Critical Thresholds in a Relaxation Model for Traffic Flows

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Abstract

In this paper, we consider a hyperbolic relaxation system arising from a dynamic continuum traffic flow model. The equilibrium characteristic speed resonates with one characteristic speed of the full relaxation system in this model. Thus the usual sub-characteristic condition only holds marginally. In spite of this obstacle, we prove global in time regularity and finite time singularity formation of solutions simultaneously by showing the critical threshold phenomena associated with the underlying relaxation system. We identify five upper thresholds for finite time singularity in solutions and three lower thresholds for global existence of smooth solutions. The set of initial data leading to global smooth solutions is large, in particular allowing initial velocity of negative slope. Our results show that the shorter the drivers' responding time to the traffic, the larger the set of initial conditions leading to global smooth solutions which correctly predicts the empirical findings for traffic flows.

Keywords. Critical thresholds, singularity formation, quasi-linear relaxation model, global regularity, traffic flow.

AMS(MOS) subject classifications. 35B30, 35B40, 35L65, 76L05, 90B20.

Short Running Title. Critical Thresholds in Relaxation Systems

1. Introduction

In this paper we continue to investigate the critical threshold phenomenon for the following quasi-linear hyperbolic system with relaxation

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x + \frac{p(\rho)_x}{\rho} = \frac{1}{\tau}(v_e(\rho) - u), \end{cases} \qquad x \in R,\ t > 0 \tag{1}$$

subject to the initial data

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in R \tag{2}$$

where $\tau > 0$ is the relaxation time, $p(\rho)$ is the pressure with $p'(\rho) > 0$ and $v_e(\rho)$ is the equilibrium velocity with $v_e'(\rho) < 0$. This system arises from a continuum model of traffic flows, see [36, 39, 40].

We are concerned with both global in time regularity and finite time singularity in solutions to such a relaxation system. As is known, the typical well-posedness result of a one dimensional system of quasi-linear hyperbolic balance laws asserts that either a solution exists for all time or else there is a finite time such that slopes of the solution become unbounded as the life span is approached, see e.g. Lax [16], John [14], Liu [32], Nishida [35], Dafermos and Hsiao [5], Wang and Chen [38], Engelberg, Liu and Tadmor [7]. In [22], we identified one lower threshold for global existence of smooth solutions and one upper threshold for the finite time breakdown. The aim of the current paper is to further our analysis to study solution behaviors for more general data. Indeed we are able to obtain three lower thresholds for global existence of smooth solutions and five upper thresholds for the finite time breakdown.

In the context of traffic flows, the first equation in (1) is a conservation law, while the second one describes drivers' acceleration behavior. It is often assumed that the desired equilibrium speed $v_e(\rho)$ is decreasing and satisfies $v_e(0) = v_f$ and $v_e(\rho_{\max}) = 0$ where v_f is the free flow speed and $\rho_{\max}$ is the maximum of concentration. $\tau > 0$ corresponds to drivers' responding time to the traffic. Several well-known traffic flow models are special cases of (1), [22]. We focus only on one physical scenario in this paper: Zhang's model [40] which is (1) with

$$p(\rho) = \frac{v_f^2}{3}\rho^3 \tag{3}$$

and

$$v_e(\rho) = v_f(1 - \rho), \tag{4}$$

where v_f is the free flow speed. The equilibrium velocity defined in (4) is rescaled from an actual measurement done by Greenshields [9]. A global weak solution of the Cauchy problem (1) (2) with (3) and (4) for initial data of bounded total variation was obtained in [17] and the L^1 stability theory was established in [18].

For hyperbolic systems with relaxation, it has been shown by Whitham [39] that a sub-characteristic type condition is necessary for linear stability of the system. A remarkable development of the stability theory for various relaxation systems have appeared in past decades, see e.g., [3, 12, 15, 26, 25, 21, 33, 34], relying on some sub-characteristic type structure conditions [33]. Nonlinear stability of the traveling wave solutions of (1) with more general $p(\rho)$ and $v_e(\rho)$ is obtained again under the subcharacteristic conditions (12) by Li and Liu [21]. The model (1) with (3) and (4) supports only a marginal subcharacteristic condition (14), that is, the equilibrium characteristic speed resonates with one characteristic speed of the full relaxation system. The phenomenon also occurs in other traffic flow models, see, e.g., [1, 8, 19]. Previous techniques of analysis relying upon such a sub-characteristic condition cannot be applied. In [22], we have developed novel techniques for analyzing the underlying nonlinear dynamics.

Following [22], we track nonlinear dynamics of slopes of the Riemann invariants along two characteristic fields. For hyperbolic balance laws such as (1), the coupling of different characteristic fields makes it difficult to detect a sharp critical threshold, as observed in [29], and further studied in [37] for a 1D Euler-poisson system with pressure effects. The situation for relaxation system (1) is more subtle. Nevertheless, for the physical scenario with (3) (4), we are able to decouple slope dynamics of one Riemann invariant from the system, and track dynamics of the whole system effectively. The genuine nonlinearity of the hyperbolic system (1) and the *a priori* estimates of solutions enable us to identify the asserted thresholds.

We state our critical threshold results as below.

Theorem 1.1 [**Global in time regularity**]

Consider the relaxation system (1) with (3) and (4), subject to initial data (2) satisfying $(\rho_0, u_0) \in C^1(R) \times C^1(R)$. Let

$$r^{\pm}(x,t) = u_x(x,t) \pm v_f \rho_x(x,t)$$

and

$$r^{\pm}(x,0) = r_0^{\pm}(x)$$

for all $x \in R$ and $t > 0$.

If one of the following is satisfied, then the Cauchy problem (1), (2) with (3) and (4) admits a unique global smooth solution.

(i) Both

$$-\frac{1}{\tau} \le r_0^+(x) \le 0 \quad \text{and} \quad r_0^-(x) \ge 0$$

hold for all $x \in R$;

(ii)

$$0 < \frac{1}{\tau} r_0^+(\alpha) < \delta (r_0^-(\beta))^2$$

for all $\alpha, \beta \in R$ and for some $0 < \delta < 1$;
(iii)

$$-\frac{1}{2}\frac{r_0^+(\alpha)}{\tau} > (r_0^-(\beta))^2, \ -\delta \le r_0^-(\beta) \le 0, \ -\frac{1}{\tau} \le r_0^+(\alpha) < 0$$

for all $\alpha, \beta \in R$ and for some $\delta > 0$.

Theorem 1.2 [**Finite time singularity**]

If one of the following is satisfied, then the solution must develop singularity at a finite time T^*, with

$$\lim_{t \to T^*} \min_{x \in R}(r^+(x,t) + r^-(x,t)) = -\infty.$$

(i) $r_0^+(x) \ge -\frac{1}{\tau}$ fails to hold at any point $x \in R$;
(ii) For all $\alpha \in R$

$$-\frac{1}{\tau} \le r_0^+(\alpha) \le 0$$

and $r_0^-(\beta) \ge -\frac{1}{\tau}$ fails to hold at any point $\beta \in R$;
(iii) For all $\alpha \in R$

$$r_0^+(\alpha) > 0$$

and $r_0^-(\beta) \ge 0$ fails to hold at any point $\beta \in R$;
(iv) For all $\alpha, \beta \in R$

$$0 \le r_0^-(\beta) < \frac{r_0^+(\alpha)}{\tau r_0^+(\alpha) + 1};$$

(v) For all $\alpha, \beta \in R$

$$-\delta(r_0^-(\beta))^2 < \frac{r_0^+(\alpha)}{\tau} < 0, \ -\frac{1}{\tau} \le r_0^-(\beta) < 0$$

for some $0 < \delta < \frac{1}{2}$.

A phase diagram is drawn in Figure 1 to describe different sets of initial data for which we have clarified the global in time behavior of corresponding solutions: initial data in $\bigcup_{i=1}^3 R_i$ lead to global in time solution, and initial data in $\bigcup_{i=1}^5 S_i$ lead to finite time singularity formation in solutions. The solution behavior for initial data from $\bigcup_{i=1}^2 U_i$ remains unknown. Nevertheless, the asserted results presented in this paper certainly indicate the existence of a critical threshold consisting of the half line

$$\{(r_0^+, r_0^-)| \quad r_0^+ = -1/\tau, \quad -1/\tau \le r_0^-\},$$

joint with a curve passing from equilibrium point $(-1/\tau, -1/\tau)$ through U_1 to equilibrium point $(0,0)$ into U_2 in the first quadrant. However, we have not been able to give a precise description of such a curve.

Concerning these theorems, several further remarks are in order.

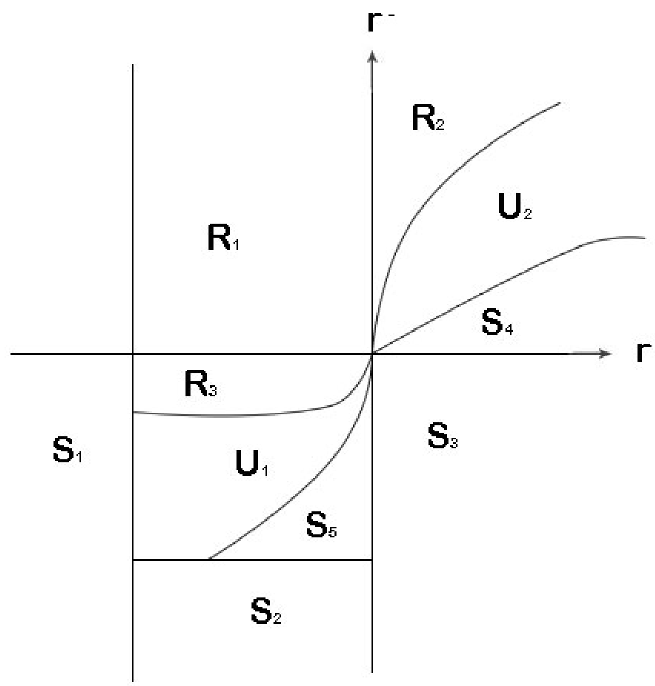

Figure 1: Qualitative diagram of thresholds

Remarks:
(i) For completeness, we have included case (i) in Theorem 1.1 and case (i) in Theorem 1.2 which were identified in our previous paper [22].
(ii) The set of initial data leading to global regularity is rich. In particular, it allows the initial Riemann invariant of negative slope. This is in sharp contrast to the generic breakdown in homogeneous hyperbolic systems , see Lax [16].
(iii) No smallness of data is assumed for the global existence of the smooth solution. The critical thresholds we identified reveal the genuine nonlinear phenomena hidden in the system.
(iv) Note that the bounds for the derivatives of the initial Riemann invariants are of order $\frac{1}{\tau}$. This implies that the smaller the relaxation time τ, the larger the set of initial data leading to global smooth solutions. This means that the shorter the drivers' reaction time, the larger the set of initial conditions leading to global smooth traffic flows. This is in agreement with the finding that in a class of optimal velocity models, the smaller the relaxation time τ is, the larger the linear stable region is, [2]. Similar phenomena occur in other problems. For example, in Euler-Poisson equations for plasma sheath problem, the small Debey length does delay the finite time breakdown [23]; also small Rossby number in rotational Euler equations helps to prevent

breakdown from happening in $O(1)$ time [31]. These results show that the equilibrium limit is highly singular.

Finally we comment on another physical scenario:

$$p(\rho) = c_0^2 \rho. \tag{5}$$

with

$$v_e(\rho) = c \ln \frac{\rho_{\max}}{\rho}, \quad 0 < \rho \leq \rho_{\max} \tag{6}$$

for some $c > 0$. This is a classical dynamic continuum model of traffic flow: the Payne [36] and Whitham [39] (PW) model. In [22], we identified one lower threshold for global existence of smooth solutions and one upper threshold for the finite time breakdown. Following similar analysis as performed in [22] and in this paper we are able to identify three lower thresholds for global in time existence of smooth solutions and five upper thresholds for finite time singularity. For brevity of the presentation, we choose to omit the details in this scenario.

We now conclude this section by outlining the rest of this paper. In Section 2 we present preliminaries about the hyperbolic relaxation system (1) and reformulation of corresponding results in terms of the Riemann invariants. This section also contains *a priori* estimates of solutions in L^∞ norm for (1). Section 3 is devoted to identifying three lower thresholds for global existence of smooth solutions, This is done by deriving the a priori estimate of the derivatives of the solution. Finally, Section 4 is devoted to detection of five upper thresholds for the finite time singularity formation.

2. Reformulation of the Problem

For the pressure and the equilibrium velocity defined in (3) (4), it is easy to check that

$$p'(\rho) = (\rho v_e'(\rho))^2 > 0. \tag{7}$$

Thus the system (1) is a strictly hyperbolic balance law, the characteristic speeds being

$$\lambda_1(\rho, u) = u + \rho v_e'(\rho) < u - \rho v_e'(\rho) = \lambda_2(\rho, u). \tag{8}$$

The corresponding right eigenvectors of the Jacobian of the flux are

$$r_i(\rho, u) = (\rho, (-1)^{i+1} v_e'(\rho))^T, \quad i = 1, \ 2.$$

Both characteristic families are genuinely nonlinear

$$\nabla \lambda_i(\rho, v) \cdot r_i(\rho, v) = (-1)^{i+1} \frac{d^2}{d\rho^2}(\rho v_e(\rho)) \neq 0, \quad i = 1, \ 2.$$

Recall that in the usual relaxation limit, $\tau \to 0^+$, the leading order of the relaxation system (1) is the LWR (Lighthill, Whitham and Richards) model

$$\rho_t + (q(\rho))_x = 0, \tag{9}$$

where

$$q(\rho) = \rho v_e(\rho) \tag{10}$$

is the equilibrium flux which is the fundamental diagram in traffic flows. The equilibrium characteristic speed is

$$\lambda_*(\rho) = q'(\rho) = v_e(\rho) + \rho v_e'(\rho). \tag{11}$$

The so-called subcharacteristic condition is

$$\lambda_1 < \lambda_* < \lambda_2 \tag{12}$$

on the equilibrium curve $u = v_e(\rho)$. (12) was shown to be a necessary condition for linear stability, Whitham [39].

It can be derived formally [33], in the same spirit as the classical Chapman-Enskog expansion, that the relaxation process is approximated by a viscous conservation law

$$\rho_t + (q(\rho))_x = (\beta(\rho)\rho_x)_x \tag{13}$$

where

$$\beta(\rho) = -\tau(\lambda_* - \lambda_1)(\lambda_* - \lambda_2).$$

Note that (13) is dissipative, $\beta(\rho) > 0$, provided that subcharacteristic condition (12) is satisfied. Similar to the diffusion, the relaxation term has smoothing and dissipative effects for the hyperbolic conservation laws. Nonlinear stability of the traveling wave solutions of (1) with more general $p(\rho)$ and $v_e(\rho)$ is obtained under the subcharacteristic conditions (12) by Li and Liu [21].

From (8) and (11) we see that, the subcharacteristic condition (12) is only satisfied marginally

$$\lambda_1 = \lambda_* < \lambda_2. \tag{14}$$

Thus the diffusion term in the Chapman-Enskog expansion of (1) vanishes,

$$\beta(\rho) = 0.$$

Hence previous stability analysis based on such a dissipation mechanism cannot be applied. We shall track the nonlinear dynamics along characteristic fields.

For the system (1) with (3) and (4) we have

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + v_f^2 \rho \rho_x = \frac{1}{\tau}(v_f(1-\rho) - u). \end{cases} \tag{15}$$

Multiplying system (15) by the left eigenvectors of the Jacobian of the flux

$$l_i(w,u) = ((-1)^i v_f, 1), \quad i = 1,\ 2,$$

we have

$$\begin{cases} R_t^- + \lambda_1 R_x^- = -\frac{1}{\tau} R^+, \\ R_t^+ + \lambda_2 R_x^+ = -\frac{1}{\tau} R^+, \end{cases} \tag{16}$$

where

$$\lambda_1 = R^- - v_f, \quad \lambda_2 = R^+ + v_f \tag{17}$$

and the Riemann invariants

$$\begin{cases} R^-(\rho,u) = u - v_f\rho + v_f \\ R^+(\rho,u) = u + v_f\rho - v_f \end{cases} \tag{18}$$

define a one-to-one mapping from (ρ, u) to (R^-, R^+) in the entire phase space.

The corresponding initial data is

$$(R^-, R^+)(x,0) = (R_0^-, R_0^+)(x) = (u_0 - v_f\rho_0 + v_f, u_0 + v_f\rho_0 - v_f)(x). \tag{19}$$

Theorem 2.1 Consider the system (16) subject to C^1 bounded initial data (19).
If

$$R_{0,x}^{\pm}(x) := r_0^{\pm}(x)$$

satisfy one of three conditions (i)-(iii) stated in Theorem 1.1, then the Cauchy problem (16) (19) has a unique smooth solution for all time $t > 0$.

Theorem 2.2 If one of the five conditions (i)-(v) stated in Theorem 1.2 is satisfied for

$$R_{0,x}^{\pm}(x) := r_0^{\pm}(x),$$

then the solution of the Cauchy problem (16) (19) must develop singularity at a finite time T^*, with

$$\lim_{t \to T^*} \min_{x \in R} (R_x^+(x,t) + R_x^-(x,t)) = -\infty.$$

The local existence of smooth solutions of hyperbolic problem is classical, see e.g. Douglis [6] and Hartman and Wintner [10]. According to the theory of first order quasilinear hyperbolic equations [4], solutions to initial value problems exist as long as one can place an *a priori* limitation on the magnitude of their first derivatives.

Equipped with the classical local existence results in [6] and [10], we need only to establish the *a priori* estimates in solutions and their derivatives, which will be presented in the following sections. Using expressions of the Riemann invariants to convert back to variables u and ρ, we prove our main results as stated in Theorem 1.1– Theorem 1.2.

We end this section by giving the desired *a priori* estimates of solutions in L^∞ norm.

Lemma 2.3 Assume that $R_0^\pm \in C^1(R)$ and that

$$\|R_0^-\|_\infty + \|R_0^+\|_\infty \le M$$

for some $M > 0$. Then the C^1 solution of the Cauchy problem (16) (19) satisfies the *a priori* estimates

$$\|R^+(\cdot,t)\|_\infty \le \|R_0^+\|_\infty e^{-\frac{t}{\tau}} \tag{20}$$

and

$$\|R^-(\cdot,t)\|_\infty + \|R^+(\cdot,t)\|_\infty \le M \tag{21}$$

for all $t \ge 0$ as long as the C^1 solution exists.

Proof. Integrating the second equation in (16) along the second characteristics $x_2(t,\alpha)$

$$\frac{dx_2}{dt} = \lambda_2 = u + c_0,\ x_2(0,\alpha) = \alpha,$$

we have

$$R^+(x_2(t,\alpha),t) = R_0^+(\alpha)e^{-\frac{t}{\tau}},$$

which leads to the asserted bound (20).

Now integrating the first equation in (16) along the first characteristics $x_1(t,\beta)$

$$\frac{dx_1}{dt} = \lambda_1 = u - c_0,\ x_1(0,\beta) = \beta,$$

we have

$$R^-(x_1(t,\beta),t) = R_0^-(\beta) - \frac{1}{\tau}\int_0^t R^+(x_1(s,\beta),s)ds$$

Using the above decay result for $\|R^+(\cdot,t)\|_\infty$, we have

$$\|R^-(\cdot,t)\|_\infty \le \|R_0^-(\cdot)\|_\infty + \|R_0^+(\cdot)\|_\infty(1 - e^{-\frac{t}{\tau}}).$$

This added upon (20) gives the desired bound (21). The proof is complete.

The uniform bound for (ρ, u) follows from (20), (21) and (18).

3. Proof of Theorem 2.1 – Lower Thresholds

In order to identify three lower thresholds for global existence of smooth solutions as claimed in Theorem 1.1 and 2.1, we derive the *a priori* estimates of the derivatives of the Riemann invariants $R^{\pm}(x,t)$ of (1) with (3), (4).

Denote $r^- = R_x^-$ and $r^+ = R_x^+$, we shall show that $R_x^{\pm}$ are bounded when initial values of them, i.e., $r_0^{\pm} := R_{0,x}^{\pm}$ are bounded by some critical thresholds, through three lemmas 3.1-3.3.

Lemma 3.1 Assume that $R_0^{\pm}(x) \in C^1(R)$ and $\|R_0^{\pm}\|_\infty$ are bounded.
If

$$0 \geq R_{0,x}^+(x) \geq -\frac{1}{\tau}, \quad x \in R$$

and

$$R_{0,x}^-(x) \geq 0, \quad x \in R,$$

then any C^1 solution of the Cauchy problem (16) (19) has the *a priori* estimates

$$0 \geq R_x^+(x,t) \geq \min_{x\in R} R_{0,x}^+(x)$$

and

$$\max_{x\in R} R_{0,x}^-(x) \geq R_x^-(x,t) \geq \min_{x\in R} \frac{R_{0,x}^-(x)}{1+R_{0,x}^-(x)t}$$

for all $x \in R$ and $t \geq 0$ as long as the C^1 solution exists.

Proof: From (17) we derive that

$$\lambda_{1,x} = r^-, \quad \lambda_{2,x} = r^+.$$

We differentiate (16) with respect to x to obtain

$$\begin{cases} r_t^- + \lambda_1 r_x^- + (r^-)^2 = -\frac{1}{\tau} r^+, \\ r_t^+ + \lambda_2 r_x^+ + (r^+)^2 = -\frac{1}{\tau} r^+. \end{cases} \tag{22}$$

Rewrite the second equation for r^+ to get

$$r_t^+ + \lambda_2 r_x^+ = -r^+ \left(\frac{1}{\tau} + r^+\right).$$

Along the second characteristics $x_2(t,\alpha)$: $\frac{dx_2}{dt} = \lambda_2,\ x_2(0,\alpha) = \alpha$, we have

$$\frac{d}{dt} r^+ = -r^+ \left(\frac{1}{\tau} + r^+\right).$$

Solving this differential equation, we obtain

$$r^+(x_2(t,\alpha),t) = \frac{r_0^+(\alpha)}{(\tau r_0^+(\alpha)+1)e^{t/\tau} - \tau r_0^+(\alpha)}, \tag{23}$$

which remains bounded

$$-\frac{1}{\tau} \le r^+(x_2(t,\alpha),t) \le \max\{0, r_0^+(\alpha)\} \tag{24}$$

if and only if

$$r_0^+(\alpha) \ge -\frac{1}{\tau}, \quad \forall \alpha \in R.$$

Now we examine $r^- = R_x^-$, which satisfies

$$r_t^- + \lambda_1 r_x^- = -\frac{r^+}{\tau} - (r^-)^2.$$

It follows from (24) that if

$$0 \ge r_0^+(\alpha) \ge -\frac{1}{\tau}, \quad \forall \alpha \in R, \tag{25}$$

then

$$0 \ge r^+(x,t) \ge -\frac{1}{\tau}, \quad (x,t) \in R \times R^+.$$

Assume (25) and let $x_1(t,\beta)$ be the first characteristics, along which we have

$$\frac{1}{\tau^2} - (r^-)^2 \ge \frac{d}{dt} r^- \ge -(r^-)^2.$$

If

$$r_0^-(\beta) \ge 0, \quad \forall \beta \in R,$$

then r^- stays bounded. Indeed

$$\frac{r_0^-(\beta)}{1 + r_0^-(\beta)t} \le r^-(x_1(t,\beta),t) \le \frac{1}{\tau} \frac{c_1 e^{\frac{2t}{\tau}} + 1}{c_1 e^{\frac{2t}{\tau}} - 1}$$

where $c_1 = \frac{r_0^- + \frac{1}{\tau}}{r_0^- - \frac{1}{\tau}}$. Note that when $r_0^- \ge 0$, the function on the right hand side is a decreasing function in time and satisfies

$$\frac{1}{\tau} \le \frac{1}{\tau} \frac{c_1 e^{\frac{2t}{\tau}} + 1}{c_1 e^{\frac{2t}{\tau}} - 1} \le r_0^-(\beta).$$

Therefore, if $r_0^-(\beta) \ge 0$ for all $\beta \in R$, then

$$\frac{r_0^-(\beta)}{1 + r_0^-(\beta)t} \le r^-(x_1(t,\beta),t) \le r_0^-(\beta), \ \beta \in R$$

which when optimizing the bounds in terms of the parameter β leads to the desired estimates. The proof of Lemma 3.1 is complete.

Lemma 3.2 Assume that $R_0^\pm(x) \in C^1(R)$ and $\|R_0^\pm\|_\infty$ are bounded.
If $R_{0,x}^\pm(x) = r_0^\pm(x)$ satisfy condition (ii) stated in Theorem 1.1, then any C^1

solution of the Cauchy problem (16), (19) has the *a priori* estimates: for some C_1 and C_2 depending only on $R^+_{0,x}$ and $R^-_{0,x}$, we have

$$C_1 \geq R^{\pm}_x(x,t) \geq C_2$$

for all $x \in R$ and $t \geq 0$ as long as the C^1 solution exists.

Proof: Assume that $R^{\pm}_{0,x}(x) = r^{\pm}_0(x)$ satisfy condition (ii) stated in Theorem 1.1.

From (23) we have

$$0 < \frac{r^+_0(\alpha)}{1+\tau r^+_0(\alpha)} e^{-\frac{t}{\tau}} \leq r^+(x_2(\alpha,t)) \leq r^+_0(\alpha) e^{-\frac{t}{\tau}}$$

for all α and for all $t \geq 0$.

Step I. *A priori* estimates

We show that there is $0 < \delta < 1$ such that if the initial data satisfies

$$0 < \frac{1}{\tau} r^+_0(\alpha) < \delta (r^-_0(\beta))^2$$

for any α and any β, then

$$\left(\frac{r^-_0(\beta)}{1+2r^-_0(\beta)t}\right)^2 > \frac{1}{\tau} r^+_0(\alpha) e^{-\frac{t}{\tau}}$$

for any α and any β and for all $t \geq 0$.

The above inequality can be proved by observing that the left and right hand sides decay to zero algebraically and exponentially, respectively, and by taking $0 < \delta < 1$ small.

For $0 < \delta < 1$ chosen above, consider initial data satisfying

$$0 < \frac{1}{\tau} r^+_0(\alpha) < \delta (r^-_0(\beta))^2$$

for any α and any β.

Step II. Local estimates

By continuity of solutions of ODEs, there is $t_0 > 0$ such that

$$\frac{1}{\tau} r^+(x_2(\alpha,t)) < (r^-(x_1(\beta,t)))^2$$

for any α and any β and for all $0 \leq t \leq t_0$.

For any $(x,t) \in R \times R^+$, there are α and β such that

$$x = x_1(\beta,t) = x_2(\alpha,t).$$

Thus

$$-(r^-(x_1(\beta,t)))^2 > \frac{d}{dt} r^-(x_1(\beta,t)) = -\frac{1}{\tau} r^+(x_2(\alpha,t)) - (r^-(x_1(\beta,t)))^2 > -2(r^-(x_1(\beta,t)))^2$$

for all $0 \le t \le t_0$. Hence

$$\frac{r_0^-(\beta)}{1+r_0^-(\beta)t} > r^-(x_1(\beta,t)) > \frac{r_0^-(\beta)}{1+2r_0^-(\beta)t}$$

for all $0 \le t \le t_0$.

Step III. Global estimates

Let

$$\Gamma = \left\{ t_0 \ge 0 \Big| \frac{r_0^-(\beta)}{1+r_0^-(\beta)t} \ge r^-(x_1(\beta,t)) \ge \frac{r_0^-(\beta)}{1+2r_0^-(\beta)t},\ 0 \le t \le t_0 \right\}.$$

We claim

$$\Gamma = [0,\ +\infty)$$

which implies the global estimates of solutions.

Now we prove the claim.

From definition of Γ, it is obvious that Γ is a closed and connected set in R. Assume that

$$\Gamma = [0,\ T]$$

for some $T < +\infty$.

Using the *a priori* estimates established in Step I and Step II and definition of Γ, we have

$$-(r^-(x_1(\beta,t)))^2 > \frac{d}{dt} r^-(x_1(\beta,t)) \ge -\frac{1}{\tau} r_0^+(\alpha) e^{-\frac{t}{\tau}} - (r^-(x_1(\beta,t)))^2$$

$$> -\left(\frac{r_0^-(\beta)}{1+2r_0^-(\beta)t}\right)^2 - (r^-(x_1(\beta,t)))^2 \ge -2(r^-(x_1(\beta,t)))^2$$

for $0 \le t \le T$. This implies that

$$\frac{r_0^-(\beta)}{1+r_0^-(\beta)t} > r^-(x_1(\beta,t)) > \frac{r_0^-(\beta)}{1+2r_0^-(\beta)t}$$

for $0 \le t \le T$.

By continuity of solutions of ODEs, we have that there exists $T_1 > T$ such that

$$\frac{r_0^-(\beta)}{1+r_0^-(\beta)t} \ge r^-(x_1(\beta,t)) \ge \frac{r_0^-(\beta)}{1+2r_0^-(\beta)t}$$

for $0 \le t \le T_1$.

Thus

$$T_1 \in \Gamma = [0, T]$$

which contradicts $T_1 > T$. Therefore the claim is proved, and we have global estimates of solutions.

Lemma 3.3 Assume that $R_0^\pm(x) \in C^1(R)$ and $\|R_0^\pm\|_\infty$ are bounded. If $R_{0,x}^\pm(x) = r_0^\pm(x)$ satisfy condition (iii) stated in Theorem 1.1, then any

C^1 solution of the Cauchy problem (16), (19) has the *a priori* estimates: for some C_1 and C_2 depending only on $R^+_{0,x}$ and $R^-_{0,x}$, we have

$$C_1 \geq R^{\pm}_x(x,t) \geq C_2$$

for all $x \in R$ and $t \geq 0$ as long as the C^1 solution exists.

Proof: Assume that the initial data satisfies

$$-\frac{1}{2}\frac{r_0^+(\alpha)}{\tau} > (r_0^-(\beta))^2, \; -\delta \leq r_0^-(\beta) \leq 0, \; -\frac{1}{\tau} \leq r_0^+(\alpha) < 0$$

for all $\alpha, \beta \in R$ and for some $\delta > 0$.

From (23) we have

$$\frac{r_0^+(\alpha)}{1+\tau r_0^+(\alpha)} e^{-\frac{t}{\tau}} \leq r^+(x_2(\alpha,t)) \leq r_0^+(\alpha) e^{-\frac{t}{\tau}} < 0$$

for all α and for all $t \geq 0$.

By continuity of solutions of ODEs, there is $t_0 > 0$ such that

$$-\frac{1}{2}\frac{r^+(x_2(\alpha,t))}{\tau} \geq (r^-(x_1(\beta,t)))^2$$

for any α and any β and for all $0 \leq t \leq t_0$.

For this fixed $t_0 > 0$, we have

$$\frac{d}{dt} r^-(x_1(\beta,t)) \geq -\frac{1}{2}\frac{r^+(x_2(\alpha,t))}{\tau} \geq -\frac{1}{2}\frac{r_0^+(\alpha)}{\tau} e^{-\frac{t}{\tau}}$$

for any α and any β and for all $0 \leq t \leq t_0$.

Integrating over $[0, t_0]$, we have

$$r^-(x_1(\beta,t_0)) \geq r_0^-(\beta) + \frac{1}{2}\min_{\alpha} |r_0^+(\alpha)|(1 - e^{-\frac{t_0}{\tau}}) > 0$$

provided that

$$-\delta < r_0^-(\beta) \leq 0$$

where

$$\delta = \min\left\{ \sqrt{\frac{1}{2}\min_{\alpha}|\frac{r_0^+(\alpha)}{\tau}|}, \; \frac{\tau}{2}\min_{\alpha}|\frac{r_0^+(\alpha)}{\tau}|(1-e^{-\frac{t_0}{\tau}}) \right\}.$$

Thus the trajectory enters region (i) at a finite time $t_1 < t_0$. Therefore we have global estimates of solutions.

This proves Lemma 3.3.

4. Proof of Theorem 2.2 – Upper Thresholds

This section is devoted to detection of five upper thresholds stated in Theorem 1.2 and 2.2 as detailed below.

(i) For initial data satisfying condition (i) as stated in Theorem 1.2, the solution r^+, expressed in (23), will becomes $-\infty$ at some time before

$$T = \tau \min_{\alpha \in R} ln \left(\frac{\tau r_0^+(\alpha)}{1 + \tau r_0^+(\alpha)} \right) < +\infty.$$

This proves (i) in Theorem 1.2 and 2.2.

(ii) For any $(x, t) \in R \times R^+$, there are α and β such that

$$x = x_1(\beta, t) = x_2(\alpha, t).$$

With initial data $(r_0^-(\beta), r_0^+(\alpha))$ in region (ii) as stated in Theorem 1.2, we have (24) for all $t \geq 0$, and thus

$$\frac{d}{dt} r^-(x_1(\beta, t)) = -\frac{r^+}{\tau} - (r^-)^2 \leq \frac{1}{\tau^2} - (r^-)^2,$$

from which it follows

$$r^-(x_1(t, \beta), t) \leq \frac{1}{\tau} \frac{c_1 e^{\frac{2t}{\tau}} + 1}{c_1 e^{\frac{2t}{\tau}} - 1},$$

where $1 > c_1 = \frac{r_0^- + \frac{1}{\tau}}{r_0^- - \frac{1}{\tau}} > 0$ for $r_0^-(\beta) < -\frac{1}{\tau}$. Thus r^- will becomes $-\infty$ in a finite time before $t^* > 0$ where $t^* = -\frac{\tau}{2} log c_1$.

(iii) If the initial data $(r_0^-(\beta), r_0^+(\alpha))$ is in region (iii) as stated in Theorem 1.2, we already have $r^+(x_2(\alpha, t)) > 0$ for all $t \geq 0$, (23). Thus $\frac{d}{dt} r^-(x_1(\beta, t)) \leq -(r^-)^2$, yielding

$$r^-(x_1(\beta, t)) \leq \frac{r_0^-(\beta)}{1 + r_0^-(\beta) t}.$$

This implies that if $r_0^-(\beta) < 0$, then r^- will become $-\infty$ in finite time before $t^* = -\frac{1}{r_0^-(\beta)}$.

(iv) If the initial data $(r_0^-(\beta), r_0^+(\alpha))$ is in region (iv) as stated in Theorem 1.2, we have

$$\frac{d}{dt} r^-(x_1(\beta, t)) \leq -\frac{1}{\tau} r^+(x_2(\alpha, t)) \leq -\frac{1}{\tau} \frac{r_0^+(\alpha)}{1 + \tau r_0^+(\alpha)} e^{-\frac{t}{\tau}}.$$

Hence,

$$r^-(x_1(\beta, t)) \leq r_0^-(\beta) - \frac{1}{\tau} \int_0^t \frac{r_0^+(\alpha)}{1 + \tau r_0^+(\alpha)} e^{-\frac{s}{\tau}} ds$$

$$= r_0^-(\beta) + \frac{r_0^+(\alpha)}{1+\tau r_0^+(\alpha)}(e^{-\frac{t}{\tau}} - 1) < 0$$

in a finite time $t = T_1 < +\infty$.

Thus

$$\frac{d}{dt} r^-(x_1(\beta, t)) \leq -(r^-)^2, \; r^-(x_1(\beta, T_1)) < 0.$$

Therefore

$$r^-(x_1(\beta, t)) \leq \frac{r^-(x_1(\beta, T_1))}{1 + r^-(x_1(\beta, T_1))(t - T_1)}$$

for $t > T_1$.

This implies that r^- will become $-\infty$ in finite time before $t^* = T_1 - \frac{1}{r^-(x_1(\beta, T_1))}$.

(v) Assume that the initial data $(r_0^-(\beta), r_0^+(\alpha))$ is in region (v) as stated in Theorem 1.2.

Step I. *A priori* estimates

We show that there is $0 < \delta < \frac{1}{2}$ such that if the initial data satisfies

$$0 < \frac{1}{\tau} r_0^+(\alpha) < \delta (r_0^-(\beta))^2$$

for any α and any β, then

$$-\frac{1}{2}\left(\frac{r_0^-(\beta)}{1 + \frac{1}{2} r_0^-(\beta) t}\right)^2 < \frac{1}{\tau} r_0^+(\alpha) e^{-\frac{t}{\tau}} < 0$$

for any α and any β and for all $t \geq 0$.

The above inequality can be proved by observing that the left and right hand sides decay to zero algebraically and exponentially, respectively, and by taking $0 < \delta < \frac{1}{2}$ small.

Step II. Local estimates

By continuity of solutions ODEs, there is $t_0 > 0$ such that

$$-\frac{1}{2}(r^-(x_1(\beta, t)))^2 < \frac{1}{\tau} r^+(x_2(\alpha, t))$$

for any α and any β and for all $0 \leq t \leq t_0$.

Thus

$$\frac{d}{dt} r^-(x_1(\beta, t)) \leq -\frac{1}{2}(r^-(x_1(\beta, t)))^2$$

for all $0 \leq t \leq t_0$ or

$$r^-(x_1(\beta, t)) \leq \frac{r_0^-(\beta)}{1 + \frac{1}{2} r_0^-(\beta) t} < 0$$

for all $0 \leq t \leq t_0$.

Step III. Finite time blow up

Let

$$\Gamma = \left\{ t_0 \geq 0 | -\frac{1}{2}(r^-(x_1(\beta,t)))^2 \leq \frac{1}{\tau} r^+(x_2(\alpha,t)),\ 0 \leq t \leq t_0 \right\}.$$

We claim

$$\Gamma = [0,\ +\infty).$$

From definition of Γ, it is obvious that Γ is a closed and connected set in R. Assume that

$$\Gamma = [0,\ T]$$

for some $T < +\infty$.

Using the *a priori* estimates established in Step I and Step II and definition of Γ, we have

$$\frac{d}{dt} r^-(x_1(\beta,t)) \leq -\frac{1}{2}(r^-(x_1(\beta,t)))^2$$

for all $0 \leq t \leq T$, leading to

$$r^-(x_1(\beta,t)) \leq \frac{r_0^-(\beta)}{1+\frac{1}{2}r_0^-(\beta)t}$$

for all $0 \leq t \leq T$.

Therefore

$$-\frac{1}{2}(r^-(x_1(\beta,t)))^2 \leq -\frac{1}{2}\left(\frac{r_0^-(\beta)}{1+\frac{1}{2}r_0^-(\beta)t}\right)^2 < \frac{1}{\tau} r^+(x_2(\alpha,t)) < 0$$

for all $0 \leq t \leq T$.

By continuity of solutions of ODEs, we have that there exists $T_1 > T$ such that

$$-\frac{1}{2}(r^-(x_1(\beta,t)))^2 \leq \frac{1}{\tau} r^+(x_2(\alpha,t))$$

for all $0 \leq t \leq T_1$.

Thus

$$T_1 \in \Gamma = [0, T]$$

which contradicts $T_1 > T$.

The claim is proved.

Thus

$$\frac{d}{dt} r^-(x_1(\beta,t)) \leq -\frac{1}{2}(r^-(x_1(\beta,t)))^2,\ r_0^-(\beta) < 0$$

for all $t \in \Gamma = [0, +\infty)$ as long as the C^1 solution exists.

Therefore

$$r^-(x_1(\beta,t)) \leq \frac{r_0^-(\beta)}{1+\frac{1}{2}r_0^-(\beta)t}.$$

This implies that r^- will become $-\infty$ in finite time before $t^* = -\frac{2}{r_0^-(\beta)}$.

The proof of Theorem 1.2 and 2.2 is thus complete.

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