,x}^-(x)t}, \quad (x,t) \in R \times R^+.$$

Theorem 2.2 Assume that $R_0^\pm(x) \in C^1(R)$ and $\|R_0^\pm\|_\infty$ are bounded. If

$$\frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}} \geq -\frac{2\sqrt{m}}{\tau}$$

fails to hold at any point $x \in R$, then the C^1 solution of the Cauchy problem (20) (23) will develop a finite time singularity. Moreover,

$$\lim_{t\to T^*} \left(\min_{x\in R} \left\{ \frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}} \right\} \right) = -\infty$$

for

$$T^* < \tau \min_{x\in R} ln \left(\frac{\tau R_{0,x}^+(x)}{2\sqrt{m\rho_0(x)} + \tau R_{0,x}^+(x)} \right).$$

The local existence of smooth solutions of hyperbolic problem is classical, see e.g. Douglis [6] and Hartman and Wintner [11]. According to the theory of first order quasilinear hyperbolic equations [4], solutions to initial value problems exist as long as one can place an *a priori* limitation on the magnitude of their first derivatives.

Equipped with the classical local existence results in [6] and [11], we need only to establish the *a priori* estimates, which will be presented in Lemma 3.1, Lemma 4.1. The finite time singularity formation follows from the proof of Lemma 3.1.

For the system (3) with (7) and (8) we have

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + v_f^2 \rho\rho_x = \frac{1}{\tau}(v_f(1-\rho) - u). \end{cases} \tag{24}$$

Multiplying system (24) by the left eigenvectors of the Jacobian of the flux

$$l_i(w,u) = ((-1)^i v_f, 1), \quad i = 1,\ 2,$$

we have

$$\begin{cases} R_t^- + \lambda_1 R_x^- = -\frac{1}{\tau} R^+, \\ R_t^+ + \lambda_2 R_x^+ = -\frac{1}{\tau} R^+, \end{cases} \tag{25}$$

where

$$\lambda_1 = R^- - v_f, \quad \lambda_2 = R^+ + v_f \tag{26}$$

and the Riemann invariants

$$\begin{cases} R^-(\rho,u) = u - v_f\rho + v_f \\ R^+(\rho,u) = u + v_f\rho - v_f \end{cases} \tag{27}$$

define a one-to-one mapping from (ρ, u) to (R^-, R^+) in the entire phase space.

The corresponding initial data is

$$(R^-, R^+)(x,0) = (R_0^-, R_0^+)(x) = (u_0 - v_f\rho_0 + v_f, u_0 + v_f\rho_0 - v_f)(x). \tag{28}$$

Theorem 2.3 Consider the system (25) subject to C^1 bounded initial data (28).
(i) If

$$0 \geq R_{0,x}^+(x) \geq -\frac{1}{\tau}, \quad x \in R$$

and

$$R_{0,x}^{-}(x) \geq 0, \quad x \in R,$$

then the Cauchy problem (25) (28) has a unique smooth solution for all time $t > 0$. Moreover,

$$0 \geq R_x^{+}(x,t) \geq -1/\tau, \quad (x,t) \in R \times R^{+},$$

and

$$\max_{x \in R} R_{0,x}^{-}(x) \geq R_x^{-}(x,t) \geq \min_{x \in R} \frac{R_{0,x}^{-}(x)}{1 + R_{0,x}^{-}(x)t}, \quad (x,t) \in R \times R^{+}.$$

(ii) If

$$R_{0,x}^{+}(x) \geq -\frac{1}{\tau}$$

fails to hold at any point $x \in R$, then the C^1 solution of the Cauchy problem (25) (28) will develop a finite time singularity. Moreover,

$$\lim_{t \to T^*} \left(\min_{x \in R} \{ R_x^{+}(x,t) \} \right) = -\infty$$

for

$$T^* < \tau \min_{x \in R} ln \left(\frac{\tau R_{0,x}^{+}(x)}{1 + \tau R_{0,x}^{+}(x)} \right).$$

This result follows from the classical local existence results and the *a priori* estimates obtained in Lemma 3.1 and Lemma 4.2. The finite time singularity formation follows from the proof of Lemma 4.2.

Using expressions of the Riemann invariants in two cases to convert back to variables u and ρ, we prove our main results as stated in Theorem 1.1–Theorem 1.3. Note that in Lemma 3.2, we showed that ρ is bounded away from zero if ρ_0 is. Therefore finite time blow up of $\frac{R_x^{+}(x,t)}{\sqrt{\rho(x,t)}}$ implies the finite time blow up of $u_x(x,t)$ or $\rho_x(x,t)$. Theorem 1.2 follows directly from Theorem 2.2, and Theorem 1.3 follows from Theorem 2.3.

3. Bounds for Smooth Solutions

We give *a priori* estimates of solutions in L^∞ norm in this section.

We first establish the uniform bounds for the Riemann invariants (R^-, R^+) of (3) with (4) (6).

Lemma 3.1 Assume that $R_0^{\pm} \in C^1(R)$ and that

$$\|R_0^-\|_\infty + \|R_0^+\|_\infty \leq M$$

for some $M > 0$. Then the C^1 solution of the Cauchy problem (20) (23) satisfies the *a priori* estimates

$$\|R^+(\cdot,t)\|_\infty \leq \|R_0^+\|_\infty e^{-\frac{t}{\tau}} \tag{29}$$

and

$$\|R^-(\cdot,t)\|_\infty + \|R^+(\cdot,t)\|_\infty \leq M \tag{30}$$

for all $t \geq 0$ as long as the C^1 solution exists.

Proof. Integrating the second equation in (20) along the second characteristics $x_2(t,\alpha)$

$$\frac{dx_2}{dt} = \lambda_2 = u + c_0, \; x_2(0,\alpha) = \alpha,$$

we have

$$R^+(x_2(t,\alpha),t) = R_0^+(\alpha)e^{-\frac{t}{\tau}},$$

which leads to the asserted bound (29).

Now integrating the first equation in (20) along the first characteristics $x_1(t,\beta)$

$$\frac{dx_1}{dt} = \lambda_1 = u - c_0, \; x_1(0,\beta) = \beta,$$

we have

$$R^-(x_1(t,\beta),t) = R_0^-(\beta) - \frac{1}{\tau}\int_0^t R^+(x_1(s,\beta),s)ds$$

Using the above decay result for $\|R^+(\cdot,t)\|_\infty$, we have

$$\|R^-(\cdot,t)\|_\infty \leq \|R_0^-(\cdot)\|_\infty + \|R_0^+(\cdot)\|_\infty(1 - e^{-\frac{t}{\tau}}).$$

This added upon (29) gives the desired bound (30). The proof is complete.

Lemma 3.2 Assume the initial data (2) are uniformly bounded with

$$-\|u_0\|_\infty \leq u_0(x) \leq \|u_0\|_\infty, \quad 0 < \nu \leq \rho_0(x) \leq 1$$

for all $x \in R$.

Let (ρ, u) be a C^1 solution of (3) with (4) (6) determined from $R^{\pm}$, then the density satisfies

$$m^{-1} \leq \rho(x,t) \leq m \tag{31}$$

for all $t \geq 0$ as long as the C^1 solution exists, where

$$m = \frac{1}{\nu} e^{\frac{\|u_0\|_\infty}{c_0}}. \tag{32}$$

The velocity is also bounded

$$\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty + c_0 |ln\nu|$$

for all $t \geq 0$ as long as the C^1 solution exists.

Proof. From the relation $w = ln\rho$ it follows that

$$e^{-\|w(\cdot,t)\|_\infty} \leq \rho(x, t) \leq e^{\|w(\cdot,t)\|_\infty}, \quad x \in R. \tag{33}$$

Applying Lemma 3.1 to estimate

$$w = \frac{R^+ - R^-}{2c_0},$$

we have

$$\|w(\cdot, t)\|_\infty \leq \frac{1}{2c_0}(\|R^-(\cdot, t)\|_\infty + \|R^+(\cdot, t)\|_\infty) \leq \frac{1}{2c_0}(\|R_0^-\|_\infty + \|R_0^+\|_\infty).$$

Using expressions of the Riemann invariants (22) we have

$$\|w(\cdot, t)\|_\infty \leq \frac{1}{2c_0}(\|u_0 - c_0 ln\rho_0\|_\infty + \|u_0 + c_0 ln\rho_0\|_\infty) \leq \frac{\|u_0\|_\infty}{c_0} + \|ln\rho_0\|_\infty.$$

Under the conditions that $0 < \nu \leq \rho_0(x) \leq 1$ for all $x \in R$,

$$\|ln\rho_0\|_\infty \leq |ln\nu|.$$

These when inserted into (33) gives the asserted bounds (31) for the density. The bound for velocity follows immediately when recalling that (22) implies $u = (R^+ + R^-)/2$.

The uniform bounds for the Riemann invariants (R^-, R^+) of (3) with (7) (8) can be obtained similarly. Indeed, the L^∞ estimates established in Lemma 3.1 still hold for the system (25) with $C^1(R)$ data (28).

4. Critical Thresholds

In order to identify the upper threshold for global existence of smooth solutions, as well as the lower threshold for the finite time singularity formation as claimed in Theorem 1.1, 1.2 and 1.3, we derive the *a priori* estimate of the derivatives of the Riemann invariants $R^{\pm}(x,t)$ of (3) with (4) (6).

Denote $r^- = R_x^-$ and $r^+ = R_x^+$, we shall show that $R_x^{\pm}$ are bounded when initial values of them, i.e., $r_0^{\pm} := R_{0,x}^{\pm}$ are bounded by some critical thresholds.

More precisely, we have the following.

Lemma 4.1 Assume that $R_0^{\pm}(x) \in C^1(R)$ and $\|R_0^{\pm}\|_\infty$ are bounded. Let m be defined in (32). If further

$$0 \geq \frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}} \geq -\frac{2}{\tau\sqrt{m}}, \quad x \in R$$

and

$$R_{0,x}^-(x) \geq 0, \quad x \in R,$$

then any C^1 solution of the Cauchy problem (20) (23) has the *a priori* estimates

$$0 \geq \frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}}$$

and

$$\max_{x\in R} \frac{R_{0,x}^-(x)}{\sqrt{\rho_0(x)}} \geq \frac{R_x^-(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R_{0,x}^-(x)}{\sqrt{\rho_0(x)} + \frac{\sqrt{m}}{2} R_{0,x}^-(x)t}$$

for all $x \in R$ and $t \geq 0$ as long as the C^1 solution exists.

Proof. From (21) we derive that

$$\lambda_{1,x} = \lambda_{2,x} = \frac{r^+ + r^-}{2}.$$

We differentiate (20) with respect to x to obtain

$$\begin{cases} r_t^- + \lambda_1 r_x^- + \frac{r^- + r^+}{2} r^- = -\frac{1}{\tau} r^+ \\ r_t^+ + \lambda_2 r_x^+ + \frac{r^- + r^+}{2} r^+ = -\frac{1}{\tau} r^+. \end{cases} \tag{34}$$

From the first equation in (3) and using $R^- = u - c_0 ln\rho$ we derive

$$\rho_t + \lambda_2 \rho_x = -\rho u_x + c_0 \rho_x = -\rho r^-. \tag{35}$$

Multiplying the second equation in (34) by $\frac{1}{\sqrt{\rho}}$ and using the above equation (35) we get

$$\left(\frac{r^+}{\sqrt{\rho}}\right)_t + \lambda_2 \left(\frac{r^+}{\sqrt{\rho}}\right)_x = -\frac{r^+}{\sqrt{\rho}}\left(\frac{1}{\tau} + \frac{r^+}{2}\right).$$

Denote $a = \frac{r^+}{\sqrt{\rho}}$. Then a satisfies

$$a_t + \lambda_2 a_x = -a\left(\frac{1}{\tau} + \frac{\sqrt{\rho}}{2}a\right).$$

Along the second characteristics $x_2(t,\alpha)$: $\frac{dx_2}{dt} = \lambda_2,\ x_2(0,\alpha) = \alpha$, we have

$$\frac{d}{dt}a = -a\left(\frac{1}{\tau} + \frac{\sqrt{\rho}}{2}a\right),$$

which when using bounds for ρ from Lemma 3.2 leads to the following

$$-a\left(\frac{1}{\tau} + \frac{\sqrt{m}}{2}a\right) \le \frac{d}{dt}a \le -a\left(\frac{1}{\tau} + \frac{1}{2\sqrt{m}}a\right). \tag{36}$$

Solving these two differential inequalities, we conclude that a remains bounded

$$-\frac{2\sqrt{m}}{\tau} \le a(x_2(t,\alpha),t) \le \max\{0, a_0(\alpha)\} \tag{37}$$

provided

$$a_0(\alpha) \ge -\frac{2}{\tau\sqrt{m}}, \quad \forall \alpha \in R.$$

On the other hand, a will blow up in a finite time if there exists a $\alpha^* \in R$ such that

$$a_0(\alpha^*) < -\frac{2\sqrt{m}}{\tau}. \tag{38}$$

More precisely, the right differential inequality in (36) enables us to obtain

$$a(x_2(t,\alpha),t) \le \frac{2\sqrt{m}a_0(\alpha)}{(2\sqrt{m} + \tau a_0(\alpha))e^{t/\tau} - \tau a_0(\alpha)}. \tag{39}$$

For initial data satisfying (38), the right hand side of (39) will become $-\infty$ at a first time

$$T = \tau \min_{\alpha\in R} ln\left(\frac{\tau a_0(\alpha)}{2\sqrt{m} + \tau a_0(\alpha)}\right) < +\infty.$$

Therefore, there exists a $T^* < T$ such that

$$\lim_{t \to T^*} (\min\{a(x,t)\}) = -\infty.$$

Now we examine $r^- = R_x^-$. From the first equation in (3) we derive

$$\rho_t + \lambda_1 \rho_x = -\rho u_x - c_0 \rho_x = -\rho r^+.$$

Multiplying the first equation in (34) by $\frac{1}{\sqrt{\rho}}$ and using the above equation we get

$$\left(\frac{r^-}{\sqrt{\rho}}\right)_t + \lambda_1 \left(\frac{r^-}{\sqrt{\rho}}\right)_x = -\frac{1}{\tau} a - \frac{(r^-)^2}{2\sqrt{\rho}}.$$

Let $b = \frac{r^-}{\sqrt{\rho}}$. Then b satisfies

$$b_t + \lambda_1 b_x = -\frac{1}{\tau} a - \frac{\sqrt{\rho}}{2} b^2.$$

It follows from (37) that if

$$0 \geq a_0(\alpha) \geq -\frac{2}{\tau\sqrt{m}}, \quad \forall \alpha \in R, \tag{40}$$

then

$$0 \geq a(x,t) \geq -\frac{2\sqrt{m}}{\tau}, \quad (x,t) \in R \times R^+.$$

Assuming (40) and let $x_1(t,\beta)$ be the first characteristics, along which we have

$$\frac{2\sqrt{m}}{\tau^2} - \frac{1}{2\sqrt{m}} b^2 \geq \frac{d}{dt} b \geq -\frac{\sqrt{m}}{2} b^2, \quad \frac{d}{dt} := \partial_t + \lambda_1 \partial_x.$$

If

$$b_0(\beta) \geq 0, \quad \forall \beta \in R,$$

then b stays bounded. Indeed

$$\frac{2\sqrt{m}}{\tau} h(t,\beta) \geq b(x_1(t,\beta),t) \geq \frac{b_0(\beta)}{1 + \frac{\sqrt{m}}{2} b_0(\beta) t}$$

where

$$h(t,\beta) = \frac{(b_0(\beta) + \frac{2\sqrt{m}}{\tau}) e^{\frac{t}{\tau}} + (b_0(\beta) - \frac{2\sqrt{m}}{\tau}) e^{-\frac{t}{\tau}}}{(b_0(\beta) + \frac{2\sqrt{m}}{\tau}) e^{\frac{t}{\tau}} - (b_0(\beta) - \frac{2\sqrt{m}}{\tau}) e^{-\frac{t}{\tau}}}, \quad \beta \in R.$$

Note that when $b_0 \geq 0$, $h(t,\beta)$ is a decreasing function in time and satisfies

$$1 \leq h(t,\beta) \leq \frac{\tau b_0(\beta)}{2\sqrt{m}}.$$

Therefore, for $b_0(\beta) \geq 0$ for all $\beta \in R$,

$$\frac{b_0(\beta)}{1+\frac{\sqrt{m}}{2}b_0(\beta)t} \leq b(x_1(t,\beta),t) \leq b_0(\beta),$$

which when optimizing the bounds in terms of the parameter β leads to the desired estimates. The proof of Lemma 4.1 is thus complete.

In our analysis above we also found a threshold condition (38) for the finite time singularity in $a(x,t) = \frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}}$, which leads to the Theorem 2.2.

We now turn to the *a priori* estimate of the derivatives of the Riemann invariants $R^{\pm}(x,t)$ of (3) with (7) (8). Denote $r^- = R_x^-$ and $r^+ = R_x^+$, we shall show that $R_x^{\pm}$ are bounded when initial values of them, i.e., $r_0^{\pm} := R_{0,x}^{\pm}$ are bounded by some critical thresholds.

Lemma 4.2 Assume that $R_0^{\pm}(x) \in C^1(R)$ and $\|R_0^{\pm}\|_\infty$ are bounded. If further

$$0 \geq R_{0,x}^+(x) \geq -\frac{1}{\tau}, \quad x \in R$$

and

$$R_{0,x}^-(x) \geq 0, \quad x \in R,$$

then any C^1 solution of the Cauchy problem (25) (28) has the *a priori* estimates

$$0 \geq R_x^+(x,t) \geq \min_{x\in R} R_{0,x}^+(x)$$

and

$$\max_{x\in R} R_{0,x}^-(x) \geq R_x^-(x,t) \geq \min_{x\in R} \frac{R_{0,x}^-(x)}{1+R_{0,x}^-(x)t}$$

for all $x \in R$ and $t \geq 0$ as long as the C^1 solution exists.

Proof. From (26) we derive that

$$\lambda_{1,x} = r^-, \quad \lambda_{2,x} = r^+.$$

We differentiate (25) with respect to x to obtain

$$\begin{cases} r_t^- + \lambda_1 r_x^- + (r^-)^2 = -\frac{1}{\tau}r^+ \\ r_t^+ + \lambda_2 r_x^+ + (r^+)^2 = -\frac{1}{\tau}r^+. \end{cases} \tag{41}$$

Rewrite the second equation for r^+ to get

$$r_t^+ + \lambda_2 r_x^+ = -r^+ \left(\frac{1}{\tau} + r^+ \right).$$

Along the second characteristics $x_2(t, \alpha)$: $\frac{dx_2}{dt} = \lambda_2,\ x_2(0, \alpha) = \alpha$, we have

$$\frac{d}{dt} r^+ = -r^+ \left(\frac{1}{\tau} + r^+ \right).$$

Solving this differential equation, we obtain

$$r^+(x_2(t, \alpha), t) = \frac{r_0^+(\alpha)}{(\tau r_0^+(\alpha) + 1)e^{t/\tau} - \tau r_0^+(\alpha)}$$

which remains bounded

$$-\frac{1}{\tau} \le r^+(x_2(t, \alpha), t) \le \max\{0, r_0^+(\alpha)\} \tag{42}$$

if and only if

$$r_0^+(\alpha) \ge -\frac{1}{\tau}, \quad \forall \alpha \in R.$$

For initial data failed to satisfy the above, the solution r^+ will becomes $-\infty$ at some time before

$$T = \tau \min_{\alpha \in R} ln \left(\frac{\tau r_0^+(\alpha)}{1 + \tau r_0^+(\alpha)} \right) < +\infty.$$

Now we examine $r^- = R_x^-$, which satisfies

$$r_t^- + \lambda_1 r_x^- = -\frac{r^+}{\tau} - (r^-)^2.$$

It follows from (42) that if

$$0 \ge r_0^+(\alpha) \ge -\frac{1}{\tau}, \quad \forall \alpha \in R, \tag{43}$$

then

$$0 \ge r^+(x, t) \ge -\frac{1}{\tau}, \quad (x, t) \in R \times R^+.$$

Assuming (40) and let $x_1(t, \beta)$ be the first characteristics, along which we have

$$\frac{1}{\tau^2} - (r^-)^2 \ge \frac{d}{dt} r^- \ge -(r^-)^2, \quad \frac{d}{dt} := \partial_t + \lambda_1 \partial_x.$$

If

$$r_0^-(\beta) \geq 0, \quad \forall \beta \in R,$$

then r^- stays bounded

$$\frac{r_0^-(\beta)}{1 + r_0^-(\beta)t} \leq r^-(x_1(t,\beta),t) \leq r_0^-(\beta)$$

which when optimizing the bounds in terms of the parameter β leads to the desired estimates. The proof of Lemma 4.2 is thus complete.

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