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Error estimates for Gaussian beam methods applied to symmetric strictly hyperbolic systems

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HIGHLIGHTS

- A complete Gaussian-Beam construction is presented for a class of linear symmetric hyperbolic systems with highly oscillatory initial data, including both strictly and non-strictly hyperbolic systems as long as they are diagonalizable.
- The evolution equations for each Gaussian beam component are derived.
- Independent of dimension and presence of caustics an optimal error estimate between the exact solution and the first order Gaussian beam superposition in terms of the high frequency parameter is obtained.
- Further applications to some non-strictly hyperbolic systems including both the acoustic equation and the system of Maxwell equations are discussed.

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ABSTRACT

In this work we construct Gaussian beam approximations to solutions of linear symmetric hyperbolic systems with highly oscillatory initial data, including both strictly and non-strictly hyperbolic systems as long as they are diagonalizable. The evolution equations for each Gaussian beam component are derived. Under some regularity assumptions of the data we obtain an error estimate between the exact solution and the first order Gaussian beam superposition in terms of the high frequency parameter ε^{-1}. The main result is that the relative local error measured in energy norm in the beam approximation decays as $\varepsilon^{\frac{1}{2}}$ independent of dimension and presence of caustics, for first order beams. This result is shown to be valid when the gradient of the initial phase is bounded away from zero. Applications to some non-strictly hyperbolic systems including both the acoustic equation and the system of Maxwell equations are discussed.

1. Introduction

In this article we are interested in the accuracy of Gaussian beam approximations to solutions of the hyperbolic system,

$$A(x)\frac{\partial u}{\partial t} + \sum_{j=1}^{n} D^j \frac{\partial u}{\partial x_j} = 0, \tag{1.1}$$

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subject to highly oscillatory initial data,

$$u(0, x) = B_0(x)e^{iS_0(x)/\varepsilon}, \tag{1.2}$$

where $x \in \mathbb{R}^n$, $S_0(x)$ is a scalar smooth function, $B_0 : \mathbb{R}^n \to \mathbb{C}^m$ is a smooth vector function, compactly supported in $K_0 \subset \mathbb{R}^n$, $A(x)$ is $m \times m$ symmetric positive definite matrix, and D^j are $m \times m$ symmetric constant coefficient matrices, $j = 1, \ldots, n$.

Symmetric systems first considered by K.O. Friedrichs can be used to describe a wide variety of physical processes. Indeed, with proper choices of A and D^j one may find in (1.1) Maxwell's equations, the elastic wave equations, and acoustic wave equations as discussed in [1]. In several areas of continuum physics including acoustic waves, the wave length parameter ϵ is often very small compared to the scale of computational domain, hence one has to solve high frequency wave propagation problems, and the research in this field can give some insight in the study of some significant physical systems. The symmetry of the hyperbolic system ensures the existence of the orthogonal basis in $\mathbb{R}^n$ formed by the associated eigenvectors, and this spectral decomposition is useful in our construction of high frequency approximate solutions. Indeed many hyperbolic equations or systems can be reduced to symmetric hyperbolic form. A typical example is a scalar hyperbolic second or higher order equation.

It is well-known that high frequency wave propagation problems create severe numerical challenges that make direct simulations unfeasible, particularly in multidimensional settings. High frequency asymptotic models, such as geometrical optics, can be found in some classical literature (see [1,2]). A main drawback of geometrical optics is that the model breaks down at caustics, where rays concentrate and the predicted amplitude becomes unbounded, therefore unphysical. As an alternative one can use the phase space based level set method, to compute multi-valued phases beyond caustics, we refer to [3–6] for the early development of this method, and [7] for a review of the level set framework for computational high frequency wave propagation. The density transport near the level set manifold produces bounded position densities everywhere except at caustics [8,9].

Gaussian beams, as another high frequency asymptotic model, are closely related to geometric optics, yet valid at caustics. The solution is concentrated near a single ray of geometric optics. In Gaussian beams, the phase function is real valued along the central ray, its imaginary part is chosen so that the solution decays exponentially away from the central ray, maintaining a Gaussian shaped profile. More general high frequency solutions can be described by superposition of Gaussian beams. In this paper we are going to use the Gaussian beam approach. This approach has gained considerable attention in recent years from both computational and theoretical points of view. A general overview of the history and the latest development of this method are given in the introduction to [10]. We remark that the phase space based level set method when combined with the Gaussian beam framework, as developed in [11–13], can both handle the crossing of Hamiltonian trajectories and yield bounded amplitudes at caustics. A systematic Gaussian beam construction using a level set formulation is presented in [14,15] together with rigorous error estimates.

Another related approach is the frozen Gaussian approximation, or the Herman–Kluk formula discovered by several authors in the chemical-physics literature in the eighties. This approach with superposition of beams in phase space is closely related to the Fourier-Integral Operator (FIO) with complex phases. The mathematical analysis of the Herman–Kluk was given only recently; see [16,17] for the semiclassical approximation of the Schrödinger equation, and [18] for the frozen Gaussian approximation to linear strictly hyperbolic systems.

In this paper we formulate a Gaussian beam superposition in physical space for symmetric hyperbolic systems. Though the Gaussian beam construction is standard after [19], we show how the Gaussian beam works in the case of systems, which requires additional care compared to scalar equations. We note, in particular, that a higher order amplitude term, $\varepsilon v_1^\top$, is needed even for first order beams to account for the spatial variations in the eigendirections. We mainly study the accuracy in terms of the high frequency parameter ε of Gaussian beams. Several such error estimates have been derived in recent years for problems modeled by several different types of PDEs: for the initial data [20], for scalar hyperbolic equations and the Schrödinger equation [10,14,15], for the acoustic wave equation with superpositions in phase space [21], for the Helmholtz equation with a singular source [22], and for the Schrödinger equation with periodic potentials [23]. The general result is that the error between the exact solution and the Gaussian beam approximation decays as $\varepsilon^{N/2}$ for Nth order beams in the appropriate Sobolev norm. For phase space based Gaussian beams with frozen Gaussians, the integral approximation decays as ε^N for Nth order beams; see [16,17,24]. We note that in the frozen Gaussian beam approximation, the extra order of accuracy is found from a symbolic calculus for FIO with complex quadratic phases. But it is no longer a superposition of asymptotic solutions, though the superposition over phase space is still an asymptotic solution. While the Gaussian beam methods benefit mainly from the asymptotic accuracy of each individual beams, therefore it is computationally more feasible. We should also point out that Gaussian beam superposition in physical space works well for highly oscillatory data of the WKB type. For non-oscillatory initial data, one can simply apply a direct discretization method to solve (1.1). In contrast, in computing the semi-classical limit of the Schrödinger equation, one would have to handle oscillatory wave fields even for non-oscillatory initial data. In such case, one may carry out Gaussian beam superpositions in phase space, we refer to [12] for related strategies for using Gaussian wave packets and corresponding beam superpositions in phase space.

The analysis of Gaussian beam superpositions for hyperbolic systems presents a few new challenges compared to the scalar wave equations previously studied in [10,14]. First, it must be clarified how beams are propagated along each wave field through some field decomposition; a specific hyperbolic system was studied in [25], where the authors investigate the stationary in time wave field that results from steady air flow over topography. Second, the distinction of the eigenvalues of the dispersion matrix is assumed to allow for a correction in the amplitude with uniform estimates. This is similar to the

Schrödinger equation with periodic potentials [26,27], in which energy gap is essentially used in [23] in the accuracy study. While the analysis is performed for this particular model (see [1]) including application systems discussed in Section 5, it can easily be carried over to a slightly more general class with $A^{-1}(x)D_j$ replaced by a smooth and bounded matrix $A_j(x)$ as long as $\sum_{j=1}^{n} k_j A_j(x)$ admits m independent eigenvectors.

During our construction we also prove several minor results, for example, the leading Gaussian beam phase is shown to be stationary (which is related to the Huygens principle) along each wave field, and the momentum does not vanish as long as it is nonzero initially.

However, some interesting physical examples of hyperbolic systems are not strictly hyperbolic. For instance, Euler's equations of gas dynamics, Maxwell's equations or the equations of elasticity are not strictly hyperbolic. In the construction of Gaussian beams, the strict hyperbolicity assumption is used at only one place, to prove the solvability of next order of amplitude, which only requires the system have a complete set of eigenvectors. Such requirement is shown to be satisfied by several non-strictly hyperbolic systems such as the linearized acoustic wave equations, and Maxwell's equations.

The first account of the results presented here can be found in the PhD thesis [28]. Related results on the system (1.1) have appeared since the present paper was submitted for publication. For example, the Gaussian beam method for non-strictly hyperbolic systems with constant multiplicity (where polarized waves arise) was investigated in [29] by L. Jefferis and S. Jin, and they studied the error arising from imperfect matching of the initial data. The main distinction between that paper and this work is that we study the effect of the "dynamic error", the error arising from the failure of the approximation to satisfy the PDE exactly. This is particularly relevant at caustics, and requires techniques like the "squeezing lemma" from [10].

The organization of this paper is as follows: In Section 2, we start with the problem formulation and state the main results, then in Section 3 we proceed with Gaussian beam construction which is new for hyperbolic systems, but quite straightforward and simple for those familiar with the Gaussian beam method. In Section 4 we prove our main results for initial phase with non-vanishing gradient everywhere in K_0 when the system is strictly hyperbolic. Finally, extensions to non-strictly hyperbolic systems are discussed in Section 5, with examples including both the acoustic equation and the system of Maxwell equations. To keep our presentation less lengthy, we consider quite simple symmetric hyperbolic systems, although we believe that our results can be generalized to a larger class of hyperbolic systems: the class of symmetrizable hyperbolic systems of constant multiplicity.

2. Problem formulation and main result

Consider the initial value problem (1.1)–(1.2). We define the dispersion matrix $L(x, k)$:

$$L(x,k) = A^{-1}(x)\left(\sum_{j=1}^{n} k_j D^j\right), \tag{2.1}$$

and introduce the inner product of two vectors u, v on $\mathbb{C}^m$:

$$\langle u, v\rangle_A := u^* A v.$$

Here u^* denotes the conjugate transpose. It is known that (see, e.g. [2,7]) L is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_A$:

$$\langle Lu, v\rangle_A = \langle u, Lv\rangle_A.$$

Hence, L has real eigenvalues $\{\lambda_j(x,k)\}_{j=1}^m$, satisfying

$$L(x,k)b_j(x,k) = \lambda_j(x,k)b_j(x,k), \quad j = 1, \ldots, m, \tag{2.2}$$

where $\{b_j(x,k)\}_{j=1}^m$ are eigenvectors, forming an orthonormal basis so that $\langle b_l, b_j\rangle_A = \delta_{lj}$, and $\lambda_l(x,k)$ are scalar smooth functions. We assume that all eigenvalues are simple (i.e., system (1.1) is strictly hyperbolic) and the following holds:

$$\lambda_{j-1}(x,k) < \lambda_j(x,k) < \lambda_{j+1}(x,k), \quad j = 2, \ldots, m-1, \tag{2.3}$$

in a neighborhood of any (x, k) in phase space.

For the initial data (1.2) we assume that the amplitude $B_0(x)$ has compact support in a bounded domain $K_0 \subset \mathbb{R}^n$, and the phase $S_0(x)$ is smooth.

Let $B_0(x)$ have the following eigenvector decomposition

$$B_0(x) = \sum_{j=1}^{m} a_j(x) b_j(x, \partial_x S_0(x)), \tag{2.4}$$

then

$$a_j(x) = \langle B_0(x), b_j(x, \partial_x S_0(x))\rangle_A, \quad j = 1, \ldots, m. \tag{2.5}$$

For each wave field associated with b_j, we construct a Gaussian beam approximation

$$u_{GB}^{j\varepsilon} = A^j(t, x; x_0)e^{i\Phi_j(t,x;x_0)/\varepsilon}, \tag{2.6}$$

where $A^j(t, x; x_0)$ and $\Phi_j(t, x; x_0)$ are Gaussian beam amplitudes and phases, respectively, concentrated on a central ray starting from $x_0 \in K_0$ with $p_0 = \partial_x S_0(x_0)$. By the linearity of the hyperbolic system, we then sum the Gaussian beam ansatz (2.6) over $j = 1, \ldots, m$ and $x_0 \in K_0$ to define the approximate solution

$$u^\varepsilon(t, x) = \frac{1}{(2\pi\varepsilon)^{\frac{n}{2}}} \int_{K_0} \sum_{j=1}^{m} u_{GB}^{j\varepsilon} dx_0, \tag{2.7}$$

where $(2\pi\varepsilon)^{-\frac{n}{2}}$ is a normalizing constant which is needed for matching the initial data (1.2).

Indeed the initial data can be approximated by the same form of the Gaussian beam superposition (2.7),

$$u^\varepsilon(0, x) = \frac{1}{(2\pi\varepsilon)^{\frac{n}{2}}} \int_{K_0} \sum_{j=1}^{m} A^j(0, x; x_0)e^{i\Phi^0(x;x_0)/\varepsilon} dx_0, \tag{2.8}$$

where Φ^0 is the initial Gaussian beam phase, which is assumed to be the same for each wave field. By the classical Gaussian beam theory [19], the initial phase can be taken of the form

$$\Phi^0(x; x_0) = S_0(x_0) + \partial_x S_0(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^\top \cdot (\partial_x^2 S_0(x_0) + iI)(x - x_0) \tag{2.9}$$

with coefficients that serve as initial data for ODEs of the Gaussian beam components. The amplitude $A^j(0, x; x_0)$ are defined later in (4.1) using $a_j(x_0)$ in (2.5).

We are going to use the following notations in this paper. The unmarked norm $\|\cdot\|$ denotes the usual L^2-norm. The energy norm $\|\cdot\|_E$ is defined as

$$\|u\|_E^2 := \int_{\mathbb{R}^n} \langle u, u\rangle_A dx. \tag{2.10}$$

L^∞ norm of function f and its derivatives:

$$|f|_{C^\alpha} := \max_x |\partial_x^\alpha f(x)|.$$

L^p matrix norm:

$$\|M\|_p := \sup_{\|\xi\|_p = 1, \xi \in \mathbb{R}^m} \|M\xi\|_p.$$

For symmetric, positive definite matrix $\|M\| := \|M\|_2 = \lambda_{\max}(M)$, the largest eigenvalue of M.

We now state the main result.

Theorem 2.1. *Let $K_0 \subset \mathbb{R}^n$ be a bounded measurable set, initial amplitude $B_0(x) \in H^1(K_0)$, initial phase $S_0(x) \in C^{n+4}(\mathbb{R}^n)$ and bounded, $|\partial_x S_0(x)|$ be bounded away from zero on K_0; eigenvectors $b_j(x, k)$ and eigenvalues $\lambda_j(x, k)$ be smooth functions satisfying (2.3), with second order derivatives of $\lambda_j(x, k)$ globally bounded in $|k| > 1$, u be the exact solution to (1.1)–(1.2) for $0 < t \le T$, and u^ε be the first order Gaussian beam superposition (2.7). Then*

$$\|u - u^\varepsilon\|_E \le C\varepsilon^{1/2}, \tag{2.11}$$

where the constant C is independent of ε, but may depend on the finite time T, the initial data and the coefficient matrices A and D.

Remark 2.1. The condition that second order derivatives of $\lambda_j(x, k)$ are globally bounded in $|k| > 1$ is sufficient for global-in-time existence of solutions for each ray ODE system. Note that there is the problem that λ_k is probably not even continuous at $k = 0$. To overcome this obstacle, we choose initial data so that $\lambda(\tilde{x}(0), p(0))$ is not zero, so that $p(t)$ stays away from zero for all $t > 0$ (see Lemma 3.1). Yet the accuracy results in Theorems 2.1 and 5.1 are effective only when T is a fixed finite time. When T becomes large, the estimate is no longer accurate due to the dependence of C on T.

We proceed to construct Gaussian beam asymptotic solutions and obtain the desired error estimate in several steps. First, we present the construction for the Gaussian beam phase components which is a straightforward extension of the Gaussian beam approach developed for high order scalar hyperbolic and Schrödinger equations, see for example, [10]. While constructing the Gaussian beam amplitude, we address some solvability difficulties and show the way to solve it using the approach developed in [27] and verifying the boundedness of the additional terms. For the error estimate, we rely on the wellposedness argument and prove initial and evolution errors separately. For the initial error, we use some techniques similar to those developed by Tanushev in [20], keeping in mind that here we have to deal with vector valued functions. As for the evolution error estimate, we rely on some phase estimates proved in [10], which is a key technique for the present proof.

3. Gaussian beam construction

Let P be the differential operator in (1.1). We look for an approximate solution to (1.1), which has the form

$$u^\varepsilon = (v_0 + \varepsilon v_1 + \cdots + \varepsilon^l v_l(t,x))e^{i\Phi(t,x)/\varepsilon}. \tag{3.1}$$

We will construct Φ by using the Hamiltonian $\lambda(x,k)$, where λ is one of the eigenvalues λ_j; the general asymptotic solution will be a superposition of this ansatz over all characteristic fields $\{b_j\}_{j=1}^m$ (see (3.7)). Inserting u^ε into (1.1), we obtain:

$$A^{-1}(x)P[u^\varepsilon] = \left(\frac{1}{\varepsilon}c_0 + c_1 + \cdots + \varepsilon^{l-1}c_l\right)e^{i\Phi/\varepsilon} \approx 0, \tag{3.2}$$

where

$$c_0 = i(\partial_t\Phi + L(x,\partial_x\Phi))v_0, \tag{3.3}$$

$$c_1 = (\partial_t + L(x,\partial_x))v_0 + i(\Phi_t + L(x,\partial_x\Phi))v_1, \tag{3.4}$$

$$c_j = (\partial_t + L(x,\partial_x))v_{j-1} + i(\Phi_t + L(x,\partial_x\Phi))v_j, \quad j = 2,\ldots,l. \tag{3.5}$$

By geometric optics, the leading term is required to vanish on the central ray γ,

$$c_0 = i(\partial_t\Phi + L(x,\partial_x\Phi))v_0 = O(d(x,\gamma)^r) \quad \text{for some } r > 0, \tag{3.6}$$

where $L(x,k)$ is the dispersion matrix defined in (2.1), $d(x,\gamma)$ is the (Euclidean) distance for x to γ, and $r = 3$ for first order beam approximations.

We set the leading amplitude as

$$v_0(t,x) = \sum_{j=1}^m a_j(t,x)b_j(x,k(t,x)), \quad k(t,x) := \partial_x\Phi(t,x), \tag{3.7}$$

to infer from (2.2) that

$$c_0 = \sum_{j=1}^m ia_j(t,x)(\partial_t\Phi + \lambda_j(x,k(t,x))b_j(x,k(t,x))),$$

which vanishes as long as for each $\lambda = \lambda_j$, there exists $\Phi = \Phi_j$ which solves the Hamilton–Jacobi equation:

$$G(t,x) := \partial_t\Phi + \lambda(x,\partial_x\Phi) = O(d(x,\gamma)^r), \tag{3.8}$$

subject to the same initial phase $S_0(x)$. From now on we shall suppress the index j, since the construction is same for each eigenvalue λ_j, $j = 1,\ldots,m$. We shall focus on a solution to one of the modes at a time, and that in the end there will be m such solutions that are added together to approximate u, the actual solution to (1.1).

3.1. Construction of the Gaussian beam phase

Let $(\tilde{x}(t;x_0), p(t;x_0))$ be the phase space trajectory governed by the Hamiltonian in (3.8), then

$$\dot{\tilde{x}} = \partial_k\lambda(\tilde{x},p), \qquad \dot{p} = -\partial_x\lambda(\tilde{x},p), \tag{3.9}$$

satisfying $\tilde{x}(0;x_0) = x_0 \in K_0$ and $p(0;x_0) = \partial_x S_0(x_0)$. For smooth $\lambda(x,k)$ such as in C^2 class, with second order derivatives globally bounded in $|k| > 1$, existence of the trajectory for $0 < t < T$ is ensured with any fixed $T > 0$. It is also well-known that (3.9) preserves the Hamiltonian $\lambda(\tilde{x},p) = \lambda(x_0,\partial_x S_0(x_0))$ globally in time. Our initial data is chosen so that $|\lambda(\tilde{x},p)| \neq 0$ implying that $p(t) > 0$ for all time (see Lemma 3.1). Next we introduce an approximation of the phase:

$$\Phi(t,x;x_0) = S(t;x_0) + p(t;x_0)\cdot(x-\tilde{x}(t;x_0)) + \frac{1}{2}(x-\tilde{x}(t;x_0))^\top M(t;x_0)(x-\tilde{x}(t;x_0)), \tag{3.10}$$

where S and M are to be chosen so that $G(t,x)$ vanishes on $x = \tilde{x}(t;x_0)$ to higher order. For this quadratic phase function, $k(t,x)$ defined in (3.7) becomes

$$k(t,x) = p + M(x-\tilde{x}),$$

which we will keep as a notation in what follows. From (3.8) and (3.10) we derive:

$$G(t,x;x_0) = \dot{S} + \dot{p}\cdot(x-\tilde{x}) - p\cdot\dot{\tilde{x}} + \frac{1}{2}(x-\tilde{x})^\top\cdot\dot{M}(x-\tilde{x}) - \dot{\tilde{x}}^\top\cdot M(x-\tilde{x}) + \lambda(x,k(t,x)). \tag{3.11}$$

Setting $G(t,x;x_0) = 0$ at $x = \tilde{x}$, we obtain

$$\dot{S} = p\cdot\partial_k\lambda(\tilde{x},p) - \lambda(\tilde{x},p). \tag{3.12}$$

Since $\lambda(x, k)$ is homogeneous of degree one in k, then

$$\lambda(x, k) = k \cdot \partial_k \lambda(x, k) \tag{3.13}$$

holds for any x, k, hence $\dot{S} = 0$. Note that $\lambda(x, k)$ is an eigenvalue of the matrix $L(x, k)$, and the matrix is bounded on $\{|k| = 1\} \times \mathbb{R}^n$, it immediately follows

$$|\lambda(x, k)| \le C \max_j |k_j| \tag{3.14}$$

uniformly for $x \in \mathbb{R}^n$.

We observe that $\partial_x G(t, \tilde{x}; x_0) = 0$ is equivalent to the p equation in (3.9). Next, we set $\partial_x^2 G(t, \tilde{x}; x_0) = 0$, to obtain

$$\dot{M} + \partial_x^2(\lambda(x, k))\Big|_{(x,k)=(\tilde{x},p)} = 0,$$

which is equivalent to

$$\dot{M} + K_1 + K_2 M + M K_2^\top + M K_3 M = 0, \tag{3.15}$$

where K_1, K_2 and K_3 are matrices with the correspondent entries:

$$K_{1lj} = \frac{\partial^2 \lambda}{\partial x_l \partial x_j}, \qquad K_{2lj} = \frac{\partial^2 \lambda}{\partial x_l \partial k_j}, \qquad K_{3lj} = \frac{\partial^2 \lambda}{\partial k_l \partial k_j},$$

which are evaluated on the ray trajectory $(\tilde{x}, p)$.

Using the Taylor expansion, we have

$$G = \sum_{|\alpha|=3} \frac{1}{\alpha!} \partial_x^\alpha G(t, \cdot; x_0)(x - \tilde{x})^\alpha = \sum_{|\alpha|=3} \frac{1}{\alpha!} \partial_x^\alpha \lambda(\cdot, \cdot)(x - \tilde{x})^\alpha, \tag{3.16}$$

which means that G vanishes up to third order on $x = \tilde{x}$.

The heart of the Gaussian beam method is to solve the nonlinear Riccati equation (3.15). It is known from [19,30] that if the initial matrix is symmetric and its imaginary part is positive definite, then a global solution M to (3.15) is guaranteed and has the following properties: (i) $M = M^\top$, and (ii) $Im(M)$ is positive definite for all $t > 0$.

In summary, we obtain evolution equations for the Gaussian beam phase components subject to appropriately chosen initial data:

$$\begin{cases} \dot{\tilde{x}} = \partial_k \lambda(\tilde{x}, p), \qquad \tilde{x}|_{t=0} = x_0, \\ \dot{p} = -\partial_x \lambda(\tilde{x}, p), \qquad p|_{t=0} = \partial_x S_0(x_0), \\ \dot{S} = 0, \qquad S|_{t=0} = S_0(x_0), \\ \dot{M} = -M K_3 M - K_2 M - M K_2^\top - K_1, \qquad M|_{t=0} = \partial_x^2 S_0 + iI. \end{cases} \tag{3.17}$$

Note that $\partial_k \lambda(\tilde{x}, p)$ may not be well defined for $|p| = 0$; for example, if $\lambda = |k|$, then $\partial_k \lambda(\tilde{x}, p) = \frac{p}{|p|}$. The following result tells that we can construct well-defined beams as long as $p(0; x_0) \neq 0$. This is a known result, say for strictly hyperbolic PDEs of higher order [10, Lemma 2.1]. The following simple and elementary proof is included for self-completeness.

Lemma 3.1. *If $p(0; x_0) \neq 0$, then*

$$|p(t; x_0)| \ge |p(0; x_0)| e^{-ct}, \quad t \in [0, T], \tag{3.18}$$

where constant c may depend on T and the data given.

Proof. First we show that $|\dot{p}| \le c|p|$. Since λ is homogeneous in k of degree 1, we have $\lambda(x, k) = |k|\lambda(x, \omega)$, where $\omega = \frac{k}{|k|}$ is a directional unit vector. Hence

$$\dot{p} = -\partial_x \lambda(\tilde{x}, p) = -\partial_x \lambda(\tilde{x}, \omega)|p|,$$

which leads to

$$|\dot{p}| \le \max_{t \le T, \omega \in \mathbb{S}^{n-1}} |\partial_x \lambda(\tilde{x}, \omega)|\,|p| := c|p|.$$

Next, we consider

$$\frac{d}{dt}(|p|^2 e^{2ct}) = (2p \cdot \dot{p} + 2c|p|^2)e^{2ct} \ge (-2c|p|^2 + 2c|p|^2)e^{2ct} = 0.$$

This proves (3.18) as claimed. □

The above construction is known to ensure that

$$\Im\Phi(t,x;x_0) \geq c|x-\tilde{x}(t)|^2, \tag{3.19}$$

for some constant c which is independent of ϵ, see [10].

3.2. Construction of the Gaussian beam amplitude

In this section we will construct the amplitude for the Gaussian beam, and, in addition, estimate the errors arising from this construction. Since Gaussian beams can always be localized to a small neighborhood of the central ray, and we are only interested in a bounded interval in time, we only need estimates in $|x| \leq B$ for some B. Note also that, since the eigenvalues $\lambda_j(x,k)$ and eigenvectors $b_j(x,k)$ are real analytic in k,

$$L(x,k)b_j(x,k) = \lambda_j(x,k)b_j(x,k)$$

holds when k has a small imaginary part. In particular, it holds for $k = k(t,x) := \partial_x\Phi(t,x)$ when x is near $\tilde{x}(t)$.

Recall that

$$c_1 = (\partial_t + L(x,\partial_x))v_0 + i(\partial_t\Phi + L(x,\partial_x\Phi))v_1 =: \tilde{c}_1 + i(\partial_t\Phi + \lambda(x,\partial_x\Phi))v_1, \tag{3.20}$$

where

$$\tilde{c}_1 = (\partial_t + L(x,\partial_x))v_0 + i(L(x,k(t,x))) - \lambda(x,k(t,x)I)v_1.$$

We have $v_0 = a_0(t,x)b(x,\partial_x\Phi(t,x))$, where b is one of the eigenvectors b_j. The Fredholm Alternative theorem of the linear algebra[1] implies that we can solve $\tilde{c}_1 = 0$ for v_1 if and only if the coefficient $a_0(t,x)$ is chosen so that

$$\langle(\partial_t + L(x,\partial_x))a_0(t,x)b, b\rangle_A = 0, \tag{3.21}$$

where b^* is the eigenvector of the adjoint of $L(x,k(t,x))$ with respect to $\langle\cdot,\cdot\rangle_A$ with eigenvalue $\overline{\lambda}(x,k(t,x))$, normalized by $\langle b,b\rangle_A = 1$. Since we are constructing first order beams, we only need impose this condition on the central ray, i.e., where $(x,\partial_x\Phi(x,t)) = (\tilde{x}(t),p(t))$. If we were constructing higher order beams, we would need to have

$$\langle(\partial_t + L(x,\partial_x))a_0(t,x)b, b\rangle_A$$

vanish to higher order on the central ray.

Near the central ray when $x \neq \tilde{x}(t)$, we have

$$\tilde{c}_1 = (\partial_t + L(x,\partial_x))v_0(t,x) + i(L(x,k(t,x)) - \lambda(x,k(t,x))I)v_1^{\perp},$$

where $v_1^{\perp}$ contains the orthogonal complement of b, satisfying $\langle v_1^{\perp}, b\rangle_A = 0$. We choose

$$v_1^{\perp} = i(L(x,k(t,x)) - \lambda(x,k(t,x))I)^{-1}((\partial_t + L(x,\partial_x))v_0 - \langle(\partial_t + L(x,\partial_x))v_0, b\rangle_A b), \tag{3.22}$$

where $(L(x,k(t,x)) - \lambda(x,k(t,x))I)^{-1}$ denotes the inverse of $L(x,k(t,x)) - \lambda(x,k(t,x))I$ defined on

$$S^{\perp}(b) = \{v : \langle v, b(x,k(t,x))\rangle_A = 0\},$$

i.e., given $w \in S^{\perp}(b)$, $z = (L-\lambda I)^{-1}w$ is the unique solution in $S^{\perp}(b)$ to $(L-\lambda)z = w$. Note that the norm of $(L-\lambda I)^{-1}$ depends only on supremum norms of the coefficients of L and the spectral gap

$$\Delta\lambda = \min_{1\leq l<j\leq m}|\lambda_l - \lambda_j| > 0.$$

Hence we have

$$\tilde{c}_1 = \langle(\partial_t + L(x,\partial_x))a_0(t,x)b, b\rangle_A b.$$

Further setting $v_1 = v_1^{\perp} + a_1 b \in \text{span}\{v_1^{\top}, b\}$, we have

$$c_1 = \langle(\partial_t + L(x,\partial_x))a_0(t,x)b, b\rangle_A b + i(\partial_t\Phi(t,x) + \lambda(x,\partial_x\Phi(t,x)))v_1. \tag{3.23}$$

Since we only require (3.21) on the central ray, we can take $a_0 = a_0(t;x_0)$. Then, when $x = \tilde{x}(t)$, (3.21) becomes

$$\partial_t a_0 + a_0 f(\tilde{x}(t),p(t)) = 0, \tag{3.24}$$

where

$$f(x,k(t,x)) = \langle b, (\partial_t + L(x,\partial_x)b)\rangle_A.$$

Solving this first order linear ODE for $a_0(t;x_0)$ completes the construction of the amplitude.

[1] The linear system $Ax = b$ has a solution if and only if $b \in Ker(A^*)^{\perp}$, i.e., $\langle y, b\rangle = 0$ for any y such that $A^*y = 0$.

While choosing the coefficient a_1 correctly is essential in the construction of higher order beams, here we can take $a_1 \equiv 0$. With these choices of v_0 and v_1 and the construction of Φ in the preceding section $c_1(x,t)$ is a smooth function defined on a neighborhood of $\gamma = \{\tilde{x}(t;x_0), 0 \le t \le T\}$ which vanishes when $x = \tilde{x}(t)$. Hence there is a neighborhood N of γ such that for $x \in N$ and $0 \le t \le T$

$$|c_1(t,x)| \le C(\Delta\lambda)d(x,\gamma), \tag{3.25}$$

where $d(x,\gamma)$ is the (Euclidean) distance for x to γ. In addition to Δ the constant in this estimate depends on the supremum norms of the terms in the definition of $v_1 = v_1^{\perp}$.

We thus obtain a Gaussian beam approximation for any fixed $x_0 \in K_0 \subset \mathbb{R}^n$,

$$u_{GB}^{\varepsilon j}(t,x;x_0) = (a_j(t;x_0)b_j(x,\partial_x\Phi_j) + \varepsilon v_1^j(t,x;x_0))e^{i\Phi_j(t,x;x_0)/\varepsilon}. \tag{3.26}$$

This can be used as a building block for approximating the solution of the initial value problem by the GB superposition over $x_0 \in K_0$ and $j = 1 \cdots, m$,

$$u^\varepsilon(t,x) = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} \sum_{j=1}^m u_{GB}^{\varepsilon j}(t,x;x_0)dx_0. \tag{3.27}$$

In passing, let us relate the present formulation to its counterpart in Eulerian framework. Let $\Omega(0)$ be the domain where we initialize Gaussian beams from the given data, then $\Omega_j(t) = X_j(t,\Omega(0))$ is the image of $\Omega(0)$ under Hamiltonian flow associated with each λ_j. Let w^j be obtained from the Liouville equation

$$\partial_t w + \nabla_k H \cdot \nabla_q w - \nabla_q H \cdot \nabla_k w = 0, \quad w(0,X) = k - \nabla_q S^{\text{in}}(q),$$

with $H(q,k) = \lambda_j(q,k)$, respectively. Our asymptotic solution then becomes

$$u^\varepsilon(t,x) = \frac{1}{(2\pi\varepsilon)^{n/2}} \sum_{j=1}^m \int_{\Omega_j(t)} u_{PGB}^{\varepsilon j}(t,X)\delta(w^j(t,q,k))dX, \tag{3.28}$$

where $X = (q,k)$ denotes variables in phase space $\mathbb{R}^{2n}$, and $u_{PGB}^{\epsilon j}(t,x,X)$ is the phase space-based Gaussian beam ansatz, which is $u_{GB}^{\varepsilon j}(t,x;x_0)$ with GB components replaced by their phase space representations obtained by solving Liouville type equations; we refer to [14,15] for further details of how to uplift quantities in physical space into phase space in this fashion. In the Eulerian formulation the superpositions are over subdomains moving with each Hamiltonian flow, and each can be written as a superposition of standard Gaussian beams composed with a time-dependent symplectic change of variables. Since the superposition over phase space is still the same asymptotic solution, our accuracy results remain valid for the superposition in Eulerian framework.

Based on our construction, we have the following residual representation

$$P(u^\varepsilon) = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} \sum_{j=1}^m A(x)\left(\frac{1}{\varepsilon}c_{0j} + c_{1j}\right) e^{i\Phi_j(t,x;x_0)/\varepsilon}dx_0, \tag{3.29}$$

where $|c_{0j}| \le Cd^3(x,\gamma)$ and c_{1j} is bounded by $Cd(x,\gamma)$.

The proof of Theorem 2.1 is based on the following well-posedness estimate.

Proposition 3.1 (*Well-Posedness*). *Let u, u^ε be an exact and approximate solution of (1.1) with initial data u_0 and u_0^ϵ, respectively. Then the following error estimate holds:*

$$\|u - u^\varepsilon\|_E \le \|u_0 - u_0^\varepsilon\|_E + C\int_0^T \|P[u^\varepsilon]\|dt, \tag{3.30}$$

where C is independent of ε, but may depend on the matrix A.

This is a classical result, which can be found, for example, in [31].

4. Error estimates

4.1. Initial error estimate

The initial condition is approximated as follows:

$$u_0^\varepsilon = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} \sum_{j=1}^m (a_j(x_0)b_j(x,\partial_x\Phi^0) + \varepsilon v_1^j(0,x;x_0))e^{i\Phi^0/\varepsilon}dx_0, \tag{4.1}$$

where we recall a_j and Φ^0 are defined in (2.5) and (2.9), respectively. Here the term $v_1^j(0, x; x_0)$ is defined to be consistent with that in (3.22). In other words it is understood to be the limit of $v_1^j(t, x; x_0)$ as $t \to 0$, therefore we have from previous estimate on v_1,

$$\max_{x,x_0} \left| \sum_{j=1}^{m} v_1^j(0, x; x_0) \right|^2 \le C. \tag{4.2}$$

We now state the initial error estimate result in the following.

Theorem 4.1. *Let u_0^ε be defined in (4.1), and*

$$u_0(x) = \sum_{j=1}^{m} a_j(x) b_j(x, \partial_x S_0(x)) e^{iS_0(x)/\varepsilon}.$$

Then the energy norm of the difference $u_0 - u_0^\varepsilon$ satisfies:

$$\|u_0 - u_0^\varepsilon\|_E \le C\varepsilon^{1/2}, \tag{4.3}$$

where the constant C depends on the data given.

Proof. The proof is based on a discussion in [20], with attention on the treatment of different wave fields through eigenvectors b_j. Set an intermediate quantity

$$\tilde{u}(x) := \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \tilde{B}(x; x_0) e^{i\Phi^0(x;x_0)/\varepsilon} dx_0, \tag{4.4}$$

where

$$\tilde{B}(x; x_0) = \sum_{j=1}^{m} a_j(x_0) b_j(x, \partial_x S_0(x)),$$

we proceed to estimate $\|\tilde{u} - u_0\|_E$ and $\|\tilde{u} - u_0^\varepsilon\|_E$, respectively.

We first show

$$\|\tilde{u} - u_0\|_E \le C\varepsilon^{1/2}. \tag{4.5}$$

Rewrite

$$\tilde{u} - u_0 = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n} (\tilde{B}(x; x_0) - B_0(x)) e^{iT_2^{x_0}[S_0]/\varepsilon} e^{-|x-x_0|^2/2\varepsilon} + B_0(x)(e^{iT_2^{x_0}[S_0]/\varepsilon} - e^{iS_0/\varepsilon}) e^{-|x-x_0|^2/2\varepsilon} dx_0 = I_1 + I_2,$$

where $T_2^{x_0}[S_0]$ is the second order Taylor polynomial of S_0 about x_0. Using the orthogonality of vectors b_j with respect to the matrix A, we have $\langle w, w\rangle_A = \sum_{j=1}^m |g_j|^2$ for $w = \sum_{j=1}^m g_j b_j(x, k)$. Hence for I_1,

$$\begin{aligned}
\|I_1\|_E^2 &= \frac{1}{(2\pi\varepsilon)^n} \sum_{j=1}^{m} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (a_j(x_0) - a_j(x)) e^{iT_2^{x_0}[S_0]/\varepsilon} e^{-|x-x_0|^2/2\varepsilon} dx_0 \right|^2 dx \\
&\le \frac{1}{(2\pi\varepsilon)^n} \sum_{j=1}^{m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |a_j(x_0) - a_j(x)|^2 e^{-|x-x_0|^2/2\varepsilon} dx_0 \int_{\mathbb{R}^n} e^{-|x-x_0|^2/2\varepsilon} dx_0 dx \\
&= \frac{1}{(2\pi\varepsilon)^{n/2}} \sum_{j=1}^{m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |a_j(x_0) - a_j(x)|^2 e^{-|x-x_0|^2/2\varepsilon} dx_0 dx \\
&= n\varepsilon \int_{\mathbb{R}^n} \sum_{j=1}^{m} |\partial_x a_j(x)|^2 dx \le C\varepsilon.
\end{aligned}$$

Here we have used the mean value theorem and the Fubini theorem. For I_2, we have

$$\|I_2\|_E^2 = \frac{1}{(2\pi\varepsilon)^n} \int_{\mathbb{R}^n} \sum_{j=1}^{m} \left| \int_{\mathbb{R}^n} a_j(x)(e^{iT_2^{x_0}[S_0]/\varepsilon} - e^{iS_0/\varepsilon}) e^{-|x-x_0|^2/2\varepsilon} dx_0 \right|^2 dx.$$

Note that $|e^{iT_2^{x_0}[S_0]/\varepsilon} - e^{iS_0/\varepsilon}| \le C|x-x_0|^3/\varepsilon$ with C depending on $|S_0|_{C^3}$. Using again the Hölder inequality, we obtain:

$$\|I_2\|_E^2 \le \frac{C}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{j=1}^m |a_j(x)|^2 \frac{|x-x_0|^6}{\varepsilon^2} e^{-|x-x_0|^2/2\varepsilon} dx_0 dx$$
$$= C\varepsilon \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{j=1}^m |a_j(x)|^2 |\xi|^6 e^{-|\xi|^2} d\xi dx \le C\varepsilon.$$

The use of the triangle inequality yields (4.5). We next show

$$\|u_0^\varepsilon - \tilde{u}\|_E \le C\varepsilon^{1/2}, \tag{4.6}$$

Using that each $a_j(x_0) = 0$ on $\mathbb{R}^n \setminus K_0$, we have

$$u_0^\varepsilon - \tilde{u} = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} \sum_{j=1}^m (a_j(x_0)K_j - \varepsilon v_1^j(0, x; x_0)) e^{i\Phi^0(x;x_0)/\varepsilon} dx_0,$$

where $K_j = b_j(x, \partial_x S_0(x)) - b_j(x, \partial_x \Phi^0(x; x_0))$. From (4.2) we see that the term involving $v_1^j(0, x; x_0)$ is smaller, we thus take it to be zero from now on.

By the boundedness of $A(x)$,

$$\|\tilde{u} - u_0^\varepsilon\|_E^2 \le \frac{C}{(2\pi\varepsilon)^n} \int_{\mathbb{R}^n} \left| \int_{K_0} \sum_{j=1}^m a_j(x_0) K_j e^{i\Phi^0(x;x_0)/\varepsilon} dx_0 \right|^2 dx$$
$$\le C\varepsilon^{-n/2} \int_{\mathbb{R}^n}\int_{K_0} \sum_{j=1}^m |a_j(x_0)|^2 |K_j|^2 e^{-|x-x_0|^2/2\varepsilon} dx_0 dx.$$

Using the fact that

$$|K_j| \le \|b\|_{C^1} |\partial_x S_0(x) - \partial_x S_0(x_0) - \partial_x^2 S_0(x_0)(x-x_0) - iI(x-x_0)|$$
$$\le \|b\|_{C^1} |x-x_0|(1 + |S_0|_{C^3}|x-x_0|),$$

we then apply the change of variables $x - x_0 = \sqrt{2\varepsilon}\xi$ to obtain

$$\|\tilde{u} - u_0^\varepsilon\|_E^2 \le C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{j=1}^m |a_j(x_0)|^2 (\varepsilon|\xi|^2 + \varepsilon^2|\xi|^4) e^{-|\xi|^2} d\xi dx_0 \le C\varepsilon \|B_0\|_E^2.$$

This gives (4.6). Combining (4.5), (4.6) with the triangle inequality we finish the proof of Theorem 4.1. □

4.2. Evolution error estimate

Estimating the evolution error involves several steps, and the key is precise norm estimates in terms of ε of the oscillatory integral operators. Instead of presenting details in every step, we only outline how to reduce to the same type of operator estimates as obtained in [10].

Recall the residual representation (3.29), i.e.,

$$P(u^\varepsilon) = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} \sum_{j=1}^m A(x) \left(\frac{1}{\varepsilon} c_{0j} + c_{1j} \right) e^{i\Phi_j(t,x;x_0)/\varepsilon} dx_0, \tag{4.7}$$

and the estimate for c_{0j} and c_{1j},

$$|A(x)c_{0j}| \le C|x - \tilde{x}(t; x_0)|^3, \qquad |A(x)c_{1j}| \le C|x - \tilde{x}(t; x_0)|,$$

then each component of the vector $P(u^\varepsilon)$ is of the form

$$\varepsilon^{1/2} \sum_{j=1}^m \sum_{|\alpha|=1,3} (Q_{\alpha,g_j} \chi_{K_0})(t, x),$$

where χ_{K_0} is the characteristic function on K_0 and $Q_{\alpha,g} : L^2(K_0) \to L^\infty([0, T]; L^2(\mathbb{R}^n))$ is an operator defined as in [10, (3.5) in page 931]: For a fixed time $t \in [0, T]$, a multi-index α, and a function $g(t, x; x_0)$, we let

$$(Q_{\alpha,g} w)(t, x) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(x_0) g(t, x; x_0)(x - \tilde{x}(t; x_0))^\alpha e^{i\Phi_j(t,x;x_0)/\varepsilon} dx_0.$$

The following norm estimate is shown in [10] to hold:

$$\sup_{t\in[0,T]} \|Q_{\alpha,g_j}\|_{L^2} \le C(T), \tag{4.8}$$

provided that for all $t \in [0, T]$, the following assumptions are met:

(A1) $\tilde{x}(t; x_0) \in C^\infty([0, T] \times K_0)$,
(A2) $\Phi_j(t, x; x_0)$ and $g_j(t, x; x_0)$ are in $C^\infty([0, T] \times \mathbb{R}^n \times K_0)$,
(A3) $\nabla_x \Phi_j(t, \tilde{x}; x_0)$ is real and there is a constant C such that for $x_0, x_0' \in K_0$,

$$|\nabla_x \Phi_j(t, \tilde{x}(t; x_0); x_0) - \nabla_x \Phi_j(t, \tilde{x}(t; x_0'); x_0')| + |\tilde{x}(t; x_0) - \tilde{x}(t; x_0')| \ge C|x_0 - x_0'|,$$

(A4) there exists a constant δ such that for all $x_0 \in K_0$,

$$\Im \Phi_j(t, x; x_0) \le \delta |x - \tilde{x}(t; x_0)|^2,$$

(A5) for any multi-index β, there exists a constant C_β, such that

$$\sup_{x_0\in K_0, x\in\mathbb{R}^n} |\partial_x^\beta g_j(t, x : x_0)| \le C_\beta.$$

The essential assumption (A3), which is the non-squeezing lemma in [10], states that the distance between two initial physical points is comparable to the distance between their Hamiltonian trajectories measured in phase space, even in the presence of caustics. Since all requirements for the non-squeezing argument are satisfied by the construction of Gaussian beam solutions in present work, we therefore omit details of the proof of (A3).

(A4) is just (3.19) which is implied by the Gaussian beam phase construction. (A1)–(A2) and (A5) are implied by the smoothness of $A(x)$, the spectral gap (2.3), and the assumption that $|\partial_x S_0(x)| \ge \delta > 0$ for some $\delta > 0$. Note that for only first order beams, no cut-off function is needed, and the smoothness requirement can be much less. For details of a direct estimate we refer the reader to the appendix.

With the operator bound in (4.8) we have

$$\|P(u^\varepsilon)\| \le C\varepsilon^{1/2}, \tag{4.9}$$

which when combined with the initial error obtained in Theorem 4.1 and the wellposedness inequality (3.30) gives the main result (2.11) stated in Theorem 2.1.

5. Extension to the non-strict hyperbolic case

In this section, we will consider the non-strictly hyperbolic case, for which the dispersion matrix admits repeating eigenvalues, yet still being diagonalizable. Suppose that $\lambda(x, k)$, one of the eigenvalues, has multiplicity ν, corresponding to ν linearly independent eigenvectors b_l, $l = 1, \dots \nu$. The evolution equation for the Gaussian beam phase is governed by the Hamilton–Jacobi equation

$$\partial_t \Phi + \lambda(x, \partial_x \Phi) = 0,$$

in the same way as for the strictly hyperbolic case.

As for the construction of the amplitude for the Gaussian beam, we rely on the Fredholm Alternative theorem of the linear algebra regarding the solvability condition for v_1, same as in (3.21). Instead of $v_0 = a_0(t, x)b(x, k(t, x))$, with $k(t, x) := \partial_x \Phi(t, x)$, we now have

$$v_0 = \sum_{l=1}^{\nu} a_l(t, x) b_l(x, k(t, x)).$$

The Fredholm Alternative theorem of the linear algebra implies that we can solve $\tilde{c}_1 = 0$ for v_1 if and only if the coefficient $a_l(t, x)$ is chosen so that

$$\left\langle (\partial_t + L(x, \partial_x)) \left(\sum_{l=1}^{\nu} a_l(t, x) b_l \right), b_l \right\rangle_A = 0, \quad l = 1, \dots, \nu, \tag{5.1}$$

where b_l^* is the eigenvector of the adjoint of $L(x, \partial_x \Phi(x, t))$ with respect to $\langle \cdot, \cdot \rangle_A$ with eigenvalue $\overline{\lambda}(x, \partial_x \Phi(x, t))$, normalized by $\langle b, b \rangle_A = 1$.

Since we are constructing first order beams, we only require (5.1) on the central ray, i.e., where $(x, \partial_x \Phi(x, t)) = (\tilde{x}(t; x_0), p(t; x_0))$. Hence, we can take $a_l = a_l(t; x_0)$, so that

$$\partial_t a_l + \sum_{j=1}^{\nu} a_j f_{lj}(\tilde{x}, p) = 0, \quad l = 1, \dots, \nu, \tag{5.2}$$

where

$$f_{lj}(x, k(t, x)) = \langle b_l, (\partial_t + L(x, \partial_x)b_j)\rangle_A.$$

Under assumptions of smoothness for all coefficients of the ODE system, this system subject to the initial value has a unique solution. To complete the construction of the amplitude, we follow the same idea as in (3.23) by choosing $v_1 = v_1^{\perp}$:

$$v_1^{\perp} = i(L(x, k(t, x)) - \lambda(x, k(t, x))I)^{-1}\left((\partial_t + L(x, \partial_x))v_0 - \sum_{l=1}^{\nu}\langle(\partial_t + L(x, \partial_x))v_0, b_l\rangle_A b_l\right), \tag{5.3}$$

where $(L(x, k(t, x)) - \lambda(x, k(t, x))I)^{-1}$ denotes the inverse of $L(x, k(t, x)) - \lambda(x, k(t, x))I$ defined on

$$S^{\perp}(b_1, \dots, b_\nu) = \{v : \langle v, b_l(x, k(t, x))\rangle_A = 0,\ l = 1, \dots, \nu\}.$$

Note that the norm of $(L - \lambda I)^{-1}$ depends only on supremum norms of the coefficients of L and the spectral gap of distinct eigenvalues

$$\Delta\lambda = \min|\lambda_l - \lambda_j| > 0.$$

We thus obtain

$$c_1 = \sum_{l=1}^{\nu}\langle(\partial_t + L(x, \partial_x))v_0, b_l\rangle_A b_l + i(\partial_t\Phi(t, x) + \lambda(x, \partial_x\Phi(x, t)))v_1. \tag{5.4}$$

Hence there is a neighborhood N of γ such that for $x \in N$ and $0 \le t \le T$

$$|c_1(t, x)| \le C(\Delta\lambda)d(x, \gamma), \tag{5.5}$$

where $d(x, \gamma)$ is the (Euclidean) distance for x to γ. The Gaussian beam approximation for any fixed $x_0 \in K_0 \subset \mathbb{R}^n$ by modes corresponding to the jth eigenvalue is the following:

$$u_{GB}^{\varepsilon j}(t, x; x_0) = \left(\left(\sum_{l=1}^{\nu_j} a_l(t; x_0)b_l(x, \partial_x\Phi_j)\right) + \varepsilon v_1^j(t, x; x_0)\right)e^{i\Phi_j(t,x;x_0)/\varepsilon}.$$

Thus, the Gaussian beam superposition formula for the non-strict hyperbolic case follows immediately:

$$u^{\varepsilon}(t, x) = \frac{1}{(2\pi\varepsilon)^{n/2}}\int_{K_0}\sum_{j=1}^{\mu} u_{GB}^{\varepsilon j}(t, x; x_0)dx_0,$$

where μ is a number of distinct eigenvalues, i.e. $m = \sum_{j=1}^{\mu}\nu_j$.

5.1. Multiple zero eigenvalues case

In some practical applications, we may have a situation when some of eigenvalues are zero. In this case, the Hamilton–Jacobi equation

$$\partial_t\Phi(t, x) + \lambda(x, \partial_x\Phi(t, x)) = 0$$

degenerates to

$$\partial_t\Phi(t, x) = 0,$$

and hence we simply take initial phase $S_0(x)$ as the phase for the asymptotic solution,

$$\Phi(t, x) = S_0(x).$$

Equations for the amplitudes $a_1, \dots, a_\nu$ are the following transport equations:

$$(\partial_t + L(x, \partial_x))(a_l(t, x)b_l(x, \partial_x S_0)) = 0, \quad l = 1, \dots, \nu.$$

Note that unlike the first order Gaussian beam amplitudes for strictly hyperbolic case, the geometric optics amplitudes actually depend on x. The approximation formula corresponding to zero eigenvalues is reduced to the following:

$$u^{\varepsilon}(t, x) = \sum_{l=1}^{\nu} a_l(t, x)b_l(x, \partial_x S_0(x))e^{iS_0(x)/\varepsilon}.$$

Suppose the hyperbolic system has zero eigenvalue $\lambda = 0$ with multiplicity ν_0, other eigenvalues λ_i with multiplicities ν_i, then the Gaussian beam superposition takes the form:

$$u^{\varepsilon}(t, x) = \sum_{l=1}^{\nu_0} a_l(t, x)b_l(x, \partial_x S_0(x))e^{iS_0(x)/\varepsilon} + \frac{1}{(2\pi\varepsilon)^{n/2}}\int_{K_0}\sum_{j=1}^{\mu} u_{GB}^{\varepsilon j}(t, x; x_0)dx_0, \tag{5.6}$$

where μ is the number of distinct nonzero eigenvalues.

5.2. Error estimates

Note that in the Gaussian beam superposition (5.6), the first part remains unchanged in time, and the second part contains the same Gaussian beam profiles as that for the strictly hyperbolic system, hence the previous error estimates can be adapted to the present case where we have μ phases and m modes and $\mu < m$. Below, we state the main theorem for the non-strictly hyperbolic case.

Theorem 5.1. *Let $K_0 \subset \mathbb{R}^n$ be a bounded measurable set, initial amplitude $B_0(x) \in H^1(K_0)$, initial phase $S_0(x) \in C^{n+4}(\mathbb{R}^n)$ and bounded, $|\partial_x S_0(x)|$ be bounded away from zero on K_0; eigenvectors $b_l(x,k)$ and eigenvalues $\lambda_j(x,k)$ be smooth and bounded functions, with second order derivatives of $\lambda_j(x,k)$ globally bounded in $|k| > 1$, u be the exact solution to (1.1)– (1.2) for $0 < t \le T$, and u^ε be the first order Gaussian beam superposition (5.6). Then*

$$\|u - u^\varepsilon\|_E \le C\varepsilon^{1/2}, \tag{5.7}$$

where the constant C is independent of ε, but may depend on the finite time T and the data given.

5.3. Acoustic waves

We now examine the applications to acoustic wave equations. Consider

$$\begin{cases} \rho(x)\partial_t v + \partial_x p = 0, \\ \kappa(x)\partial_t p + \partial_x \cdot v = 0, \end{cases} \tag{5.8}$$

where $\rho(x)$ is density, $\kappa(x)$ is compressibility, both $\rho(x)$ and $\kappa(x)$ are smooth and positive functions, we require that

$$\min_x(\rho(x), \kappa(x)) \ge \gamma > 0$$

for some γ. The vector-valued function $v(x) = (v_1(x), v_2(x), v_3(x))$ denotes velocity, and scalar-valued function $p(x)$ denotes pressure, $x \in \mathbb{R}^3$, $u = (v, p) \in \mathbb{R}^4$.

We consider a high frequency initial condition:

$$u(0,x) = B_0(x)e^{iS_0(x)/\varepsilon},$$

with B_0 and S_0 being smooth and bounded functions, B_0 is compactly supported.

This system has the following eigenvalues:

$$\lambda_{1,2} = 0, \qquad \lambda_{3,4} = \pm\frac{|k|}{\sqrt{\kappa(x)\rho(x)}}.$$

The associated eigenvectors are the following:

$$b_1 = \frac{1}{\sqrt{\rho(x)(k_1^2+k_2^2)}}(k_2, -k_1, 0, 0),$$
$$b_2 = \frac{1}{|k|\sqrt{\rho(x)(k_1^2+k_2^2)}}(k_1k_3, k_2k_3, -k_1^2-k_2^2, 0),$$
$$b_3 = \frac{1}{|k|}\left(\frac{k_1}{\sqrt{2\rho(x)}}, \frac{k_2}{\sqrt{2\rho(x)}}, \frac{k_3}{\sqrt{2\rho(x)}}, \frac{|k|}{\sqrt{2\kappa(x)}}\right),$$
$$b_4 = \frac{1}{|k|}\left(-\frac{k_1}{\sqrt{2\rho(x)}}, -\frac{k_2}{\sqrt{2\rho(x)}}, -\frac{k_3}{\sqrt{2\rho(x)}}, \frac{|k|}{\sqrt{2\kappa(x)}}\right).$$

The Gaussian beam superposition for the acoustic waves takes the form:

$$u^\varepsilon = (a_1(t,x)b_1(x,\partial_x S_0(x)) + a_2(t,x)b_2(x,\partial_x S_0(x)))e^{iS_0(x)/\varepsilon} + \frac{1}{(2\pi\varepsilon)^{3/2}}\int_{K_0}(u_{GB}^3(t,x;x_0) + u_{GB}^4(t,x;x_0))dx_0.$$

5.4. Maxwell equations

We next examine the applications to Maxwell equations of the form

$$\begin{cases} \partial_t E = \nabla_x \times B, \\ \partial_t B = -\nabla_x \times E, \\ \nabla_x \cdot E = \nabla_x \cdot B = 0, \end{cases} \tag{5.9}$$

where $x \in \mathbb{R}^3$, $u = (E, B) \in \mathbb{R}^6$. E and B are electric and magnetic fields, respectively. Suppose we have a highly oscillatory initial condition: $u(0, x) = A_0(x)e^{iS_0(x)/\varepsilon}$. The dispersive matrix L does not depend on x,

$$L(x, k) = \begin{pmatrix} 0 & 0 & 0 & 0 & k_3 & -k_2 \\ 0 & 0 & 0 & -k_3 & 0 & k_1 \\ 0 & 0 & 0 & k_2 & -k_1 & 0 \\ 0 & -k_3 & k_2 & 0 & 0 & 0 \\ k_3 & 0 & -k_1 & 0 & 0 & 0 \\ -k_2 & k_1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

Computing eigenvalues of the matrix L we obtain:

$$\lambda_{1,2} = 0, \qquad \lambda_{3,4} = |k|, \qquad \lambda_{5,6} = -|k|.$$

The corresponding eigenvectors $b_1, \dots, b_6$ can be chosen to be orthonormal. For $\lambda = |k|$, we have

$$\begin{cases} \dot{\tilde{x}} = \dfrac{p}{|p|}, & \tilde{x}|_{t=0} = x_0, \\ \dot{p} = 0, & p|_{t=0} = \nabla_x S_0(x_0), \\ \dot{S} = 0, & S|_{t=0} = S_0(x_0), \\ \dot{M} = \Big(\dfrac{1}{|p|^3}P - \dfrac{1}{|p|}I\Big)M^2, & M|_{t=0} = \nabla_x(\nabla_x S_0) + iI, \end{cases} \tag{5.11}$$

where $P = (p_i p_j)$, $i, j = 1, 2, 3$, and the equations for the amplitudes satisfy (5.2). Equations for $\lambda = -|k|$ are computed in the same way.

The Gaussian beam superposition for the Maxwell equations is the following:

$$u^\varepsilon = \left(\sum_{l=1}^{2} a_l(t, x) b_l(x, \partial_x S_0(x))\right) e^{iS_0(x)/\varepsilon} + \frac{1}{(2\pi\varepsilon)^{3/2}} \int_{K_0} (u_{GB}^{\varepsilon,|p|}(t, x; x_0) + u_{GB}^{\varepsilon,-|p|}(t, x; x_0)) dx_0,$$

where

$$u_{GB}^{\varepsilon,|p|}(t, x; x_0) = \left(\left(\sum_{l=3}^{4} a_l(t; x_0) b_l(x, \partial_x \Phi_{|p|})\right) + \varepsilon v_1^{|p|}(t, x; x_0)\right) e^{i\Phi_{|p|}(t,x;x_0)/\varepsilon},$$

$$u_{GB}^{\varepsilon,-|p|}(t, x; x_0) = \left(\left(\sum_{l=5}^{6} a_l(t; x_0) b_l(x, \partial_x \Phi_{-|p|})\right) + \varepsilon v_1^{-|p|}(t, x; x_0)\right) e^{i\Phi_{-|p|}(t,x;x_0)/\varepsilon},$$

where $v_1^{\pm|p|}$ satisfy (5.3).

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Appendix

We follow the idea in [10] to carry out the details of the estimate of $\|P(u^\varepsilon)\|$. We begin with

$$\|P(u^\varepsilon)\| \le \sum_{j=1}^{m} (\|I_{0j}\| + \|I_{1j}\|),$$

where

$$I_{lj} := \frac{\varepsilon^{l-1}}{(2\pi\varepsilon)^{n/2}} \int_{K_0} A(x) c_{lj} e^{i\Phi_j/\varepsilon} dx_0$$

is vector-valued. Since the estimate for each wave field is similar, we thus omit the index j using only $I_l(t, x; x_0)$ for $l = 0, 1$ in what follows. Let $'$ denote quantities defined on the ray radiating from x_0' such as $\tilde{x}'$, c_l' and Φ'. Then we can represent the L^2 norm of I_l by

$$\begin{aligned} \|I_l\|^2 &= \int_{\mathbb{R}^n} I_l(t, x; x_0) \cdot \overline{I_l(t, x; x_0')} dx \\ &= \int_{\mathbb{R}^n} \int_{K_0} \int_{K_0} J_l(t, x, x_0, x_0') dx_0 dx_0' dx, \end{aligned}$$

where

$$J_l = \frac{\varepsilon^{-n+2l-2}}{(2\pi)^n} A(x)c_l(t, x; x_0) \cdot \overline{A(x)c_l(t, x, x_0')} e^{i\psi(t,x;x_0,x_0')/\varepsilon} \tag{A.12}$$

with

$$\psi(t, x, x_0, x_0') = \Phi(t, x; x_0) - \overline{\Phi(t, x; x_0')}. \tag{A.13}$$

The rest is to establish the following

$$\left| \int_{\mathbb{R}^n} \int_{K_0} \int_{K_0} J_l dx_0 dx_0' dx \right| \le C\varepsilon. \tag{A.14}$$

With this estimate we have $\|I_l\| \le C\varepsilon^{\frac{1}{2}}$, leading to the desired estimate (4.9).

In order to estimate (A.14), we note that for any x_0, x_0', there exists a constant δ independent of ϵ such that

$$\Im\psi = \Im\Phi + \Im\Phi' \ge \frac{\delta}{2}(|x - \tilde{x}|^2 + |x - \tilde{x}'|^2),$$

which follows directly from (3.19). Hence

$$|J_l| \le C\varepsilon^{-n+2l-2} |c_l(t, x; x_0)| \cdot |c_l(t, x, x_0')| e^{-\frac{\delta}{2\varepsilon}(|x-\tilde{x}|^2 + |x-\tilde{x}'|^2)}, \tag{A.15}$$

with $C = (2\pi)^{-n}|A|_\infty^2$, and $l = 0, 1$.

Let $\rho_j(x, x_0, x_0') \in C^\infty$ be a partition of unity such that

$$\rho_2 = \begin{cases} 1, & |x - \tilde{x}| \le \eta \cap |x - \tilde{x}'| \le \eta, \\ 0, & |x - \tilde{x}| \ge 2\eta \cup |x - \tilde{x}'| \ge 2\eta, \end{cases} \tag{A.16}$$

and $\rho_1 + \rho_2 = 1$. Moreover, let

$$J_l^1 = \rho_1 J_l(t, x, x_0, x_0'), \qquad J_l^2 = \rho_2 J_l(t, x, x_0, x_0'),$$

so that $J_l(t, x, x_0, x_0') = J_l^1 + J_l^2$.

We first estimate c_0: using (3.11) with (3.13) and (3.9), we have

$$G(t, x; x_0) = \lambda(x, k) - \lambda(\tilde{x}, p) - \partial_x\lambda(\tilde{x}, p) \cdot (x - \tilde{x}) - \partial_k\lambda(\tilde{x}, p)M(x - \tilde{x}) + \frac{1}{2}(x - \tilde{x})^\top \cdot \dot{M}(x - \tilde{x}), \tag{A.17}$$

with $\dot{M} = -\partial_x^2\lambda(\tilde{x}, p)$. Also from (3.14) and $k = p + M(x - \tilde{x})$, we thus obtain

$$|c_0| = |aGb| \le C(1 + |x - \tilde{x}|^2), \quad |x - \tilde{x}| \ge 2\eta, \tag{A.18}$$

$$|c_0| = |aGb| \le C|x - \tilde{x}|^3, \quad |x - \tilde{x}| \le 2\eta, \tag{A.19}$$

provided η is sufficiently small.

From the expression

$$c_1 = a(f(x, \partial_x\Phi) - f(\tilde{x}, p))b + iGv_1,$$

we can verify the following bounds

$$|c_1| \le C(1 + |x - \tilde{x}|)(1 + |x - \tilde{x}|^2), \quad |x - \tilde{x}| \ge 2\eta, \tag{A.20}$$

$$|c_1| \le C|x - \tilde{x}|, \quad |x - \tilde{x}| \le 2\eta. \tag{A.21}$$

A.1. Estimate of j_l^1

Denote

$$s = |x - \tilde{x}|, \qquad s' = |x - \tilde{x}'|,$$

then from (A.12) using (A.18) and (A.20) it follows that

$$|J_l^1| \le C\rho_1\varepsilon^{-n+2l-2}(1 + s)(1 + s^2)(1 + s')(1 + (s')^2)e^{-\frac{\delta}{2\varepsilon}(s^2 + (s')^2)}.$$

Using the estimate

$$s^q e^{-cs^2} \le \left(\frac{q}{e}\right)^{q/2} c^{-p/2} e^{-cs^2/2}, \tag{A.22}$$

with $c = \frac{\delta}{\varepsilon}$, we have

$$(1 + s)(1 + s^2)e^{-\frac{\delta}{2\varepsilon}s^2} \le C(1 + \varepsilon^{1/2} + \varepsilon^1 + \varepsilon^{3/2})e^{-\frac{\delta}{4\varepsilon}s^2} \le 4Ce^{-\frac{\delta}{4\varepsilon}s^2}.$$

Hence

$$|J_l^1| \le C\varepsilon^{-n+2l-2} e^{-\frac{\delta}{4\varepsilon}(s^2+(s')^2)} \le C\varepsilon^{-n+2l-2} e^{-\frac{\delta}{4\varepsilon}s^2} e^{-\frac{\eta^2\delta}{\varepsilon}},$$

where we have assumed $s' > 2\eta$ due to the definition of ρ_1, we thus obtain an exponential decay

$$\left|\int_{\mathbb{R}^n}\int_{K_0}\int_{K_0} J_0^1 dx_0 dx_0' dx\right| \le C\varepsilon^{2l-2-\frac{n}{2}} |K_0|^2 e^{-\frac{\eta^2\delta}{\varepsilon}} \le C\varepsilon^r \quad \forall r.$$

A.2. Estimate of j_l^2

For $|x-\tilde{x}| \le \eta$, both (A.19) and (A.21) imply that $|c_l| \le C|x-\tilde{x}|^{3-2l}$, then from (A.15) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} |J_l^2| dx &\le C\varepsilon^{-n+2l-2}\int_{\mathbb{R}^n} \rho_2 |c_l(t,x;x_0)| \cdot |c_l(t,x,x_0')| e^{-\frac{\delta}{2\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx \\
&\le C\varepsilon^{-n+2l-2}\int_{\mathbb{R}^n} |x-\tilde{x}|^{3-2l}|x-\tilde{x}'|^{3-2l} e^{-\frac{\delta}{2\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx \\
&\le C\varepsilon^{-n+1}\int_{\mathbb{R}^n} e^{-\frac{\delta}{4\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx.
\end{aligned}$$

Using the identity

$$|x-\tilde{x}|^2 + |x-\tilde{x}'|^2 = 2\left|x - \frac{\tilde{x}+\tilde{x}'}{2}\right|^2 + \frac{1}{2}|\tilde{x}-\tilde{x}'|^2, \tag{A.23}$$

we obtain

$$\int_{\mathbb{R}^n} |J_l^2| dx \le C\varepsilon^{-n+1}\int_{\mathbb{R}^n} e^{-\frac{\delta}{2\varepsilon}\left|x-\frac{\tilde{x}+\tilde{x}'}{2}\right|^2} dx e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2}.$$

Hence,

$$\left|\int_{\mathbb{R}^n}\int_{K_0}\int_{K_0} J_l^2 dx_0 dx_0' dx\right| \le C\varepsilon^{-\frac{n}{2}+1}\int_{K_0}\int_{K_0} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2} dx_0 dx_0'. \tag{A.24}$$

In order to obtain (A.14), we need to recover an extra $\varepsilon^{\frac{n}{2}}$ from the integral on the right hand side of (A.24), which is difficult when $|\tilde{x}-\tilde{x}'|$ is small.

Following [10], we split the set $K_0 \times K_0$ into

$$D_1(t,\theta) = \left\{(x_0,x_0') : |\tilde{x}-\tilde{x}'| \ge \theta|x_0-x_0'|\right\},$$

which corresponds to the non-caustic region of the solution, and the set associated with the caustic region

$$D_2(t,\theta) = \left\{(x_0,x_0') : |\tilde{x}-\tilde{x}'| < \theta|x_0-x_0'|\right\}.$$

For the former we have

$$\int_{D_1} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2} dx_0 dx_0' \le \int_{D_1} e^{-\frac{\delta\theta^2}{8\varepsilon}|x_0-x_0'|^2} dx_0 dx_0'.$$

Changing to spherical coordinates, we obtain

$$\begin{aligned}
\int_{D_1} e^{-\frac{\delta\theta^2}{8\varepsilon}|x_0-x_0'|^2} dx_0 dx_0' &\le C\int_0^\infty s^{n-1} e^{-\frac{\delta\theta^2}{8\varepsilon}s^2} ds \\
&\le C\varepsilon^{\frac{n-1}{2}}\int_0^\infty e^{-\frac{\delta\theta^2}{8\varepsilon}s^2} ds \le C\varepsilon^{\frac{n}{2}}
\end{aligned}$$

as needed.

To estimate J_l^2 restricted on D_2, we need the following result on phase estimate.

Lemma A.1 (*Phase Estimate*)**.** *For $(x_0,x_0') \in D_2$, it holds*

$$|\nabla_x \psi(t,x,x_0,x_0')| \ge C(\theta,\eta)|x_0-x_0'|, \tag{A.25}$$

where $C(\theta,\eta)$ is independent of x and positive if θ and η are sufficiently small.

The proof of this result is due to [10], where the non-squeezing lemma is crucial. Since all requirements for the non-squeezing argument are satisfied by the construction of Gaussian beam solutions in present work, we therefore omit details of the proof.

To continue, we note that the phase estimate ensures that for $(x_0, x_0') \in D_2$, $x_0 \neq x_0'$, $\nabla_x \psi(t, x, x_0, x_0') \neq 0$. Therefore, in order to estimate $J_l^2|_{D_2}$ we shall use the following non-stationary phase lemma.

Lemma A.2 (*Non-Stationary Phase Lemma*)**.** *Suppose that $u(x, \xi) \in C_0^\infty(\Omega \times Z)$ where Ω and Z are compact sets and $\psi(x; \xi) \in C^\infty(O)$ for some open neighborhood O of $\Omega \times Z$. If $\partial_x \psi$ never vanishes in O, then for any $K = 0, 1, \ldots,$*

$$\left| \int_\Omega u(x; \xi) e^{i\psi(x;\xi)/\varepsilon} dx \right| \leq C_K \varepsilon^K \sum_{|\beta|=1}^{K} \int_\Omega \frac{|\partial_x^\beta u(x; \xi)|}{|\partial_x \psi(x; \xi)|^{2K-|\beta|}} e^{-\Im \psi(x;\xi)/\varepsilon} dx,$$

where C_K is a constant independent of ξ.

Using the non-stationary lemma, (A.12), (A.15) and the lower bound for ψ in (A.25), we obtain for $(x_0, x_0') \in D_2$,

$$\left| \int_{\mathbb{R}^n} J_l^2 dx \right| \leq C\varepsilon^{K-n+2l-2} \int_{\mathbb{R}^n} \sum_{|\beta|=1}^{K} \frac{|L_\beta^l|}{|\partial_x \psi|^{2K-|\beta|}} e^{-\frac{\delta}{2\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx$$
$$\leq C \sum_{|\beta|=1}^{K} \frac{\varepsilon^{K-n+2l-2}}{\inf_x |\partial_x \psi|^{2K-|\beta|}} \int_{\mathbb{R}^n} |L_\beta^l| e^{-\frac{\delta}{2\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx,$$

where we have used the notation

$$L_\beta^l := \partial_x^\beta [\rho_2 A(x) c_l(t, x, x_0) \cdot \overline{A(x) c_l(t, x, x_0')}].$$

We claim the following estimate for L_β^l,

$$|L_\beta^l| \leq C \sum_{|\beta_1|+|\beta_2|=|\beta|} |x - \tilde{x}|^{(3-2l-|\beta_1|)_+} |x - \tilde{x}'|^{(3-2l-|\beta_2|)_+}. \tag{A.26}$$

Therefore,

$$\left| \int_{\mathbb{R}^n} J_l^2 dx \right| \leq C \sum_{|\beta|=1}^{K} \frac{\varepsilon^{K-n+2l-2}}{\inf_x |\partial_x \psi|^{2K-|\beta|}} \int_{\mathbb{R}^n} \sum_{|\beta_1|+|\beta_2|=|\beta|} |x - \tilde{x}|^{(3-2l-|\beta_1|)_+} |x - \tilde{x}'|^{(3-2l-|\beta_2|)_+} e^{-\frac{\delta}{2\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx$$
$$\leq C \sum_{|\beta|=1}^{K} \frac{\varepsilon^{K-n-|\beta|/2+1}}{\inf_x |\partial_x \psi|^{2K-|\beta|}} \int_{\mathbb{R}^n} e^{-\frac{\delta}{4\varepsilon}(|x-\tilde{x}|^2+|x-\tilde{x}'|^2)} dx.$$

Using (A.23) we have

$$\left| \int_{\mathbb{R}^n} J_l^2 dx \right| \leq C \sum_{|\beta|=1}^{K} \frac{\varepsilon^{K-n-|\beta|/2+1}}{\inf_x |\partial_x \psi|^{2K-|\beta|}} \int_{\mathbb{R}^n} e^{-\frac{\delta}{4\varepsilon}\left(2\left|x-\frac{\tilde{x}+\tilde{x}'}{2}\right|^2+\frac{1}{2}|\tilde{x}-\tilde{x}'|^2\right)} dx$$
$$\leq C \sum_{|\beta|=1}^{K} \frac{\varepsilon^{K-n/2-|\beta|/2+1}}{\inf_x |\partial_x \psi|^{2K-|\beta|}} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2}.$$

Hence,

$$\left| \int_{\mathbb{R}^n} \int_{D_2} J_l^2 dx_0 dx_0' dx \right| \leq C\varepsilon^{1-\frac{n}{2}} \int_{D_2} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2} \sum_{|\beta|=1}^{K} \frac{1}{\inf |\partial_x \psi/\sqrt{\varepsilon}|^{2K-|\beta|}} dx_0 dx_0'.$$

The last estimate together with (A.24) yields:

$$\left| \int J_l^2 1_{D_2} \right| \leq C\varepsilon^{1-\frac{n}{2}} \int_{D_2} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2} \min\left[1, \sum_{|\beta|=1}^{K} \frac{1}{\inf |\partial_x \psi/\sqrt{\varepsilon}|^{2K-|\beta|}}\right] dx_0 dx_0'$$
$$\leq C\varepsilon^{1-\frac{n}{2}} \int_{D_2} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2} \sum_{|\beta|=1}^{K} \min\left[1, \frac{1}{\inf |\partial_x \psi/\sqrt{\varepsilon}|^{2K-|\beta|}}\right] dx_0 dx_0'$$
$$\leq C\varepsilon^{1-\frac{n}{2}} \int_{K_0} \int_{K_0} e^{-\frac{\delta}{8\varepsilon}|\tilde{x}-\tilde{x}'|^2} \sum_{|\beta|=1}^{K} \frac{2}{1 + \inf |\partial_x \psi/\sqrt{\varepsilon}|^{2K-|\beta|}} dx_0 dx_0'$$

$$\leq C\varepsilon^{1-\frac{n}{2}} \int_{K_0}\int_{K_0}\sum_{|\beta|=1}^{K} \frac{1}{1+(C(\theta,\eta)|x_0-x_0'|/\sqrt{\varepsilon})^{2K-|\beta|}} dx_0 dx_0',$$

where we have used the inequality $\min\{1, \frac{1}{b}\} \leq \frac{2}{1+b}$ for any $b > 0$. Taking $K = n+1$ and changing variable $\xi = \frac{x_0 - x_0'}{\sqrt{\varepsilon}}$, we compute

$$\left|\int J_l^2 1_{D_2}\right| \leq C\varepsilon^{1-\frac{n}{2}} \int_{K_0\times K_0} \frac{1}{1+\left(|x_0-x_0'|/\sqrt{\varepsilon}\right)^{n+1}} dx_0 dx_0'$$
$$\leq C\varepsilon \int_0^\infty \frac{1}{1+\xi^{n+1}} d\xi = C\varepsilon.$$

which gives (A.14) when restricted to the caustic region. This completes the proof of (A.14), except the claim (A.26), which we show below.

We assume smoothness and boundedness of any component contributing to

$$\partial_x^\beta[\rho_2 A(x) c_l(t,x,x_0)\overline{A(x)c_l(t,x,x_0')}].$$

Note that the typical term in L_β^0 has form $\partial_x^\beta[\rho_2 A(x) b \cdot \overline{A(x)b'} g\overline{g'}(x-\tilde{x})^\alpha (x-\tilde{x}')^\alpha]$, where g is a third order partial derivative of λ and α is a multiindex, $|\alpha| = 3$. For the sake of brevity, we denote

$$h := \rho_2 A(x) b \cdot \overline{A(x)b'} g\overline{g'}.$$

Hence

$$|L_\beta^0| \leq C|\partial_x^\beta[h(x-\tilde{x})^\alpha (x-\tilde{x}')^\alpha]| = C\left|\sum_{|\beta_1|+|\beta_2|=|\beta|} \partial_x^{\beta_1} h \partial_x^{\beta_2}[(x-\tilde{x})^\alpha (x-\tilde{x}')^\alpha]\right|$$
$$= C\left|\sum_{|\beta_1|+|\beta_2|=|\beta|} \partial_x^{\beta_1} h \sum_{|\beta_{21}|+|\beta_{22}|=|\beta_2|} (x-\tilde{x})^{(\alpha-\beta_{21})_+}(x-\tilde{x}')^{(\alpha-\beta_{22})_+}\right|.$$

In the "worst" case, i.e., when $|\beta_1| = 0$ we obtain the lowest power of $(x-\tilde{x})(x-\tilde{x}')$ and since x is near the ray, then the higher order terms are controlled by lower order terms, and (A.26) is satisfied for $l = 0$.

As for $l = 1$ case, we only need to take care of the lower order term,

$$|L_\beta^1| \leq C\left|\partial_x^\beta[\rho_2 A(x) a D_x f(\cdot,\cdot)\cdot(x-\tilde{x}) b \cdot \overline{A(x) a D_x f(\cdot,\cdot)\cdot(x-\tilde{x}')\overline{b}}]\right|,$$

so that (A.26) follows for $l = 1$ too.

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