

THE CAUCHY–DIRICHLET PROBLEM FOR THE FINITELY EXTENSIBLE NONLINEAR ELASTIC DUMBBELL MODEL OF POLYMERIC FLUIDS*

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Abstract. The finitely extensible nonlinear elastic (FENE) dumbbell model consists of the incompressible Navier–Stokes equation for the solvent and the Fokker–Planck equation for the polymer distribution. In such a model, the polymer elongation cannot exceed a limit m_0 which yields all interesting features of solutions near this limit. This work is concerned with the sharpness of boundary conditions in terms of the dimensionless parameter $b = \frac{Hm_0^2}{k_B T}$. Through a careful analysis of the Fokker–Planck operator coupled with the Navier–Stokes equation, we establish a local well-posedness for the full coupled FENE dumbbell model under a class of Dirichlet-type boundary conditions dictated by the parameter b . For each $b > 0$ we identify a sharp boundary requirement for the underlying density distribution, while the sharpness follows from the existence result for each specification of the boundary behavior. It is shown that the probability density governed by the Fokker–Planck equation approaches zero near the boundary, necessarily faster than the distance function d for $b > 2$, faster than $d|\ln d|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Moreover, the sharp boundary requirement for $b \geq 2$ is also sufficient for the distribution to be a probability density.

Key words. Fokker–Planck equation, Navier–Stokes equation, finitely extensible nonlinear elastic model, boundary conditions, well-posedness

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1. Introduction. Let $N \geq 2$ be an integer. We consider a dimer—an idealized polymer chain—as an elastic dumbbell consisting of two beads joined by a spring that can be modeled by an elongation vector $m \in \mathbb{R}^N$ (see, e.g., [6]) with the elastic spring potential given by

$$(1.1) \quad \Psi(m) = -\frac{Hm_0^2}{2} \log \left(1 - \frac{|m|^2}{m_0^2} \right), \quad |m| \leq m_0.$$

A spring with this potential will get stiffer as the spring is extended, but the spring cannot be extended beyond a separation m_0 . In kinetic theory for the finitely extensible nonlinear elastic (FENE) dumbbells, the dimensionless ratio

$$b = \frac{Hm_0^2}{k_B T_a}$$

is a useful symbol which measures the nonlinearity of the spring, where T_a is the absolute temperature and k_B is the Boltzmann constant. In [6], some experimental comparisons are recorded for different b 's ranging from small (near zero) to large

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(>1000). We note that if b is allowed to approach infinity, then the formulas for the Hookean dumbbell are recovered. A general bead-spring chain model may contain more than two beads coupled with elastic springs to represent a polymer chain.

Polymers as such when immersed in an incompressible, viscous, isothermal Newtonian solvent are modeled by a system coupling the incompressible Navier–Stokes equation for the macroscopic velocity field $v(t, x)$ with the Fokker–Planck equation for the probability distribution function $f(t, x, m)$:

$$(1.2) \quad \partial_t v + (v \cdot \nabla)v + \nabla p = \nabla \cdot \tau + \nu_k \Delta v,$$

$$(1.3) \quad \nabla \cdot v = 0,$$

$$(1.4) \quad \partial_t f + (v \cdot \nabla)f + \nabla_m \cdot (\nabla v m f) = \frac{2}{\zeta} \nabla_m \cdot (\nabla_m \Psi(m)f) + \frac{2k_B T_a}{\zeta} \Delta_m f,$$

where $x \in \mathbb{R}^N$ is the macroscopic Eulerian coordinate and $m \in B(0, m_0) \subset \mathbb{R}^N$ is the microscopic molecular configuration variable. The model describes diluted solutions of polymeric liquids with noninteracting polymer chains (dimers). Note that the Fokker–Planck equation can be conveniently augmented to incorporate other effects such as inertial forces (see [14]).

In Navier–Stokes equation (1.2), p is the hydrostatic pressure, ν_k is the kinematic viscosity coefficient, and τ is a tensor representing the polymer contribution to stress,

$$\tau = \lambda_p \int_{|m| \leq |m_0|} m \otimes \nabla_m \Psi(m) f dm,$$

where λ_p is the polymer density constant. In the Fokker–Planck equation (1.4), ζ is the friction coefficient of the dumbbell beads. We refer to [6, 13, 43] for a comprehensive survey of the physical background and [42] for the computational aspect.

Notice that the term $\nabla_m \Psi$ in (1.4) becomes singular (unbounded) on $\partial B_{m_0} = \{m : |m| = m_0\}$, where $B_{m_0} := B(0, m_0)$ is a ball in \mathbb{R}^N with radius m_0 . The question arises as to what kind of boundary condition needs to be imposed on ∂B_{m_0} . When we consider the evolution of

$$\int_{\mathbb{R}^N} \int_{B_{m_0}} f dm dx,$$

we find that the evolution rate $\frac{d}{dt} \int_{\mathbb{R}^N} \int_{B_{m_0}} f dm dx$ depends on both values of $f v$ at the far field in x through $-\lim_{r \rightarrow \infty} \int_{|x|=r} \int_{B_{m_0}} f v \cdot \frac{x}{r} dm dS_x$ and the total flux on ∂B . For mass conservation, we therefore might expect that $f v$ should decay for large $|x|$ and should have zero flux on ∂B , i.e.,

$$\left[\frac{2}{\zeta} (k_B T_a \nabla_m f + \nabla_m \Psi f) - \nabla v m f \right] \cdot \frac{m}{|m|} \Big|_{\partial B_{m_0}} = 0.$$

This is indeed the condition which has been frequently adopted in previous works. Notice that here m is an end-to-end vector, and this zero flux condition does not seem to have a definite physical interpretation. Instead, since the Fokker–Planck equation (1.4) is singular at ∂B_{m_0} , it is natural to ask what kind of boundary behavior one should impose or one would expect.

The singularity in the potential requires at least zero Dirichlet boundary condition

$$f|_{\partial B_{m_0}} = 0.$$

This is consistent with the result in [22], which states that the trajectory does not reach the boundary almost surely if $b \geq 2$. However, the above condition is insufficient for uniqueness; construction of infinitely many solutions for the case of $b > 2$ was presented in [36] through simpler forms of the Fokker–Planck equation. In [35], C. Liu and H. Liu examined the ratio of the distribution f and the equilibrium f_{eq} , i.e.,

$$w = f/f_{eq},$$

for the microscopic FENE model, and by the method of the Fichera function they showed that $b = 2$ is a threshold in the sense that for $b \geq 2$ any preassigned boundary value of w will become redundant, and for $b < 2$ that value has to be a priori given. As a side note we point out that there is a misprint in the statement of this result, Theorem 1.1 in [35], where the correct assertion should be about the boundary condition for w rather than for f —the proof is otherwise correct.

Our main quest in this article is what the least boundary requirement for f is so that both existence and uniqueness of the solution to the FENE model can be established; also the solution be a probability density. Upon pursuing this, we shall achieve two main objectives:

- (1) to identify some novel Dirichlet type boundary conditions for all $b > 0$ for the well-posedness of the coupled Navier–Stokes–Fokker–Planck system,
- (2) to show that the identified boundary condition is optimal.

To illustrate the main idea, we consider (2.1), which derives from (1.2)–(1.4) through a rescaling, for which the domain for m becomes $B := \{m : |m| \leq \sqrt{b}\}$. In order to describe the behavior of f near ∂B , we identify a rate function ν , which approaches zero near the boundary at a certain rate. We also use another function q to quantify the relative ratio of f/ν near the boundary. More precisely, we impose the following boundary condition:

$$(1.5) \quad f(t, x, m)\nu^{-1}|_{\partial B} = q(t, x, m)|_{\partial B}.$$

This boundary condition is a mathematical description of the boundary behavior of f , instead of the Dirichlet data in a physical sense. Nevertheless, the pair (ν, q) , once known, determines the behavior of solutions near ∂B , including the physically relevant solution.

We shall investigate solvability of the coupled system (1.2)–(1.4) subject to (1.5) and the initial data

$$(1.6) \quad v(0, x) = v_0(x),$$

$$(1.7) \quad f(0, x, m) = f_0(x, m).$$

In fact, for each b , we are able to identify the form of ν

$$(1.8) \quad \nu = \begin{cases} \rho^{b/2}, & 0 < b < 2, \\ \rho \ln \frac{\rho}{\rho}, & b = 2, \\ \rho, & b > 2, \end{cases}$$

where $\rho = b - |m|^2$ plays a role of the distance function $d = \sqrt{b} - |m|$. We should point out that $\nu = \rho$ when $b > 2$ was identified in [36], where the Fokker–Planck equation (1.4) alone was studied. In this work, with some regularity requirement on q as well as on initial data, we prove local well-posedness for the above Cauchy–Dirichlet problem in a weighted Sobolev space for each given q . Our results indicate that simply requiring $f = 0$ on the boundary does not guarantee uniqueness of solutions.

As is known, the singularity of the Fokker–Planck equation near ∂B makes the boundary issue rather subtle and presents various challenges. To address the difficulties caused, several transformations relating to the equilibrium solution have been introduced in the literature (see, e.g., [16, 24, 35, 36]). For the Dirichlet-type boundary condition (1.5), our strategy is to study a transformed problem via

$$w = \frac{f}{\nu} - q$$

with ν defined in (1.8) so that $w|_{\partial B} = 0$. Once w is known, one can extract f from $\nu(w + q)$. Inspired by [38], for the coupled FENE system we use a weak norm in m and a strong norm in x ; this enables us to prove well-posedness for all cases of $b > 0$ and any given smooth q .

We identify a sharp boundary requirement for each $b > 0$ for the underlying density distribution, while the sharpness is a consequence of the existence result for each q . For $b \geq 2$, we show that f is a density distribution if and only if $q|_{\partial B} = 0$. In particular, our result asserts that near the boundary the probability density governed by the Fokker–Planck equation approaches zero, necessarily faster than the distance function d for $b > 2$, faster than $d|\ln d|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Unfortunately, within our current framework we have not been able to identify a non-trivial q for $0 < b < 2$ such that the corresponding solution is a density distribution.

Among difficulties encountered in our investigation, the main one is to find the appropriate rate ν and weighted function spaces so that (i) the trace and embedding theorems hold; (ii) well-posedness for the Fokker–Planck problem is ensured; (iii) the relation between fluid velocity and polymer distribution is well defined; and (iv) the rate is the least among all that ensure (i)–(iii).

We remark that the sharp boundary condition presented in this work provides a threshold on the boundary requirement: subject to the sharp requirement or a stronger condition incorporated through some weighted function spaces [47, 38], the Fokker–Planck dynamics will select the physically relevant solution, which is a probability density; any weaker boundary requirement can lead to many solutions, each depending on the ratio of f/ν near the boundary.

This article is organized as follows. In section 2, we state our main results and main ideas of the proofs. In section 3, we study the Fokker–Planck operator and well-posedness of the initial boundary value problem for the Fokker–Planck equation alone. This part improves upon our previous work [36]. The main result is summarized in Theorem 13. The Fokker–Planck problem involving spatial variable x is investigated in section 4. Well-posedness of the coupled system is proved in section 5. In section 6, we sketch the proof of well-posedness for the coupled system with $b \geq 6$ in a different function space than what was used in section 5. Some concluding remarks are drawn in section 7.

We conclude this section with some bibliographical remarks.

Existence results for the FENE model are usually limited to local in time existence and uniqueness of strong solutions. We refer to [44] for the local existence on some related coupled systems, [22] for the FENE model (in the setting where the Fokker–Planck equation is formulated by a stochastic differential equation) with $b > 6$, and [17] for a polynomial force. More related to this paper are the works by Zhang and Zhang [47] for the FENE model when $b > 76$ and Masmoudi [38] for $b > 0$. Global existence results are usually limited to solutions near equilibrium (see [28, 33]) or to some two-dimensional simplified models [10, 12, 27, 41]. For results concerning the existence of weak solutions to the coupled FENE system we refer to [2, 3, 4, 5, 34, 39, 45, 48].

Boundary behavior of the polymer distribution governed by the FENE model is also essential in several other aspects, including the study of large time behavior (see [1, 20, 23, 45]) and development of numerical methods (see, e.g., [8, 9, 16, 24, 37, 46]). We also refer to [21] for references on numerical aspects of polymeric fluid models.

There are also some interesting works on non-Newtonian fluid models derived through a closure of the linear Fokker–Planck equation (see, e.g., [15, 16]). We can refer to the pioneering work [18, 19] and to the more recent [11, 29, 30, 31, 32].

However, none of these works is concerned with the sharpness of boundary conditions in terms of the dimensionless elongation parameter.

2. Main results. After a suitable scaling and choice of parameters we arrive at the following Cauchy–Dirichlet problem for the coupled system:

$$(2.1a) \quad \partial_t v + (v \cdot \nabla)v + \nabla p = \nabla \cdot \tau + \Delta v, \quad x \in \mathbb{R}^N, \quad t > 0,$$

$$(2.1b) \quad \nabla \cdot v = 0,$$

$$(2.1c) \quad \partial_t f + (v \cdot \nabla)f + \nabla_m \cdot (\nabla v m f) = \frac{1}{2} \nabla_m \cdot \left(\frac{b m}{\rho} f \right) + \frac{1}{2} \Delta_m f, \quad m \in B,$$

$$(2.1d) \quad \tau = \int_B m \otimes \frac{b m}{\rho} f dm,$$

$$(2.1e) \quad v(0, x) = v_0(x),$$

$$(2.1f) \quad f(0, x, m) = f_0(x, m),$$

$$(2.1g) \quad f(t, x, m) \nu^{-1}|_{\partial B} = q(t, x, m)|_{\partial B}.$$

To present our main results we first fix notation to be used throughout this article. We fix an exponent s , which is an integer in the range $s > N/2 + 1$. We use C to denote various constants depending on s , b , and some other quantities which we will indicate in what follows. A b -dependent weight function is defined as

$$(2.2) \quad \mu = \begin{cases} \rho^{b/2}, & 0 < b < 2, \\ \rho(\ln \frac{e}{\rho})^2, & b = 2, \\ \rho^{2-b/2}, & b > 2. \end{cases}$$

For $b \geq 6$, we also use

$$(2.3) \quad \mu_0 = \rho^\theta, \quad -1 < \theta < 1, \quad b \geq 6.$$

Other notation is as follows:

- $L_\mu^2 = \{\phi : \int_B \phi^2 \mu dm < \infty\}$.
- $H_\mu^1 = \{\phi : \phi, \partial_{m_j} \phi \in L_\mu^2, j = 1 \dots N\}$.
- $\overset{\circ}{H}_\mu^1$ denotes the completion of C_c^∞ with H_μ^1 norm.
- H^* is a dual space of H .
- H_x^s is the usual Sobolev space with respect to x .

•

$$\begin{aligned}
 |v|_s^2 &= \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} |\partial^\alpha v|^2 dx, \\
 |w|_{0,s}^2 &= \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} \int_B |\partial^\alpha w|^2 \mu dm dx, \\
 |w|_{1,s}^2 &= |w|_{0,s}^2 + |\nabla_m w|_{0,s}^2, \\
 \|w\|_{1,1,s}^2 &= \sup_t (|w|_{1,s}^2 + |\partial_t w|_{1,s}^2), \\
 \|q\| &= \|q\|_{H_\mu^1} + \|\partial_t q\|_{H_\mu^1}.
 \end{aligned}$$

- $H_x^s L_\mu^2 = \{\phi : |\phi|_{0,s} < \infty\}$, $H_x^s H_\mu^1 = \{\phi : |\phi|_{1,s} < \infty\}$.
- $L_t^2 H = L^2((0, T); H)$, $C_t H = C([0, T]; H)$, $C_t^1 H = C^1([0, T]; H)$ for a Hilbert space H .
- $\mathcal{H} = \{\phi : \|\phi\|_{L_t^2 H_\mu^1} + \|\phi_t\|_{L_t^2 (H_\mu^1)^*} < \infty\}$,
- $\mathring{\mathcal{H}} = \{\phi(t, \cdot) \in \mathring{H}_\mu^1 : \|\phi\|_{L_t^2 H_\mu^1} + \|\phi_t\|_{L_t^2 (\mathring{H}_\mu^1)^*} < \infty\}$.

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$$\mathbf{X}_\mu = [C_t H_x^s \cap L_t^2 H_x^{s+1}] \times [C_t H_x^s L_\mu^2 \cap L_t^2 H_x^s H_\mu^1].$$

- For a generic constant C , independent of T and $a \in L_t^2$, we define

$$(2.4) \quad F(a) = C \left(T + \int_0^T |a(t)|^2 dt \right).$$

Due to such a constant, any two instances of F should be presumed to be with different constants.

We now state our main theorem as follows.

THEOREM 1. *Let $b > 0$ and s be an integer such that $s > N/2 + 1$. Suppose that $v_0 \in H_x^s$, $f_0 \nu^{-1} \in H_x^s L_\mu^2$, and $q \in C_t^1 H_x^{s+1} H_\mu^1$. Then, for some $T > 0$ there exists a unique solution (v, f) to the coupled problem (2.1) such that*

$$(v, f \nu^{-1}) \in \mathbf{X}_\mu.$$

It is known from [25] that $\mathring{H}_\mu^1 = H_\mu^1$ for $b \geq 6$ with μ defined in (2.2). Thus, boundary condition (2.1g) is nothing but the zero Dirichlet boundary condition under the assumption on q in Theorem 1. For nontrivial q when $b \geq 6$, we show the well-posedness in a different weighted Sobolev space. The result is summarized as below.

THEOREM 2. *Let $b \geq 6$ and s be an integer such that $s > N/2 + 1$. Suppose that $v_0 \in H_x^s$, $f_0 \nu^{-1} \in H_x^s L_{\mu_0}^2$, and $q \in C_t^1 H_x^{s+1} H_{\mu_0}^1$ with μ_0 defined in (2.3). Then, for some $T > 0$ there exists a unique solution (v, f) to the coupled problem (2.1) such that*

$$(v, f \nu^{-1}) \in \mathbf{X}_{\mu_0}.$$

Theorems 1 and 2 tell us that for each given q , which denotes the rate of f approaching to zero relative to ν near ∂B , there exists a unique solution (v, f) . Also, they indicate that any weaker boundary requirement may lead to more than one solutions to (2.1). For instance, the boundary condition

$$f \nu^{-1} \rho^\varepsilon|_{\partial B} = 0, \quad \varepsilon > 0,$$

gives infinitely many solutions to (2.1). Precisely we state the following nonuniqueness result.

COROLLARY 3. *Let $\tilde{\nu}$ be a smooth function of ρ such that*

$$(2.5) \quad \lim_{\rho \rightarrow 0} \frac{\nu}{\tilde{\nu}} = 0.$$

Then, the coupled problem (2.1) with (2.1g) replaced by

$$(2.6) \quad f(t, x, m)\tilde{\nu}^{-1}|_{\partial B} = 0$$

has infinitely many solutions in \mathbf{X}_μ and \mathbf{X}_{μ_0} for $0 < b < 6$ and $b \geq 6$, respectively.

A natural question is for what q the obtained distribution f is a probability distribution. The answer when $b \geq 2$ is given in the following theorem.

THEOREM 4. *Suppose that $b \geq 2$ and $q|_{\partial B} \geq 0$. Under the assumption of Theorem 1 or 2, the unique solution f to the Cauchy–Dirichlet problem (2.1) is a probability distribution if and only if $q|_{\partial B} = 0$. That is, $f \geq 0$ if $f_0 \geq 0$, and for any $t > 0$, $x \in \mathbb{R}^N$,*

$$(2.7) \quad \int_B f(t, x, m)dm = \int_B f_0(x, m)dm.$$

Theorem 1 is proved by a fixed point argument, which is now outlined. Given (u, g) , we first solve the Navier–Stokes equation:

$$(2.8a) \quad \partial_t v + (u \cdot \nabla)v + \nabla p = \nabla \cdot \tau + \Delta v,$$

$$(2.8b) \quad \nabla \cdot v = 0,$$

$$(2.8c) \quad v(0, x) = v_0(x),$$

$$(2.8d) \quad \tau = \int_B m \otimes \frac{bm}{\rho} g dm.$$

With the obtained v we solve the Fokker–Planck equation:

$$(2.9a) \quad \partial_t f + (v \cdot \nabla)f + \nabla_m \cdot (\nabla v m f) = \frac{1}{2} \nabla_m \cdot \left(\frac{bm}{\rho} f \right) + \frac{1}{2} \Delta_m f,$$

$$(2.9b) \quad f(0, x, m) = f_0(x, m),$$

$$(2.9c) \quad f(t, x, m)\nu^{-1}|_{\partial B} = q(t, x, m)|_{\partial B}.$$

The above two systems define a mapping $(u, g) \rightarrow (v, f)$, and the existence of problem (2.1) is equivalent to the existence of a fixed point of this mapping.

The main difficulty lies in monitoring the boundary behavior of f . Our strategy is to apply the transformation

$$(2.10) \quad f = \nu(w + q)$$

to (2.9) to obtain a w-problem,

$$(2.11a) \quad \mu(\partial_t + v \cdot \nabla)w + L[w] = \mu h,$$

$$(2.11b) \quad w(0, x, m) = w_0(x, m),$$

$$(2.11c) \quad w(t, x, m)|_{\partial B} = 0.$$

Here the operator L is induced from the Fokker–Planck operator, and ν and μ are weights depending on the distance functions defined in (1.8) and (2.2), respectively. The source term is obtained from q ,

$$(2.12) \quad h = -\partial_t q - (v \cdot \nabla)q - \mu^{-1}L[q],$$

and the initial data is given by

$$(2.13) \quad w_0(x, m) := f_0(x, m)\nu^{-1} - q(0, x, m).$$

For given (u, ϖ) with $g = \nu(\varpi + q)$, we arrive at a map \mathcal{F} :

$$\begin{aligned} \mathcal{F} : \mathbf{M} &\rightarrow \mathbf{M}, \\ (u, \varpi) &\mapsto (v, w). \end{aligned}$$

Here \mathbf{M} is a subset of

$$C_t H_x^s \times [C_t H_x^s L_\mu^2 \cap L_t^2 H_x^s \mathring{H}_\mu^1]$$

such that

$$\mathbf{M} = \left\{ (v, w) : \sup_{0 \leq t \leq T} |v|_s^2 \leq A_1, \sup_{0 \leq t \leq T} |w|_{0,s}^2 + \frac{1}{2} \int_0^T |\nabla_m w|_{0,s}^2 dt \leq A_2 \right\}.$$

The strategy for the fixed point proof, which we implement in sections to follow, is to first prove that \mathcal{F} is well defined for some T, A_1 , and A_2 , then to show that \mathcal{F} is actually a contraction map in a weak norm. Moreover, we will show that

$$(2.14) \quad \mathcal{F}(\mathbf{M}) \subset \mathbf{X}_\mu.$$

These prove Theorem 1 for

$$q \in C_t^1 H_x^{s+1} H_\mu^1 \subset [C_t H_x^s L_\mu^2 \cap L_t^2 H_x^s H_\mu^1].$$

Theorem 2 is proved in the same manner. A sketch of the proof is presented in section 6.

In order to prove Corollary 3, we pick $q(t, x, \cdot) \in C^\infty(B) \cap C(\overline{B})$ and $q|_{\partial B} \neq 0$ such that

$$q \in \begin{cases} C_t^1 H_x^{s+1} H_\mu^1, & 0 < b < 6, \\ C_t^1 H_x^{s+1} H_{\mu_0}^1, & b \geq 6. \end{cases}$$

Note that the existence of such a q follows from the density of the weighted Sobolev space (see [25] for details). Then for each q we have a unique solution (v, f) to the coupled problem (2.1) from Theorems 1 and 2. Finally, we turn to examine the boundary condition (2.6),

$$f\tilde{\nu}^{-1}|_{\partial B} = f\nu^{-1}\frac{\nu}{\tilde{\nu}}|_{\partial B} = q\frac{\nu}{\tilde{\nu}}|_{\partial B},$$

which vanishes since $q|_{\partial B}$ is bounded and condition (2.5) holds. This proves Corollary 3.

Theorem 4 follows from Propositions 16 and 17 via a flow map to be described in section 4. The case for $b \geq 6$ can be proved by a simple modification, which is also sketched in section 6.

3. The Fokker–Planck operator. We start with (2.9) when x is not involved. In such a case it reduces to the following problem:

$$(3.1a) \quad \partial_t f + \mathcal{L}[f] = 0, \quad m \in B, t > 0,$$

$$(3.1b) \quad f(0, m) = f_0(m),$$

$$(3.1c) \quad f(t, m)\nu^{-1}|_{\partial B} = q(t, m)|_{\partial B}.$$

Here

$$(3.2) \quad \mathcal{L}[f] := \nabla \cdot (\kappa m f) - \frac{1}{2} \nabla \cdot \left(\frac{b m}{\rho} f \right) - \frac{1}{2} \Delta f;$$

$\kappa = \kappa(t)$ is a square integrable matrix function such that $\text{Tr}(\kappa) = 0$. We omit m from ∇_m in (3.2) for notational convenience.

The goal of this section is twofold:

- (1) to provide tools for subsequent sections,
- (2) to elaborate on this model alone as an extension of our previous work [36].

3.1. Transformed operator. The transformation (2.10) leads to

$$(3.3a) \quad \partial_t w \mu + L[w] = \mu h, \quad m \in B, t > 0,$$

$$(3.3b) \quad w(0, m) = w_0,$$

$$(3.3c) \quad w(t, m)|_{\partial B} = 0,$$

with the transformed operator L determined by

$$(3.4) \quad L[w] = \mu \nu^{-1} \mathcal{L}[\nu w].$$

The source term is given as $h = -\partial_t q - \mu^{-1} L[q]$ and initial data for w is $w_0 = f_0 \nu^{-1} - q(0, m)$.

From a direct calculation with the choice of μ in (2.2) and ν in (1.8), (3.4) can be expressed as

$$(3.5) \quad L[w] = -\frac{1}{2} \nabla \cdot (\nabla w \mu) + \nabla \cdot (\kappa m w \mu) - K w,$$

where

$$(3.6) \quad K = \begin{cases} 0, & 0 < b < 2, \\ (N + 2\kappa m \cdot m) \ln \frac{e}{\rho}, & b = 2, \\ (N + 2\kappa m \cdot m)(b/2 - 1)\rho^{1-b/2}, & b > 2. \end{cases}$$

Associated with the operator L , we define its time-dependent bilinear form

$$(3.7) \quad \mathcal{B}[w, \phi; t] := \int_B \left(\frac{1}{2} \nabla w \cdot \nabla \phi \mu - w \mu \kappa m \cdot \nabla \phi - K w \phi \right) dm$$

for $\phi, w \in \mathring{H}_\mu^1$ and fixed $t > 0$.

We now describe the weak solution which we are looking for.

DEFINITION 5. A function $w \in \mathring{\mathcal{H}}$ is a weak solution of w -problem (3.3), provided

- (1) for each $\phi \in \mathring{H}_\mu^1$ and almost every $0 \leq t \leq T$,

$$(3.8) \quad (\partial_t w, \phi)_{\mathring{H}_\mu^1} + \mathcal{B}[w, \phi; t] = (h, \phi)_{\mathring{H}_\mu^1};$$

(2) $w(0, m) = w_0(m)$ in L^2_μ sense, i.e.,

$$\int_B |w(0, m) - w_0(m)|^2 \mu dm = 0.$$

Remark 6. In (3.8), $(\psi, \phi)_{\mathring{H}^1_\mu}$ is a dual pair for $\psi \in (\mathring{H}^1_\mu)^*$ and $\phi \in \mathring{H}^1_\mu$, and can be regarded as L^2_μ inner product. Indeed, from the Riesz representaiton theorem, for each $\psi \in (\mathring{H}^1_\mu)^*$ there exists a unique $u \in \mathring{H}^1_\mu$ such that

$$(\psi, \phi)_{\mathring{H}^1_\mu} = \int_B (\nabla u \cdot \nabla \phi + u\phi) \mu dm.$$

Formally, the right-hand side will be

$$\int_B (\nabla \cdot (\nabla u \mu) \mu^{-1} + u) \phi \mu dm.$$

We identify ψ as $\nabla \cdot (\nabla u \mu) \mu^{-1} + u$ and the dual pair will be the L^2_μ inner product.

Remark 7. With the weight function μ so chosen as (2.2), we observe that if $\phi \in H^1_\mu$, then $\phi \in W^{1,1}$ since

$$\int_B (|\phi| + |\nabla \phi|) dm \leq \sqrt{2} \left(\int_B (|\phi|^2 + |\nabla \phi|^2) \mu dm \right)^{1/2} \left(\int_B \mu^{-1} dm \right)^{1/2} < \infty.$$

From the standard trace theorem, the map

$$\begin{aligned} \mathcal{T} : H^1_\mu(B) &\rightarrow L^1(\partial B), \\ \phi &\mapsto \phi|_{\partial\Omega} \end{aligned}$$

is well defined. Thus, the element in \mathring{H}^1_μ is characterized by the zero trace, and the Dirichlet data (3.3c) makes sense.

The well-posedness of problem (3.3) is stated in the following.

THEOREM 8. *Suppose that $w_0 \in L^2_\mu$, $h \in L^2_t(\mathring{H}^1_\mu)^*$, and $\kappa \in L^2_t$ with $\text{Tr}(\kappa) = 0$. Then the w -problem (3.3) has a unique weak solution in \mathcal{H} such that*

$$(3.9) \quad \|w\|_{\mathcal{H}}^2 \leq e^{F(|\kappa|)} \left(\|w_0\|_{L^2_\mu}^2 + \|h\|_{L^2_t(\mathring{H}^1_\mu)^*}^2 \right)$$

with F defined in (2.4).

This result when $b > 2$ and $q = 0$ was proved in [36]. For the general case we proceed in several steps.

An embedding theorem. We define

$$(3.10) \quad \mu^* = \begin{cases} \rho^{b/2-2}, & 0 < b < 2, \\ \rho^{-1}, & b = 2, \\ \rho^{-b/2}, & b > 2. \end{cases}$$

We call μ^* the conjugate of μ due to the Sobolev inequalities in the following lemma.

LEMMA 9. If $\phi \in \mathring{H}^1_\mu$, then

$$(3.11) \quad \int_B |\phi|^2 \mu^* dm \leq C \int_B (|\phi|^2 + |\nabla \phi|^2) \mu dm.$$

Also, if $\phi \in H^1_{\rho^\theta}$ for $\theta \leq 1$, then for any $\delta > 0$

$$(3.12) \quad \int_B |\phi|^2 \rho^{-1+\delta} dm \leq C \int_B (|\phi|^2 + |\nabla \phi|^2) \rho^\theta dm.$$

Proof. We refer to [25] for a proof of (3.11) when $b \neq 2$, as well as (3.12). Here, we prove only the case $b = 2$.

First for $C = \max_{1 \leq \rho \leq 2} (\rho \mu)^{-1} = \frac{1}{4} (\ln \frac{\epsilon}{2})^{-2}$ we have

$$\begin{aligned} \int_B |\phi|^2 / \rho dm &\leq C \int_{1 \leq \rho \leq 2} |\phi|^2 \mu dm + \int_{0 \leq \rho \leq 1} |\phi|^2 / \rho dm \\ &\leq C \int_B |\phi|^2 \mu dm + \int_0^1 \frac{G^2}{\rho} d\rho, \end{aligned}$$

where we have used the spherical coordinate representation with $\rho = 2 - r^2$ and

$$(3.13) \quad G^2(\rho) = - \int_{|\xi|=1} |\phi(r\xi)|^2 r^{N-1} dS_\xi \cdot \left(\frac{d\rho}{dr}\right)^{-1} = \frac{1}{2} \int_{|\xi|=1} |\phi(r\xi)|^2 r^{N-2} dS_\xi.$$

Here $1 \leq \rho \leq 2$ corresponds to $1 \leq r \leq \sqrt{2}$. Note that from $\phi \in \mathring{H}^1_\mu$ one can verify that $G(0) = 0$. It is known (see [26]) that

$$\int_0^1 \left(\int_0^x g(t) dt\right)^2 \frac{1}{x} dx \leq C \int_0^1 g^2(x) x |\ln x|^2 dx.$$

Thus,

$$(3.14) \quad \int_0^1 \frac{G^2}{\rho} d\rho \leq C \int_0^1 (G')^2 \rho |\ln \rho|^2 d\rho \leq C \int_0^1 \frac{(\partial_r G)^2}{r^2} \mu d\rho \leq C \int_0^1 (\partial_r G)^2 \mu d\rho,$$

where we have used the fact that $\rho |\ln \rho|^2 \leq \mu = \rho (\ln \frac{\epsilon}{\rho})^2$. Differentiation of (3.13) in terms of r leads to

$$2G\partial_r G = \int_{|\xi|=1} \phi \nabla \phi \cdot \xi r^{N-2} dS_\xi + \frac{N-2}{2} \int_{|\xi|=1} |\phi(r\xi)|^2 r^{N-3} dS_\xi.$$

Squaring both sides and using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} 4G^2(\partial_r G)^2 &\leq 2 \int_{|\xi|=1} \phi^2 r^{N-2} dS_\xi \int_{|\xi|=1} |\nabla \phi|^2 r^{N-2} dS_\xi \\ &\quad + \frac{(N-2)^2}{2} \left(\int_{|\xi|=1} \phi^2 r^{N-2} dS_\xi \right)^2, \end{aligned}$$

where we have used the fact $r \geq 1$. Hence

$$(\partial_r G)^2 \leq \int_{|\xi|=1} |\nabla \phi(r\xi)|^2 r^{N-2} dS_\xi + \frac{(N-2)^2}{2} G^2,$$

which inserted into (3.14) ensures that the term $\int_0^1 \frac{G^2}{\rho} d\rho$ is also bounded by $C \|\phi\|_{H^1_\mu}^2$.

The proof is now complete. \square

Energy estimates. We return now to the bilinear operator \mathcal{B} .

LEMMA 10 (energy estimates). *For any $t > 0$, there exists a constant C which is dependent on N, b such that*

(1) for $w(t, \cdot) \in \dot{H}_\mu^1$

$$(3.15) \quad \frac{1}{4} \int_B |\nabla w|^2 \mu dm \leq \mathcal{B}[w, w; t] + C(1 + |\kappa|^2) \int_B w^2 \mu dm;$$

(2) for $\psi(t, \cdot) \in H_\mu^1$ and $\phi \in \dot{H}_\mu^1$,

$$(3.16) \quad |\mathcal{B}[\psi, \phi; t]| \leq C(1 + |\kappa|) \|\psi\|_{H_\mu^1} \|\phi\|_{H_\mu^1}.$$

Proof. From (3.7) it follows that

$$(3.17) \quad \frac{1}{2} \int_B \nabla w \cdot \nabla \phi \mu dm = \mathcal{B}[w, \phi; t] + \int_B \kappa m \cdot \nabla \phi w \mu dm + \int_B K w \phi dm,$$

where K is given in (3.6).

Case 1. If $0 < b < 2$, then $K = 0$; hence

$$(3.18) \quad \begin{aligned} \frac{1}{2} \int_B |\nabla w|^2 \mu dm &= \mathcal{B}[w, w; t] + \int_B \kappa m \cdot \nabla w w \mu dm \\ &\leq \mathcal{B}[w, w; t] + \frac{1}{4} \int_B |\nabla w|^2 \mu dm + b|\kappa|^2 \int_B w^2 \mu dm \end{aligned}$$

and

$$\begin{aligned} |\mathcal{B}[\psi, \phi; t]| &\leq \frac{1}{2} \int_B |\nabla \psi| |\nabla \phi| \mu dm + \sqrt{b} |\kappa| \int_B |\psi| |\nabla \phi| \mu dm \\ &\leq C(1 + |\kappa|) \|\psi\|_{H_\mu^1} \|\nabla \phi\|_{L_\mu^2}. \end{aligned}$$

Case 2. For $b \geq 2$, it suffices to estimate the K -related term. If $b = 2$, we have

$$K = (N + 2\kappa m \cdot m) \ln \frac{e}{\rho} \leq (N + 2b|\kappa|) \sqrt{\mu \mu^*}.$$

If $b > 2$, we have

$$\begin{aligned} K &= \left(\frac{b}{2} - 1 \right) \rho^{1-b/2} (N + 2\kappa m \cdot m) \\ &\leq \left(\frac{b}{2} - 1 \right) (N + 2b|\kappa|) \sqrt{\mu \mu^*}. \end{aligned}$$

Hence for $b \geq 2$ we have

$$\begin{aligned} \int_B K w^2 dm &\leq C(1 + |\kappa|) \int_B w^2 \sqrt{\mu \mu^*} dm \\ &\leq \varepsilon \int_B w^2 \mu^* dm + C_\varepsilon (1 + |\kappa|^2) \int_B w^2 \mu dm. \end{aligned}$$

This when added upon the right side of (3.18) using (3.11) with some small ε leads to (3.15). Using (3.11) again we have

$$\left| \int_B K \psi \phi dm \right| \leq C(1 + |\kappa|) \int_B |\psi| |\phi| \sqrt{\mu \mu^*} dm \leq C(1 + |\kappa|) \|\psi\|_{H_\mu^1} \|\phi\|_{H_\mu^1},$$

which when combined with the above estimate for $b < 2$ gives (3.16). \square

A priori estimate.

LEMMA 11 (a priori estimates). *Let w be a weak solution to (3.3). Then*

$$(3.19) \quad \sup_t \|w(t, \cdot)\|_{L^2_\mu}^2 + \frac{1}{2} \|w\|_{L^2_t H^1_\mu}^2 \leq e^{F(|\kappa|)} \left(\|w_0\|_{L^2_\mu}^2 + \|h\|_{L^2_t(\dot{H}^1_\mu)^*}^2 \right)$$

with F defined in (2.4), and furthermore

$$(3.20) \quad \|w\|_{\mathcal{H}}^2 \leq e^{F(|\kappa|)} \left(\|w_0\|_{L^2_\mu}^2 + \|h\|_{L^2_t(\dot{H}^1_\mu)^*}^2 \right).$$

Proof. From the weak solution definition in (3.8) we have for any $\phi \in \dot{H}^1_\mu$

$$(3.21) \quad (\partial_t w, \phi)_{\dot{H}^1_\mu} + \mathcal{B}[w, \phi; t] = (h, \phi)_{\dot{H}^1_\mu}.$$

By (3.16), $(\partial_t w, \phi)_{\dot{H}^1_\mu}$ is bounded by

$$\|h\|_{(\dot{H}^1_\mu)^*} \|\phi\|_{H^1_\mu} + C(1 + |\kappa|) \|w\|_{H^1_\mu} \|\phi\|_{H^1_\mu}.$$

Hence

$$(3.22) \quad \|\partial_t w\|_{(\dot{H}^1_\mu)^*} \leq \|h\|_{(\dot{H}^1_\mu)^*} + C(1 + |\kappa|) \|w\|_{H^1_\mu}.$$

Next we set $\phi = w$ in (3.21) and use (3.15) to have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_{L^2_\mu}^2 + \frac{1}{4} \int_B |\nabla w|^2 \mu dm &\leq \|h\|_{(\dot{H}^1_\mu)^*} \|w\|_{H^1_\mu} + C(1 + |\kappa|^2) \|w\|_{L^2_\mu}^2 \\ &\leq 2 \|h\|_{(\dot{H}^1_\mu)^*}^2 + \frac{1}{8} \|w\|_{H^1_\mu}^2 + C(1 + |\kappa|^2) \|w\|_{L^2_\mu}^2. \end{aligned}$$

Hence

$$(3.23) \quad \frac{d}{dt} \|w\|_{L^2_\mu}^2 + \frac{1}{4} \|w\|_{H^1_\mu}^2 \leq C(1 + |\kappa|^2) \|w\|_{L^2_\mu}^2 + 4 \|h\|_{(\dot{H}^1_\mu)^*}^2,$$

and therefore by Gronwall's inequality,

$$\sup_t \|w(t, \cdot)\|_{L^2_\mu}^2 + \frac{1}{2} \|w\|_{L^2_t H^1_\mu}^2 \leq e^{C(T + \int_0^T |\kappa|^2 dt)} \left(\|w_0\|_{L^2_\mu}^2 + \|h\|_{L^2_t(\dot{H}^1_\mu)^*}^2 \right),$$

which together with (3.22) yields (3.20). \square

Proof of Theorem 8. We construct a weak solution to (3.3) using the Galerkin approximation. Let $\{\phi_i\}$ be a basis of \dot{H}^1_μ and L^2_μ . Then an approximate solution w_l in a finite dimensional space is defined as $w_l = \sum_{i=1}^l d_i^l(t) \phi_i$. Here $d_i^l(t)$ is a unique solution to a system of linear differential equations,

$$\begin{aligned} (\partial_t w_l, \phi_j)_{\dot{H}^1_\mu} + \mathcal{B}[w_l, \phi_j; t] &= (h, \phi_j)_{\dot{H}^1_\mu}, \\ d_i^l(0) &= \left((\phi_i, \phi_j)_{L^2_\mu}^{-1} (w_0, \vec{\phi})_{L^2_\mu} \right)_i, \end{aligned}$$

where $\vec{\phi} = (\phi_1, \dots, \phi_l)^\top$. Using the same argument as that in the proof of Lemma 11, we obtain

$$\|w_l\|_{L_t^2 H_\mu^1}^2 + \|\partial_t w_l\|_{L_t^2(\dot{H}_\mu^1)^*}^2 \leq e^{F(|\kappa|)} \left(\|w_0\|_{L_\mu^2}^2 + \|h\|_{L_t^2(\dot{H}_\mu^1)^*}^2 \right).$$

Extracting a subsequence and passing to the limit give a weak solution w in \mathcal{H} . The uniqueness follows from the a priori estimate (3.20). \square

To return to the Fokker–Planck problem (3.1) we will also need the following lemma.

LEMMA 12. *Let $h = -\partial_t q - \mu^{-1}L[q]$. If $q \in C_t^1 H_\mu^1$ and $\kappa \in L_t^2$ with $Tr(\kappa) = 0$, then*

$$(3.24) \quad \|h\|_{L_t^2(\dot{H}_\mu^1)^*}^2 \leq C \int_0^T (1 + |\kappa|^2) \|q(t)\|^2 d\tau.$$

Proof. For $q \in C_t^1 H_\mu^1$, it is obvious that $\partial_t q \in L_t^2(\dot{H}_\mu^1)^*$ since $H_\mu^1 \subset (H_\mu^1)^* \subset (\dot{H}_\mu^1)^*$. In order to show $\mu^{-1}L[q] \in L_t^2(\dot{H}_\mu^1)^*$, we use integration by parts and (3.16) to get

$$\left| \int_B \mu^{-1}L[q]\phi \mu dm \right| = |\mathcal{B}[q, \phi; t]| \leq C(1 + |\kappa|) \|q(t, \cdot)\|_{H_\mu^1} \|\phi\|_{H_\mu^1} \quad \forall \phi \in C_c^\infty.$$

Since C_c^∞ is a dense subset of \dot{H}_μ^1 , for any $\phi \in \dot{H}_\mu^1$ with $\|\phi\|_{H_\mu^1} = 1$, we have

$$(3.25) \quad |(\mu^{-1}L[q], \phi)_{\dot{H}_\mu^1}| \leq C(1 + |\kappa|) \|q(t, \cdot)\|_{H_\mu^1}.$$

Taking the L^2 norm in t leads to the desired estimate. \square

Theorem 8 and Lemma 12 lead to the following result for problem (3.1) with a general Dirichlet boundary condition.

THEOREM 13. *Suppose that $f_0 \nu^{-1} \in L_\mu^2$, $q \in C_t^1 H_\mu^1$, and $\kappa \in L_t^2$ with $Tr(\kappa) = 0$. Then for any $T > 0$ the Fokker–Planck problem (3.1) has a unique solution f such that*

$$(3.26) \quad f = \nu(w + q) \quad \text{with } w \in \mathcal{H} \text{ for } 0 < t \leq T.$$

Moreover, for F defined in (2.4),

$$(3.27) \quad \sup_t \|w(t, \cdot)\|_{L_\mu^2}^2 + \frac{1}{2} \|w\|_{L_t^2 H_\mu^1}^2 \leq e^{F(|\kappa|)} \left(\|w_0\|_{L_\mu^2}^2 + \int_0^T (1 + |\kappa(t)|^2) \|q(t)\|^2 dt \right).$$

Proof. The estimate (3.27) follows from (3.19) and the estimate in Lemma 12 with Fe^F replaced by e^F .

We now prove the uniqueness of f in terms of $q|_{\partial B}$. Let $f_i (i = 1, 2)$ be two solutions with q_i such that $q_1|_{\partial B} = q_2|_{\partial B}$ and initial data f_0 . Set $w = (f_2 - f_1)\nu^{-1}$; then w solves w-problem (3.3) with $w_0 \equiv h \equiv 0$. Hence $w \equiv 0$, leading to $f_1 = f_2$. \square

Remark 14. As mentioned in section 2, $\dot{H}_\mu^1 = H_\mu^1$ if $b \geq 6$, i.e., the trace of $q \in H_\mu^1$ vanishes if $b \geq 6$. Thus, boundary condition (3.1c) is nothing but a zero Dirichlet

boundary condition. In section 6, we show the well-posedness with a nonzero Dirichlet boundary condition for $b \geq 6$ using yet a different transformation.

Remark 15. The condition $Tr(\kappa) = 0$ comes from the divergence-free velocity field of the Navier–Stokes equations. Many of the above arguments, however, do not use this condition explicitly.

3.2. Probability density function. So far we have discussed well-posedness of the initial-boundary value problem (3.1) for $b > 0$ and any given q . We now turn to the question of which q corresponds to the probability density, i.e., nonnegative solution with constant mass for all time.

PROPOSITION 16. *Let $f(t, m)$ be the solution to problem (3.1) obtained in Theorem 13. If $f_0 \geq 0$ and $q(t, m)|_{\partial B} \geq 0$ almost everywhere, then f remains nonnegative for $t > 0$.*

Proof. We adapt an idea from [7]. Let f^\pm be the positive and negative parts of the solution f such that $f = f^+ - f^-$. Obviously, $w^\pm := f^\pm \nu^{-1} \in H^1_\mu$ and $q|_{\partial B} \geq 0$. This implies that the trace of w^- at the boundary vanishes, so

$$w^- \in \mathring{H}^1_\mu.$$

From the equation

$$\partial_t w \mu + L[w] = 0,$$

which is transformed from (3.1a), it follows that

$$(\partial_t w, w^-)_{\mathring{H}^1_\mu} + \mathcal{B}[w, w^-; t] = 0.$$

Since $(\partial_t w^+, w^-)_{\mathring{H}^1_\mu}$ and $\int L[w^+] w^- dm$ vanish, hence

$$\frac{1}{2} \frac{d}{dt} \left(\int_B |w^-|^2 \mu dm \right) + \mathcal{B}[w^-, w^-; t] = 0.$$

The coercivity of \mathcal{B} , (3.15), gives

$$\frac{1}{2} \frac{d}{dt} \left(\int_B |w^-|^2 \mu dm \right) + \frac{1}{4} \int_B |\nabla w^-|^2 \mu dm \leq C(1 + |\kappa|^2) \int_B |w^-|^2 \mu dm.$$

Hence

$$\sup_t \|w^-(t, \cdot)\|_{L^2_\mu}^2 \leq \|w^-_0\|_{L^2_\mu}^2 e^{F(|\kappa|)}$$

for $T > 0$. Since $w^-_0 = 0$, $\|w^-(t, \cdot)\|_{L^2_\mu}^2 = 0$ for all $0 \leq t \leq T$. \square

PROPOSITION 17. *Let f be a solution to the Fokker–Planck problem (3.1) obtained in Theorem 13. Suppose $b \geq 2$ and $q(t, m)|_{\partial B} \geq 0$. If $q|_{\partial B} = 0$ for all $t \in [0, T]$, then*

$$\int_B f(t, \cdot) dm = \int_B f_0 dm, \quad t \in [0, T],$$

and vice versa.

Proof. It suffices to prove the claim for smooth enough f since the general case can be treated by an approximation as in [36]. We rewrite (3.1a) as

$$\partial_t f = -\nabla \cdot (\kappa m f) + \nabla \cdot \left(\rho^{b/2} \nabla \frac{f}{\rho^{b/2}} \right).$$

First, we take a test function $\phi_\varepsilon(m) = \phi_\varepsilon(|m|) \in C_c^\infty(\mathbb{R}^N)$ converging to χ_B as $\varepsilon \rightarrow 0$ such that

$$\phi_\varepsilon(|m|) = \begin{cases} 1, & |m| \leq \sqrt{b} - \varepsilon, \\ 0, & |m| \geq \sqrt{b} - \varepsilon/2, \end{cases} \quad |\nabla \phi_\varepsilon| \leq C \frac{1}{\varepsilon},$$

and for any smooth g

$$(3.28) \quad \int_{\sqrt{b}-\varepsilon}^{\sqrt{b}-\varepsilon/2} g(r) \phi'_\varepsilon(r) dr \rightarrow -g(\sqrt{b}) \quad \text{as } \varepsilon \rightarrow 0,$$

where $\phi'_\varepsilon(r) = \nabla \phi_\varepsilon \cdot \frac{m}{|m|}$.

One can construct such a ϕ_ε by mollifiers, for example,

$$\phi_\varepsilon(m) = \int_{B_{\sqrt{b}-3\varepsilon/4}} \eta_{\varepsilon/4}(m - m') dm',$$

where

$$\eta_\varepsilon(m) = \frac{1}{\varepsilon^N} \eta(m/\varepsilon), \quad \eta(m) = \begin{cases} C e^{-\frac{1}{1-|m|^2}}, & |m| < 1, \\ 0, & |m| \geq 1, \end{cases}$$

and C is a normalizing constant.

Since $\nabla \phi_\varepsilon$ is supported in $B^\varepsilon := B_{\sqrt{b}-\varepsilon/2} \setminus B_{\sqrt{b}-\varepsilon}$,

$$(3.29) \quad \frac{d}{dt} \int_B f \phi_\varepsilon dm = \int_{B^\varepsilon} f \kappa m \cdot \nabla \phi_\varepsilon dm - \int_{B^\varepsilon} \rho^{b/2} \nabla \left(\frac{f}{\rho^{b/2}} \right) \cdot \nabla \phi_\varepsilon dm.$$

By $w = f\nu^{-1}$, the right-hand side reduces to

$$(3.30) \quad \int_{B^\varepsilon} (w \kappa m - \nabla w) \cdot \nabla \phi_\varepsilon \nu dm - \int_{B^\varepsilon} w \rho^{b/2} \nabla \phi_\varepsilon \cdot \nabla (\nu \rho^{-b/2}) dm.$$

The first term converges to 0. Indeed,

$$\left| \int_{B^\varepsilon} (w \kappa m - \nabla w) \cdot \nabla \phi_\varepsilon \nu dm \right| \leq \left(\int_{B^\varepsilon} |w \kappa m - \nabla w|^2 \mu dm \right)^{1/2} \left(\int_{B^\varepsilon} |\nabla \phi_\varepsilon|^2 \frac{\nu^2}{\mu} dm \right)^{1/2}.$$

Since $\nu^2/\mu = \rho^{b/2}$ for $b \geq 2$, by mean value theorem there exists $r \in (\sqrt{b}-\varepsilon, \sqrt{b}-\varepsilon/2)$ such that

$$\int_{B^\varepsilon} |\nabla \phi_\varepsilon|^2 \frac{\nu^2}{\mu} dm = \frac{\varepsilon}{2} \int_{\partial B_r} |\nabla \phi_\varepsilon|^2 \rho^{b/2} dS \leq C \varepsilon^{b/2-1},$$

which is uniformly bounded for $b \geq 2$. Using $w \in H_\mu^1$, we obtain $\int_{B^\varepsilon} |w \kappa m - \nabla w|^2 \mu dm \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence the first term in (3.30) converges to 0.

On the other hand, for $C_0 = \begin{cases} -2, & b=2 \\ 2-b, & b>2 \end{cases}$

$$\begin{aligned} - \int_{B^\varepsilon} w \rho^{b/2} \nabla \phi_\varepsilon \cdot \nabla (\nu \rho^{-b/2}) dm &= C_0 \int_{B^\varepsilon} w \nabla \phi_\varepsilon \cdot m dm \\ &= C_0 \int_{\sqrt{b}-\varepsilon}^{\sqrt{b}-\varepsilon/2} \int_{\partial B_r} w r \phi'_\varepsilon(r) dS dr \\ &= C_0 \int_{\sqrt{b}-\varepsilon}^{\sqrt{b}-\varepsilon/2} \left(r \int_{\partial B_r} w dS \right) \phi'_\varepsilon(r) dr. \end{aligned}$$

Due to (3.28) this converges to

$$-C_0\sqrt{b} \int_{\partial B} w dS = -C_0\sqrt{b} \int_{\partial B} q dS.$$

Since $C_0 \neq 0$, this shows that $\frac{d}{dt} \int_B f dm = 0$ if and only if $\int_{\partial B} q dS = 0$ or $q|_{\partial B} = 0$. \square

Remark 18. In Proposition 17, the assumption $b \geq 2$ is sharp. In the case $b < 2$, we need to consider nontrivial $q \neq 0$ since the equilibrium profile $f_{eq} = \rho^{b/2}$ satisfies

$$q|_{\partial B} = \rho^{b/2}\nu^{-1}|_{\partial B} = 1.$$

This requirement is also consistent with [35], in which it was shown that when $b < 2$, $f\nu^{-1}|_{\partial B} = q|_{\partial B}$ is necessarily prescribed and each solution depends on the choice of q . It would be interesting to figure out a particular q for which the corresponding solution when $b < 2$ is a probability density.

4. The Fokker–Planck equation. In this section, we show the well-posedness of the Fokker–Planck equation (2.9) including the x variable. The result is stated as follows.

THEOREM 19. *Suppose that for $b > 0$ and any integer $s > N/2 + 1$, $\nabla \cdot v = 0$ and*

$$(4.1) \quad v \in C_t H_x^s \cap L_t^2 H_x^{s+1}, \quad f_0\nu^{-1} \in H_x^s L_\mu^2, \quad q \in C_t^1 H_x^{s+1} H_\mu^1, \quad 0 < t < T,$$

for any $T > 0$. Then (2.9) has a unique solution $f = \nu(w + q)$ satisfying

$$(4.2) \quad \sup_t |w|_{0,s}^2 + \frac{1}{2} \int_0^T |\nabla_m w|_{0,s}^2 dt \leq e^{F(|v|_{s+1})} (|w_0|_{0,s}^2 + \|q\|_{1,1,s+1}^2),$$

where F was defined in (2.4).

The proof of Theorem 19 consists of two parts: first we show the existence of the solution f to problem (2.9) by using the flow map, followed by proving regularity in x inductively such that $w \in C_t H_x^s L_\mu^2 \cap L_t^2 H_x^s H_\mu^1$ with v, f_0 , and q given in (4.1). In the second step, we derive estimate (4.2) directly from (2.9) to control f in terms of the given data. The uniqueness can be obtained from the estimation (4.2) as performed in the proof of Theorem 13.

First, we state a technical lemma.

LEMMA 20. *Suppose that $\psi \in H_\mu^1$ and $\phi \in \mathring{H}_\mu^1$. Then for the trace map $\mathcal{T} : W^{1,1}(B) \rightarrow L^1(\partial B)$*

$$(4.3) \quad \mathcal{T}(\psi\phi\mu) = 0.$$

Proof. Since C_c^∞ is a dense subset of \mathring{H}_μ^1 , it suffices to show that for a fixed $\psi \in H_\mu^1$ and any $\phi \in C_c^\infty$

$$(4.4) \quad \|\psi\phi\mu\|_{W^{1,1}} \leq C\|\phi\|_{H_\mu^1}.$$

Then, the standard trace theorem asserts that $\mathcal{T}(\psi\phi\mu)$ is well defined in $L^1(\partial B)$ and it vanishes, and also \mathcal{T} is a continuous map with respect to ϕ ; we can thus conclude (4.3) for any $\phi \in \mathring{H}_\mu^1$ by passing to the limit of sequence $\phi_n \in C_c^\infty$ such that $\phi_n \rightarrow \phi$.

Equation (4.4) is indeed the case. It is obvious that $\psi\phi\mu, \nabla_m \psi\phi\mu$, and $\psi\nabla_m \phi\mu$ are integrable. For $b \neq 2$, $|\nabla_m \mu| \leq C\sqrt{\mu\mu^*}$ and (3.11) yield

$$\int_B |\psi\phi\nabla_m \mu| dm \leq C\|\psi\|_{L_\mu^2} \|\phi\|_{H_\mu^1}.$$

For $b = 2$,

$$|\nabla_m \mu| \leq C \left(\left(\ln \frac{e}{\rho} \right)^2 + \ln \frac{e}{\rho} \right) \leq C \left(\left(\ln \frac{e}{\rho} \right)^2 + \sqrt{\mu \mu^*} \right).$$

Using (3.12) and $\psi \in H_\mu^1$, we obtain $\psi \in L_{-1+\delta}^2$ for any $\delta > 0$. Hence

$$\left| \int_B \psi \phi \left(\ln \frac{e}{\rho} \right)^2 dm \right| \leq C \left(\sqrt{\int_B \psi^2 \rho^{-1+\delta} dm} \sqrt{\int_B \phi^2 \rho^{1-\delta} \ln^4 \left(\frac{e}{\rho} \right) dm} \right).$$

It follows that for any $b > 0$

$$\int_B |\psi \phi \mu| + |\nabla_m(\psi \phi \mu)| dm < C \|\psi\|_{H_\mu^1} \|\phi\|_{H_\mu^1}$$

as we desired. \square

The main ingredient for the proof of Theorem 19 is to use the calculus inequalities in the Sobolev spaces (see Appendix 3.5 of [40]): for any positive integer $r > 0$ and $u, v \in L_x^\infty \cap H_x^r$,

$$(4.5) \quad \sum_{|\gamma| \leq r} \|\partial^\gamma (uv) - u \partial^\gamma v\|_{L^2} \leq C (\|\nabla u\|_{L^\infty} \|v\|_{H^{r-1}} + \|u\|_{H^r} \|v\|_{L^\infty}),$$

$$(4.6) \quad \|uv\|_{H^r} \leq C (\|u\|_{L^\infty} \|v\|_{H^r} + \|u\|_{H^r} \|v\|_{L^\infty}).$$

Note that (4.5) remains valid when ∂^γ on the left side is replaced by the corresponding difference operator.

Proof of Theorem 19.

Step 1 (well-posedness). Let a particle path be defined by

$$\partial_t x(t, y) = v(t, x(t, y)), \quad x(0, y) = y,$$

along which the distribution function $\tilde{f}(t, y, m) := f(t, x(t, y), m)$ solves

$$(4.7a) \quad \partial_t \tilde{f} + \mathcal{L}[\tilde{f}] = 0,$$

$$(4.7b) \quad \tilde{f}(0, y, m) = f_0(y, m),$$

$$(4.7c) \quad \tilde{f}(t, y, m) \nu^{-1}|_{\partial B} = \tilde{q}(t, y, m)|_{\partial B}.$$

Here \mathcal{L} is defined in (3.2) with κ replaced by $\tilde{\kappa}(t, y) = \nabla v(t, x(t, y))$, and $\tilde{q}(t, y, m) := q(t, x(t, y), m)$.

In order to show existence of a solution to (2.9) under the conditions $v \in C_t H_x^s \cap L_t^2 H_x^{s+1}$ and $\nabla \cdot v = 0$, it suffices to show that (4.7) has a solution $\tilde{f} = \nu(\tilde{w} + \tilde{q})$ such that

$$\tilde{w} := w(t, x(t, y), m) \in C_t H_y^s L_\mu^2 \cap L_t^2 H_y^s H_\mu^1,$$

assuming that

$$(4.8) \quad \tilde{\kappa} \in L_t^2 H_y^s, \quad w_0 \in H_y^s L_\mu^2, \quad \tilde{q} \in C_t^1 H_y^s H_\mu^1.$$

These follow from (4.1) since $|\tilde{\kappa}(t)|_s \leq C|v(t)|_{s+1}$ for $t > 0$, $w_0(x, m) = f_0 \nu^{-1} - \tilde{q}(t = 0) = w_0(y, m)$, and $\|\tilde{q}\|_{1,1,s} \leq C\|q\|_{1,1,s+1}$, for which we have used $\partial_t \tilde{q} = \partial_t q + v \cdot \nabla q$.

Using Theorem 13 for each y , there exists a unique solution \tilde{f} such that

$$\tilde{f} = \nu(\tilde{w} + \tilde{q})$$

with \tilde{w} satisfying (3.27), i.e.,

$$(4.9) \quad \sup_t \|\tilde{w}(t, y, \cdot)\|_{L_\mu^2}^2 + \frac{1}{2} \|\tilde{w}(\cdot, y, \cdot)\|_{L_t^2 H_\mu^1}^2 \leq e^{F(|\tilde{\kappa}(\cdot, y)|)} \left(\|w_0(y, \cdot)\|_{L_\mu^2}^2 + \int_0^T (1 + |\tilde{\kappa}(\cdot, y)|^2) \|\tilde{q}(t, y, \cdot)\|^2 dt \right).$$

Integration of (4.9) with respect to y , upon exchanging the order of integration in y and m and using the Sobolev inequality, $\sup_y |\tilde{\kappa}| \leq C|\tilde{\kappa}|_{s-1}$, gives

$$(4.10) \quad \sup_t |\tilde{w}|_{0,0}^2 + \frac{1}{2} \int_0^T |\tilde{w}|_{1,0}^2 dt \leq e^{F(|\tilde{\kappa}|_{s-1})} (|w_0|_{0,0}^2 + \|\tilde{q}\|_{1,1,0}^2).$$

Hence $\tilde{w} \in C_t L_y^2 L_\mu^2 \cap L_t^2 L_y^2 H_\mu^1$. On the other hand, the right-hand side of (4.9) is uniformly bounded in y , and taking \sup_y of (4.9) gives

$$(4.11) \quad \sup_{t,y} \|\tilde{w}(t, y, \cdot)\|_{L_\mu^2}^2 \leq e^{F(|\tilde{\kappa}|_{s-1})} (|w_0|_{0,s-1}^2 + \|\tilde{q}\|_{1,1,s-1}^2).$$

We now use an induction argument to prove that $\tilde{w} \in C_t H_y^r L_\mu^2 \cap L_t^2 H_y^r H_\mu^1$ for $0 \leq r \leq s$, and

$$(4.12) \quad \sup_t |\tilde{w}|_{0,r}^2 + \frac{1}{2} \int_0^T |\tilde{w}|_{1,r}^2 dt \leq e^{F(|\tilde{\kappa}|_s)} (|w_0|_{0,s}^2 + \|\tilde{q}\|_{1,1,s}^2).$$

The case $r = 0$ has been proved as shown in (4.10). Suppose (4.12) holds for $r = k$; we only need to show (4.12) for $r = k + 1 \leq s$.

To prove regularity of \tilde{f} in the y variable, we use difference quotients. Define the difference operator in the y variable as

$$\delta^\gamma := \delta_1^{\gamma_1} \cdots \delta_N^{\gamma_N}, \quad \delta_i u(y) := \frac{1}{\eta} [u(y + \eta e_i) - u(y)].$$

Apply δ^γ to (4.7) with $|\gamma| \leq s$; then

$$(4.13a) \quad \partial_t \delta^\gamma \tilde{f} + \mathcal{L}[\delta^\gamma \tilde{f}] = \nabla_m \cdot J,$$

$$(4.13b) \quad \delta^\gamma \tilde{f}(0, y, m) = \delta^\gamma f_0(y, m),$$

$$(4.13c) \quad \delta^\gamma \tilde{f}(t, y, m) \nu^{-1}|_{\partial B} = \delta^\gamma \tilde{q}(t, y, m)|_{\partial B},$$

where

$$(4.14) \quad J = \tilde{\kappa} m \delta^\gamma \tilde{f} - \delta^\gamma (\tilde{\kappa} m \tilde{f}).$$

This when transformed into the w-problem of form (3.3) involves the following non-homogeneous term:

$$(4.15) \quad h = -\partial_t \delta^\gamma \tilde{q} - \mu^{-1} L[\delta^\gamma \tilde{q}] + \nabla_m \cdot J \nu^{-1}.$$

Using Theorem 13 again for each y , $\delta^\gamma \tilde{f}$ is the unique solution to (4.13) as long as $h \in L^2_t(\mathring{H}^1_\mu)^*$. Moreover,

$$\delta^\gamma \tilde{f} = \nu(\delta^\gamma \tilde{w} + \delta^\gamma \tilde{q}),$$

where $\delta^\gamma \tilde{w}$, using (3.19), satisfies

$$\sup_t \|\delta^\gamma \tilde{w}(t, y, \cdot)\|_{L^2_\mu}^2 + \frac{1}{2} \|\delta^\gamma \tilde{w}(\cdot, y, \cdot)\|_{L^2_t H^1_\mu}^2 \leq e^{F(\tilde{\kappa}(\cdot, y))} \left(\|\delta^\gamma w_0\|_{L^2_\mu}^2 + \|h\|_{L^2_t(\mathring{H}^1_\mu)^*}^2 \right).$$

Integration in y gives

$$\begin{aligned} \sup_t |\delta^\gamma \tilde{w}|_{0,0}^2 + \frac{1}{2} \int_0^T |\delta^\gamma \tilde{w}|_{1,0}^2 dt &\leq e^{F(\sup_y |\tilde{\kappa}(\cdot, y)|)} \left(|\delta^\gamma w_0|_{0,0}^2 + \|h\|_{L^2_t L^2_y(\mathring{H}^1_\mu)^*}^2 \right) \\ (4.16) \qquad \qquad \qquad &\leq e^{F(|\tilde{\kappa}|_{s-1})} \left(|w_0|_{0,s}^2 + \|h\|_{L^2_t L^2_y(\mathring{H}^1_\mu)^*}^2 \right). \end{aligned}$$

We now turn to bound the last term in the above inequality. For any $\phi \in \mathring{H}^1_\mu$ and J defined in (4.14), Lemma 20 allows the use of integration by parts. Hence,

$$\begin{aligned} \left| \int_B \nabla_m \cdot J\nu^{-1} \phi \mu dm \right| &\leq \left(\int_B |J\nu^{-1}| |\nu \nabla_m \frac{\mu}{\nu}| |\phi| dm + \int_B |J\nu^{-1}| |\nabla_m \phi| \mu dm \right) \\ &\leq C \|J\nu^{-1}\|_{L^2_\mu} (\|\phi\|_{L^2_{\mu^*}} + \|\nabla_m \phi\|_{L^2_\mu}) \\ &\leq C \|J\nu^{-1}\|_{L^2_\mu} \|\phi\|_{H^1_\mu}. \end{aligned}$$

Here we have used $|\nu \nabla_m \frac{\mu}{\nu}| \leq C\sqrt{\mu^* \mu}$ and the embedding theorem (3.11). This together with Lemma 12 and (4.15) yields

$$(4.17) \quad \|h\|_{L^2_t L^2_y(\mathring{H}^1_\mu)^*}^2 \leq C \int_0^T (1 + \sup_y |\tilde{\kappa}(t, y)|^2) \int_{\mathbb{R}^N} \|\delta^\gamma \tilde{q}(t, y, \cdot)\|^2 dy dt + C \int_0^T |J\nu^{-1}|_{0,0}^2 dt.$$

For $|\gamma| \leq s$, the first term on the right side is bounded by

$$(4.18) \quad F(|\tilde{\kappa}|_{s-1}) \|\delta^\gamma \tilde{q}\|_{1,1,0}^2 \leq F(|\tilde{\kappa}|_{s-1}) \|\tilde{q}\|_{1,1,s}^2.$$

To obtain (4.12) for $r = k + 1 \leq s$, it remains to estimate the last term in (4.17) with $|\gamma| = k + 1$. In fact,

$$\begin{aligned} |J\nu^{-1}|_{0,0}^2 &= |(\delta^\gamma(\tilde{\kappa} m \tilde{f}) - \tilde{\kappa} m \delta^\gamma \tilde{f}) \nu^{-1}|_{0,0}^2 \\ &\leq C(\sup_y |\nabla_y \tilde{\kappa}|^2 |\tilde{f} \nu^{-1}|_{0,k}^2 + |\tilde{\kappa}|_{k+1}^2 \sup_y \|\tilde{f} \nu^{-1}\|_{L^2_\mu}^2) \\ &\leq C|\tilde{\kappa}|_s^2 (|\tilde{w}|_{0,k}^2 + \sup_y \|\tilde{w}\|_{L^2_\mu}^2 + \|\tilde{q}\|_{1,1,s}^2), \end{aligned}$$

where we have used (4.5) with ∂^γ replaced by δ^γ .

Using (4.12) for $r = k$ and (4.11) we have

$$\int_0^T |J\nu^{-1}|_{0,0}^2 dt \leq e^{F(|\tilde{\kappa}|_s)} (|w_0|_{0,s}^2 + \|\tilde{q}\|_{1,1,s}^2).$$

This and (4.18) when inserted into (4.17) give a bound for $\|h\|_{L_t^2 L_y^2 (H_\mu^1)^*}^2$. That bound combined with (4.16) yields

$$\sup_t |\delta^\gamma \tilde{w}|_{0,0}^2 + \frac{1}{2} \int_0^T |\delta^\gamma \tilde{w}|_{1,0}^2 dt \leq e^{F(|\tilde{\kappa}|_s)} (|w_0|_{0,s}^2 + \|\tilde{q}\|_{1,1,s}^2) < \infty, \quad |\gamma| = k + 1.$$

Sending $\eta \rightarrow 0$ we obtain (4.12) with $r = k + 1$. Hence, (4.12) holds for any $r \leq s$, and thus the solution f to (2.9) exists, and

$$\sup_t |w|_{0,s}^2 + \frac{1}{2} \int_0^T |w|_{1,s}^2 dt < \infty.$$

One may obtain an upper bound from (4.12) with $r = s$ using the inverse map of $x = x(t, y)$. Nevertheless, the next step gives the claimed bound in (4.2).

Step 2 (a priori estimate). For a priori estimate, we consider the w-problem (2.11)

$$(4.19) \quad \mu(\partial_t + v \cdot \nabla)w + L[w] = -\mu(\partial_t + v \cdot \nabla)q - L[q].$$

Recall that

$$L[w] = -\frac{1}{2} \nabla_m \cdot (\nabla_m w \mu) + \nabla_m \cdot (\kappa m w \mu) - Kw.$$

Take the γ derivative in the x variable. Then, the left- and right-hand sides of (4.19) will be

$$(4.20) \quad I = \mu(\partial_t + v \cdot \nabla)\partial^\gamma w - \frac{1}{2} \nabla_m \cdot (\nabla_m \partial^\gamma w \mu)$$

$$(4.21) \quad + \mu[\partial^\gamma((v \cdot \nabla)w) - (v \cdot \nabla)\partial^\gamma w]$$

$$(4.22) \quad + \nabla_m \cdot (\partial^\gamma(\kappa m w \mu))$$

$$(4.23) \quad - \partial^\gamma(Kw),$$

$$(4.24) \quad II = -\mu\partial_t\partial^\gamma q + \frac{1}{2} \nabla_m \cdot (\nabla_m \partial^\gamma q \mu)$$

$$(4.25) \quad - \mu\partial^\gamma((v \cdot \nabla)q)$$

$$(4.26) \quad - \nabla_m \cdot (\partial^\gamma(\kappa m q \mu))$$

$$(4.27) \quad + \partial^\gamma(Kq).$$

We now estimate term by term

$$(4.28) \quad \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B I \partial^\gamma w \mu dx = \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B II \partial^\gamma w \mu dx.$$

Since v is divergence free, the first two terms on the left-hand side will be

$$\frac{1}{2} \frac{d}{dt} |w|_{0,s}^2 + \frac{1}{2} |\nabla_m w|_{0,s}^2.$$

Indeed, the Young inequality shows that the term related to (4.21) is bounded by

$$\varepsilon |w|_{0,s}^2 + C_\varepsilon \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B |\partial^\gamma((v \cdot \nabla)w) - (v \cdot \nabla)\partial^\gamma w|^2 \mu dx.$$

Now, we exchange the order of integration in x and m and apply (4.5) to obtain

$$\begin{aligned} & \varepsilon |w|_{0,s}^2 + C_\varepsilon \int_B \left(\|\nabla v\|_{L_x^\infty}^2 \|\nabla w(\cdot, m)\|_{H_x^{s-1}}^2 + \|v\|_{H_x^s}^2 \|\nabla w(\cdot, m)\|_{L_x^\infty}^2 \right) \mu dm \\ & \leq \varepsilon |w|_{0,s}^2 + C_\varepsilon |v|_s^2 |w|_{0,s}^2, \end{aligned}$$

where the Sobolev inequality, $|u|_0 \leq C|u|_{s-1}$ for any $u \in H_x^{s-1}$, is invoked in the last inequality. Similarly, the term with (4.22) will be estimated as follows due to (4.6);

$$\begin{aligned} & \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B |\partial^\gamma(\kappa m w)|^2 \mu dm dx \\ & \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon \int_B \left(|\kappa|_{L_x^\infty}^2 |w(\cdot, m)|_s^2 + |\kappa|_s^2 |w(\cdot, m)|_{L_x^\infty}^2 \right) \mu dm \\ & \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2 |w|_{0,s}^2. \end{aligned}$$

Recall that

$$K = \begin{cases} 0, & 0 < b < 2, \\ (N + 2\kappa m \cdot m) \ln \frac{\rho}{\rho}, & b = 2, \\ (N + 2\kappa m \cdot m)(b/2 - 1)\rho^{1-b/2}, & b > 2. \end{cases}$$

Thus, we can express K as

$$(4.29) \quad K = c_1 \sqrt{\mu \mu^*} + c_2 \kappa m \cdot m \sqrt{\mu \mu^*}$$

for some positive constant c_i depending on N and b . We now estimate the last term on the left-hand side by using

$$\partial^\gamma(Kw)\partial^\gamma w = c_1 |\partial^\gamma w|^2 \sqrt{\mu \mu^*} + c_2 \partial^\gamma(\kappa m \cdot m w) \partial^\gamma w \sqrt{\mu \mu^*}.$$

The Young inequality and the embedding theorem (3.11) give

$$\begin{aligned} c_1 \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B |\partial^\gamma w|^2 \sqrt{\mu \mu^*} dm dx &= c_1 \int_B |w(t, \cdot, m)|_s^2 \sqrt{\mu \mu^*} dm \\ &\leq \varepsilon \int_B |w(t, \cdot, m)|_s^2 \mu^* dm + C_\varepsilon \int_B |w(t, \cdot, m)|_s^2 \mu dm \\ &\leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |w|_{0,s}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} c_2 \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B |\partial^\gamma(\kappa m \cdot m w) \partial^\gamma w| \sqrt{\mu \mu^*} dm dx &\leq \varepsilon |\nabla_m w|_{0,s}^2 \\ &\quad + C_\varepsilon \int_{\mathbb{R}^N} \int_B |\partial^\gamma(\kappa m \cdot m w)|^2 \mu dm dx. \end{aligned}$$

The last term, using (4.6) and the Sobolev inequality for $\kappa = \nabla v$, is then bounded by

$$C_\varepsilon |v|_{s+1}^2 |w|_{0,s}^2.$$

Hence,

$$\left| \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} \int_B \partial^\gamma(Kw)\partial^\gamma w dm dx \right| \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon (|v|_{s+1}^2 + 1) |w|_{0,s}^2.$$

Now we turn to the right-hand side, related to (4.24)–(4.27). The estimation is similar to that for the left-hand side, except that here we have to assume higher regularity of q in x than that of w since $\int_{\mathbb{R}^N} v \cdot \int_B \nabla \partial^\gamma q \partial^\gamma w \mu dm dx$ does not vanish as $\int_{\mathbb{R}^N} v \cdot \int_B \nabla \partial^\gamma w \partial^\gamma w \mu dm dx$. Indeed, the first two terms, related to (4.24), are bounded by

$$\varepsilon |w|_{0,s}^2 + C_\varepsilon |\partial_t q|_{0,s}^2 + \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |\nabla_m q|_{0,s}^2,$$

and the other terms are estimated as follows:

$$\begin{aligned} \sum_{|\gamma| \leq s} \left| \int_{\mathbb{R}^N} \int_B \partial^\gamma (v \cdot \nabla q) \partial^\gamma w \mu dm dx \right| &\leq \varepsilon |w|_{0,s}^2 + C_\varepsilon |v|_s^2 |q|_{0,s+1}^2, \\ \sum_{|\gamma| \leq s} \left| \int_{\mathbb{R}^N} \int_B \partial^\gamma (\kappa m q) \nabla_m \partial^\gamma w \mu dm dx \right| &\leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2 |q|_{0,s}^2, \\ \sum_{|\gamma| \leq s} \left| \int_{\mathbb{R}^N} \int_B \partial^\gamma (K q) \partial^\gamma w \mu dm dx \right| &\leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |q|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2 |q|_{0,s}^2. \end{aligned}$$

We combine all estimates for sufficiently small ε to obtain

$$(4.30) \quad \partial_t |w|_{0,s}^2 + \frac{1}{2} |\nabla_m w|_{0,s}^2 \leq C(|v|_{s+1}^2 + 1) (|w|_{0,s}^2 + (|q|_{1,s+1}^2 + |\partial_t q|_{1,s+1}^2)).$$

We deduce that

$$|w|_{0,s}^2 + \frac{1}{2} \int_0^t |\nabla_m w|_{0,s}^2 dt \leq e^{F(|v|_{s+1})} (|w_0|_{0,s}^2 + F(|v|_{s+1}) \|q\|_{1,1,s+1}^2).$$

Replacing $F e^F$ by e^F leads to (4.2). □

5. Coupled system. In this section, we prove Theorem 1 by the fixed point argument as described in section 2.

We begin with a key lemma, which will be used to estimate the stress τ .

LEMMA 21. *Suppose that $\phi \in \dot{H}_\mu^1$. For any $\varepsilon > 0$ there exists C_ε such that*

$$(5.1) \quad \left| \int_B \phi \nu \rho^{-1} dm \right|^2 \leq C_\varepsilon \int_B |\phi|^2 \mu dm + \varepsilon \int_B |\nabla_m \phi|^2 \mu dm.$$

Proof. For $b > 2$, the Cauchy–Schwarz inequality yields

$$\left| \int_B \phi dm \right|^2 \leq \int_B |\phi|^2 \mu dm \int_B \mu^{-1} dm.$$

For any $\varepsilon > 0$, taking $C_\varepsilon = \int_B \mu^{-1} dm < \infty$, we obtain (5.1) for $b > 2$.

For $b \leq 2$, we define for fixed M ,

$$G = \left\{ \phi \in \dot{H}_\mu^1 : \int_B \phi \nu \rho^{-1} dm = 1, \|\phi\|_{H_\mu^1} \leq M \right\}.$$

It suffices to prove

$$l := \inf_{\phi \in G} \int_B |\phi|^2 \mu dm > 0.$$

Let $\{\phi_n\} \subset G$ be a sequence such that

$$\lim_{n \rightarrow \infty} \int_B |\phi_n|^2 \mu dm = \inf_{\phi \in G} \int_B |\phi|^2 \mu dm.$$

Since $\{\phi_n\}$ is bounded in H_μ^1 , by embedding theorem (3.11), there exists a subsequence $\{\phi_{n_k}\}$ such that

$$\begin{aligned} \phi_{n_k} &\rightharpoonup \phi^* && \text{in } H_\mu^1, \\ \phi_{n_k} &\rightharpoonup \phi^* && \text{in } L_\mu^2, \\ \phi_{n_k} &\rightharpoonup \phi^* && \text{in } L_{\mu^*}^2. \end{aligned}$$

Furthermore, since $\sqrt{\frac{\mu}{\mu^*}} \in L_{\mu^*}^2$ for $b \leq 2$

$$\begin{aligned} \int_B \phi^* \nu \rho^{-1} dm &= \int_B \phi^* \sqrt{\frac{\mu}{\mu^*}} \mu^* dm \\ &= \lim_{n_k \rightarrow \infty} \int_B \phi_{n_k} \sqrt{\frac{\mu}{\mu^*}} \mu^* dm = 1. \end{aligned}$$

This shows that $\phi^* \in G$. On the other hand,

$$\int_B |\phi^*|^2 \mu dm \leq \lim_{n_k \rightarrow \infty} \int_B |\phi_{n_k}|^2 \mu dm = l.$$

If $l = 0$, then $\phi^* = 0$, which is a contradiction to $\phi^* \in G$. \square

The zero trace of ϕ is essential for the estimate (5.1). For the general case, i.e., for $\phi \in H_\mu^1$, one can only have a weaker estimate.

LEMMA 22. *If $\phi \in H_\mu^1$, then there exists C such that*

$$(5.2) \quad \left| \int_B \phi \nu \rho^{-1} dm \right|^2 \leq C \|\phi\|_{H_\mu^1}^2.$$

Proof. For $b > 2$, we have

$$\left| \int_B \phi \nu \rho^{-1} dm \right|^2 \leq C \int_B |\phi|^2 \mu dm, \quad C := \int_B \mu^{-1} dm < \infty.$$

For $b \leq 2$,

$$\left| \int_B \phi \nu \rho^{-1} dm \right|^2 \leq C_\delta \int_B |\phi|^2 \rho^{-1+\delta} dm, \quad C_\delta := \left(\int_B \nu^2 \rho^{-1-\delta} dm \right).$$

We choose $\delta > 0$ small enough so that C_δ is bounded. On the other hand, by (3.12) in Lemma 9 we have

$$\begin{aligned} \int_B |\phi|^2 \rho^{-1+\delta} dm &\leq C \int_B (|\phi|^2 + |\nabla_m \phi|^2) \rho^{b/2} dm = C \int_B (|\phi|^2 + |\nabla_m \phi|^2) \mu dm, \quad b < 2, \\ \int_B |\phi|^2 \rho^{-1+\delta} dm &\leq C \int_B (|\phi|^2 + |\nabla_m \phi|^2) \rho dm \leq C \int_B (|\phi|^2 + |\nabla_m \phi|^2) \mu dm, \quad b = 2. \end{aligned}$$

This completes the proof. \square

We now turn to the map

$$\mathcal{F} : \mathbf{M} \rightarrow \mathbf{M},$$

$$(u, \varpi) \mapsto (v, w),$$

and

$$\mathbf{M} = \left\{ (v, w) : \sup_{0 \leq t \leq T} |v|_s^2 \leq A_1, \sup_{0 \leq t \leq T} |w|_{0,s}^2 + \frac{1}{2} \int_0^T |\nabla_m w|_{0,s}^2 dt \leq A_2 \right\}.$$

We first prove that, given $v_0 \in H_x^s, f_0 \nu^{-1} \in H_x^s L_\mu^2$, and $q \in C_t^1 H_x^{s+1} H_\mu^1$, the map \mathcal{F} is well defined, i.e., $\mathcal{F}(\mathbf{M}) \subset \mathbf{M}$ for some A_1, A_2, T .

Let $(u, \varpi) \in \mathbf{M}$. It is now well known that (2.8) has a unique solution v such that

$$(5.3) \quad \sup_t |v|_s^2 + \int_0^T |v|_{s+1}^2 dt \leq |v_0|_s^2 + C \int_0^T |u|_s |v|_s^2 dt + \int_0^T |\tau|_s^2 dt, \quad s > N/2 + 1.$$

By Gronwall's inequality and $\sup_{0 \leq t \leq T} |u|_s^2 \leq A_1$, we have

$$(5.4) \quad \sup_t |v|_s^2 + \int_0^T |v|_{s+1}^2 dt \leq \left(|v_0|_s^2 + \int_0^T |\tau|_s^2 dt \right) e^{C\sqrt{A_1}T}.$$

We proceed to estimate the stress term

$$\int_0^T |\tau|_s^2 dt = \int_0^T \sum_{|\gamma| \leq s} \int_{\mathbb{R}^N} |\partial^\gamma \tau|^2 dx dt,$$

where using Lemma 21,

$$|\partial^\gamma \tau|^2 = b^2 \left| \int_B m \otimes m \partial^\gamma (\varpi + q) \nu \rho^{-1} dm \right|^2$$

$$\leq C_\varepsilon \int_B |\partial^\gamma \varpi|^2 \mu dm + \frac{\varepsilon}{2} \int_B |\partial^\gamma \nabla_m \varpi|^2 \mu dm + 2b^4 \left| \int_B \partial^\gamma q \nu \rho^{-1} dm \right|^2.$$

Using (5.2) the last term is uniformly bounded by

$$C \|\partial^\gamma q(t, x, \cdot)\|_{H_\mu^1}^2 \leq C \|q\|_{1,1,s+1}^2.$$

Hence for $(u, \varpi) \in \mathbf{M}$ we obtain

$$(5.5) \quad \int_0^T |\tau|_s^2 dt \leq C_\varepsilon T A_2 + \varepsilon A_2 + CT \|q\|_{1,1,s+1}^2 \leq CT(A_2 + \|q\|_{1,1,s+1}^2) + \varepsilon A_2,$$

where we have used the assumption $q \in C_t^1 H_x^{s+1} H_\mu^1$.

We choose A_1 as

$$(5.6) \quad A_1 = 2|v_0|_s^2 e$$

and A_2 as

$$(5.7) \quad A_2 = (|w_0|_{0,s}^2 + \|q\|_{1,1,s+1}^2) e^{C(T+A_1)}$$

for $T \leq 1/(C\sqrt{A_1})$.

Hence, if T and ε are chosen small enough so that

$$CT(A_2 + \|q\|_{1,1,s+1}^2) + \varepsilon A_2 \leq \frac{1}{2e} A_1,$$

we get

$$(5.8) \quad e^{C\sqrt{A_1}T} (|v_0|_s^2 + CT(A_2 + \|q\|_{1,1,s+1}^2) + \varepsilon A_2) \leq e \left(|v_0|_s^2 + \frac{1}{2e} A_1 \right) \leq A_1.$$

This together with (5.4), (5.5) gives

$$(5.9) \quad \sup_t |v|_s^2 + \int_0^{T_1} |v|_{s+1}^2 dt \leq A_1.$$

Estimate (4.2) in Theorem 19, (5.7), and (5.9) yield

$$(5.10) \quad \sup_t |w|_{0,s}^2 + \frac{1}{2} \int_0^T |\nabla_m w|_{0,s}^2 dt \leq A_2.$$

So the map \mathcal{F} is well defined in \mathbf{M} .

Next, we show that \mathcal{F} is a contraction mapping for small enough T using a weak norm on \mathbf{M} , i.e.,

$$(5.11) \quad \|(v, w)\|_{\mathbf{M}}^2 := \sup_t |v|_0^2 + \sup_t |w|_{0,0}^2 + \frac{1}{2} \int_0^T |\nabla_m w|_{0,0}^2 dt.$$

Suppose that $v_i (i = 1, 2)$ are solutions of the Navier–Stokes equation (2.8) with $u_i (i = 1, 2)$ and $\tau_i (i = 1, 2)$ computed from $\varpi_i (i = 1, 2)$, respectively. Then we obtain

$$(5.12) \quad \partial_t v + (u_2 \cdot \nabla)v + (u \cdot \nabla)v_1 + \nabla p = \nabla \cdot \tau + \Delta v, \quad v(0, \cdot) = 0,$$

where $v = v_2 - v_1, u = u_2 - u_1, p = p_2 - p_1, \tau = \tau_2 - \tau_1$, and $\varpi = \varpi_2 - \varpi_1$. Multiplication by v to (5.12) and integration with respect to x yield

$$\frac{1}{2} \frac{d}{dt} |v|_0^2 + \int_{\mathbb{R}^N} (u \cdot \nabla v_1) v dx = - \int_{\mathbb{R}^N} \tau \nabla v dx - \int_{\mathbb{R}^N} |\nabla v|^2 dx.$$

Hence

$$(5.13) \quad \begin{aligned} \frac{d}{dt} |v|_0^2 + |\nabla v|_0^2 &\leq |u|_0^2 + |\tau|_0^2 + \sup_x |\nabla v_1|^2 |v|_0^2 \\ &\leq |u|_0^2 + |\tau|_0^2 + A_1 |v|_0^2. \end{aligned}$$

Let f_i be the solutions to (2.9) associated with $v_i (i = 1, 2)$. Then

$$w = (f_2 - f_1)\nu^{-1} =: w_2 - w_1$$

solves

$$(5.14a) \quad \partial_t w \mu + v_2 \cdot \nabla w \mu + L_2[w] = -v \cdot \nabla w_1 \mu - \nabla_m \cdot (\nabla v m \tilde{w}_1 \nu) \frac{\mu}{\nu},$$

$$(5.14b) \quad w(0, x, m) = 0,$$

$$(5.14c) \quad w(t, x, m)|_{\partial B} = 0,$$

where $L_2[w] = L[w]$ defined in (3.5) with $\kappa = \nabla v_2$. Note that $w_i|_{\partial B} = q|_{\partial B}$, i.e., $w_i(t, x, \cdot) \in H^1_\mu$, so $w(t, x, \cdot) \in H^1_\mu$.

We deduce from (5.14a) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |w|_{0,0}^2 + \frac{1}{2} |\nabla_m w|_{0,0}^2 \\ & \leq \int_{\mathbb{R}^N} \int_B |\nabla v_2 m \cdot \nabla_m w \nu w| dm dx + \int_{\mathbb{R}^N} \int_B |K w^2| dm dx \\ & \quad + \int_{\mathbb{R}^N} |v \cdot \int_B \nabla w_1 w \mu dm| dx + \int_{\mathbb{R}^N} \left| \int_B \nabla_m \cdot (\nabla v m w_1 \nu) \frac{\mu}{\nu} w dm \right| dx. \end{aligned}$$

By techniques similar to those used to derive (4.2), the first two terms on the right-hand side are bounded by

$$C_\varepsilon (|v_2|_s^2 + 1) |w|_{0,0}^2 + \varepsilon |\nabla_m w|_{0,0}^2$$

and the third term by

$$\begin{aligned} \int_{\mathbb{R}^N} |v \cdot \int_B \nabla w_1 w \mu dm| dx & \leq C \int_{\mathbb{R}^N} |v|^2 \int_B |\nabla w_1|^2 \mu dm dx + \int_{\mathbb{R}^N} \int_B |w|^2 \mu dm dx \\ & \leq C |v|_0^2 |w_1|_{0,s}^2 + |w|_{0,0}^2. \end{aligned}$$

The last term, using integration by parts with vanished boundary term due to Lemma 20, is bounded by

$$\begin{aligned} \int_{\mathbb{R}^N} \left| \int_B \nabla_m \cdot (\nabla v m w_1 \nu) \frac{\mu}{\nu} w dm \right| dx & = \int_{\mathbb{R}^N} \left| \int_B \nabla v m w_1 \nu \cdot \nabla_m \left(\frac{\mu}{\nu} w \right) dm \right| dx \\ & \leq C_\varepsilon |\nabla v|_0^2 |w_1|_{0,s}^2 + \varepsilon |\nabla_m w|_{0,0}^2. \end{aligned}$$

Putting it all together we have

$$\begin{aligned} \frac{d}{dt} |w|_{0,0}^2 + \frac{1}{2} |\nabla_m w|_{0,0}^2 & \leq C (|v_2|_s^2 + 1) |w|_{0,0}^2 + C |w_1|_{0,s}^2 (|v|_0^2 + |\nabla v|_0^2) \\ & \leq C(A_1 + 1) |w|_{0,0}^2 + CA_2 (|v|_0^2 + |\nabla v|_0^2). \end{aligned}$$

Substitution of the estimates of $|\nabla v|_0^2$ and $\frac{d}{dt} |v|_0^2$ in (5.13) gives

$$(5.15) \quad \frac{d}{dt} (|v|_0^2 + |w|_{0,0}^2) + \frac{1}{2} |\nabla_m w|_{0,0}^2 \leq D (|v|_0^2 + |w|_{0,0}^2) + D |u|_0^2 + D |\tau|_0^2,$$

where D is a large constant depending on C, A_1, A_2 ; for example, we may choose

$$D = C(A_1 + 1)(A_2 + 1).$$

The Gronwall inequality gives

$$\sup_t (|v|_0^2 + |w|_{0,0}^2) + \frac{1}{2} \int_0^{T^*} |\nabla_m w|_{0,0}^2 dt \leq D e^{DT} \int_0^{T^*} |u|_0^2 + |\tau|_0^2 dt$$

for any $0 < T^* \leq T$. Due to the similar estimate for τ as (5.5), the right-hand side is bounded by

$$D e^{DT} \left(T^* \sup_t |u|_0^2 + C_\varepsilon T^* \sup_t |\varpi|_{0,0}^2 + \varepsilon \int_0^{T^*} |\nabla_m \varpi|_{0,0}^2 dt \right).$$

We choose $\varepsilon = \frac{1}{4De^{DT}}$, $T^* = \frac{1}{2} \min\{T, \frac{1}{(C_\varepsilon+1)De^{DT}}\}$ and redefine $T = T^*$ to obtain

$$(5.16) \quad \|(v_2, w_2) - (v_1, w_1)\|_{\mathbf{M}}^2 = \|(v, w)\|_{\mathbf{M}}^2 \leq \frac{1}{2} \|(u_2, \varpi_2) - (u_1, \varpi_1)\|_{\mathbf{M}}^2.$$

This shows that \mathcal{F} has a fixed point (v, w) in \mathbf{M} , which is a solution to the coupled problem (2.1). Since $\mathcal{F}(v, w) = (v, w)$, (5.3) and Theorem 13 imply that $(v, w) \in \mathbf{X}_\mu$.

The uniqueness follows from the same computation of estimates for the contraction mapping. Let $(v_i, f_i \nu^{-1}) (i = 1, 2)$ be solutions of the coupled problem (2.1). Then $v = v_2 - v_1$ solves (5.12) with $u_i = v_i$, $u = v$, and $\tau = \tau_2 - \tau_1$ computed from f_i . $w = (f_2 - f_1)\nu^{-1}$ also solves (5.14) with $w_1 = f_1 \nu^{-1}$. Similar to (5.15), we obtain

$$\frac{d}{dt}(|v|_0^2 + |w|_{0,0}^2) + \frac{1}{2}|\nabla_m w|_{0,0}^2 \leq D(|v|_0^2 + |w|_{0,0}^2 + |\tau|_0^2).$$

It follows from the estimate for τ and the Gronwall inequality that $(v, w) \equiv (0, 0)$, which gives the uniqueness of problem (2.1).

6. A further look at $b \geq 6$. In this section, we sketch proofs of Theorems 2 and 4 for the case of $\mu = \mu_0$.

Consider (2.9) when x is not involved, i.e., (3.1). The corresponding w-problem for $w = f\nu^{-1} - q$ with $\mu = \mu_0$ solves (3.3) with the operator L replaced by

$$(6.1) \quad L_0[w] = -\frac{1}{2}\nabla \cdot (\nabla w \mu_0) + \left(2 - \frac{1}{2}b - \theta\right) m \cdot \nabla w \rho^{\theta-1} + \nabla \cdot (\kappa m w \mu_0) - K_0 w,$$

where

$$(6.2) \quad K_0 = [N(b/2 - 1) + 2\kappa m \cdot m(1 - \theta)] \rho^{\theta-1}.$$

Define the conjugate of μ_0 as (3.10), $\mu_0^* = \rho^{\theta-2}$; then K_0 can be rewritten as

$$(6.3) \quad K_0 = [N(b/2 - 1) + 2\kappa m \cdot m(1 - \theta)] \sqrt{\mu_0 \mu_0^*}.$$

To ensure well-posedness of (3.3), we need to check the coercivity of $\mathcal{B}_0[w, w; t]$, which is defined as

$$\begin{aligned} \frac{1}{2} \int_B |\nabla w|^2 \mu_0 dm &= \mathcal{B}_0[w, w; t] - \left(2 - \frac{1}{2}b - \theta\right) \int_B m \cdot \nabla w w \rho^{\theta-1} dm \\ &\quad - \int_B \nabla \cdot (\kappa m w \mu_0) w dm + \int_B K_0 w^2 dm. \end{aligned}$$

From the proof of Lemma 10, the last two terms are bounded by

$$C_\varepsilon \int_B w^2 \mu_0 dm + \varepsilon \int_B |\nabla w|^2 \mu_0 dm,$$

where the embedding theorem (3.11) has been used. For small enough ε , this estimate yields

$$\frac{1}{4} \int_B |\nabla w|^2 \mu_0 dm \leq \mathcal{B}_0[w, w; t] + C \int_B w^2 \mu_0 dm,$$

as long as

$$\int_B \left(2 - \frac{1}{2}b - \theta\right) m \cdot \nabla w w \rho^{\theta-1} dm \geq 0,$$

for $w \in \overset{\circ}{H}_{\mu_0}^1$. This is indeed the case, as shown below.

LEMMA 23. Let $w \in \overset{\circ}{H}^1_{\mu_0}$. Then

$$(6.4) \quad \int_B \left(2 - \frac{1}{2}b - \theta\right) m \cdot \nabla w w \rho^{\theta-1} dm \geq 0.$$

Proof. From $-1 < \theta < 1$ and $b \geq 6$, we see that $(2 - b/2 - \theta) < 0$. It suffices to show

$$\int_B m \cdot \nabla w w \rho^{\theta-1} dm = \frac{1}{2} \int_B m \cdot \nabla w^2 \rho^{\theta-1} dm \leq 0.$$

Integration by parts gives

$$\begin{aligned} \int_B m \cdot \nabla w^2 \rho^{\theta-1} dm &= - \int_B w^2 (N \rho^{\theta-1} + 2(1 - \theta) |m|^2 \rho^{\theta-2}) dm \\ &\quad + \int_{\partial B} w^2 \rho^{\theta-1} m \cdot \frac{m}{|m|} dS \\ &\leq \sqrt{b} \int_{\partial B} w^2 \rho^{\theta-1} dS = 0. \end{aligned}$$

Here we use the fact that $w^2 \rho^{\theta-1} \in W^{1,1}$ and $w^2 \rho^{\theta-1}|_{\partial B} = 0$. To see this, for any $w \in \overset{\circ}{H}^1_{\mu_0}$, we estimate

$$\begin{aligned} \int_B w^2 \rho^{\theta-1} + |\nabla(w^2 \rho^{\theta-1})| dm &\leq \int_B w^2 \rho^{\theta-1} + 2|w \nabla w| \rho^{\theta-1} + 2(1 - \theta) |m w^2| \rho^{\theta-2} dm \\ &\leq C \int_B w^2 \sqrt{\mu_0 \mu_0^*} + |w| |\nabla w| \sqrt{\mu_0 \mu_0^*} + w^2 \mu_0^* dm \\ &\leq C \|w\|_{\overset{\circ}{H}^1_{\mu_0}}^2, \end{aligned}$$

due to the embedding theorem (3.11). Thus $w^2 \rho^{\theta-1}|_{\partial B} \in L^1(\partial B)$ from the trace theorem and it is zero from the fact that C_c^∞ is a dense subset of $\overset{\circ}{H}^1_{\mu_0}$. Thus (6.4) follows. \square

We now turn to the Fokker–Planck equation problem including the x variable. The first step in the proof of Theorem 19 remains valid for $\mu = \mu_0$. To check the second part of the proof, we need only look at two extra terms beyond those in (4.28):

$$\begin{aligned} & - \left(2 - \frac{1}{2}b - \theta\right) \int_B m \cdot \nabla_m \partial^\gamma w \rho^{\theta-1} \partial^\gamma w dm, \\ & - \left(2 - \frac{1}{2}b - \theta\right) \int_B m \cdot \nabla_m \partial^\gamma q \rho^{\theta-1} \partial^\gamma w dm. \end{aligned}$$

The first term is nonpositive from Lemma 23, and the second term is bounded by

$$C \left| \int_B m \cdot \nabla_m \partial^\gamma q \rho^{\theta-1} \partial^\gamma w dm \right| \leq C_\varepsilon \int_B |\nabla_m \partial^\gamma q|^2 \mu_0 dm + \varepsilon \int_B |\partial^\gamma w|^2 \mu_0^* dm.$$

These ensure the same estimate (4.30) and thus (4.2).

For the well-posedness for the coupled problem, we utilize $\theta < 1$ and Lemma 23. For example, for the proof of Lemma 21 with μ_0

$$\left| \int_B \phi \nu \rho^{-1} dm \right|^2 = \left| \int_B \phi dm \right|^2 \leq \int_B \phi^2 \mu_0 dm \int_B \mu_0^{-1} dm.$$

Since $\theta < 1$ we have $\int \mu_0^{-1} dm < \infty$, hence (5.1). Verification of other terms is omitted.

It remains to show Theorem 4; the solution f is a probability distribution if and only if $q|_{\partial B} = 0$ for $\mu = \mu_0$, and positivity of f follows as in Proposition 16. For the conservation of mass, as in Proposition 17, we only have to check (3.30),

$$\int_{B^\varepsilon} (w\kappa m - \nabla w \cdot \nabla \phi_\varepsilon \nu) dm - \int_{B^\varepsilon} w \rho^{b/2} \nabla \phi_\varepsilon \cdot \nabla (\nu \rho^{-b/2}) dm.$$

Since $\nu^2/\mu_0 = \rho^{2-\theta}$ and $2 - \theta > 1$

$$\frac{\varepsilon}{2} \int_{\partial B_r} |\nabla \phi_\varepsilon|^2 \rho^{2-\theta} dS$$

converges to 0 as $\varepsilon \rightarrow 0$. Thus the first term converges to 0 as well. On the other hand, the same argument shows that the second term converges to $C \int_{\partial B} q dS$ for some nonzero constant C . Hence, we conclude Theorem 4 under the assumption of Theorem 2.

7. Conclusion. In this paper, we have analyzed the FENE dumbbell model which is of bead-spring type Navier–Stokes–Fokker–Planck models for dilute polymeric fluids, with a focus on developing a local well-posedness theory subject to a class of Dirichlet-type boundary conditions

$$f\nu^{-1} = q \quad \text{on } \partial B$$

for the polymer distribution f , where ν depends on $b > 0$ through the distance function, and q is a given smooth function measuring the relative ratio of f/ν near boundary. We have thus identified a sharp Dirichlet-type boundary requirement for each $b > 0$, while the sharpness of the boundary requirement is a consequence of the existence result for each specification of the boundary behavior. It has been shown that the probability density governed by the Fokker–Planck equation approaches zero near the boundary, necessarily faster than the distance function d for $b > 2$, faster than $d|\ln d|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Moreover, the sharp boundary requirement for $b \geq 2$ is also sufficient for the distribution to remain a probability density.

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