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On Invariant-Preserving Finite Difference Schemes for the Camassa-Holm Equation and the Two-Component Camassa-Holm System

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Abstract. The purpose of this paper is to develop and test novel invariant-preserving finite difference schemes for both the Camassa-Holm (CH) equation and one of its 2-component generalizations (2CH). The considered PDEs are strongly nonlinear, admitting soliton-like peakon solutions which are characterized by a slope discontinuity at the peak in the wave shape, and therefore suitable for modeling both short wave breaking and long wave propagation phenomena. The proposed numerical schemes are shown to preserve two invariants, momentum and energy, hence numerically producing wave solutions with smaller phase error over a long time period than those generated by other conventional methods. We first apply the scheme to the CH equation and showcase the merits of considering such a scheme under a wide class of initial data. We then generalize this scheme to the 2CH equation and test this scheme under several types of initial data.

AMS subject classifications: 65M60, 65M12, 35Q53

Key words: Camassa-Holm equation, Peakon solutions, energy-preserving methods.

1 Introduction

This paper is concerned with the numerical approximation of a completely integrable nonlinear evolutionary partial differential equation as well as one of its two-component generalizations which often arises in various applications of shallow water wave theory. In particular, a main goal of this paper is to develop a new invariant-preserving finite difference scheme for both the Camassa-Holm (CH) equation and the two-component Camassa-Holm equation (2CH) as well as showcase the merits of using invariant-preserving methods to simulate solutions to these equations under a wide class of initial data.

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1.1 The CH equation

The CH equation is given by

\[ m_t + um_x + 2mu_x = 0, \quad m = u - u_{xx}, \quad (1.1) \]

which is subjected to the initial data:

\[ m(x,0) = m_0(x), \quad (1.2) \]

with periodic boundary conditions. Here, \( m \) is the momentum related to the fluid velocity \( u \) by the one-dimensional (1-D) Helmholtz operator (see (1.1)).

Eq. (1.1) arises in a wide range of scientific applications and, for example, can be described as a bi-Hamiltonian model in the context of shallow water waves, see [4, 13, 14]. It can also be used to quantify growth and other changes in shape, such as those which occur in a beating heart, by providing the transformative mathematical path between two shapes (for instance, see [23, page 420]). Recalling that \( m = u - u_{xx} \), the two compatible Hamiltonian descriptions of the CH equation are given by

\[ m_t = -(m \partial_x + \partial_x m) \frac{\delta \mathcal{H}_1}{\delta m} = -\partial_x (1 - \partial_x^2) \frac{\delta \mathcal{H}_2}{\delta u} = -\frac{\delta \mathcal{H}_2}{\delta u}, \quad (1.3) \]

with the following conserved quantities:

\[ \mathcal{H}_0 = \int_R m(x,t) dx, \quad \mathcal{H}_1 = \frac{1}{2} \int_R (u^2 + u_x^2) dx \quad \text{and} \quad \mathcal{H}_2 = \frac{1}{2} \int_R (u^3 + uu_{xx}^2) dx. \quad (1.4) \]

The CH equation is integrable with an infinite number of conservation laws, admits soliton-like peakon solutions, and can be viewed as a model of shallow water waves.

The CH equation (1.1) can also be written as the system of equations:

\[ u_t + uu_x + p_x = 0, \quad p = (1 - \partial_x^2)^{-1} \left( u^2 + \frac{1}{2}(u_x)^2 \right), \]

where \( p \) is the dimensionless pressure or surface tension.

For this nonlocal conservation law with any initial data \( u_0 \in H^1(\mathbb{R}) \), several authors have studied the global existence of solutions, conservative or dissipative, c.f. [3, 11, 33, 36, 42]. Uniqueness is a delicate issue because in general the flow map has less regularity than usually needed to justify the uniqueness. Recently, it was proved by Bressan et al. [2] that the Cauchy problem with general initial data \( u_0 \in H^1(\mathbb{R}) \) has a unique conservative solution, globally in time; using a direct approach based on characteristics for the uniqueness of conservative solutions. Our goal is to compute such conservative solutions.

Simulating these peakon solutions numerically poses quite a challenge – especially if one is interested in considering a peakon-antipeakon interaction (i.e., the interaction...
between positive and negative peakons). Several sophisticated numerical methods in finite-difference, finite-element, and spectral settings have been proposed for accurately resolving the CH equation – in particular, peakon interactions. For example, in [16], a self-adaptive mesh method was proposed, whereas in [27, 28], a spectral projection method was used to simulate solutions to the CH equation. In [10], the authors used multi-symplectic integration, and in [34], an energy-conserving Galerkin scheme was proposed. In [8], the authors considered a dispersion-relation-preserving algorithm. For additional numerical schemes proposed for solving the CH equation, we refer the reader to [1, 5, 22, 39, 40] and references therein. Many of these methods are computationally intensive and require very fine grids along with adaptivity techniques in order to model the peakon behavior. By preserving some key invariants inherent in the CH equation, our method can successfully resolve the complicated interaction among two positive peakons as well as the peakon-antipeakon interactions.

1.2 The two-component CH system

Recently, the CH equation has been extended to a two-component integrable system (2CH) which includes both velocity and density variables in the dynamics (see e.g. [12, 24, 26, 32]). In this regard, the 2CH equation is given by the following system of equations:

$$
m_t + u m_x + 2mu_x = -g \rho \rho_x,
$$
$$
\rho_t + (\rho u)_x = 0,
$$

subjected to the following initial condition

$$
m(x,0) = m_0(x),
$$
$$
\rho(x,0) = \rho_0(x).
$$

This system can still be interpreted in the context of shallow water waves: $m$ is the momentum related to the fluid velocity $u$ by $m = u - u_{xx}$, $\rho$ is the total depth of the water column, and $g > 0$ is the gravitational constant. This multicomponent generalization of the CH equation has been studied extensively. In [12], Eq. (1.5) was shown to be completely integrable, having a Lax pair formulation and being bi-Hamiltonian. Many authors studied a modified version of (1.5) which supports peakon solutions (see e.g. [15, 37] and references therein). However, a hallmark feature of (1.5) is that the system was shown to be physically relevant in shallow water theory [12], where it is derived from the Green-Naghdi [18] equations using appropriate expansions in terms of physical parameters. In [12], it was also shown that the only way for singularities to occur in solutions is through wave breaking – a similar occurrence for the CH equation. In addition, they were able to establish the global existence of small amplitude solutions of (1.5) and large amplitude traveling wave solutions with initial data that has a sufficient rate of decay at far field. From their investigation, it was determined that unlike the CH equation, the solitary waves generated from (1.5) must be smooth and hence cannot be referred to as
peakon solutions. Similar to the CH equation (1.1), we note that 2CH equation conserves the following quantities,

\[ H_0 = \int_R m \, dx, \quad H_1 = \frac{1}{2} \int_R \left( u^2 + u_x^2 + g \rho^2 \right) \, dx \quad \text{and} \quad H_2 = \int_R \rho \, dx. \]  

(1.7)

While the multicomponent generalization of the CH equation has been studied extensively, the only numerical scheme we are aware of for (1.5) is the multi-symplectic numerical integrator introduced in [9], and the numerical simulations presented in [25].

In this paper, we propose a new conservative finite difference scheme which conserves the quantities \( H_0 \) and \( H_1 \) given in (1.4) for the CH equation and a conservative scheme which conserves three global quantities given in (1.7) for the 2CH equation. Once we show that our proposed schemes have the desired conservation properties, we numerically solve (1.1) and (1.5) under a variety of situations to showcase the merit of using an invariant-preserving scheme to simulate solutions.

The organization of this paper is as follows. The schemes for the CH equation and the preservation of conserved properties are given in Section 2, and numerical tests to validate such schemes are given in Section 3. The invariant-preserving schemes for the two component CH equation are given in Section 4, with corresponding numerical tests given in Section 5. Finally, we end this paper with some concluding remarks.

2 The scheme for the CH equation

2.1 Semi-discrete scheme

To derive a semi-discrete energy-conserving finite difference scheme, we solve (1.1)-(1.2) with periodic boundary conditions on a computational domain \([a,b]\). For simplicity, we discretize the domain using \( N \) equally spaced points of the form \( x_i = a + ih \), where \( i = 1, \cdots, N \) and the step size \( h = \Delta x = (b-a)/N \). We then have that \( m_i(t) \sim m(x_i,t) \) and \( u_i(t) \sim u(x_i,t) \), and the derivatives of \( m \) and \( u \) are approximated in such a way that the total momentum \( H_0 \) and the total energy \( H_1 \) in (1.4) are conserved. To this extent, we propose the following semi-discrete energy-conserving finite difference scheme (ECFD):

\[
\frac{d}{dt} m_i(t) + \frac{m_{i+1}(t)u_{i+1}(t) - m_{i-1}(t)u_{i-1}(t)}{2h} + m_i(t) \frac{u_{i+1}(t) - u_{i-1}(t)}{2h} = 0 \quad 1 \leq i \leq N, \\
m_i(t) = u_i(t) - \frac{u_{i+2}(t) - 2u_i(t) + u_{i-2}(t)}{4h^2} \quad 1 \leq i \leq N. 
\]  

(2.1)

We observe that for certain values of \( i \), \( u_{i+2}, u_{i+1}, \cdots, u_{i-2} \) are not defined. To resolve this issue, we impose the following condition on our velocity \( u \) for any \( i \geq 0 \):

\[ u_{N+i} = u_i, \quad i = -2, \cdots, 2. \]  

(2.2)
We note that this imposition on $u$ will ensure that the boundary conditions associated with the momentum $m$ are periodic. This is true since
\[
    m_{N+1} = u_{N+1} - \frac{u_{N+3} - 2u_{N+1} + u_{N-1}}{4h^2} = u_1 - \frac{u_3 - 2u_1 + u_{-1}}{4h^2} = m_1.
\]

We claim that this newly proposed scheme is energy preserving in the following sense.

**Theorem 2.1.** Consider the numerical approximation of the CH equation (1.1) given by (2.1). Then the total momentum and energy of (2.1) is conserved in the following sense:
\[
    \frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = 0, \quad \text{and} \quad \frac{d}{dt} \left[ \sum_{i=1}^{N} u_i^2(t) + \frac{(u_{i+1}(t) - u_{i-1}(t))^2}{4h^2} \right] = 0, \tag{2.3}
\]
for any $t > 0$.

**Proof.** We begin by showing that $\frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = 0$ as follows. We note that
\[
    \frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = \sum_{i=1}^{N} \frac{d}{dt} m_i(t)
    = \sum_{i=1}^{N} \left[ m_{i-1}(t)u_{i-1}(t) - m_{i+1}(t)u_{i+1}(t) + m_i(t)\frac{u_{i-1}(t) - u_{i+1}(t)}{2h} \right].
\]

Observing that the boundary conditions for $m$ and $u$ are periodic, we have
\[
    \sum_{i=1}^{N} \frac{m_{i-1}(t)u_{i-1}(t) - m_{i+1}(t)u_{i+1}(t)}{2h} = 0.
\]

By a similar argument,
\[
    \sum_{i=1}^{N} m_i(t)\frac{u_{i-1}(t) - u_{i+1}(t)}{2h} = 0.
\]

Thus,
\[
    \frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = 0.
\]

To show that $\frac{d}{dt} E(t) = 0$, where
\[
    E(t) = \left[ \sum_{i=1}^{N} u_i^2(t) + \frac{(u_{i+1}(t) - u_{i-1}(t))^2}{4h^2} \right],
\]
we begin by observing that if we define \( v_i = (u_{i+1} - u_{i-1})/(2h) \) so that \( m_i(t) = u_i(t) - (v_{i+1}(t) - v_{i-1}(t))/(2h) \), then
\[
2 \sum_{i=1}^{N} u_i(t) \dot{m}_i(t) = 2 \sum_{i=1}^{N} u_i(t) u_i(t) - u_i(t) \left( \frac{v_{i+1}(t) - v_{i-1}(t)}{2h} \right)
\]
\[
= \frac{d}{dt} \sum_{i=1}^{N} (u_i^2(t) + v_i^2(t)) = \dot{\mathcal{E}}(t).
\]
(2.4)

Thus, it suffices to show that \( \sum_{i=1}^{N} u_i(t) \dot{m}_i(t) = 0 \). Indeed we have that
\[
\sum_{i=1}^{N} \dot{m}_i(t) u_i(t) = - \sum_{i=1}^{N} \left( \frac{m_{i+1}(t) u_{i+1}(t) - m_{i-1}(t) u_{i-1}(t)}{2h} + m_i(t) \frac{u_{i+1}(t) - u_{i-1}(t)}{2h} \right) u_i(t)
\]
\[
= - \sum_{i=1}^{N} \left( \frac{m_{i+1}(t) u_{i+1}(t) u_i(t) - m_{i-1}(t) u_{i-1}(t) u_i(t)}{2h} + m_i(t) v_i(t) u_i(t) \right)
\]
\[
= - \sum_{i=1}^{N} \left( \frac{m_i(t) u_i(t) u_{i-1}(t) - m_i(t) u_i(t) u_{i+1}(t)}{2h} + m_i(t) v_i(t) u_i(t) \right)
\]
\[
= - \sum_{i=1}^{N} \left( (m_i(t) v_i(t) u_i(t)) + m_i(t) v_i(t) u_i(t) \right)
= 0,
\]
where we have used the periodic boundary conditions on \( m \) obtained by our restriction on \( u \) to shift the sums appropriately. \( \square \)

2.2 Time-discretization

In the previous section, we obtained a semi-discrete finite difference approximation to (1.1). In order to numerically simulate solutions to (1.1), we need a way to evolve the solution in time without destroying the conservation properties. To this extent, we discretize (2.1) implicitly in time as follows:
\[
\frac{m_i^* - m_i^n}{\Delta t/2} + \frac{m_{i+1}^{*} u_{i+1}^{*} - m_{i-1}^{*} u_{i-1}^{*}}{2h} + m_i^{*} \frac{u_{i+1}^{*} - u_{i-1}^{*}}{2h} = 0,
\]
\[
m_i^{n+1} = 2m_i^n - m_i^*,
\]
(2.5)
where \( m_i^n \sim m(x_i, t^n) \), \( t^n = n\Delta t \) and \( i = 1, \ldots, N \). Note that this generates a system of non-linear equations in \( m^* \) and \( u^* \) that one must solve for in order to advance to the next time...
step. We also observe that by using the fact that \( m = u - u_{xx} \), one can rewrite (2.5) in terms of \( m^* \) only. Recalling from (2.1) that
\[
m_i = u_i - \frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2},
\]
we can express the relationship between \( m \) and \( u \) as a linear system \( m = Au \), where \( A \) is given by
\[
A = \begin{bmatrix}
1 + \frac{1}{2h^2} & 0 & -\frac{1}{4h^2} & 0 & \cdots & 0 & -\frac{1}{4h^2} & 0 \\
0 & 1 + \frac{1}{2h^2} & 0 & -\frac{1}{4h^2} & 0 & \cdots & -\frac{1}{4h^2} & 0 \\
-\frac{1}{4h^2} & 0 & 1 + \frac{1}{2h^2} & 0 & -\frac{1}{4h^2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -\frac{1}{4h^2} & 0 & 1 + \frac{1}{2h^2} & 0 & -\frac{1}{4h^2} \\
-\frac{1}{4h^2} & 0 & \cdots & 0 & -\frac{1}{4h^2} & 0 & 1 + \frac{1}{2h^2} & 0 \\
0 & -\frac{1}{4h^2} & 0 & \cdots & 0 & -\frac{1}{4h^2} & 0 & 1 + \frac{1}{2h^2} \\
\end{bmatrix},
\] (2.7)
where the periodic boundary conditions have been incorporated. Since \( A \) is diagonally dominant, it is invertible and one may recover \( u \) from \( m \) by \( u = A^{-1}m \). This allows us to recast (2.5) entirely in terms of \( m^* \) as follows
\[
m_i^{n+1} = m_i^n + \frac{m_i^{n+1} - m_i^{n-1}}{2h} + \frac{2h}{m_i^n - m_i^{n+1}},
\]
(2.8)
We are left with a system of nonlinear equations in \( m^* \) that we can solve via Newton’s method. Our initial guess \( m_i^{0,n} \) is chosen in such a way so that \( \sum m_i^{0,n} = \sum m_i^n \) in order to conserve the total momentum. Once \( m^* \) is known, we may evolve the solution \( m \) and \( u \) to the next time step as described above. One may show that with this time discretization, one maintains the important conservation properties established in the previous section.

**Theorem 2.2.** Consider the time discretization for (2.1) given by (2.8). Then the total momentum and energy of (2.1) is conserved in time in the following sense:
\[
M^n = M^{n+1}, \quad E^n = E^{n+1},
\]
(2.9)
where
\[
M = \sum_{j=1}^{N} m_j \quad \text{and} \quad E = \sum_{j=1}^{N} \left[ u_j^2 + \frac{(u_{j+1} - u_{j-1})^2}{4h^2} \right].
\]
(2.10)
Thus $E_n = M_n^{n+1}$. Note that $m_j^{n+1} = 2m_j^n - m_j^n \implies (m_j^{n+1} + m_j^n)/2 = m_j^n$. Thus, from (2.5), we have that
\[
0 = \frac{m_j^{n+1} - m_j^n}{\Delta t} + \frac{m_{j+1}^n u_{j+1}^n - m_{j-1}^n u_{j-1}^n}{2h} + m_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h}.
\] (2.11)

Summation of (2.11) over $j = 1, \ldots, N$ yields
\[
0 = \sum_{j=1}^N \frac{m_j^{n+1} - m_j^n}{\Delta t} + \frac{m_{j+1}^n u_{j+1}^n - m_{j-1}^n u_{j-1}^n}{2h} + m_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h}
= \sum_{j=1}^N \frac{m_j^{n+1} - m_j^n}{\Delta t}
= \frac{1}{\Delta t} (M^{n+1} - M^n).
\] (2.12)

Thus $M^{n+1} = M^n$. To show that $E^{n+1} = E^n$ we observe that if we take $v_j = (u_{j+1} - u_{j-1})/(2h)$ as before then:
\[
0 = \sum_{j=1}^N \frac{m_j^{n+1} - m_j^n}{\Delta t} u_j^* + \left( \frac{m_{j+1}^n u_{j+1}^n - m_{j-1}^n u_{j-1}^n}{2h} + m_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h} \right) u_j^*
= \sum_{j=1}^N \frac{m_j^{n+1} - m_j^n}{\Delta t} u_j^*
= \sum_{j=1}^N \frac{u_j^{n+1} - u_j^n}{\Delta t} u_j^* - \frac{1}{2h\Delta t} \left( v_j^{n+1} - v_j^n - v_j^{n+1} - v_j^n + v_j^{n+1} + v_j^n \right) u_j^*
= \sum_{j=1}^N \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{v_j^{n+1} - v_j^n}{\Delta t} \left( u_j^{n+1} - u_j^n \right)
= \sum_{j=1}^N \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{v_j^{n+1} - v_j^n}{\Delta t} \left( u_j^{n+1} - u_j^n \right)
= \sum_{j=1}^N \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{v_j^{n+1} - v_j^n}{\Delta t} \left( u_j^{n+1} - u_j^n \right)
= \frac{E^{n+1} - E^n}{2\Delta t}.
\] (2.13)

Thus $E^{n+1} = E^n$, where $E$ is given by (2.10).

\[ \square \]

Remark 2.1. It is worth noting that the time discretization given in (2.8) for (1.1) is the usual implicit midpoint method, which may be called a symplectic integrator when applied to a Hamiltonian system. However, this implicit treatment is more computationally
involved than the explicit approaches, such as a SSP Runge-Kutta Method, c.f. [17]. Nevertheless, we show in the next section that one of the key benefits for using (2.8) is the preservation of \( \|u(t, \cdot)\|_{H^1} \), which in turn yields a better numerical solution as shown in Fig. 4.

3 Numerical examples for the CH equation

3.1 Accuracy test

Recalling that (1.1) admits traveling wave solutions of the form

\[
u(x, t) = ce^{-|x-ct|},
\]

where \( c \) is any real number, if we assume that the initial condition associated with (1.1) is of the form of a single peakon, then (1.1) assumes this traveling wave solution. To test the accuracy of our method, we perform a grid refinement study on our energy-conserving finite difference scheme (ECFD) applied to (1.1). To this extent, we follow the methods of [40] by considering a periodized version of (3.1). In what follows, we consider (3.1) coupled with the following initial condition

\[
u_0(x) = \begin{cases} \frac{c}{\cosh(a/2)} \cosh(x - x_0), & |x - x_0| \leq a/2, \\ \frac{c}{\cosh(a/2)} \cosh(a - (x - x_0)), & |x - x_0| > a/2, \end{cases}
\]

(3.2)

to obtain the exact solution \( u(x, t) = u_0(x - t) \). Here, \( x_0 \) is the position of the trough and \( a \) is the period. The computational domain is taken to be \([0, a]\). We begin with a grid refinement analysis for the ECFD scheme. Below are the results for \( t = 5 \). To generate the numerical results, we note that although we are given \( u_0(x) = u(x, 0) \), we may easily recover \( m(x, 0) = u(x, 0) - u_{xx}(x, 0) \) using the fact that \( u = A^{-1}m \), where \( A \) is given by (2.7). Once \( m(x, 0) \) is known, we apply our ECFD scheme to (1.1) with the initial condition \( m(x, 0) \) by placing \( N = 200, 400, 800, 1600 \) equidistant points in the interval \([0, 30]\) at \( t = 0 \). We then evolve the velocity \( u \) and momentum \( m \) according to (2.1) using the time discretization in (2.8) assuming periodic boundary conditions. In Fig. 1, we see that the ECFD scheme is capable of accurately resolving a single peakon solution associated with the CH equation.

Remark 3.1. We note that once the solution to (1.1) becomes less smooth, numerical oscillations may appear. In order to suppress these oscillations, we propose an adaptive strategy by adding local numerical viscosity as necessary. The form is given as

\[
m^n_i - m^n_{i+1} \Delta t/2 + \frac{m^n_{i+1}u^n_{i+1} - m^n_{i-1}u^n_{i-1}}{2h} + m^n_i u^n_{i+1} - u^n_{i-1} = \frac{\epsilon_i}{2h} (u^n_{i+1} - 2u^n_i + u^n_{i-1}),
\]

(3.3)
Figure 1: The velocity $u$ for the CH equation at $t = 5$, for $N = 200, 400, 800, 1600$ points.

Figure 2: The velocity $u$, total momentum and total energy for the CH equation at $t = 5$, for $N = 400$ points with numerical viscosity.
where
\[ \epsilon_i = \begin{cases} 1, & |u^*_{i+1} - 2u^*_i + u^*_{i-1}| \geq 10^{-2}h \\ 0, & \text{otherwise.} \end{cases} \]

Note that here the factor $10^{-2}$, which may vary for another case, is a threshold to detect where the solution’s slope can become unbounded. In Fig. 2, we show the result of adding numerical viscosity to the single peakon example. In this case, the total energy is no longer preserved but rather is non increasing.

### 3.2 Multi-peakon interaction

In this example, we consider the interaction among three peakons. To this extent, we follow [40] in which we consider the three-peakon interaction of the CH equation with the initial condition
\[ u_0(x) = \phi_1(x) + \phi_2(x) + \phi_3(x), \tag{3.4} \]
where
\[ \phi_i(x) = \begin{cases} \frac{c_i}{\cosh(a/2)} \cosh(x - x_i), & |x - x_i| \leq a/2, \\ \frac{c_i}{\cosh(a/2)} \cosh(a - (x - x_i)), & |x - x_i| > a/2, \end{cases} \tag{3.5} \]
for $i = 1, 2, 3$. The parameters are given by $c_1 = 2$, $c_2 = 1$, $c_3 = 0.7$, $x_1 = -5$, $x_2 = -3$, $x_3 = -1$, $a = 30$. The computational domain is $[0, a]$. Similarly to the accuracy test above, we generate our numerical results by first computing $m(x, 0)$ from $u(x, 0)$. Once $m(x, 0)$ is known, we apply our ECFD scheme to (1.1) by placing $N = 400$ equidistant points in the interval $[0, 30]$ at $t = 0$. We then evolve the velocity $u$ and momentum $m$ according to (2.1) using the time discretization in (2.8) assuming periodic boundary conditions. In Fig. 3, we show the interaction for times $t = 1, 2, 4, 6$. We see that the ECFD performs well in resolving the complex interaction among multiple peakons.

### 3.3 Advantages of Hamiltonian conservation

In this section, we showcase the advantages of conserving the Hamiltonian for the long term propagation of solutions to (1.1). We first compare our semi-discrete ECFD scheme (2.1) with a semi-discrete central-upwind finite volume scheme [31]. In both approaches, we use the time discretization given by the 3rd order SSP Runge-Kutta method (see e.g. [17]), which for solving $\frac{d}{dt}q = L(q)$ is given by
\[
q^{(1)} = q^n + \Delta t L(q^{(n)}), \\
q^{(2)} = \frac{3}{4} q^n + \frac{1}{4} q^{(1)} + \frac{1}{4} \Delta t L(q^{(1)}), \\
q^{(n+1)} = \frac{1}{3} q^n + \frac{2}{3} q^{(2)} + \frac{2}{3} \Delta t L(q^{(2)}). \tag{3.6}
\]
The advantage of conserving the quantities $H_0$ and $H_1$ becomes quite apparent in Fig. 6, where we see that even in short time, the performance of the ECFD scheme is vastly superior to that of the semi-discrete central upwind scheme. In terms of $H_1$-energy, at time $t=100$, the velocity computed from the semi-discrete central upwind scheme yields $\|u(100,\cdot)\|_{H_1} = 19.956125709691534$.

Next we compare the performance of the ECFD scheme with two ways of time-discretization, one is the third-order strong stability preserving (SSP) Runge-Kutta method which does not conserve the $H_0$ and $H_1$ in (1.4), and the other is the second-order time discretization given in (2.8) which does indeed preserve these associated quantities. In Fig. 4, we see that both methods of time discretization perform well in short time. However, as the solitary wave propagates for longer periods of time, the difference in the solutions becomes more pronounced. We observe that the ECFD scheme under the time discretization given in (2.8) performs better than the same scheme using the time discretization in (3.6) as the wave associated with (2.8) moves faster and hence has a corresponding higher wave length. This observation is realized through comparing the scheme’s ability to preserve $\|u(t,\cdot)\|_{H_1}$. Indeed, we observe that $\|u(0,\cdot)\|_{H_1} = 102.1104856085890$, where $u(0,x)$ is the initial velocity over our computational domain. At time $t = 100$, the velocity $u$ computed with the time discretization given in (2.8) yields $\|u(100,\cdot)\|_{H_1} = 102.1104868293117$, which agrees with the initial energy up to the 5th decimal place, see Fig. 5. On the other hand, at time $t = 100$, the velocity $u$ computed using the time discretization in (3.6) yields $\|u(100,\cdot)\|_{H_1} = 99.346623467994519$, which does not agree to any decimal place. These results suggest that the order of convergence in time is less
Figure 4: A comparison of different time discretization approaches for the ECFD scheme applied to a single peakon for the CH equation with periodic boundary conditions on a uniform grid with $N = 400$ at various times.

Figure 5: The total energy for the CH equation solved using our ECFD Scheme.

important than the conservation of the Hamiltonian. These two examples highlight the importance of conserving the quantities $H_0$ and $H_1$ in (1.4) for generating an accurate portrayal of the long-time behavior of solutions to (1.1).

3.4 Peakon-anti-peakon interaction

In this section, we consider the interaction of a peakon and antipeakon (a peakon with a negative initial weight) with the same magnitude. To this extent, we follow [4], by considering an initial velocity of the form:

$$ u(x,0) = p_1e^{-|x-x_1|} + p_2e^{-|x-x_2|}, $$

(3.7)
where \( x_1 = -10 \) and \( x_2 = 10 \) and have momenta of equal magnitude but opposite signs so that the total momentum is zero, i.e., \( p_1 = 1 \) and \( p_2 = -1 \). We move the peakons exactly in time according to (2.5) on the domain \([a,b] = \{-20,20\}\). During the simulation, we note that the total momentum remains zero. At some finite time, \( t^* \), the peakon and antipeakon will collide. Since the total momentum of the system is zero, we expect that the solution will be zero at the collision time \( t^* \). However, due to the inherent symmetry of the problem, \( u(x,t) \rightarrow -u(-x,-t) \), (c.f, [6]), peakons may develop after the collision time and propagate in opposite directions, thus exhibiting an elastic collision indicative of a dissipative solution (c.f, [7] and references therein). This peakon-antipeakon interaction is shown in Fig. 7.

### 4 The scheme for the 2CH equation

#### 4.1 Semi-discrete scheme

To derive a semi-discrete energy-conserving finite difference scheme, we follow a similar approach to that in the previous sections for which we solve (1.5)-(1.6) with periodic boundary conditions on a computational domain \([a,b]\). We discretize the domain using \( N \) equally spaced points of the form \( x_i = a + i\Delta x \), where \( i = 1, \ldots, N \) and \( \Delta x = (b-a)/N \) as before. We then have that \( m_i(t) \sim m(x_i,t) \) and \( u_i(t) \sim u(x_i,t) \) and \( \rho_i(t) \sim \rho(x_i,t) \). We approximate the derivatives of \( m, u \) and \( \rho \) in such a way that the quantities in (1.7) are preserved. To this extent, we propose the following semi-discrete energy-conserving fi-
Figure 7: The peakon-antipeakon interaction of the CH equation with periodic boundary conditions on a uniform grid with $N = 800$ at various times.

difference scheme (ECFD2) for the 2CH equation:

$$\begin{align*}
\frac{dm_i(t)}{dt} + \frac{m_{i+1}(t)u_{i+1}(t) - m_{i-1}(t)u_{i-1}(t)}{2h} + m_i(t)\frac{u_{i+1}(t) - u_{i-1}(t)}{2h} &= P_i(t), \\
\frac{d\rho_i(t)}{dt} + \frac{(u_{i+1}(t) + u_i(t))(\rho_{i+1}(t) + \rho_i(t)) - (u_{i-1}(t) + u_i(t))(\rho_{i-1}(t) + \rho_i(t))}{2h} &= 0, \\
m_i(t) &= u_i(t) - \frac{u_{i+2}(t) - 2u_i(t) + u_{i-2}(t)}{4h^2} = u_i(t) - \frac{v_{i+1}(t) - v_{i-1}(t)}{2h},
\end{align*}$$

where

$$P_i(t) = \frac{g(\rho_{i-1}(t))^2 - g(\rho_{i+1}(t))^2}{4h}$$

and

$$v_i(t) = \frac{u_{i+1}(t) - u_{i-1}(t)}{2h},$$

for $i = 1, \cdots, N$. Similar to the ECFD scheme for the CH equation, we observe that for certain values of $i$, $u_{i+2}, u_{i+1}, \cdots, u_{i-2}$ are not defined. To resolve this issue, we impose the condition given in (2.2) on our velocity $u$ for any $t \geq 0$. We remark that this ensures that...
The boundary conditions on the momentum \( m \) are periodic. We are now in a position to prove that the ECFD2 scheme given in (4.1) conserves quantities in (1.7) in the following sense.

**Theorem 4.1.** Consider the ECFD scheme given in (4.1) with periodic boundary conditions for the 2CH equation (1.5). Then the total momentum, density, and energy \( \mathcal{H}_1 \) of the 2CH equation is conserved in the following sense:

\[
\frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = \frac{d}{dt} \left[ \sum_{i=1}^{N} \rho_i(t) \right] = 0, \quad \text{and} \\
\frac{d}{dt} \left[ \sum_{i=1}^{N} E_2(t) \right] = 0,
\]

where

\[
E_2(t) = \left[ \sum_{i=1}^{N} u_i^2(t) + \frac{(u_{i+1}(t) - u_{i-1}(t))^2}{4h^2} + g\rho_i(t)^2 \right], \quad (4.2)
\]

for any \( t > 0 \).

**Proof.** We begin by showing that \( \frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = 0 \). Using (4.1), our previous results from Theorem 2.1, and the periodicity of our boundary conditions, we observe that

\[
\frac{d}{dt} \left[ \sum_{i=1}^{N} m_i(t) \right] = \sum_{i=1}^{N} \frac{m_{i-1}(t)u_{i-1}(t) - m_{i+1}(t)u_{i+1}(t)}{2h} + m_i(t) \frac{u_{i-1}(t) - u_{i+1}(t)}{2h} + \frac{g\rho_{i-1}^2(t) - g\rho_{i+1}^2(t)}{4h} \\
= \sum_{i=1}^{N} \frac{g\rho_{i-1}^2(t) - g\rho_{i+1}^2(t)}{4h} \\
= 0.
\]

Similarly, we may show that \( \frac{d}{dt} \left[ \sum_{i=1}^{N} \rho_i(t) \right] = 0 \) by noting that

\[
\frac{d}{dt} \left[ \sum_{i=1}^{N} \rho_i(t) \right] = \sum_{i=1}^{N} \frac{(\rho_{i+1}(t) + \rho_{i-1}(t))(u_{i+1}(t) + u_{i-1}(t)) - (\rho_i(t) + \rho_{i+1}(t))(u_i(t) + u_{i+1}(t))}{4h} \\
= \sum_{i=1}^{N} \frac{(\rho_{i+1}(t) + \rho_{i-1}(t))(u_{i+1}(t) + u_{i-1}(t)) - (\rho_i(t) + \rho_{i+1}(t))(u_i(t) + u_{i+1}(t))}{4h} \\
= 0,
\]
where we have once again used the periodicity of our boundary conditions. Finally, we show that \( \frac{d}{dt} \left[ \sum_{i=1}^{N} u_i(t)m_i(t) + g\rho_i^2(t) \right] = \dot{E}_2(t) = 0 \). We begin by observing that by using (4.1), we obtain

\[
\sum_{i=1}^{N} u_i(t) \dot{m}_i(t) + g\rho_i(t) \dot{\rho}_i(t) = \sum_{i=1}^{N} u_i(t) \dot{u}_i(t) - u_i(t) \left( \frac{v_{i+1}(t) - v_{i-1}(t)}{2h} \right) + \frac{g(\rho_i(t))^2}{2}
\]

where we have used the fact that \( \frac{d}{dt} \left[ \sum_{i=1}^{N} u_i(t)m_i(t) + g\rho_i^2(t) \right] = \dot{E}_2(t) = \frac{d}{dt} \left[ \sum_{i=1}^{N} v_i^2(t) + g\rho_i^2(t) \right] = \frac{\dot{E}_2(t)}{2} \),

where we have used the fact that

\[
\sum_{i=1}^{N} \frac{v_i(t)u_{i+1}(t) - u_{i-1}(t)}{2h} = \sum_{i=1}^{N} v_i(t)v_i(t) = \frac{d}{dt} \sum_{i=1}^{N} v_i^2(t) = \frac{\dot{E}_2(t)}{2}.
\]

We also note (using (4.1)) that

\[
\sum_{i=1}^{N} u_i(t) \dot{m}_i(t) + g\rho_i(t) \dot{\rho}_i(t) = \sum_{i=1}^{N} u_i(t) \left( \frac{m_{i-1}(t)u_{i-1}(t) - m_{i+1}(t)u_{i+1}(t)}{2h} \right) - m_i(t) \left( \frac{u_{i+1}(t) - u_{i-1}(t)}{2h} \right) + \frac{g(\rho_{i-1}^2(t)) - g(\rho_{i+1}^2(t))}{4h}
\]

\[
+ \sum_{i=1}^{N} \frac{g(\rho_i(t))}{4h} \left[ (\rho_i(t) + \rho_{i-1}(t))(u_i(t) + u_{i-1}(t)) - (\rho_{i+1}(t) + \rho_i(t))(u_{i+1}(t) + u_i(t)) \right]
\]

\[
= \sum_{i=1}^{N} \frac{g(\rho_{i-1}^2(t))u_i(t) - g(\rho_{i+1}^2(t))u_i(t) + g(\rho_i(t))(\rho_{i+1}(t) + \rho_{i-1}(t))(u_{i+1}(t) + u_{i-1}(t))}{4h}
\]

\[
- \sum_{i=1}^{N} \frac{g(\rho_i(t))(\rho_{i+1}(t) + \rho_{i+1}(t))(u_{i+1}(t) + u_i(t))}{4h}
\]

\[
= \sum_{i=1}^{N} \frac{g(\rho_{i-1}^2(t))u_i(t) - g(\rho_{i+1}^2(t))u_i(t) + g(\rho_i(t))(\rho_{i+1}(t) - \rho_i(t))^2(u_{i+1}(t) + u_i(t))}{4h}
\]

\[
= \sum_{i=1}^{N} \frac{g(\rho_{i-1}^2(t))u_i(t) - g(\rho_{i+1}^2(t))u_i(t) + g(\rho_i(t))(\rho_{i+1}(t) - \rho_i(t))^2(u_{i+1}(t) + u_i(t))}{4h}
\]

Thus, \( \dot{E}_2(t) = 0 \).

### 4.2 Time discretization

From the previous section, we obtained a semi-discrete finite difference approximation for (1.5). In order to numerically simulate solutions to (1.5), we need a way to evolve
the solution in time without destroying the conservation properties given in (1.7). To this extent, we discretize (4.1) implicitly in time as follows:

\[
\frac{m_{j}^{n+1} - m_{j}^{n}}{\Delta t} + \frac{m_{i+1}^{n}u_{i+1}^{n} - m_{i-1}^{n}u_{i-1}^{n}}{2h} + m_{j}^{n}u_{i+1}^{n} - u_{i-1}^{n} = P_{i}^{n},
\]

\[
\frac{\rho_{j}^{n+1} - \rho_{j}^{n}}{\Delta t} + \frac{(u_{i+1}^{n} + u_{i}^{n})(\rho_{i+1}^{n} + \rho_{i}^{n}) - (u_{i-1}^{n} + u_{i}^{n})(\rho_{i-1}^{n} + \rho_{i}^{n})}{2h} = 0,
\]

\[
\frac{m_{j}^{n+1}}{\Delta t/2} + \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2h} = 2m_{j}^{n} - m_{j}^{n},
\]

\[
\rho_{j}^{n+1} = 2\rho_{j}^{n} - \rho_{j}^{n},
\]

where \(P_{i}^{n} = (g(\rho_{i-1}^{n})^2 - g(\rho_{i+1}^{n})^2)/(4h), m_{j}^{n} \sim m(x_{j}, t^{n}), \rho_{j}^{n} \sim \rho(x_{j}, t^{n})\) and \(t^{n} = n\Delta t\) for \(i = 1, \cdots, N\).

Similar to the ECFD scheme for the CH equation, we may recast (4.3) entirely in terms of \(u^{*}\) and \(\rho^{*}\), by recalling that \(m = Au\) given in (1.7). In this form, we may prove the following conservation in time property.

**Theorem 4.2.** Consider the time discretization for (4.1) given by (4.3). Then the total momentum, density and energy of (1.5) are conserved in time in the following sense:

\[
M^{n} = M^{n+1}, \quad D^{n} = D^{n+1}, \quad E_{2}^{n} = E_{2}^{n+1},
\]

where

\[
M = \sum_{j=1}^{N} m_{j}, \quad D = \sum_{j=1}^{N} \rho_{j}, \quad \text{and} \quad E_{2} = \sum_{j=1}^{N} \left[ u_{j}^{2} + \frac{(u_{j+1}^{n} - u_{j-1}^{n})^{2}}{4h^{2}} + g\rho_{j}^{2} \right].
\]

**Proof.** We begin by showing that \(M^{n} = M^{n+1}\). Indeed, we follow a similar process to that for the CH equation by noting that

\[
m_{j}^{n+1} = 2m_{j}^{n} - m_{j}^{n} \implies \frac{m_{j}^{n+1} + m_{j}^{n}}{2} = m_{j}^{n}.
\]

Thus, from (4.3), we have that

\[
\frac{m_{j}^{n+1} - m_{j}^{n}}{\Delta t} + \frac{m_{i+1}^{n}u_{i+1}^{n} - m_{i-1}^{n}u_{i-1}^{n}}{2h} + m_{j}^{n}u_{i+1}^{n} - u_{i-1}^{n} = \frac{P_{i}^{n}}{\Delta t/2} + \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2h} = 0.
\]

If we sum (4.6) over our computational domain, we obtain

\[
\sum_{j=1}^{N}\frac{m_{j}^{n+1} - m_{j}^{n}}{\Delta t} + \frac{m_{i+1}^{n}u_{i+1}^{n} - m_{i-1}^{n}u_{i-1}^{n}}{2h} + \frac{m_{j}^{n}u_{i+1}^{n} - u_{i-1}^{n}}{2h} + \frac{g(\rho_{i+1}^{n})^{2} - g(\rho_{i-1}^{n})^{2}}{4h}
\]

\[
= \sum_{j=1}^{N}\frac{m_{j}^{n+1} - m_{j}^{n}}{\Delta t}
\]

\[
= \frac{M^{n+1} - M^{n}}{\Delta t}
\]

\[
= 0.
\]

(4.7)
Thus, $M^{n+1} = M^n$. In a similar manner, we can show that $D^n = D^{n+1}$ by observing that

$$
\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{(u_{j+1}^* + u_j^*) (\rho_{j+1}^* + \rho_j^*) - (u_{j-1}^* + u_j^*) (\rho_{j-1}^* + \rho_j^*)}{2h} = 0.
$$

(4.8)

If we sum (4.8) over our computational domain, we obtain

$$
\sum_{j=1}^{N} \frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{(u_{j+1}^* + u_j^*) (\rho_{j+1}^* + \rho_j^*) - (u_{j-1}^* + u_j^*) (\rho_{j-1}^* + \rho_j^*)}{2h}
= \sum_{j=1}^{N} \frac{\rho_j^{n+1} - \rho_j^n}{\Delta t}
= \frac{D^{n+1} - D^n}{\Delta t}
= 0.
$$

(4.9)

Thus, $D^{n+1} = D^n$. Finally, to show that $E_2^{n+1} = E_2^n$, we begin with the following observation and use the results in (4.3) as well as (2.13) to obtain

$$
0 = \sum_{j=1}^{N} \frac{m_j^{n+1} - m_j^n}{\Delta t} u_j^* + \left( \frac{m_{j+1}^* u_{j+1}^* - m_j^* u_j^*}{2h} + m_j^* u_{j+1}^* - u_{j+1}^* \right) u_j^* + \frac{g(\rho_{j+1}^*)^2 - g(\rho_{j+1}^*)^2}{4h} u_j^*
+ \sum_{j=1}^{N} \frac{u_j^{n+1} - u_j^n}{\Delta t} u_j^* + \left( \frac{v_{j+1}^{n+1} - v_j^n}{\Delta t} \right) \left( \frac{v_{j+1}^{n+1} + v_j^n}{2} \right) + \frac{g(\rho_{j+1}^*)^2 - g(\rho_{j+1}^*)^2}{4h} u_j^*
+ \sum_{j=1}^{N} \frac{u_j^{n+1} - u_j^n}{\Delta t} u_j^* + \left( \frac{\rho_{j+1}^* + \rho_j^*}{2} \right) \left( \frac{\rho_{j+1}^* + \rho_j^*}{2} \right) \rho_j^*
+ \sum_{j=1}^{N} \frac{u_j^{n+1} - u_j^n}{\Delta t} u_j^* + \left( \frac{v_{j+1}^{n+1} - v_j^n}{\Delta t} \right) \left( \frac{v_{j+1}^{n+1} + v_j^n}{2} \right) + \frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} \left( \frac{\rho_j^{n+1} + \rho_j^n}{2} \right)
= \frac{E_2^{n+1} - E_2^n}{2\Delta t}.
$$

(4.10)

Thus $E_2^{n+1} = E_2^n$, where $E_2$ is given by (4.5).
5 Numerical examples for the 2CH equation

5.1 Accuracy test

In this section, we apply our new ECFD2 scheme to the 2CH equation subjected to the following dam-break initial conditions:

\[ \rho(x,0) = 1 + \tanh(x+4) - \tanh(x-4), \]
\[ u(x,0) = 0, \]

(5.1)

on the interval \([-12\pi, 12\pi]\). Following the authors in [24], the boundary conditions are taken to be periodic with \(g = 1\). In this case, we do not have an exact solution for the 2CH equation with the initial data given by (5.1). Instead, we must construct an appropriate reference solution. To do this, we compare the exact solution of the propagation of a single peakon for the CH equation with reference solutions with different meshes. From here, we observe that using at least 20,000 grid points with a semi discrete central upwind scheme (using a third order SSP method for the time discretization, see (3.6)) is sufficient for generating a reference solution for such a case. We can then use this number of grid points or more for producing reference solutions for other cases when an exact solution is not available. To this regard, a reference solution for the 2CH equation with the initial data given by (5.1) is generated by taking 25,000 grid points using a semi-discrete finite volume method. The computed solutions can then be compared to the reference solution using a spline interpolation. To generate our computed solutions, we begin by first recovering \(m(x,0)\), which is easily done by recalling that \(u = A^{-1}m\), where \(A\) is given by (2.7). Once the initial momentum, \(m(x,0)\), is known, we apply our ECFD2 scheme to (1.5) by placing \(N=200,400,800,1600\) equidistant points in the interval \([-12\pi, 12\pi]\) at \(t=0\). We then evolve the velocity \(u\), the momentum \(m\), and the density \(\rho\) according to (4.1) using the time discretization (4.3) assuming periodic conditions. In Figs. 8-9, we show the behavior of the solution for time \(t=2\). We observe that the ECFD2 scheme performs well in accurately depicting the dam break solution to the 2CH equation. From Tables 1 and 2, we observe what appears to be second order accuracy for our scheme.

<table>
<thead>
<tr>
<th>(N)</th>
<th>Error</th>
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<tr>
<td>100</td>
<td>.1950</td>
<td>–</td>
</tr>
<tr>
<td>200</td>
<td>.0924</td>
<td>1.10</td>
</tr>
<tr>
<td>400</td>
<td>.0295</td>
<td>1.69</td>
</tr>
<tr>
<td>800</td>
<td>.0083</td>
<td>1.84</td>
</tr>
<tr>
<td>1600</td>
<td>.0021</td>
<td>1.98</td>
</tr>
</tbody>
</table>

Table 1: Grid Refinement Analysis, \(||u(x,t) - u_{FD}(x,t)||_{L_{\infty}}||\)

<table>
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<tr>
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<th>Error</th>
<th>Order</th>
</tr>
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<td>.3192</td>
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<td>.1863</td>
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</tr>
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<td>400</td>
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<td>800</td>
<td>.0247</td>
<td>1.68</td>
</tr>
<tr>
<td>1600</td>
<td>.0066</td>
<td>1.94</td>
</tr>
</tbody>
</table>

Table 2: Grid Refinement Analysis, \(||\rho(x,t) - \rho_{FD}(x,t)||_{L_{\infty}}||\)
Figure 8: The velocity $u$ for the 2CH equation at $t=2$, for $N=200,400,800,1600$ points.

Figure 9: The density $\rho$ for the 2CH equation at $t=2$, for $N=200,400,800,1600$ points.
5.2 Preservation of Hamiltonian

In this numerical experiment, we check the preservation of $H_1$ in (1.7) using our ECFD2 scheme proposed in (4.1). To this regard, we follow [9] by considering the following initial condition

$$u_0(x) = e^{-|x|},$$
$$\rho_0(x) = 0.5,$$

(5.2)

where $u_0(x)$ is taken as a form of a peakon on the interval $[0,20]$. We begin by generating our numerical results by first computing $m_0(x)$ (recalling that $m(x,t) = u(x,t) - u_{xx}(x,t)$). Once the initial momentum is known we can then apply our ECFD2 scheme to (1.5) by placing $N = 1200$ equidistant points in the interval $[0,20]$ at time, $t=0$. We then evolve the momentum $m$ and the density $\rho$ according to (4.1) using the time discretization in (4.3) assuming periodic boundary conditions. In Fig. 10, we show the results of the numerical simulation for the velocity $u$, density $\rho$, and the discretized Hamiltonian $H_1$ given in (1.7).

![Figure 10: The velocity $u$, density $\rho$, and the discretized Hamiltonian $H_1$ for the 2CH equation at $t = 5$, for $N = 1200$.](image)

5.3 Peakon anti-peakon interaction

Since the 2CH equation reduces to the CH equation with $\rho \equiv 0$, it is natural to explore the effects of $\rho$ on the peakon-antipeakon interaction given in Section 3.4. To this extent, we
Figure 11: The velocity $u$ for the 2CH equation at various times for $N = 800$ points.

Consider the following initial data:

$$u_0(x) = p_1 e^{-|x - x_1|} + p_2 e^{-|x - x_2|},$$
$$\rho_0(x) = 0.5,$$  \hspace{1cm} (5.3)

where $x_1 = -10$ and $x_2 = 10$ and have momenta of equal magnitude but opposite signs so that the total momentum is zero, i.e., $p_1 = 1$ and $p_2 = -1$. As usual, we begin by first computing $m_0(x)$. Once the initial momentum is known we can then apply our ECFD2 scheme to (1.5) by placing $N = 800$ equidistant points in the interval $[-20, 20]$ at time, $t = 0$. We then evolve the momentum $m$ and the density $\rho$ according to (4.1) using the time discretization in (4.3) assuming periodic boundary conditions. It is known that if $\rho_0 > 0$, then $\rho(t)$ remains strictly positive and is bounded for all time, c.f. [9]. Similar to the peakon-antipeakon interaction given in Section 3.4, we observe an elastic collision among the peakon solutions for a sufficiently large $t$. That is, we obtain a dissipative solution for the 2CH equation. We also observe that the energy $\mathcal{H}_f$ as well as the total momentum and total density are all well preserved as shown in Fig. 13. Finally, we observe a phenomenon first observed in [20], in which if we consider initial data in the form of a peakon and antipeakon coupled with a constant density, then one may observe a considerable part of the total energy $\int u^2 + u_0^2 + \rho^2 \, dx$ being transferred from the term $u^2 + u_0^2$ to $\rho^2$ near the time of collision. We illustrate this phenomenon in Fig. 14. The peakon-antipeakon interaction and the corresponding densities are shown in Figs. 11-12.
Figure 12: The density $\rho$ for the 2CH equation at various times for $N = 800$ points.

Figure 13: The total velocity $u$, total density $\rho$, and the discretized Hamiltonian $H_1$ for the 2CH equation at $t = 10$, for $N = 800$.

6 Concluding remarks

We have developed finite difference schemes for the CH equation and one of its two-component generalizations arising in the context of shallow water wave theory. The
Figure 14: We observe an initial concentration of the part of the energy $H_1$ given by $u^2 + u_x^2$. However, as we approach the time $t$ where the peakon and antipeakon collide (exchange momentum), we note a transfer of energy from $u^2 + u_x^2$ to $\rho^2$.

schemes are shown to preserve both the momentum and total energy of the system. The results in this paper suggest that the preservation of these invariants may be essential to producing numerical solutions which exhibit properties conducive to satisfying long time behavior, thus revealing the desired wave pattern. Numerical examples are shown to illustrate the accuracy for various types of initial data and underline the efficiency to preserve the large-time behavior of the solution. Our future work will extend the schemes presented here to higher order invariants-preserving methods.

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References

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