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GLOBAL DYNAMICS OF BOSE–EINSTEIN CONDENSATION FOR A MODEL OF THE KOMPANEETS EQUATION*

C. DAVID LEVERMORE†, HAILIANG LIU‡, AND ROBERT L. PEGO§

Abstract. The Kompaneets equation describes a field of photons exchanging energy by Compton scattering with the free electrons of a homogeneous, isotropic, nonrelativistic, thermal plasma. This paper strives to advance our understanding of how this equation captures the phenomenon of Bose–Einstein condensation through the study of a model equation. For this model we prove existence and uniqueness theorems for global weak solutions. In some cases a Bose–Einstein condensate will form in finite time, and we show that it will continue to gain photons forever afterward. Moreover we show that every solution approaches a stationary solution for large time. Key tools include a universal super solution, a one-sided Oleinik type inequality, and an L^1 contraction.

Key words. Kompaneets equation, Bose–Einstein condensate, quantum entropy, LaSalle invariance principle

AMS subject classifications. 35K55, 35B40, 35Q85

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1. Introduction.

Photons can play a major role in the transport of energy in a fully ionized plasma through the processes of emission, absorption, and scattering. At high temperatures or low densities, the dominant process can be Compton scattering off free electrons. We make the simplification that the plasma is spatially uniform, isotropic, nonrelativistic, and thermal at temperature T. We also neglect the heat capacity of the photons and assume that T is fixed. If the photon field is also spatially uniform and isotropic, then it can be described by a nonnegative number density $f(x,t)$ over the unitless photon energy variable $x \in (0,\infty)$ given by

$$x = \frac{\hbar |k| c}{k_B T},$$

where $\hbar$ is Planck's constant, c is the speed of light, k_B is Boltzmann's constant, and k is the photon wave vector. Because x is a unitless radial variable, the total photon number and (unitless) total photon energy associated with $f(x,t)$ are then given by

$$N[f] = \int_0^\infty f\, x^2\, dx\,, \qquad E[f] = \int_0^\infty f\, x^3\, dx\,.$$

When the only energy exchange mechanism is Compton scattering of the photons by the free electrons in the plasma, then the evolution of f is governed by the *Kompaneets equation* [20]

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†Department of Mathematics, University of Maryland, College Park, MD 20742-4015 (lvrmr@math.umd.edu).

‡Department of Mathematics, Iowa State University, Ames, IA 50011 (hliu@iastate.edu). The research of this author was supported by grants DMS 0907963 and DMS 1312636.

§Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890 (rpego@cmu.edu). The research of this author was partially supported by the Center for Nonlinear Analysis (CNA) under National Science Foundation PIRE grant OISE-0967140 and by grants DMS 0905723, DMS 121161 and DMS 1515400.

$$\partial_t f = \frac{1}{x^2}\,\partial_x\left[x^4\big(\partial_x f + f + f^2\big)\right]\,. \tag{1.1}$$

This Fokker–Planck approximation to a quantum Boltzmann equation is justified physically by arguing that little energy is exchanged by each photon-electron collision.

Because x is a radial variable, the associated divergence operator has the form $x^{-2}\partial_x x^2$. Thereby we see from (1.1) that the diffusion coefficient in the Kompaneets equation is x^2, which vanishes at $x = 0$. This singular behavior allows the f^2 convection term to drive the creation of a photon concentration at $x = 0$. This hyperbolic mechanism models the phenomenon of Bose–Einstein condensation. Our goal is to better understand how the Kompaneets equation generally describes the process of relaxation to equilibrium over large time and how it captures the phenomenon of Bose–Einstein condensation in particular.

Rather than addressing these questions for the Kompaneets equation (1.1) we will consider the model Fokker–Planck equation

$$\partial_t f = \frac{1}{x^2}\,\partial_x\left[x^4\big(\partial_x f + f^2\big)\right]\,, \tag{1.2}$$

posed over $x \in (0, 1)$ and subject to a zero flux boundary condition at $x = 1$. This model is obtained by simply dropping the f term that appears in the flux of the Kompaneets equation (1.1) and reducing the x-domain to $(0, 1)$. As we will see, this model shares many structural features with the Kompaneets equation. In particular, it shares the x^2 diffusion coefficient and the f^2 convection term that allow the onset of Bose–Einstein condensation. The neglect of the f term in the flux of the Kompaneets equation is a reasonable approximation during the onset of Bose–Einstein condensation when we expect f to be large. The advantage of model (1.2) is that we know some estimates for it that have no known analogues for (1.1) and which facilitate the study of condensate dynamics and equilibration. A disadvantage of (1.2) is that its equilibrium solutions differ from those of (1.1), so we may expect the long-time behavior of its solutions to be similar to that of solutions of (1.1) only in a qualitative sense.

1.1. Structure of the Kompaneets equation. Here we describe some structural features of the Kompaneets equation that will be shared by our model. First, solutions of (1.1) formally conserve total photon number $N[f]$. Indeed, we formally compute that

$$\frac{d}{dt}N[f] = x^4\big(\partial_x f + f + f^2\big)\Big|_0^\infty = 0$$

under the expectation that the flux vanishes as x approaches 0 and ∞. Second, solutions of (1.1) formally dissipate quantum entropy $H[f]$ given by

$$H[f] = \int_0^\infty h(f, x)\, x^2 dx\,, \qquad h(f, x) = f\log(f) - (1 + f)\log(1 + f) + xf\,.$$

Indeed, because

$$\begin{aligned} h_f(f, x) &= \log(f) - \log(1 + f) + x = \log\left(\frac{e^x f}{1 + f}\right)\,, \\ \partial_x h_f &= h_{ff}\partial_x f + 1 = \frac{1}{f(1 + f)}(\partial_x f + f + f^2)\,, \end{aligned}$$

we formally compute that

$$\frac{d}{dt}H[f] = \int_0^\infty h_f(f,x)\,(\partial_t f)\,x^2 dx = -\int_0^\infty x^4 f(1+f)\big(\partial_x h_f(f,x)\big)^2 dx \le 0\,.$$

By this "H theorem," we expect solutions to approach an equilibrium for which $\partial_x h_f(f,x) = 0$. These equilibria have the Bose–Einstein form

$$f = f_\mu(x) = \frac{1}{e^{x+\mu}-1} \qquad \text{for some } \mu \ge 0\,.$$

At this point a paradox arises. The total photon number for the equilibrium f_μ is

$$N[f_\mu] = \int_0^\infty \frac{x^2}{e^{x+\mu}-1}\,dx\,.$$

This is a decreasing function of μ over $[0,\infty)$ and is thereby bounded above by $N[f_0]$, which is finite. Because total number is supposed to be conserved, we expect any solution to relax to an equilibrium f_μ with the same total number as the initial data, satisfying $N[f_\mu] = N[f^{\mathrm{in}}]$. But if the initial number $N[f^{\mathrm{in}}] > N[f_0]$, then no such equilibrium exists!

1.2. Bose–Einstein condensation. The foregoing paradox indicates that there must be a breakdown in the expectations given above. Previous studies ([6] in particular) have shown that a breakdown in the no-flux condition at $x = 0$ can occur. A physical interpretation of a nonzero photon flux at $x = 0$ is that the photon distribution forms a concentration of photons at zero energy (i.e., energy that is negligible on the scales described by the model). This Bose–Einstein condensate accounts for some of the total photon number. See especially the works [31, 3, 6, 12] and the discussion of related literature in subsection 1.4 below. As massless, chargeless particles of integer spin, photons are the simplest bosons. Indeed, S. N. Bose had photons in mind in 1924 when he proposed his new way of counting indistinguishable particles, work soon followed by Einstein's prediction of the existence of the condensate. Yet it was not until 2010 that the first observation of a photon condensate was reported by Martin Weitz and colleagues [19].

In the present context, we can gain insight into this phenomenon by dropping the diffusion term in (1.1), as discussed by Zel'dovich and Levich [31]. In this case the Kompaneets equation simplifies to the first-order hyperbolic equation

$$\partial_t f = \frac{1}{x^2}\,\partial_x\left[x^4\big(f+f^2\big)\right]\,.$$

Letting $n = x^2 f$, this becomes

$$\partial_t n = \partial_x\left[x^2 n + n^2\right]\,, \tag{1.3}$$

whose characteristic equations are

$$\dot{x} = -x^2 - 2n\,, \qquad \dot{n} = 2xn\,.$$

Because $n \ge 0$, the origin $x = 0$ is an outflow boundary, and no boundary condition can be specified there. Clearly any nonzero entropy solution will develop a nonzero flux of photons into the origin in finite time, leading to the formation of a condensate.

The fact the f^2 convection term plays an essential role in the formation of Bose–Einstein condensates is illustrated by considering what happens when that term is dropped from the Kompaneets equation (1.1). This leads to the linear degenerate parabolic equation

$$\partial_t f = \frac{1}{x^2}\,\partial_x\left[x^4\big(\partial_x f + f\big)\right]\,. \tag{1.4}$$

This equation is the analogue of the Kompaneets equation for classical statistics. Its solutions formally conserve $N[f]$ and dissipate the associated entropy

$$H[f] = \int_0^\infty h(f,x)\,x^2 dx\,, \qquad \text{where} \quad h(f,x) = f\log(f) - f + xf\,.$$

Its family of equilibria is

$$f_\mu(x) = e^{-x-\mu} \qquad \text{for some } \mu \in \mathbb{R}\,.$$

The initial-value problem for (1.4) is well-posed in cones of nonnegative densities f such that

$$\int_0^\infty (e^x f)^p e^{-x} x^2 dx < \infty \qquad \text{for some } p \in (1,\infty)\,.$$

Kavian and Levermore showed [18] (see [17]) that these solutions

- are smooth over $\mathbb{R}^+ \times \mathbb{R}^+$,
- are positive over $\mathbb{R}^+ \times \mathbb{R}^+$ provided that f^{in} is nonzero,
- satisfy all the expected boundary conditions,
- conserve $N[f]$ and dissipate $H[f]$ as expected,
- approach f_μ as $t \to \infty$, where $N[f^{\text{in}}] = N[f_\mu]$.

In particular, the no-flux boundary condition is satisfied at $x = 0$ without being imposed! Therefore, no Bose–Einstein concentration happens!

1.3. Present investigation. Of course, solutions of the hyperbolic model (1.3) may develop shocks at any location. However, the diffusion term in the Kompaneets equation (1.1) prevents shock formation for $x > 0$. The results of Escobedo, Herrero, and Velazquez [6] prove that the degeneracy of its diffusion does *not* prevent shock formation at $x = 0$. These authors proved that there exist solutions of (1.1) that are regular and satisfy no-flux conditions for t on a bounded interval $0 < t < T_c$ (which is solution-dependent), but at time $t = T_c$ the flux at $x = 0$ becomes nonzero. Such solutions exist for an arbitrarily small initial photon number. Moreover, global existence and uniqueness of solutions of (1.1) was proved subject to a boundedness condition for $x^2 f$ for $x \in [0, 1]$.

A number of interesting questions about solutions to the Kompaneets equation remain unanswered by previous studies: What happens to a condensate once it forms? Can it lose photons as well as gain them? Are there any boundary conditions at all that we can impose near $x = 0$ that yield different condensate dynamics, allowing the condensate to interact with other photons? Can we identify the long-time limit of any initial density of photons?

In order to focus clearly on these questions, we have found it convenient to drop the linear term $x^2 f$ from the Kompaneets flux and consider the model equation (1.2), which retains the essential features of nonlinearity and degenerate diffusion. The equilibria of (1.2) are

$$f_\mu(x) = \frac{1}{x+\mu} \qquad \text{for some } \mu \geq 0\,. \tag{1.5}$$

We include these solutions in the class of functions considered by restricting our attention to the interval $0 < x < 1$ and imposing a no-flux boundary condition at $x = 1$. For these equilibria the maximal total photon number is $N[f_0] = \frac{1}{2}$.

For this model problem, we shall assemble a fairly detailed description of well-posedness and long-time dynamics. We establish existence and uniqueness in a natural class of nonnegative weak solutions for initial data that simply has *some* finite moment

$$\int_0^\infty x^p f^{\text{in}}\, dx\,, \quad p \geq 2\,.$$

These results are proved with essential use of estimates for hyperbolic (first-order) equations and establish that while the model Kompaneets equation (1.4) is parabolic for $x > 0$, the point $x = 0$ remains an *outflow* boundary at which no boundary condition can be specified.

The solution map is *nonexpansive* in L^1 norm with weight x^2. Therefore the total photon number $N[f(t)]$ is nonincreasing in time. A condensate can gain photons but never lose them and must form in finite time whenever $N[f^{\text{in}}] > N[f_0]$. Moreover, once it starts growing it never stops. Every solution relaxes to some equilibrium state f_μ in the long-time limit $t \to \infty$. We cannot identify the limiting state in general, but the solution must approach the maximal steady state f_0 if the initial data $f_{\text{in}} \geq f_0$ everywhere.

The proofs of the results on long-time behavior are greatly facilitated by two features of the model problem (1.2). First, the problem admits a *universal supersolution* f_{super} determined by

$$x^2 f_{\text{super}}(x,t) = x + \frac{1-x}{t} + \frac{2}{\sqrt{t}}\,. \tag{1.6}$$

By consequence, for every solution, $x^2 f$ is in fact bounded in x for each $t > 0$, and moreover one has $\limsup_{t\to\infty} x^2 f(x,t) \leq x = x^2 f_0(x)$ for every solution, for example. Also, every solution satisfies

$$\partial_x(x^2 f) \geq -\frac{4}{t}\,,$$

which is Oleinik's inequality for admissible solutions of the conservation law (1.3) after dropping the linear flux term $x^2 n$.

1.4. Literature on related problems. As indicated above, the Kompaneets equation is derived from a Boltzmann–Compton kinetic equation for photons interacting with a gas of electrons in thermal equilibrium—see [20], and especially [8] for a derivation and links to some of the physical literature. Regarding the analysis of the Boltzmann–Compton equation itself, when a simplified regular and bounded kernel is adopted, Escobedo and Mischler [7] studied the asymptotic behavior of the solutions and showed that the photon distribution function may form a condensate at zero energy asymptotically in infinite time. Further, Escobedo, Mischler, and Velazquez [9] showed that the asymptotic behavior of solutions is sensitive not only to the total mass of the initial data but also to its precise behavior near the origin. In some cases, solutions develop a Dirac mass at the origin for long times (in the limit $t \to \infty$) in a self-similar manner. For the Boltzmann–Compton equation with a physical kernel, some results concerning both global existence and nonexistence, depending on the size of initial data, were obtained by Ferrari and Nouri [11].

A natural question is whether results analogous to those obtained in the present paper concerning the development of condensates may hold for other kinetic equations that govern boson gases, such as Boltzmann–Nordheim (aka Uehling–Uhlenbeck) quantum kinetic equations. Concerning these issues we refer to the studies of Semikoz and Tkachev [28, 27] and Lacaze et al. [22], the analyses of Spohn [29] and Lu [23], the recent analysis of blowup and condensation formation by Escobedo and Velazquez [10], and references cited therein. Higher-order Fokker–Planck-type approximations to the Boltzmann–Nordheim equation were derived formally by Josserand, Pomeau, and Rica [12], and an analysis of the behavior of solutions has been performed recently by Jüngel and Winkler [13, 14].

Bose–Einstein equilibria and condensation phenomena also appear in classical Fokker–Planck models that incorporate a quantum-type exclusion principle [15, 16]. Concerning mathematical results on blowup and condensates for these models, we refer to work of Toscani [30] and Carrillo, Di Francesco, and Toscani [4] and references therein.

1.5. Plan of the paper. In section 2 we introduce our notion of weak solutions for (1.2) together with relevant notations, followed by precise statements of the main results and a discussion of related literature. In section 3 we prove the uniqueness of weak solutions for initial data with some finite moment. Existence is proved in section 4 by passing to the limit in a problem regularized by truncating the domain away from $x = 0$.

In section 5 we establish that condensation must occur if the initial photon number $N[f_{\mathrm{in}}] > N[f_0]$, and we show that once a shock forms at $x = 0$ in finite time, it will persist and continue growing for all later time. Large time convergence to equilibrium is proved for every solution in section 6, using arguments related to LaSalle's invariance principle.

The paper concludes with three appendices that deal with several technical but less central issues. A simple, self-contained treatment of some anisotropic Sobolev embedding estimates used in our analysis is contained in Appendix A. The truncated problem used in section 4 requires a special treatment due to the fact that the zero-flux boundary condition at $x = 1$ is nonlinear—this treatment is carried out in Appendix B. A proof of interior regularity of the solution, sufficient to provide a classical solution away from $x = 0$ but up to the boundary $x = 1$, is established in Appendix C.

2. Main results.

2.1. Model initial-value problem. In light of the foregoing discussion, it is convenient to work with the densities

$$n = x^2 f\,, \qquad n^{\mathrm{in}} = x^2 f^{\mathrm{in}}\,. \tag{2.1}$$

The flux in our model equation (1.2) can be expressed as

$$J = x^2 \partial_x n + n^2 - 2xn\,. \tag{2.2}$$

The initial-value problem for our model equation (1.2) that we will consider is

$$\partial_t n - \partial_x J = 0\,, \qquad 0 < x < 1\,,\ t > 0\,, \tag{2.3a}$$

$$J(1,t) = 0\,, \qquad t > 0\,, \tag{2.3b}$$

$$n(x,0) = n^{\mathrm{in}}(x), \qquad 0 < x < 1\,. \tag{2.3c}$$

Here we have imposed the no-flux boundary condition at $x = 1$ but do not impose any boundary condition at $x = 0$, where the diffusion coefficient x^2 vanishes.

We work with a weak formulation of the initial-value problem (2.3). We require the initial data $n^{\rm in}$ to satisfy

$$n^{\rm in} \geq 0\,, \qquad x^p n^{\rm in} \in L^1((0,1]) \quad \text{for some } p \geq 0\,. \tag{2.4}$$

Let $Q = (0,1] \times (0,\infty)$. We say n is a *weak solution* of the initial-value problem (2.3) if

$$n \geq 0\,, \qquad n\,,\ \partial_x n \in L^2_{\rm loc}(Q)\,, \tag{2.5a}$$

$$x^p n \in L^1((0,1] \times (0,T)) \quad \text{for every } T > 0\,, \tag{2.5b}$$

$$n(\cdot,t) \to n^{\rm in} \quad \text{in } L^1_{\rm loc}((0,1]) \text{ as } t \to 0^+\,, \tag{2.5c}$$

$$\int_Q \big(n\,\partial_t\psi - J\,\partial_x\psi\big)\,dX = 0 \quad (dX = dx\,dt) \tag{2.5d}$$

for every C^1 test function ψ with compact support in Q. Condition (2.5a) is needed to make sense of the weak formulation (2.5d). Condition (2.5b) is an admissibility condition we need to establish uniqueness. Condition (2.5c) gives the sense in which the initial data is recovered.

2.2. Uniqueness, existence, and regularity. The following results establish the basic uniqueness, existence, and regularity properties of weak solutions to (2.3). Henceforth we will use $N[n]$ to denote the total photon number,

$$N[n] = \int_0^1 n\,dx\,,$$

replacing the earlier notation $N[f]$. We will also denote the positive part of a number a by $a_+ = \max\{a,0\}$.

THEOREM 1 (stability and comparison). *Let $n^{\rm in}$ and $\bar n^{\rm in}$ satisfy* (2.4) *for some $p \geq 0$. Let n and $\bar n$ be weak solutions of* (2.3) *associated with the initial data $n^{\rm in}$ and $\bar n^{\rm in}$, respectively, as defined by* (2.5). *Set $c_p = p(p+3)$. Then*

$$\int_0^1 x^p (n-\bar n)_+(x,t)\,dx \leq e^{c_p t} \int_0^1 x^p (n^{\rm in} - \bar n^{\rm in})_+\,dx \quad \text{a.e. } t > 0\,. \tag{2.6}$$

Furthermore, if $n^{\rm in} \geq \bar n^{\rm in}$ a.e. on (0,1), *then $n \geq \bar n$ a.e. on Q. In particular, if $n^{\rm in} = \bar n^{\rm in}$ a.e. on* (0,1), *then $n = \bar n$ a.e. on Q.*

From (2.6) we draw immediately the following conclusion on uniqueness.

COROLLARY 2 (uniqueness). *Let n and $\bar n$ be two weak solutions to (2.3), subject to initial data $n^{\rm in}$, $\bar n^{\rm in}$, respectively, with $x^p n^{\rm in}, x^p \bar n^{\rm in} \in L^1((0,1])$. Then*

$$\int_0^1 x^p |n(x,t) - \bar n(x,t)|\,dx \leq e^{c_p t} \int_0^1 x^p |n^{\rm in} - \bar n^{\rm in}|\,dx \quad \text{a.e. } t > 0. \tag{2.7}$$

For each initial data $n^{\rm in}$ satisfying $x^p n^{\rm in} \in L^1(0,1)$ for some $p \geq 0$, there exists at most one weak solution of (2.3).

Remark 3. Because $c_0 = 0$, if (2.4) holds with $p = 0$, then (2.7) is the L^1 contraction property

$$\|(n - \bar n)(t)\|_{L^1(0,1)} \leq \|n^{\rm in} - \bar n^{\rm in}\|_{L^1(0,1)}\,. \tag{2.8}$$

In particular, the total photon number $N[n]$ is nonincreasing in time.

THEOREM 4 (existence and global bounds). *Let n^{in} satisfy* (2.4) *for some $p \geq 0$. Then there exists a unique global weak solution n of* (2.3) *as defined by* (2.5). *Moreover $x^p n \in C([0,\infty); L^1(0,1))$ and we have the following bounds:*

(i) *(A universal upper bound) For every $t > 0$,*

$$n \leq x + \frac{1-x}{t} + \frac{2}{\sqrt{t}} \qquad \textit{for a.e. } x \in (0,1)\,,$$

(ii) *(Oleinik-type inequality) For a.e. $(x,t) \in Q$,*

$$\partial_x n \geq -\frac{4}{t}\,.$$

(iii) *(Energy estimate) $n \in C((0,\infty), L^2(0,1))$, and whenever $0 < s < t$,*

(2.9)
$$\int_0^1 n^2(x,t)\,dx + \int_s^t \int_0^1 [n^2 + x^2(\partial_x n)^2]\,dx\,d\tau \leq \int_0^1 n^2(x,s)\,dx + \frac{8}{3}(t-s)\,.$$

Note that the Oleinik-type inequality allows for the formation of "shock waves" in n at $x = 0$ but rules out oscillations.

THEOREM 5 (regularity away from $x = 0$). *For the global weak solution n from Theorem* 4, *the quantities n, $\partial_x n$, $\partial_t n$, and $\partial_x^2 n$ are locally Hölder-continuous on Q. Furthermore, n is smooth in the interior of Q.*

2.3. Dynamics of solutions. Next we state our main results concerning the formation of condensates and the large time behavior of solutions. Observe that the bounds in (i) and (ii) of Theorem 4 imply the existence of the right limit $n(0^+,t)$ for each $t > 0$.

THEOREM 6 (formation and growth of condensates). *Let n^{in} satisfy* (2.4) *for some $p \geq 0$. Let n be the unique global weak solution to* (2.3) associated with n^{in}. Then

(i) (Conservation of photons) For every $t > s > 0$ we have

$$\int_0^1 n(x,t)\,dx = \int_0^1 n(x,s)\,dx - \int_s^t n(0^+,\tau)^2\,d\tau\,.$$

(ii) (Persistence) There exists $t_* \in [0,\infty]$ such that $n(0^+,t) > 0$ whenever $t > t^*$ and $n(0^+,t) = 0$ whenever $0 \leq t < t_*$.

(iii) (Formation) If $N[n^{\text{in}}] > \frac{1}{2}$, then $n(0^+,t) > 0$ whenever

$$\frac{1}{2\sqrt{t}} < \sqrt{1+\delta} - 1\,, \quad \text{where} \quad 2\delta = N[n^{\text{in}}] - \frac{1}{2}\,.$$

(iv) (Absence) If $n^{\text{in}} \leq x$, then $t_* = \infty$. That is, for every $t > 0$ we have $n(0^+,t) = 0$ and $N[n(\cdot,t)] = N[n^{\text{in}}]$.

The formula in part (i) justifies a physical description of the photon energy distribution that contains a Dirac delta mass at $x = 0$, corresponding to a condensate of photons at zero energy that keeps total photon number conserved. By the formula in part (i), the quantity

$$\int_s^t n(0^+,\tau)^2\,d\tau$$

is the number of photons that have entered the condensate between times s and t. This quantity is nonnegative, meaning the condensate behaves like a "black hole"—photons go in but do not come out. Part (ii) shows that a condensate never stops growing once it starts. Part (iii) states that a condensate must develop in finite time for any initial data $n^{\rm in}$ with more photons than the maximal equilibrium $n_0 = x$. Part (iv) means that for initial data bounded above by n_0, a condensate does not form.

According to Theorem 1 and Corollary 2, however, we see that the notion of a condensate is not strictly required for mathematically discussing existence and uniqueness. The solution is determined by the conditions imposed for $x \in (0,1]$, and it so happens that the total photon number can decrease due to an outward flux at $x = 0^+$.

THEOREM 7 (large time convergence). *Let $n^{\rm in}$ satisfy* (2.4) *for some $p \geq 0$. Let n be the unique global weak solution to* (2.3) *associated with $n^{\rm in}$. Then there exists $\mu \geq 0$ such that*

$$\lim_{t\to\infty} \|n(\cdot,t) - n_\mu\|_1 = 0\,, \qquad \textit{where} \quad n_\mu(x) = \frac{x^2}{x+\mu}\,.$$

The equilibrium n_μ to which a solution converges depends not only on $N[n^{\rm in}]$ but also on details of $n^{\rm in}$. In some special cases, μ can be explicitly determined.

COROLLARY 8. *Let n be the global solution to* (2.3)*, subject to initial data satisfying $n^{\rm in}(x) \geq x$ for $x \in (0,1]$. Then*

$$\lim_{t\to\infty} n(x,t) = n_0(x) = x.$$

Moreover,

$$|n(x,t) - x| \leq \frac{1}{t} + \frac{2}{\sqrt{t}} \qquad \textit{for every } t > 0\,. \tag{2.10}$$

If $n^{\rm in}(x) \leq x$ for $x \in (0,1]$, then

$$\lim_{t\to\infty} n(x,t) = n_\mu(x) = \frac{x^2}{x+\mu}$$

with μ uniquely determined by the relation

$$N[n^{\rm in}] = N[n_\mu] = \frac{1}{2} - \mu + \mu^2 \log\left(1 + \frac{1}{\mu}\right)\,. \tag{2.11}$$

Remark 9. For the model equation (1.2), these results provide a definite answer to the main issues of concern. The main assertions are expected to hold true for the full Kompaneets equation (1.1) and may be partially true for some extensions of the Kompaneets equation [26, 5]. Theorem 1 and Corollary 2 improve upon Theorems 1 and 2 of [6, p. 3839] for the Kompaneets equation, in the sense that we impose no fixed growth condition near $x = 0$. However for a model equation, Theorems 6 and 7 provide a theoretical justification of observations made previously, including the detailed singularity analysis given in [6], the self-similar blowup of the Kompaneets equation's solution in finite time [12], as well as the classical result of Zel'dovich and Levich [31] on shock waves in photon spectra.

Remark 10. We remark that the quantum entropy defined by

$$H[n] = \int_0^1 [xn - x^2 \log(n)]\, dx$$

satisfies

$$H[n(t)] + \int_0^t \int_0^1 n^2 \left(1 - \partial_x \left(\frac{x^2}{n}\right)\right)^2 dx \le H[n^{\text{in}}] \;\forall t > 0,$$

provided $H[n^{\text{in}}] < \infty$. As we have no need for this entropy dissipation inequality in this paper, we omit the proof. We mention, however, that the entropy $H[n]$ is not sensitive to the presence of the Bose–Einstein condensate.

3. Uniqueness of weak solutions. This section is primarily devoted to the proof of Theorem 1. At the end of this section we include an additional result, a strict L^1 contraction property, which will be used in section 6.

3.1. Proof of Theorem 1. Let $w = n - \bar{n}$ and $\hat{n} = n + \bar{n} - 2x$. Then from (2.5d) for both n and $\bar{n}$ it follows that

$$\int_Q (w\,\partial_t\psi - (x^2\partial_x w + \hat{n}w)\,\partial_x\psi) = 0. \tag{3.1}$$

The estimate (2.6) can be derived formally by using a test function of form $\psi = x^p \mathbb{1}_{[0,t]} H(w)$, where $H(w)$ is the usual Heaviside function and $\mathbb{1}_E$ is the characteristic function of a set E. This is not an admissible test function, however, and instead we need several approximation steps.

For use below, we fix a smooth, nondecreasing cut-off function $\chi : \mathbb{R} \to [0,1]$ with the property $\chi(x) = 0$ for $x \le 1$, $\chi(x) = 1$ for $x \ge 2$ and set $\chi_\epsilon(x) = \chi(x/\epsilon)$ for $\epsilon > 0$. For any interval $I \subset [0,\infty)$ we define the space-time domains

$$Q_I = (0,1] \times I, \qquad \text{so}\;\; Q = Q_{(0,\infty)} = (0,1] \times (0,\infty). \tag{3.2}$$

1. (Steklov average in t.) For $h \ne 0$, the Steklov average u_h of a continuous function u on Q is defined by extending $u(x,t)$ to be zero for $t < 0$ and setting

$$u_h(x,t) = \frac{1}{h}\int_t^{t+h} u(x,s)\,ds, \qquad (x,t) \in Q.$$

By density arguments, the Steklov average extends to an operator with the following properties. First, for $1 \le p < \infty$, if $u \in L^p_{\text{loc}}(Q)$, then $u_h \in L^p_{\text{loc}}(Q)$ with weak derivative

$$\partial_t u_h = \frac{u(\cdot,\cdot+h) - u(\cdot,\cdot)}{h} \in L^p_{\text{loc}}(Q).$$

Moreover, one has $u_h \to u$ in $L^p_{\text{loc}}(Q)$ as $h \to 0$.

Since

$$n, \bar{n} \in B_+ := \{n \mid n,\; \partial_x n \in L^2_{\text{loc}}(Q) \text{ with } n \ge 0\},$$

it follows $\partial_x^j \partial_t^k w_h \in L^2_{\text{loc}}(Q)$ for $j,k = 0,1$, whence w_h is continuous on Q. If ψ is a C^1 test function with compact support in Q, the same is true for ψ_{-h} if $|h|$ is sufficiently small, and a simple calculation with integration by parts and justified by density of smooth functions shows that

$$\int_Q w\,\partial_t(\psi_{-h}) = \int_Q w(\partial_t\psi)_{-h} = \int_Q w_h\partial_t\psi = -\int_Q (\partial_t w_h)\psi.$$

Substitution of this into (3.1) and treating the other term similarly, one finds

$$\int_Q \Big((\partial_t w_h)\psi + (x^2\partial_x w + \hat{n}w)_h\,\partial_x\psi\Big) = 0. \tag{3.3}$$

Recall $n, \bar{n} \in B_+$, hence $\hat{n}w$ and $\partial_x(\hat{n}w)$ are in $L^1_{\rm loc}(Q)$, whence $(\hat{n}w)_h$ is continuous in Q. By replacing $\psi(x,t)$ by $\psi(x,t)\chi_\epsilon(t-\sigma)\chi_\epsilon(\tau-t)$ and taking $\epsilon \to 0$ using dominated convergence, we find that for any C^1 function ψ with compact support in $Q_{[0,\infty)}$,

$$\int_{Q_{[\sigma,\tau]}} \Big((\partial_t w_h)\psi + (x^2\partial_x w + \hat{n}w)_h\, \partial_x \psi \Big) = 0 \qquad \text{whenever } [\sigma,\tau] \subset (0,\infty). \tag{3.4}$$

By approximation, (3.4) holds for any $\psi \in W^{1,2}(Q)$ supported in $Q_{[0,\infty)}$.

2. (Integrate in t.) Define $\zeta(a) = \int_0^a \chi_{\bar\epsilon}(u)\,du$ as a smooth, convex approximation to the function $a \mapsto a_+$. Since ζ is Lipshitz with $\zeta'(a) = 1$ for $a > 2\bar\epsilon$, the composition $\zeta(w_h) \in W^{1,2}_{\rm loc}(Q)$, with the weak derivatives

$$\partial_t \zeta(w_h) = \zeta'(w_h)\partial_t w_h, \quad \partial_x \zeta(w_h) = \zeta'(w_h)\partial_x w_h.$$

We may now set $\psi(x,t) = \eta(x)(\zeta' \circ w_h)(x,t)$ in (3.4), where η is any C^2 function with compact support in $(0,1]$. In what follows, we assume also that $\eta \ge 0$ and $\eta' \ge 0$ on $(0,1]$. The function $t \mapsto \int_0^1 \eta(x)\zeta(w_h(x,t))\,dx$ is absolutely continuous for $t > 0$, with

$$\int_0^1 \eta(x)\zeta(w_h(x,t))\,dx \Big|_{t=\sigma}^{t=\tau} = \int_{Q_{[\sigma,\tau]}} \eta\zeta'(w_h)\partial_t w_h = -\int_{Q_{[\sigma,\tau]}} (x^2\partial_x w + \hat{n}w)_h\, \partial_x(\eta\zeta'(w_h)) \tag{3.5}$$

whenever $[\sigma,\tau] \subset (0,\infty)$.

3. (Take $h \to 0$.) As $h \to 0$, the hypotheses for weak solutions imply that n, $\bar{n} \in L^1_{\rm loc}(Q_{[0,\infty)})$. By consequence, in $L^1_{\rm loc}([0,\infty))$ we have

$$\int_0^1 \eta(x)\zeta(w_h(x,\cdot))\,dx \to \int_0^1 \eta(x)\zeta(w(x,\cdot))\,dx. \tag{3.6}$$

In fact, we will show that the right-hand side here is absolutely continuous for $t > 0$, from studying the terms on the right-hand side of (3.5). First, as $h \to 0$, in $L^2_{\rm loc}(Q)$ we have

$$x^2\partial_x w_h \to x^2\partial_x w. \tag{3.7}$$

And along a subsequence $h_j \to 0$, $\zeta'(w_h) \to \zeta'(w)$ and $\zeta''(w_h) \to \zeta''(w)$ boundedly a.e. on compact subsets of Q. Hence in $L^2_{\rm loc}(Q)$ we have

$$\partial_x(\eta\,\zeta'(w_h)) = \eta'\zeta'(w_h) + \eta\,\zeta''(w_h)\partial_x w_h \quad \to \quad \eta'\zeta'(w) + \eta\,\zeta''(w)\partial_x w.$$

Since $\zeta'(w)\partial_x w = \partial_x \zeta(w)$, we find

$$\int_{Q_{[\sigma,\tau]}} (x^2\partial_x w_h)\partial_x(\eta\,\zeta'(w_h)) \to \int_{Q_{[\sigma,\tau]}} \Big(x^2\eta'\partial_x\zeta(w) + x^2\eta\,\zeta''(w)(\partial_x w)^2 \Big). \tag{3.8}$$

Next we deal with the nonlinear term in (3.5). Observe

$$\int_{Q_{[\sigma,\tau]}} (\hat{n}w)_h \partial_x(\eta\,\zeta'(w_h))$$
$$= \int_0^\tau \eta\,\zeta'(w_h)(\hat{n}w)_h(1,t)\,dt - \int_{Q_{[\sigma,\tau]}} \eta\,\zeta'(w_h)(\hat{n}\partial_x w + w\partial_x \hat{n})_h. \tag{3.9}$$

Because $n, \bar{n} \in B_+$ we have $\hat{n}w$, $\partial_x(\hat{n}w) \in L^1_{\text{loc}}(Q)$, and it follows that $(\hat{n}w)_h(1,\cdot) \to \hat{n}w(1,\cdot)$ in $L^1_{\text{loc}}((0,\infty))$. Moreover, $w(1,\cdot) \in L^2_{\text{loc}}((0,\infty))$ and $\zeta'(w_h(1,\cdot)) \to \zeta'(w(1,\cdot))$ boundedly a.e. on compact subsets of $(0,\infty)$ along a sub-subsequence of $h \to 0$. Thus we may pass to the limit on the right-hand side of (3.9), integrate back by parts, and infer that the limit is

$$\begin{aligned} &\int_\sigma^\tau \eta\,\zeta'(w)(\hat{n}w)(1,t)\,dt - \int_{Q_{[\sigma,\tau]}} \eta\,\zeta'(w)(\hat{n}\partial_x w + w\partial_x \hat{n}) \\ &\qquad = \int_{Q_{[\sigma,\tau]}} (\hat{n}w)(\eta'\zeta'(w) + \eta\,\zeta''(w)\partial_x w). \end{aligned} \tag{3.10}$$

To justify this equality requires an additional argument that involves approximating n and $\bar{n}$ by smooth functions in B_+: The equality is true for the smooth approximations, and one may pass to the limit along subsequences by essentially the same arguments as were used to take $h \to 0$. Note that direct integration by parts does not work—the intermediate term $\int_{Q_{[\sigma,\tau]}} (\hat{n}w)\partial_x(\eta\,\zeta'(w))$ does not make sense with only the regularity assumed for w, due to insufficient integrability in time (L^1 for one factor, L^2 for the other. On the right-hand side of (3.10), however, $\zeta'(w)$ and $w\zeta''(w)$ are bounded on the support of the integrand, and one can pass to the limit a.e. along subsequences from smooth approximations to deduce (3.10).

In sum, we find that in $L^1_{\text{loc}}((0,\infty))$ and for a.e. $\tau > \sigma > 0$,

$$\begin{aligned} \int_0^1 \eta(x)\zeta(w(x,t))\,dx\Big|_{t=\sigma}^{t=\tau} &= -\int_{Q_{[\sigma,\tau]}} \Big(x^2\eta'\partial_x\zeta(w) + x^2\eta\,\zeta''(w)(\partial_x w)^2\Big) \\ &\quad - \int_{Q_{[\sigma,\tau]}} (\hat{n}w)(\eta'\zeta'(w) + \eta\,\zeta''(w)\partial_x w). \end{aligned} \tag{3.11}$$

4. (Take $\bar{\epsilon} \to 0$.) Note that since $\zeta(w)$, $\partial_x\zeta(w) \in L^2_{\text{loc}}(Q)$, we have $\zeta(w(1,\cdot)) \in L^2_{\text{loc}}((0,\infty))$ and

$$\begin{aligned} -\int_{Q_{[\sigma,\tau]}} x^2\eta'\partial_x\zeta(w) &= -\int_\sigma^\tau \eta'(1)\zeta(w(1,t))\,dt + \int_{Q_{[\sigma,\tau]}} \zeta(w)\partial_x(x^2\eta') \\ &\le \int_{Q_{[\sigma,\tau]}} \zeta(w)\partial_x(x^2\eta'). \end{aligned}$$

Since $\eta, \eta' \ge 0$, $\hat{n} \ge -2x$, and $w\zeta'(w) \ge 0$, therefore

$$\int_0^1 \eta(x)\zeta(w(x,t))\,dx\Big|_{t=\sigma}^{t=\tau} \le \int_{Q_{[\sigma,\tau]}} \Big(\zeta(w)\partial_x(x^2\eta') + 2x\eta' w\zeta'(w) - \hat{n}w\eta\,\zeta''(w)(\partial_x w)\Big). \tag{3.12}$$

Now we take the limit $\bar{\epsilon} \downarrow 0$, for which we have $\zeta \circ w \uparrow w_+$ and $w(\zeta' \circ w) \uparrow w_+$ pointwise. Moreover, $w\zeta'' \circ w = (w/\bar{\epsilon})\chi'(w/\bar{\epsilon})$ is bounded and converges to zero a.e. Since $\eta\hat{n}\partial_x w \in L^1(Q)$, by dominated convergence the last term in (3.12) tends to zero, and we derive

$$\int_0^1 \eta\, w_+(x,\tau)\,dx \le \int_0^1 \eta\, w_+(x,\sigma)\,dx + \int_{Q_{[\sigma,\tau]}} (x^2\eta'' + 4x\eta')w_+ \tag{3.13}$$

for a.e. $\tau > \sigma > 0$. Due to assumption (2.5c) on weak solutions, now we can take $\sigma \to 0$ and conclude that this inequality holds also with $\sigma = 0$.

5. (Make Gronwall estimate.) Finally, we take η of the form $\eta(x) = x^p\chi_\epsilon(x)$, where $p \geq 0$ is the exponent for which we assume $x^p n^{\rm in} \in L^1(0,1)$. Observe that

$$x\eta' = x^p(p\chi + (x/\epsilon)\chi'), \quad x^2\eta'' = x^p(p(p-1)\chi + 2p(x/\epsilon)\chi' + (x/\epsilon)^2\chi''), \tag{3.14}$$

where the arguments of χ, χ', and χ'' are x/ϵ. As $\epsilon \to 0$, since we assume $x^p n$ and $x^p\bar{n}$ are in $L^1(Q_{[0,T]})$ for any $T > 0$, we infer by monotone and dominated convergence that

$$\int_0^1 x^p w_+(x,\tau)\,dx \leq \int_0^1 x^p w_+^{\rm in}(x)\,dx + c_p\int_0^\tau\int_0^1 x^p w_+(x,t)\,dx\,dt \tag{3.15}$$

for a.e. $\tau > 0$, with $c_p = p(p-1)+4p = p(p+3)$. Denoting the (absolutely continuous) right-hand side of (3.15) by $U(\tau)$, we have

$$U(\tau) = U(0) + \int_0^\tau U'(s)\,ds \leq U(0) + c_p\int_0^\tau U(s)ds,$$

and Gronwall's inequality implies that

$$U(\tau) \leq e^{c_p\tau}\int_0^1 x^p\, w_+^{\rm in}(x)\,dx$$

for all $\tau > 0$. This proves (2.6). Clearly, $n^{\rm in} \leq \bar{n}^{\rm in}$ implies $n \leq \bar{n}$, by virtue of (2.6).

3.2. Strict L^1 contraction. From Corollary 2 it easily follows that weak solutions of (2.3) enjoy the L^1 contraction property mentioned in (2.8). For use in section 6 below, we strengthen this to the following *strict* L^1 contraction property for C^1 solutions that *cross transversely*.

LEMMA 11. *Let $n, \bar{n}$ be nonnegative solutions to* (2.3) *with respect to initial data $n^{\rm in}, \bar{n}^{\rm in}$ that are in $L^1(0,1) \cap L^\infty(0,1)$; then for a.e. $t > 0$,*

$$\|n(\cdot,t) - \bar{n}\|_{L^1(0,1)} \leq \|n^{\rm in} - \bar{n}^{\rm in}\|_{L^1(0,1)}. \tag{3.16}$$

Moreover, assuming the solutions n and $\bar{n}$ are C^1 in $(0,1)\times[0,\infty)$ and that for some $t_0 \geq 0$, $n(\cdot,t_0)$ and $\bar{n}(\cdot,t_0)$ cross transversely at least once on $(0,1)$, then for all $t > t_0$ we have

$$\|n(\cdot,t) - \bar{n}(\cdot,t)\|_{L^1(0,1)} < \|n(\cdot,t_0) - \bar{n}(\cdot,t_0)\|_{L^1(0,1)}. \tag{3.17}$$

Proof. The L^1 contraction estimate (3.16) follows directly from Corollary 2 with $p = 0$. In order to prove (3.17), it suffices to treat the case $t_0 = 0$ for $t > 0$ sufficiently small and assume the right-hand side is finite. Let $w = n - \bar{n}$ and $\hat{n} = n + \bar{n} - 2x$. If n crosses $\bar{n}$ transversely at $(x_0, 0)$, then the regularity of the solution implies that there exists a nondegenerate rectangle $\Sigma_0 = [x_0 - \hat{\delta}, x_0 + \hat{\delta}] \times [0,\delta]$ such that $w(x_0,0) = 0$ and $\partial_x w \neq 0$ in Σ_0. We suppose $\partial_x w(x_0,0) > 0$ (relabeling n and $\bar{n}$ if necessary), whence $\partial_x w \geq c_1 > 0$ in Σ_0, so $w(x_0+\hat{\delta},t) > c_1\hat{\delta} > 0$ and $w(x_0-\hat{\delta},t) < -c_1\hat{\delta} < 0$.

We follow the proof of Theorem 1 up to (3.12), finding that for $0 < \sigma < \tau < \delta$,

$$\begin{aligned}&\int_0^1 \eta(x)\zeta(w(x,t))\,dx\Big|_{t=\sigma}^{t=\tau}\\ &\quad\leq \int_{Q_{[\sigma,\tau]}} \Big(-x^2\eta\zeta''(w)(\partial_x w)^2 + \zeta(w)\partial_x(x^2\eta') + 2x\eta' w\zeta'(w) - \hat{n}w\eta\,\zeta''(w)(\partial_x w)\Big).\end{aligned} \tag{3.18}$$

Here we include a term $-x^2\eta\zeta''(w)(\partial_x w)^2$ from (3.11) that was dropped in (3.12). This identity is valid for any C^2 function $\eta \geq 0$ with compact support in $(0,1]$ and with $\eta' \geq 0$. We may require $x^2\eta \geq c_2 > 0$ in Σ_0. Therefore, taking $\bar\epsilon \downarrow 0$, we find

$$\begin{aligned}\int_{Q_{[\sigma,\tau]}} x^2\eta\zeta''(w)(\partial_x w)^2 &\geq \int_{\Sigma_0\cap Q_{[\sigma,\tau]}} c_1c_2\zeta''(w)\partial_x w \\ &= \int_\sigma^\tau c_1c_2\zeta'(w)\Big|_{x_0-\hat\delta}^{x_0+\hat\delta}\,dt \to c_1c_2(\tau-\sigma) > 0\,.\end{aligned}$$

When taking the limit $\bar\epsilon \downarrow 0$,we also have $\zeta\circ w \uparrow w_+$ and $w(\zeta'\circ w)\uparrow w_+$ pointwise. Moreover, $w\zeta''\circ w = (w/\bar\epsilon)\chi'(w/\bar\epsilon)$ is bounded and converges to zero a.e. Since $\eta\hat n\partial_x w \in L^1(Q)$, by dominated convergence the last term in (3.18) tends to zero, and we derive

$$\int_0^1 \eta\, w_+(x,\tau)\,dx \leq \int_0^1 \eta\, w_+(x,\sigma)\,dx + \int_{Q_{[\sigma,\tau]}} (x^2\eta''+4x\eta')w_+ - c_1c_2(\tau-\sigma)\,. \tag{3.19}$$

Finally, we take η of the form $\eta(x) = \chi_\theta(x)$ with $\theta < x_0 - \hat\delta$. Observe that

$$x\eta' = (x/\theta)\chi', \quad x^2\eta'' = (x/\theta)^2\chi'', \tag{3.20}$$

where the arguments of χ, χ', and χ'' are x/θ. As $\theta\to 0$, since we assume n and $\bar n$ are in $L^1(Q)$, we infer by monotone and dominated convergence that

$$\int_0^1 w_+(x,\tau)\,dx \leq \int_0^1 w_+(x,\sigma)\,dx - c_1c_2(\tau-\sigma) < \int_0^1 w_+^{\text{in}}(x)\,dx\,, \tag{3.21}$$

where the last inequality follows by applying Theorem 1 with $t=\sigma$ and $p=0$. Adding this result together with (2.6) with $p=0$ and n interchanged with $\bar n$, we obtain (3.17). ∎

4. Existence of weak solutions. The existence result in Theorem 4 is proved through three main approximation steps:

(i) Approximate the rough initial data $n^{\text{in}} \in L^1(x^p dx)$ by smooth data n_κ^{in} that is strictly positive and bounded.

(ii) Truncate the problem (2.3) to $x\in[\epsilon,1]$ with $\epsilon > 0$, resulting in a strictly parabolic problem at the cost of needing to impose an additional boundary condition at $x=\epsilon$.

(iii) Further approximate by cutting off the nonlinearity in the flux near the boundary $x=1$, resulting in a problem with linear boundary conditions.

Passing to the limit in the various approximations involves compactness arguments and uniform estimates that are based on energy estimates and Gronwall inequalities. Step (iii) is comparatively straightforward and its analysis is relegated to Appendix B. We deal with steps (i) and (ii) in the remainder of this section.

4.1. Smoothing the initial data. Consider fixed initial data n^{in} in $L^1(x^p dx)$. We regularize the given initial data to obtain a family of functions n_κ^{in} for small $\kappa > 0$, which are smooth on $[0,1]$ and positive on $(0,1]$, with the following properties:

$$\int_0^1 x^p|n_\kappa^{\text{in}} - n^{\text{in}}|\,dx \to 0 \quad \text{as } \kappa\to 0\,, \tag{4.1}$$

$$n_\kappa^{\text{in}}(x) = \kappa x^2\,, \qquad 0 < x < \kappa\,, \tag{4.2}$$

$$n_\kappa^{\text{in}}(x) = \frac{\kappa x^2}{\kappa x+1}\,, \qquad 1-\kappa < x < 1\,. \tag{4.3}$$

(The properties in (4.2) and (4.3) are conveniences so that we get compatible initial data in the approximation steps to follow below.) The desired regularization can be achieved through mollification: Let ρ be a smooth, nonnegative function on $\mathbb{R}$ with support contained in $(-1, 1)$ and total mass one. Define

$$\rho_\kappa(x) = \kappa^{-1}\rho(x/\kappa), \qquad \chi(x) = \int_{-\infty}^{x} \rho(z)\, dz \tag{4.4}$$

(note $\chi(x) = 0$ for $x < -1$ and $\chi(x) = 1$ for $x > 1$), and require that

$$x^p n_\kappa^{\mathrm{in}}(x) = \int_{2\kappa}^{1-2\kappa} \rho_\kappa(x-y)\, y^p n^{\mathrm{in}}(y)\, dy + \frac{\kappa x^{2+p}}{1 + \kappa x\, \chi(4x-2)}\ .$$

The integral term vanishes when $x < \kappa$ or $x > 1 - \kappa$, and there is no singularity near $x = 0$.

For this regularized initial data, our goal is to prove the following result.

Proposition 12. *For every small enough $\kappa > 0$, there exists a weak solution n_κ of* (2.3) *with initial data $n^{\mathrm{in}} = n_\kappa^{\mathrm{in}}$, having $n_\kappa \in C([0,\infty), L^1((0,1]))$.*

4.2. Truncation. To obtain n_κ, we regularize by truncating the domain away from the origin, thus removing the degenerate parabolic nature of the problem. In other words, we will study classical solutions of the following problem for small $\epsilon > 0$: In terms of the (left-oriented) flux

$$J_\epsilon = x^2 \partial_x n_\epsilon - 2x n_\epsilon + n_\epsilon^2\,, \tag{4.5}$$

we seek a solution to the problem

$$\partial_t n_\epsilon = \partial_x J_\epsilon\,, \qquad x \in (\epsilon, 1),\ t \in (0,\infty)\,, \tag{4.6a}$$

$$n_\epsilon = n_\kappa^{\mathrm{in}}\,, \qquad x \in (\epsilon, 1)\,,\ t = 0\,, \tag{4.6b}$$

$$0 = J_\epsilon\,, \qquad x = 1\,,\ t \in [0,\infty)\,, \tag{4.6c}$$

$$0 = \epsilon^2 \partial_x n_\epsilon - 2\epsilon n_\epsilon\,, \qquad x = \epsilon\,,\ t \in [0,\infty)\,. \tag{4.6d}$$

The boundary condition (4.6d) says $J_\epsilon = n_\epsilon^2$ at $x = \epsilon$. As will be seen in section 5 below, this boundary condition is well adapted to proving the conservation identity for photon number in Theorem 6. An important point to note, however, is that the uniqueness result of Theorem 1 shows that the solution of (2.3) *does not depend* on the choice of this boundary condition in (4.6d).

For fixed small $\epsilon > 0$, the following global existence result for classical solutions of (4.6) is proved in Appendix B. Note that due to (4.2) and (4.3), the boundary conditions (4.6c)–(4.6d) hold at $t = 0$ whenever $0 < \epsilon < \kappa$.

Proposition 13. *Let n_κ^{in} be smooth and positive on* $(0, 1]$ *and satisfy* (4.2)–(4.3). *Then for any sufficiently small $\epsilon > 0$, there is a global classical solution n_ϵ of* (4.6), *smooth in the domain*

$$Q^\epsilon := (\epsilon, 1) \times (0, \infty)\,, \tag{4.7}$$

with n_ϵ, J_ϵ and $\partial_x n_\epsilon$ globally bounded and continuous on $\bar{Q}^\epsilon = [\epsilon, 1] \times [0, \infty)$.

From this result, we will derive Proposition 12 by taking $\epsilon \downarrow 0$ after establishing a number of uniform bounds on the solution n_ϵ of (4.6). The global bounds stated in Theorem 4 will follow directly from corresponding uniform bounds on n_ϵ, which are proved in Lemmas 15 and 16 and are inherited by n_κ.

4.3. Uniform estimates for the truncation. The first few uniform estimates that we establish on the solution n_ϵ of (4.6) are pointwise estimates that arise from comparison principles.

LEMMA 14. *We have $n_\epsilon(x,t) > 0$ for every $(x,t) \in [\epsilon,1] \times [0,\infty)$.*

Proof. Recall $\min_{[\epsilon,1]} n_\kappa^{\text{in}} > 0$. If we suppose the claim fails, then $0 < t^* < \infty$, where

$$t^* = \sup\{t \mid n_\epsilon(x,t) > 0 \quad \forall x \in [\epsilon,1]\}.$$

By continuity, there exists $X^* = (x^*, t^*)$ with $x^* \in [\epsilon,1]$ such that $n_\epsilon(X^*) = 0$. We claim first that $x^* \neq \epsilon$ or 1. If $x^* = \epsilon$ or 1, by the strong maximum principle [25], we must have $0 \neq \partial_x n_\epsilon(X^*)$, but this violates the boundary conditions (4.6c)–(4.6d).

Thus $\epsilon < x^* < 1$, but this is also not possible due to rather standard comparison arguments: There exists $\delta > 0$ such that δ is less than the minimum of $n_\epsilon(\epsilon,t)$, $n_\epsilon(1,t)$, and $n_\epsilon(x,0)$ whenever $0 \le t \le t^*$ and $x \in [\epsilon,1]$. Setting $w = e^{3t} n_\epsilon$, we find that $w > \delta$ at $t = 0$, and there is some first time $\hat{t} \in (0,t^*)$ when $w(\hat{X}) = \delta$ for some $\hat{X} = (\hat{x},\hat{t})$ with $\hat{x} \in (\epsilon,1)$. Then $\partial_t w \le 0$, $\partial_x w = 0$, and $\partial_x^2 w \ge 0$ at $\hat{X}$, but computation then shows $\partial_t w \ge w = \delta > 0$. This finishes the proof. ∎

Next we establish a universal upper bound on our solution of (4.6). We do this by establishing that the function defined by

$$S(x,t) = x + \frac{1-x}{t} + \frac{2}{\sqrt{t}} \tag{4.8}$$

is a universal supersolution. This fact depends essentially on the hyperbolic nature of our problem at large amplitude—note that the middle term $(1-x)/t$ is a centered rarefaction wave solution of the equation $\partial_t n - 2n\partial_x n = 0$.

LEMMA 15. *We have*

$$n_\epsilon(x,t) < S(x,t) \quad \forall (x,t) \in \bar{Q}^\epsilon.$$

Furthermore, there exists $\tau_1 > 0$, depending only on $\sup n_\kappa^{\text{in}}$, such that

$$n_\epsilon(x,t) < S(x,t+\tau_1) \quad \forall (x,t) \in \bar{Q}^\epsilon.$$

Proof. Let us write $L[n] := \partial_t n - x^2\partial_x^2 n - 2n(\partial_x n - 1)$. Then a simple calculation gives

$$L[S] = \frac{1-x}{t^2} + \frac{2x}{t} + 3t^{-3/2} > 0.$$

Hence with $v = S - n_\epsilon$, we have

$$L[S] - L[n_\epsilon] = \partial_t v - x^2\partial_x^2 v - \partial_x((n_\epsilon + S)v) + 2v > 0. \tag{4.9}$$

By continuity we have $\min_x v(x,t) > 0$ for small $t > 0$, and we claim that this continues to hold for all $t > 0$. If not, there is a first time $\hat{t}$ when it fails, and some $\hat{X} = (\hat{x},\hat{t})$ with $\hat{x} \in [\epsilon,1]$ where $v(\hat{X}) = 0$. By (4.9) it is impossible that $\hat{x} \in (\epsilon,1)$. If $\hat{x} = \epsilon$, then $v = 0$ and $\partial_x v \ge 0$ at $\hat{X}$. But due to the boundary condition (4.6d) we find that at $(x,t) = (\epsilon,\hat{t})$,

$$0 \le \epsilon\partial_x v = \epsilon\partial_x S - 2S = \epsilon\left(1 - \frac{1}{t}\right) - 2\left(\epsilon + \frac{1-\epsilon}{t} + \frac{2}{\sqrt{t}}\right) < 0.$$

So $\hat{x} \neq \epsilon$. On the other hand, if $\hat{x} = 1$, we would have $v = 0$ and $\partial_x v \leq 0$ at $\hat{X}$. But then, at $(x,t) = (1,\hat{t})$ we find by (4.6c) that

$$0 \geq \partial_x v = \partial_x S + S^2 - 2S = 1 - \frac{1}{t} + \left(1 + \frac{2}{\sqrt{t}}\right)^2 - 2\left(1 + \frac{2}{\sqrt{t}}\right) = \frac{3}{t} > 0\,.$$

Thus $\hat{x} \neq 1$, and the result $S - n_\epsilon > 0$ follows. Furthermore, if $\sup n_\kappa^{\mathrm{in}} \leq 2/\sqrt{\tau_1}$, then

$$\min_x (S(x,\tau_1) - n_\kappa^{\mathrm{in}}(x)) > \frac{2}{\sqrt{\tau_1}} - \|n_\kappa^{\mathrm{in}}\|_{L^\infty} \geq 0\,.$$

Then the above procedure shows that $S(x, t+\tau_1)$ is also a supersolution. ◻

The next result bounds $\partial_x n_\epsilon$ from below. This is again a typical kind of estimate for the hyperbolic equation $\partial_t n - 2n\partial_x n = 0$.

LEMMA 16 (Oleinik-type inequality). *We have*

$$\partial_x n_\epsilon(x,t) \geq -\frac{4}{t} \qquad \forall (x,t) \in \bar{Q}^\epsilon\,.$$

Furthermore, there exists $\tau_2 > 0$, depending only on $\inf \partial_x n_\kappa^{\mathrm{in}}$, *such that*

$$\partial_x n_\epsilon(x,t) \geq -\frac{4}{t+\tau_2} \qquad \forall (x,t) \in \bar{Q}^\epsilon\,.$$

Proof. Let $w = \partial_x n_\epsilon$ with n_ϵ being the solution of (4.6). Differentiation of (4.6a) shows that w satisfies

$$\partial_t w = x^2 \partial_x^2 w + 2(n_\epsilon + x)\partial_x w + 2w(w-1)\,, \qquad (x,t) \in Q^\epsilon\,, \tag{4.10a}$$

$$\epsilon^2 w(\epsilon,t) = 2\epsilon n_\epsilon(\epsilon,t)\,, \qquad t > 0\,, \tag{4.10b}$$

$$w(1,t) = 2n_\epsilon(1,t) - n_\epsilon(1,t)^2\,, \qquad t > 0\,. \tag{4.10c}$$

We claim that $z = -4/t$ is a subsolution of this problem. Set $U = w - z = w + 4/t$. A direct calculation gives

$$\partial_t U - x^2\partial_x^2 U - 2(n_\epsilon + x)\partial_x U - 2(w - 4/t)U + 2U = \frac{4}{t^2}(7 + 2t) > 0\,.$$

Note that $U(x,t) > 0$ for $t > 0$ small, because $\partial_x n_\epsilon$ is continuous on $\bar{Q}^\epsilon$. Then $U(x,t) > 0$ for all x and t as long as it is so at $x = \epsilon$ and $x = 1$. At $x = \epsilon$, we have

$$U(\epsilon,t) = w(\epsilon,t) + \frac{4}{t} = \frac{2}{\epsilon} n_\epsilon(\epsilon,t) + \frac{4}{t} > 0\,.$$

On the other hand, at $x = 1$, we have

$$U(1,t) = w(1,t) + \frac{4}{t} = 2n_\epsilon(1,t) - n_\epsilon(1,t)^2 + \frac{4}{t}\,.$$

If $0 \leq n_\epsilon(1,t) \leq 2$, then $U(1,t) \geq 4/t$, and otherwise, $2 < n_\epsilon(1,t) \leq S(1,t) = 1 + 2t^{-1/2}$, hence

$$U(1,t) \geq 2S(1,t) - S(1,t)^2 + \frac{4}{t} = 1\,.$$

Therefore $U(1,t) > 0$.

Provided $\tau_2 > 0$ is sufficiently small so that $\partial_x n_\kappa^{\text{in}} > -4/\tau_2$, we have

$$\min_x \left\{ w(x,0) + \frac{4}{\tau_2} \right\} > 0\,,$$

hence the above procedure shows that $w(x,t) \geq -4/(t+\tau_2)$ for all (x,t) under consideration. This concludes the proof. $\square$

We now turn to obtain some compactness estimates that will be needed to establish convergence as $\epsilon \downarrow 0$. First we establish equicontinuity in the mean for solutions of (4.6).

LEMMA 17. *For each $t > 0$, we have*

$$\int_\epsilon^1 |\partial_x n_\epsilon|\,dx \leq K_1(t), \qquad K_1(t) = 1 + \frac{2}{\sqrt{t+\tau_1}} + \frac{8}{t+\tau_2}\,, \tag{4.11}$$

and for all $t > 0$ and all small $h > 0$,

$$\int_\epsilon^1 |n_\epsilon(x,t+h) - n_\epsilon(x,t)|\,dx \leq K_2(t) h^{1/2}\,, \tag{4.12}$$

where $K_2(t)$ is a decreasing function of t with $K_2(0)$ bounded by a constant depending only on $\sup n_\kappa^{\text{in}}$ *and* $\inf \partial_x n_\kappa^{\text{in}}$.

Proof. With τ_1 and τ_2 determined by the previous two lemmas, set

$$u(x,t) = n_\epsilon(x,t) + \frac{4x}{t+\tau_2}.$$

Then by Lemma 16, u is a nondecreasing function of x, satisfying $\partial_x u > 0$. We have

$$\begin{aligned}\int_\epsilon^1 |\partial_x n_\epsilon|\,dx &= \int_\epsilon^1 \left| \partial_x u - \frac{4}{t+\tau_2} \right| dx \leq \frac{4}{t+\tau_2} + \int_\epsilon^1 \partial_x u\,dx \\ &\leq \frac{8}{t+\tau_2} + n_\epsilon(1,t) \leq \frac{8}{t+\tau_2} + S(1, t+\tau_1) = K_1(t)\,.\end{aligned} \tag{4.13}$$

This proves the first estimate of the lemma.

We next prove the bound (4.12). Fix any $t > 0$ and consider $h > 0$ small. Suppressing the dependence on t and h, we set

$$v(x) = n_\epsilon(x,t+h) - n_\epsilon(x,t)\,, \qquad x \in [\epsilon, 1]\,,$$

and observe that

$$\|v\|_\infty \leq 2\|S(\cdot, t+\tau_1)\|_\infty\,, \qquad \int_\epsilon^1 |\partial_x v|\,dx \leq 2K_1(t)\,. \tag{4.14}$$

We proceed by approximating $|v(x)|$ by $\phi(x)v(x)$, where ϕ is obtained by mollifying $\operatorname{sgn} v(x)$ as follows. Let ρ be a smooth, nonnegative function on $\mathbb{R}$ with support contained in $(-1,1)$ and total mass one, and let $\alpha > 0$ be a parameter. (We will take $\alpha = \frac{1}{2}$ below.) We define $\rho_h(x) = h^{-\alpha}\rho(x/h^\alpha)$ and set

$$\phi(x) = \int_\epsilon^1 \rho_h(x-z) \operatorname{sgn} v(z)\,dz\,. \tag{4.15}$$

To bound the integral of $|v(x)|$ over $[\epsilon,1]$, we bound integrals over the sets

$$I_h = [\epsilon + h^\alpha, 1-h^\alpha]\,, \qquad \hat{I}_h = [\epsilon, \epsilon+h^\alpha] \cup [1-h^\alpha, 1]\,,$$

writing

$$\begin{aligned}\int_\epsilon^1 |v(x)|\,dx = \int_\epsilon^1 \phi(x)v(x)\,dx + \int_{I_h} \Big(|v(x)| - \phi(x)v(x)\Big)\,dx \\ + \int_{\hat{I}_h} \Big(|v(x)| - \phi(x)v(x)\Big)\,dx\,.\end{aligned} \tag{4.16}$$

Since $|\phi| \le 1$, the third term is bounded using the first estimate in (4.14) as

$$\int_{\hat{I}_h} \Big||v(x)| - \phi(x)v(x)\Big|\,dx \le 8h^\alpha \|S(\cdot,t)\|_\infty\,. \tag{4.17}$$

We next estimate the middle term in (4.16). For $x \in I_h$, we compute

$$|v(x)| - \phi(x)v(x) = \int_{\mathbb{R}} \rho_h(x-z)\Big(|v(x)| - v(x)\operatorname{sgn} v(z)\Big)\,dz\,.$$

Noting that $||a| - a\operatorname{sgn} b| \le 2|a-b|$ for any real a, b, we have

$$\Big||v(x)| - v(x)\operatorname{sgn} v(z)\Big| \le 2|v(x) - v(z)|\,.$$

Integrating over I_h, we find

$$\begin{aligned}\int_{I_h} \Big||v(x)| - \phi(x)v(x)\Big|\,dx &\le 2\int_{I_h}\int_{|y|<h^\alpha} \rho_h(y)|v(x) - v(x-y)|\,dy\,dx \\ &\le 2\int_{I_h}\int_{|y|<h^\alpha} \rho_h(y)|y|\int_0^1 |\partial_x v(x-ys)|\,ds\,dy\,dx\,.\end{aligned}$$

Now we integrate first over x, note $x-ys \in [\epsilon,1]$ and use (4.14), and note that $|y| \le h^\alpha$ and ρ_h has unit integral. We infer that

$$\int_{I_h} \Big||v(x)| - \phi(x)v(x)\Big|\,dx \le 2h^\alpha \int_\epsilon^1 |\partial_x v|\,dx \le 4h^\alpha K_1(t)\,. \tag{4.18}$$

Finally, we bound the first term in (4.16). Multiply (4.6a) by ϕ and integrate over $(\epsilon,1)\times(t,t+h)$. Integration by parts yields

(4.19)
$$\int_\epsilon^1 \phi v(x)\,dx = \int_t^{t+h}\int_\epsilon^1 (\partial_x\phi)\left(-x^2\partial_x n_\epsilon - n_\epsilon^2 + 2xn_\epsilon\right)\,dx\,d\tau - \int_t^{t+h} \phi(\epsilon)n_\epsilon^2(\epsilon,\tau)\,d\tau\,.$$

Note that $|\phi| \le 1$ and $|\partial_x\phi| \le h^{-\alpha}\|\rho'\|_1$. By virtue of $0 \le n_\epsilon \le S$, we have

$$\begin{aligned}\int_\epsilon^1 \phi v\,dx &\le h^{-\alpha}\|\rho'\|_1 \int_t^{t+h}\int_\epsilon^1 (|\partial_x n_\epsilon| + S^2 + 2S)\,dx\,d\tau + \int_t^{t+h} S(\epsilon,\tau)^2\,d\tau \\ &\le h^{1-\alpha}\|\rho'\|_1(K_1(t) + 3\|S(\cdot,t+\tau_1)\|_\infty^2) + h\|S(\cdot,t+\tau_1)\|_\infty^2\,.\end{aligned}$$

Assembling all the bounds on the terms in (4.16) above, we obtain

$$\int_\epsilon^1 |v(x)|\,dx \le \|S(\cdot,t+\tau_1)\|_\infty^2(8h^\alpha + h + 3h^{1-\alpha}\|\rho'\|_1) + K_1(t)(4h^\alpha + h^{1-\alpha}\|\rho'\|_1)\,.$$

Choosing $\alpha = \frac{1}{2}$ and determining $K_2(t)$ to correspond, the result in the lemma follows. $\square$

Finally, we have the following energy estimate.

LEMMA 18. *For any $t > s > 0$,*

$$\int_\epsilon^1 n_\epsilon^2(x,t)\,dx + \int_s^t \int_\epsilon^1 [n_\epsilon^2 + x^2(\partial_x n_\epsilon)^2]\,dx\,d\tau \le \int_\epsilon^1 n_\epsilon^2(x,s)\,dx + \frac{8}{3}(t-s)\,. \tag{4.20}$$

Proof. From (4.6a) and the boundary conditions (4.6c)–(4.6d) it follows that

$$\frac{d}{dt}\int_\epsilon^1 n_\epsilon^2\,dx = -2\int_\epsilon^1 (\partial_x n_\epsilon)J_\epsilon\,dx - 2n_\epsilon^3(\epsilon,t) = -2\int_\epsilon^1 [n_\epsilon^2 + x^2(\partial_x n_\epsilon)^2]dx + \Gamma(t)$$

with

$$\Gamma(t) = -\frac{2}{3}n_\epsilon^3(1,t) - \frac{4}{3}n_\epsilon^2(\epsilon,t) + 2n_\epsilon^2(1,t) - 2\epsilon n_\epsilon^2(\epsilon,t) \le \max_{u>0}\left(-\frac{2}{3}u^3 + 2u^2\right) = \frac{8}{3}\,.$$

Hence, the claimed estimate follows by integration in time. ∎

4.4. Proof of Proposition 12. We now show a solution of (2.3) does exist for initial data n_κ^{in} as prepared in subsection 4.1. Let n_ϵ be our solution of (4.6) for small $\epsilon > 0$.

Recalling the uniform estimates $0 < n_\epsilon \le S(x, t+\tau_1)$ from Lemmas 14 and 15, and using Lemma 17, we see the family$\{n_\epsilon\}$ is uniformly bounded and equicontinuous in the mean on any compact subset of $(0,1]\times[0,\infty)$. Consequently, we may extract a sequence $\epsilon_k \downarrow 0$ as $k \to \infty$, such that for each $a \in (0,1)$ and $T > 0$, n_{ϵ_k} converges to some function n, boundedly almost everywhere in $[a,1]\times[0,\infty)$ and in $C([0,T];L^1([a,1]))$, with

$$\int_a^1 |n(x,t+h) - n(x,t)|\,dx \le Ch^{1/2}\,. \tag{4.21}$$

Actually, this C is independent of a, so (4.21) holds also with $a = 0$. Moreover, due to (4.20) we can ensure that

$$x\partial_x n_{\epsilon_k} \to x\partial_x n \qquad \text{weakly in } \; L^2_{\text{loc}}(Q)\,.$$

We claim that n is a weak solution of (2.3). Multiply (4.6a) by a smooth test function ψ with compact support in $(0,1]\times(0,\infty)$, and integrate over $(\epsilon,1)\times(0,\infty)$ with integration by parts to obtain, for small enough ϵ,

$$\int_0^\infty \int_\epsilon^1 \Big(n_\epsilon \partial_t\psi - (x^2\partial_x n_\epsilon + n_\epsilon^2 - 2xn_\epsilon)\partial_x\psi\Big) = 0\,. \tag{4.22}$$

Setting $\epsilon = \epsilon_k$ and letting $k \to \infty$, we conclude that

$$\int_0^\infty \int_0^1 \Big(n\partial_t\psi - (x^2\partial_x n + n^2 - 2xn)\partial_x\psi\Big) = 0 \tag{4.23}$$

for all smooth test functions ψ. By completion, we infer that (4.23) holds for all ψ merely in H^1 with compact support in Q. Hence n is a weak solution as claimed.

Since there may exist at most one such solution of (2.3), we conclude that the whole family $\{n_\epsilon\}$ converges to n, as $\epsilon \to 0$. This ends the proof of Proposition 12.

4.5. Proof of Theorem 4. Our next task is to complete the proof of Theorem 4 by studying the solutions n_κ from Proposition 12 in the limit $\kappa \to 0$.

By the Gronwall inequality from (2.6), for any fixed $T > 0$, and small $\kappa_1, \kappa_2 > 0$,

$$\sup_{t\in[0,T]} \int_0^1 x^p |(n_{\kappa_1} - n_{\kappa_2})(x,t)|\, dx \le e^{c_p t} \int_0^1 x^p |(n_{\kappa_1}^{\text{in}} - n_{\kappa_2}^{\text{in}})(x)|\, dx \,. \tag{4.24}$$

This with (4.1) implies that n_κ is a Cauchy sequence in $C([0,T]; L^1(x^p dx))$, and therefore there is a function $n \in C([0,T]; L^1(x^p dx))$ such that $\lim_{\kappa\to 0} n_\kappa = n$. From the local-in-time energy estimate (4.20) we have

$$\int_0^1 n_\kappa^2(x,t)\, dx + \int_s^t \int_0^1 [n_\kappa^2 + x^2(\partial_x n_\kappa)^2]\, dx\, d\tau \le \int_0^1 n_\kappa^2(x,s)\, dx + \frac{8}{3}(t-s)\,.$$

The right-hand side, by virtue of $0 \le n_\kappa(x,t) \le S(x,t)$, is bounded for any $s > 0$ by

$$\int_0^1 S(x,s)^2 dx + \frac{8}{3}(t-s) < \infty\,.$$

Taking the limit $\kappa \to 0$, we deduce that n and $\partial_x n$ lie in $L^2_{\text{loc}}(Q_{(0,T]})$, and the limit n is nonnegative and satisfies (2.5d). This proves that n is indeed a weak solution of (2.3), as claimed. Moreover, from Lemma 15 it follows that the limit n has the universal upper bound that for every $t > 0$,

$$n \le S(x,t) = x + \frac{1-x}{t} + \frac{2}{\sqrt{t}} \quad \text{for a.e. } x \in (0,1)\,,$$

and Lemma 16 implies that for almost every $(x,t) \in Q_{(0,T]}$, its slope has the one-sided bound

$$\partial_x n \ge -\frac{4}{t}\,.$$

Finally, the limit function $n \in C((0,T], L^2)$, due to the following estimate. With

$$\omega(h) = \sup_{s\le t\le s+h} \int_0^1 x^p |n(x,t) - n(x,s)|\, dx\,, \qquad C_s = \sup_{x\in[0,1]} S(x,s)\,,$$

whenever $0 < t - s < h$ is so small that $\alpha = \omega(h)^{1/(p+1)} < 1$ we have

$$\int_0^1 |n(x,t) - n(x,s)|^2\, dx \le C_s^2 \alpha + C_s \alpha^{-p} \int_\alpha^1 x^p |n(x,t) - n(x,s)|\, dx \le (C_s^2 + C_s)\alpha\,.$$

By consequence, the energy estimate (2.9) follows from the one for n_κ. This finishes the proof of Theorem 4.

5. Finite time condensation. The results of this section establish Theorem 6, demonstrating that loss of photons is due to the generation of a nonzero flux at $x = 0^+$, that such a flux persists if ever formed, and that photon loss does occur if the initial photon number exceeds the maximum attained in steady state.

Throughout this section, we let n be any global weak solution of (2.3).

5.1. Formula for loss of photon number. First, we show how any possible decrease of photon number in time is related to the nonvanishing of $n(0,t)^2 = n(0^+,t)^2$, which is the formal limit of the flux J at the origin $x=0$. The following result implies part (i) of Theorem 6 in particular.

LEMMA 19. *For any fixed $t > s > 0$,*

$$\int_{0^+}^1 n(x,t)\,dx = \int_{0^+}^1 n(x,s)\,dx - \int_s^t n^2(0,\tau)\,d\tau\,. \tag{5.1}$$

Moreover, for any $t > 0$

$$\int_{0^+}^1 n(x,t)\,dx \le \frac{1}{2} + \frac{1}{2t} + \frac{2}{\sqrt{t}}\,. \tag{5.2}$$

Proof. Integration of (4.6a) over $(x,1)\times(s,t)$, using $J_\epsilon(1,t)=0$, gives

$$\int_x^1 n_\epsilon(y,\tau)\,dy\Big|_{\tau=s}^{\tau=t} = -\int_s^t (x^2\partial_x n_\epsilon(x,\tau) + n_\epsilon^2(x,\tau) - 2xn_\epsilon(x,\tau))\,d\tau\,.$$

Taking an average in x over (ϵ,a) with $0<\epsilon<a<1$, we find

$$\fint_\epsilon^a \int_x^1 n_\epsilon(y,\tau)\,dy\,dx\Big|_s^t + \int_s^t \fint_\epsilon^a n_\epsilon^2\,dx\,d\tau = \int_s^t \fint_\epsilon^a (2xn_\epsilon - x^2\partial_x n_\epsilon)\,dx\,d\tau\,. \tag{5.3}$$

For the first term on the left-hand side, integrating on $[x,1]=[x,a]\cup[a,1]$ we note

$$\fint_\epsilon^a \int_x^1 n_\epsilon(y,\tau)\,dy\,dx = \int_a^1 n_\epsilon(y,\tau)\,dy + R,$$

where, because $\epsilon \le x$,

$$\begin{aligned} R = \fint_\epsilon^a \int_x^a n_\epsilon(y,\tau)\,dy &\le \left(\fint_\epsilon^a \int_\epsilon^a n_\epsilon(y,\tau)^2\,dy\,dx\right)^{1/2} \left(\fint_\epsilon^a \int_\epsilon^a 1\,dy\,dx\right)^{1/2} \\ &\le \left(\int_\epsilon^a n_\epsilon^2\,dy\right)^{1/2} (a-\epsilon)^{1/2} \\ &\le C_s a^{1/2}\,, \end{aligned}$$

due to the energy estimate in Theorem 5. The right-hand side in (5.3) is bounded above by

$$\begin{aligned} &\left(\int_s^t \fint_\epsilon^a (2n_\epsilon - x\partial_x n_\epsilon)^2\,dx\,d\tau\right)^{1/2} \left(\int_s^t \fint_\epsilon^a x^2\,dx\,d\tau\right)^{1/2} \\ &\le \left(\int_s^t \fint_\epsilon^a 8n_\epsilon^2 + 2x^2(\partial_x n_\epsilon)^2\,dx\,d\tau\right)^{1/2} \left((t-s)a^2\right)^{1/2} \\ &\le \left(\int_s^t \int_\epsilon^1 n_\epsilon^2 + x^2(\partial_x n_\epsilon)^2\,dx\,d\tau\right)^{1/2} \left(\frac{8(t-s)a^2}{a-\epsilon}\right)^{1/2} \\ &\le C_{t,s}\left(\frac{a^2}{a-\epsilon}\right)^{1/2}\,. \end{aligned}$$

Passing to the limit $\epsilon \downarrow 0$ first, we have

$$\int_a^1 n(x,\tau)\,dx\Big|_s^t + \int_s^t \fint_0^a n^2(x,\tau)\,dx\,d\tau = O(1)a^{1/2}\,.$$

The desired equality follows from further taking $a \downarrow 0$. Moreover, by virtue of $n \le S$, we have

$$\int_0^1 n(x,t)\,dx \le \int_0^1 S(x,t)\,dx = \frac{1}{2} + \frac{1}{2t} + 2t^{-1/2}$$

for any $t > 0$. The proof is complete. ∎

Because n is a classical solution of $\partial_t n = \partial_x J$ for $x, t > 0$, by integration over $x \in [a,1]$, $\tau \in [s,t]$ we can infer that the loss term in (5.1) arises from the exit flux from the interval $[a,1]$ in the limit $a \to 0$. Thus the following result provides a precise sense in which the flux $J(a,t)$ converges to $n^2(0,t)$ as $a \to 0$.

COROLLARY 20. *Whenever $t > s > 0$ we have*

$$\lim_{a\to 0^+} \int_s^t J(a,\tau)\,d\tau = \int_s^t n^2(0,\tau)\,d\tau. \tag{5.4}$$

5.2. Persistence of condensate growth. Next we prove part (ii) of Theorem 6, showing that $n(0,t)$ once positive will remain positive for all time. More precisely, we have the following.

LEMMA 21. *If $n(0,t^*) > 0$ for some $t^* > 0$, then $n(0,t) \ge b(t)\hat{x}$ for all $t > t^*$, where*

$$b(t) = \left((1 + t^*/4)e^{2(t-t^*)} - 1\right)^{-1}, \quad \hat{x} = \min\left\{\frac{t^* n(0,t^*)}{4}, 1\right\}.$$

Proof. From $\partial_x n(x,t^*) \ge -4/t^*$ we have

$$n(x,t^*) \ge \left(n(0,t^*) - \frac{4x}{t^*}\right)_+ \ge \frac{4}{t^*}(\hat{x} - x)_+ = b(t^*)(\hat{x} - x)_+\,.$$

Hence $n(x,t^*) \ge \hat{n}(x,t^*)$ with

$$\hat{n}(x,t) = b(t)(\hat{x} - x)_+\,.$$

Note that $\hat{n}(1,t) = 0$, and for $0 < x < \hat{x}$ we have

$$L[\hat{n}] := \partial_t \hat{n} - x^2\partial_x^2 \hat{n} + 2\hat{n} - 2\hat{n}\partial_x \hat{n} = (\hat{x} - x)(b'(t) + 2b(t) + 2b(t)^2) = 0\,.$$

We claim $n \ge \hat{n}$ on $(0, \hat{x}]$ for $t > t^*$. Let $\epsilon > 0$. Substitution of $n = \hat{n} + v + \epsilon\Psi$ into the equation $L[n] = 0$ gives

$$\hat{L}[v] := \partial_t v - x^2\partial_x^2 v + 2v - 2v\partial_x \hat{n} - 2n\partial_x v = -\epsilon\hat{L}[\Psi]\,.$$

Choosing $\Psi = -t + \log x$, we have $\Psi < 0$ and

$$\hat{L}[\Psi] = 2\Psi + 2\Psi b - \frac{2n}{x} < 0\,,$$

hence $\hat{L}[v] > 0$ for $0 < x < \hat{x}$, $t \ge t^*$. For $0 < x \le \sigma$ for σ sufficiently small (depending on ϵ),

$$v = n - \hat{n} - \epsilon\Psi \ge -\hat{n} + \epsilon(t - \log\sigma) > 0 \quad \forall t > t^*\,.$$

Moreover, at $x = \hat{x}$ we have

$$v(\hat{x}, t) = n(\hat{x}, t) + \epsilon(t - \log \hat{x}) > 0\,.$$

These facts, together with the fact $v(x, t^*) \geq \epsilon(t - \log x) > 0$, ensure that

$$v(x, t) > 0 \quad \forall t > t^*,\ x \in (0, \hat{x}]\,.$$

Since $\epsilon > 0$ is arbitrary, we infer

$$n(x, t) \geq \hat{n}(x, t) \quad \forall t \geq t^*,\ x \in (0, 1]\,.$$

This gives the desired estimate upon taking $x \to 0$. ◻

5.3. Formation of condensates. The next result shows that photon loss will occur—meaning a condensate will form—in finite time if the initial photon number $N[n^{\mathrm{in}}] > \frac{1}{2}$. This proves part (iii) of Theorem 6. Note that $\frac{1}{2} = N[x]$ is the maximum photon number for any steady state.

PROPOSITION 22. *If $N[n^{\mathrm{in}}] > \frac{1}{2}$, then photon loss begins in finite time. More precisely, we have $n(0, t) > 0$ whenever*

$$\frac{1}{2\sqrt{t}} < \sqrt{1 + \delta} - 1\,, \quad \textit{where} \quad 2\delta = N[n^{\mathrm{in}}] - \frac{1}{2}\,. \tag{5.5}$$

Proof. From the supersolution obtained in Lemma 15 it follows that

$$n(x, t) \leq x + \frac{1 - x}{t} + 2t^{-1/2}\,.$$

Integration in x over $(0, 1)$ leads to

$$N[n(\cdot, t)] \leq \frac{1}{2} + \frac{1}{2t} + \frac{2}{\sqrt{t}}\,.$$

Using Lemma 19 we have

$$\int_0^t n(0, \tau)^2\, d\tau \geq N[n^{\mathrm{in}}] - \frac{1}{2} - \frac{2}{\sqrt{t}} - \frac{1}{2t} = 2\delta + 2 - 2\left(1 + \frac{1}{2\sqrt{t}}\right)^2\,.$$

The right-hand side becomes positive when (5.5) holds. The conclusion then follows from Lemma 21. ◻

5.4. Absence of condensates. Part (iv) of Theorem 6 follows from a simple comparison: If $n^{\mathrm{in}}(x) \leq x$, then since $x = n_0(x)$ is a steady weak solution, the comparison property from Theorem 1 implies $n(x, t) \leq x$ for all $t \geq 0$. Then $n(0^+, t) = 0$, so by part (a), no condensate is formed and we have

$$N[n(\cdot, t)] = N[n^{\mathrm{in}}] \quad \forall t > 0\,.$$

6. Large time convergence. We now investigate the large time behavior of solutions with nontrivial initial data. In the system (2.3), the flux vanishes for any equilibrium:

$$0 = J = n^2 \partial_x \left(x - \frac{x^2}{n}\right)\,. \tag{6.1}$$

Consequently $n = n_\mu$ for some constant $\mu \geq 0$, where

$$n_\mu(x) = \frac{x^2}{x+\mu}\,.$$

Our main goal in this section is to prove Theorem 7, which means that for every solution of (2.3) provided by Theorem 4 with nonzero initial data n^{in}, there exists $\mu \geq 0$ such that

$$\|n(\cdot,t) - n_\mu\|_1 = \int_0^1 |n(x,t) - n_\mu(x)|\,dx \to 0 \quad \text{as } t \to \infty. \tag{6.2}$$

It will be convenient to denote by

$$n(\cdot,t) = U(t)a$$

the solution of (2.3) with initial data $n^{\text{in}}(x) = a(x)$, $x \in (0,1)$. Due to the bound $n(x,t) \leq S(x,t)$ that holds by Theorem 4(i), for $t \geq 1$ any solution $U(t)n^{\text{in}}$ will lie in the set

$$A := \{a \in L^\infty(0,1) : 0 \leq a(x) \leq 3 \ \text{ for a.e. } x \in (0,1)\}$$

since $S(x,1) \equiv 3$. The set A is positively invariant under the semiflow induced by the solution operator:

$$U(t)A \subset A\,, \quad t \geq 0\,.$$

With the metric induced by the L^1 norm,

$$\rho(n_1,n_2) = \|n_1 - n_2\|_1\,,$$

the set A is a complete metric space, and by Lemma 11, $U(t)$ is a contraction: We have

$$\|U(t)a - U(t)b\|_1 \leq \|a-b\|_1$$

whenever $t \geq 0$ and a, $b \in A$.

For present purposes it is important that a stronger contractivity property also holds, as shown in Lemma 11: Namely, if the functions a and b are C^1 and *cross transversely*, then for $t > 0$, $U(t)$ strictly contracts the L^1 distance between a and b. Based on these contraction properties and the one-sided Oleinik bound in Theorem 4(ii), we proceed to establish the large time convergence (6.2).

We introduce the usual ω-limit set of any element $a \in A$ as

$$\omega(a) = \cap_{s>0}\,\overline{\{U(t)a \mid t \geq s\}}\,.$$

We have $b \in \omega(a)$ if and only if there is a sequence $\{t_j\} \to \infty$ such that $\|U(t_j)a - b\|_1 \to 0$.

Lemma 23 (the ω-limit set). *Let $a \in A$. Then $\omega(a)$ is not empty, and is invariant under $U(t)$, with*

$$U(t)\omega(a) = \omega(a)\,, \quad t > 0\,. \tag{6.3}$$

Moreover, for any $b \in \omega(a)$, b is smooth (at least C^2 on $(0,1]$) and satisfies

$$\partial_x b(x) \geq 0\,, \quad 0 \leq b(x) \leq x\,, \qquad 0 < x < 1\,. \tag{6.4}$$

Proof. To show $\omega(A)$ is not empty, note that for any sequence $t_j \to \infty$, the estimates from Lemma 17 show that $\{U(t_j)a\}$ is bounded in BV. By virtue of the Helley compactness theorem, some subsequence converges in L^1, and this limit belongs to $\omega(a)$.

Next we prove (6.3). Given $b \in \omega(a)$, there exists t_j such that

$$\|U(t_j)a - b\|_1 \to 0\,, \quad j \to \infty\,.$$

From L^1 contractivity and the semigroup property it follows that $\|(U(t+t_j)a - U(t)b\|_1 \to 0$, hence $U(t)b \in \omega(a)$. On the other hand, if $b \in U(t)\omega(a)$, we have $b = U(t)b^*$ with $b^* \in \omega(a)$. Then for some sequence $t_j \to \infty$,

$$\|U(t+t_j)a - b\|_1 = \|U(t)U(t_j)a - U(t)b^*\|_1 \le \|U(t_j)a - b^*\|_1 \to 0$$

as $j \to \infty$, hence $b \in \omega(a)$.

By relation (6.3), for each $b \in \omega(a)$ and $t > 0$, $b = U(t)b^*$ for some $b^* \in \omega(a)$. From this it follows b is smooth and that $\partial_x b \ge -4/t$ and $0 \le b(x) \le S(x,t)$ by Theorem 4. Taking $t \to \infty$, since $S(x,t) \to x$ we infer (6.4). □

LEMMA 24 (equilibria and $\omega(a)$). (i) *If $n_\mu \in \omega(a)$ for some $\mu \ge 0$, then*

$$\lim_{t\to\infty} \|U(t)a - n_\mu\|_1 = 0\,. \tag{6.5}$$

(ii) *Let $b \in \omega(A)$. Then for any $\mu \ge 0$,*

$$\|b - n_\mu\|_1 = \|U(t)b - n_\mu\|_1\,. \tag{6.6}$$

(iii) *If $a \not\equiv 0$, then $0 \notin \omega(a)$.*

Proof. (i) By definition, for any $\epsilon > 0$, $\|U(t_j)a - n_\mu\|_1 < \epsilon$ for large t_j. This ensures that for any $t > t_j$,

$$\|U(t)a - n_\mu\|_1 = \|U(t-t_j)U(t_j)a - U(t-t_j)n_\mu\| \le \|U(t_j)a - n_\mu\|_1 < \epsilon\,,$$

hence (6.5).

(ii) Since $b \in \omega(A)$, there exists $a \in A$ and a sequence $\{t_j\}$ such that $t_j \to \infty$ as $j \to \infty$ and

$$\lim_{t\to\infty} \|U(t_j)a - b\|_1 = 0\,.$$

Given any $\mu \ge 0$, by contraction of $U(t)$ we know that

$$\|U(t)a - n_\mu\|_1 = \|U(t)a - U(t)n_\mu\|_1$$

is decreasing in time and thus admits a limit $c_\mu \ge 0$ as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \|U(t)a - n_\mu\|_1 = c_\mu\,, \quad t \to \infty\,.$$

Letting $t = t_j$ in the above equation and passing to the limit, we have

$$\|b - n_\mu\|_1 = c_\mu\,.$$

Note that if $b \in \omega(a)$, then $U(t)b \in \omega(a)$; thereby

$$\|U(t)b - n_\mu\|_1 = c_\mu\,.$$

Therefore (6.6) holds for for all $t > 0, \mu \geq 0$.

(iii) Suppose $a \not\equiv 0$, so that $N[a] > 0$. We claim $0 \notin \omega(a)$. Supposing $0 \in \omega(a)$ instead, we write $n(\cdot, t) = U(t)a$. Then $N[n(\cdot, t)] = \|U(t)a - 0\|_1$ is nonincreasing and approaches zero as $t \to \infty$. By Lemmas 19 and 21, then, a condensate forms and $n(0^+, t) > 0$ for all large t.

From the Oleinik-type lower bound of Theorem 4(ii), $x < z < 1$ entails $n(x,t) - \frac{4}{t} \leq n(z,t)$. After integration from $1-x$ to 1 we find

$$(1-x)\left(n(x,t) - \frac{4}{t}\right) \leq N[n(\cdot,t)] \,.$$

For t large enough we have $N[n(\cdot,t)] < \frac{1}{16}$ and $t > 32$, and this ensures that

$$\forall x \in \left[\frac{1}{4}, \frac{1}{2}\right] \,, \qquad n(x,t) \leq 2N[n(\cdot,t)] + \frac{4}{t} < \frac{1}{4} \leq x \,.$$

Then, because $n(0^+, t) > 0$, the last crossing point defined by

$$x_1 = \max\left\{x \in \left(0, \frac{1}{4}\right] : n(x,t) = x\right\} \tag{6.7}$$

is well defined. Using again Theorem 4(ii), it now follows

$$\begin{aligned} 0 \leq n(x,t) \leq x_1 + \frac{4}{t}x_1 \quad &\text{for } 0 < x < x_1 \,, \\ x \geq n(x,t) \geq x_1 - \frac{4}{t}x_1 \quad &\text{for } x_1 < x < 2x_1 \,. \end{aligned}$$

From these inequalities, we deduce respectively that

$$\begin{aligned} \int_0^{x_1} |x - n(x,t)| \, dx &\leq x_1^2\left(1 + \frac{4}{t}\right) \,, \\ \int_{x_1}^{2x_1} |x - n(x,t)| \, dx &\leq \int_{x_1}^{2x_1} x \, dx - x_1^2\left(1 - \frac{4}{t}\right) \,. \end{aligned}$$

We may also assume t is so large that $S(x,t) < 2x$ for $\frac{1}{2} \leq x \leq 1$. Then since $0 \leq n(x,t) \leq S(x,t)$, it follows that

$$\int_{2x_1}^{1} |x - n(x,t)| \, dx \leq \int_{2x_1}^{1} x \, dx \,.$$

Because $x_1^2(8/t) < \int_0^{x_1} x \, dx$, after adding the last three inequalities we find $\|x - U(t)a\|_1 < \|x - 0\|_1$. But then since $\|x - U(t)a\|_1$ is nonincreasing in t, it is impossible that $\|U(t)a - 0\|_1 \to 0$ as $t \to \infty$. This proves $0 \notin \omega(a)$. $\square$

The following restatement of the result in Lemma 11 plays a critical role in proving (6.2).

LEMMA 25. *If $a, b \in A \cap C^1((0,1))$ and a and b cross transversely at least once on $(0,1)$, then*

$$\|U(t)a - U(t)b\|_1 < \|a - b\|_1 \,, \quad t > 0 \,.$$

We are now ready to prove (6.2). Let $a \in A$ with $a \not\equiv 0$. By Lemma 23 we know that $\omega(a)$ is not empty. Let $b \in \omega(a)$. We need to show there exists a $\mu \geq 0$ such that

$$b = n_\mu\,. \tag{6.8}$$

Since $b \not\equiv 0$ and b is nondecreasing, the quantity

$$g(x) = x - \frac{x^2}{b(x)}\,,$$

which is the first variation $\delta H/\delta n$ of entropy, is well defined in some nonempty interval $(x_0, 1)$. If g is not a constant, there exists some $x^* \in (x_0, 1)$ such that $g'(x^*) \neq 0$. Then it follows that at $x = x^*$, with $\mu^* = -g(x^*)$ we have

$$b = \frac{x^2}{x - g(x)} = n_{\mu^*}\,, \qquad \partial_x b = \partial_x n_{\mu*} + \frac{x^2 g'}{(x - g(x))^2} \neq \partial_x n_{\mu^*}\,.$$

In other words, b and n_{μ^*} cross transversely at x^*. Therefore by Lemma 25 we have

$$\|U(t)b - U(t)n_{\mu^*}\|_1 < \|b - n_{\mu^*}\|_1\,.$$

This contradicts (6.6). We conclude that g must be a constant, i.e., $g(x) = -\mu$, which gives (6.8). From $b \neq 0$ and $b \leq x$ we see that $\mu \geq 0$.

Remark 26. Due to loss of mass, determining μ quantitively for each given initial data is not straightforward, except for some special cases as treated in Corollary 8.

Proof of Corollary 8. If $n^{\text{in}} \geq x$, by the comparison result in Theorem 1, we have

$$x \leq n(x,t)\,, \quad t > 0\,.$$

On the other hand, the supersolution bound from Theorem 4(i) ensures that

$$n(x,t) \leq x + \frac{1-x}{t} + 2t^{-1/2}.$$

These together lead to (2.10), hence $\lim_{t\to\infty} n(x,t) = x$.

In the case of $n^{\text{in}} \leq x$, we have $n(x,t) \leq x$ for all t. Then there is no mass loss, hence the limiting equilibrium state n_μ satisfies

$$\int_0^1 n_\mu\,dx = \int_0^1 n^{\text{in}}\,dx = N[n^{\text{in}}]\,.$$

Integration of the left-hand side yields (2.11). ∎

Appendix A. Anisotropic Sobolev estimates. For use in section 4 and Appendices B and C, we need some basic anisotropic Sobolev estimates that are not easy to find in the extensive literature on the subject. The results that we need appear to be related to embedding results for anisotropic Besov spaces contained in [2]. For the reader's convenience, however, we provide a self-contained treatment based on simple estimates for Fourier transforms.

If $\Omega \subset \mathbb{R}^2$, the typical anisotropic Sobolev space is

$$u \in W_2^{2k,k}(\Omega) = \{u \mid D_x^s D_t^r u \in L^2(\Omega),\ 0 \leq 2r + s \leq 2k\}.$$

As usual, if a function $u \in W_2^{2k,k}(\Omega)$, it will automatically belong to certain other spaces, which depend on k and the dimension. One such space is $C^{\gamma,\gamma/2}(\Omega)$. We say $u \in C^{\gamma,\gamma/2}(\Omega)$ if there is a constant K such that

$$|u(x,t) - u(y,\tau)| \le K(|x-y|^2 + |t-\tau|)^{\gamma/2} \qquad \forall (x,t),(y,\tau) \in \Omega.$$

The space $C^{\gamma,\gamma/2}(\Omega)$ is a Banach space with norm given by

$$\|u\|_{C^{\gamma,\gamma/2}(\Omega)} = \max_{(x,t)\in\Omega} |u(x,t)| + \sup_{(x,t),(y,\tau)\in\Omega} \frac{|u(x,t)-u(y,\tau)|}{(|x-y|^2+|t-\tau|)^{\gamma/2}}.$$

The results we need are contained in the following result.

THEOREM 27. *Let $D = (a,b) \times (c,d)$ be a rectangular domain in $\mathbb{R}^2$. Suppose that u and its distributional derivatives $\partial_t u$ and $\partial_x^2 u$ are in $L^2(D)$, i.e., $u \in W_2^{2,1}(D)$. Then $u \in C^{1/2,1/4}(D)$, and there is a constant C depending on D and s such that*

$$\|\partial_x u\|_{L^s(D)} \le C\|u\|_{W_2^{2,1}(D)}, \quad 2 \le s < 6.$$

This result is proved in the remainder of this section. It seems interesting to point out, however, that by using the same techniques and with very little more work, one can discuss higher-order embeddings and arbitrary space dimensions.

THEOREM 28. *Let D be a bounded parabolic cylinder in $\mathbb{R}^{n+1}$ with C^1 spatial boundary. Suppose that $u \in W_2^{2k,k}(D)$; then*

$$\text{(i)} \qquad W_2^{2k,k} \to \begin{cases} C^{\gamma,\gamma/2}, & \gamma = 2k - \frac{n+2}{2}, \qquad k > \frac{n+2}{4}, \\ L^s, & 2 \le s < \infty, \qquad k = \frac{n+2}{4}, \\ L^s, & 2 \le s < \frac{2(n+2)}{(n+2)-4k}, \quad k < \frac{n+2}{4}, \end{cases}$$

$$\text{(ii)} \qquad C\|u\|_{W_2^{2k,k}} \ge \begin{cases} \|\partial_x u\|_\infty, & k > \frac{n}{4}+1, \\ \|\partial_x u\|_{L^s(D)}, \quad 2 \le s < \infty, & k = \frac{n}{4}+1, \\ \|\partial_x u\|_{L^s(D)}, \quad 2 \le s < \frac{2(n+2)}{(n+4)-4k}, & k < \frac{n}{4}+1. \end{cases}$$

The proof of Theorem 28 is a rather straightforward modification of the proof of Theorem 27 that is given below. We omit the complete proof, however, since we make no use of this theorem in this paper except for those cases contained already in Theorem 27. Those cases correspond to $n = 1$, $k = 1$, and the cases $\gamma = 1/2$ in part (i) and $s \in [2,6)$ in part (ii). The results of Theorem 28 are again related to results in the comprehensive work [2], but it is not easy to cite precise statements with complete proofs.

A.1. Fourier estimates in $\mathbb{R}^2$. The Fourier transform for $u \in L^1(\mathbb{R}^2)$ is

$$\text{(A.1)} \qquad \hat{u}(\xi,l) = \int_{\mathbb{R}^2} u(x,t) e^{-2\pi i(x\xi+tl)} dx\, dt,$$

which extends to a bounded linear map $u \to \hat{u}$ from L^p to $L^{p'}$, for $1 \le p \le 2$ and $1/p + 1/p' = 1$. Moreover, the Hausdorff–Young inequality holds:

$$\text{(A.2)} \qquad \|\hat{u}\|_{p'} \le \|u\|_p$$

for $u \in L^p$. This simply interpolates $\|\hat{u}\|_\infty \le \|u\|_1$ and the Plancherel theorem, $\|\hat{u}\|_2 = \|u\|_2$. The continuity of $\hat{u}$ follows from the dominated convergence theorem. In case $\hat{u}$ is integrable, one may recover u from $\hat{u}$ by

$$\text{(A.3)} \qquad u(x,t) = \int_{\mathbb{R}^2} \hat{u}(\xi,l) e^{2\pi i(x\xi+tl)} d\xi dl.$$

We will deduce Theorem 27 from the corresponding result on all of $\mathbb{R}^2$.

THEOREM 29. *Suppose* $u \in W_2^{2,1}(\mathbb{R}^2)$. *Then* $u \in C^{1/2,1/4}(\mathbb{R}^2)$. *Moreover,* $\partial_x u \in L^s(\mathbb{R}^2)$ *for* $2 \le s < 6$, *with*

$$\|\partial_x u\|_{L^s(\mathbb{R}^2)} \le C\|u\|_{W_2^{2,1}(\mathbb{R}^2)}.$$

To proceed, we first recall a characterization of $W_2^{2k,k}$.

LEMMA 30 (characterization of $W_2^{2k,k}(\mathbb{R}^2)$ by Fourier transform). *Let k be a non-negative integer, and set*

$$m(\xi,l) = (1 + l^2 + |\xi|^4)^{1/2}.$$

Then $u \in W_2^{2k,k}(\mathbb{R}^2)$ *if and only if* $m^k\hat{u} \in L^2(\mathbb{R}^2)$. *In addition, there exists a constant C such that*

$$C^{-1}\|u\|_{W_2^{2k,k}} \le \|m^k\hat{u}\|_{L^2} \le C\|u\|_{W_2^{2k,k}}.$$

The following two technical lemmas will be used as well.

LEMMA 31. *For* $0 \le \alpha < 2\beta$ *and* $\beta \ge 1$, *we have*

$$A_{\alpha,\beta} := |\xi|^\alpha/m^\beta \in L^s(\mathbb{R}^2) \qquad \textit{if any only if} \qquad s > \max\left\{\frac{3}{2\beta-\alpha}, \frac{1}{\beta}\right\}.$$

Proof. A direct calculation using the substitution $l = y(1+|\xi|^4)^{1/2}$ gives

$$\begin{aligned}
\|A_{\alpha,\beta}\|_s^s &= \int_{\mathbb{R}^2} \frac{|\xi|^{\alpha s}}{(1+l^2+|\xi|^4)^{\beta s/2}} d\xi\, dl \\
&= \int_{\mathbb{R}^2} \frac{|\xi|^{\alpha s}(1+|\xi|^4)^{1/2}}{(1+y^2)^{\beta s/2}(1+|\xi|^4)^{\beta s/2}} d\xi\, dy \\
&= \int_{\mathbb{R}} \frac{dy}{(1+y^2)^{\beta s/2}} \int_{\mathbb{R}} \frac{|\xi|^{\alpha s}(1+|\xi|^4)^{1/2}}{(1+|\xi|^4)^{\beta s/2}} d\xi.
\end{aligned}$$

This is bounded if and only if $\beta s > 1$ and $2\beta s - 2 - \alpha s > 1$. That is, $s\beta > 1$ and $s(2\beta - \alpha) > 3$. ∎

LEMMA 32. *Let* $V(x) = |x| \wedge 1 := \min\{|x|, 1\}$. *Then for some constant* $C > 0$,

$$\|m^{-1}V(r\xi)\|_2 + \|m^{-1}V(r^2 l)\|_2 \le Cr^{1/2} \quad \forall r > 0. \tag{A.4}$$

Proof. For the first term, substituting $l = y(1+|\xi|^4)^{1/2}$ again, we find

$$\begin{aligned}
\|m^{-1}V(r\xi)\|_2^2 &= \int_{\mathbb{R}^2} \frac{(|r\xi| \wedge 1)^2}{(1+l^2+|\xi|^4)} d\xi\, dl \\
&\le \int_{\mathbb{R}^2} \frac{(|r\xi| \wedge 1)^2(1+|\xi|^4)^{1/2}}{(1+y^2)(1+|\xi|^4)} d\xi\, dy \\
&= \int_{\mathbb{R}} \frac{dy}{(1+y^2)} \int_{\mathbb{R}} \frac{(|r\xi| \wedge 1)^2}{(1+|\xi|^4)^{1/2}} d\xi.
\end{aligned}$$

The first factor is finite. We proceed to decompose the last integral into two parts, one over $\{\xi : \ |\xi| < r^{-1}\}$ and the other over $\{\xi : \ |\xi| > r^{-1}\}$: The integrand is even, and

$$\int_0^\infty \frac{(|r\xi| \wedge 1)^2}{(1+|\xi|^4)^{1/2}} d\xi = \int_0^{r^{-1}} \frac{(r|\xi|)^2}{(1+|\xi|^4)^{1/2}} d\xi + \int_{r^{-1}}^\infty \frac{1}{(1+|\xi|^4)^{1/2}} d\xi$$
$$\leq r^2 \int_0^{r^{-1}} d\xi + \int_{r^{-1}}^\infty |\xi|^{-2} d\xi = 2r.$$

In a similar fashion, we estimate, using $\xi = (1+l^2)^{1/4}\eta$,

$$\begin{aligned}\|m^{-1}V(r^2 l)\|_2^2 &= \int_{\mathbb{R}^2} \frac{||r^2 l| \wedge 1|^2}{1+l^2+|\xi|^4} d\xi\, dl \\ &\leq \int_{\mathbb{R}^2} \frac{(|r^2 l| \wedge 1)^2}{(1+l^2)^{3/4}(1+|\eta|^4)} d\eta\, dl \\ &= \int_{\mathbb{R}} \frac{d\eta}{(1+|\eta|^4)} \int_{\mathbb{R}} \frac{(|r^2 l| \wedge 1)^2}{(1+|l|^2)^{3/4}} dl.\end{aligned}$$

The first integral is bounded; the second integral is further estimated by

$$\begin{aligned}\int_0^\infty \frac{(|r^2 l| \wedge 1)^2}{(1+|l|^2)^{3/4}} dl &\leq \int_0^{r^{-2}} \frac{(r^2|l|)^2}{(1+|l|^2)^{3/4}} dl + \int_{r^{-2}}^\infty \frac{1}{(1+|l|^2)^{3/4}} dl \\ &\leq r^4 \int_0^{r^{-2}} |l|^{1/2} dl + \int_{r^{-2}}^\infty |l|^{-3/2} dl \\ &= \left(\frac{2}{3} + 2\right) r = \frac{8}{3} r.\end{aligned}$$

These estimates together yield the bound (A.4) as claimed. ∎

Proof of Theorem 29. From the inversion formula (A.3) it follows that

$$\|u\|_\infty \leq \|\hat{u}\|_1 \leq \|m\hat{u}\|_2 \|m^{-1}\|_2 \leq C\|u\|_{W_2^{2,1}},$$

where the bound on $\|m^{-1}\|_2 = \|A_{0,1}\|_2$ is ensured by Lemma 31.

(i) Fix $(x,t) \neq (y,\tau)$ so that $r = \sqrt{|y-x|^2 + |\tau - t|} > 0$. Using the inequalities

$$|e^{2ia} - e^{2ib}| \leq 2|a-b| \wedge 2 = 2V(a-b),$$
$$|(y-x)\cdot\xi + (\tau - t)l| \leq r|\xi| + r^2|l|,$$

we obtain from the inversion formula and Lemma 32 that

$$\begin{aligned}|u(x,t) - u(y,\tau)| &\leq 2\pi \int (V(r\xi) + V(r^2 l))|\hat{u}(\xi,l)| d\xi\, dl \\ &\leq 2\pi(\|m^{-1}V(r\xi)\|_2 + \|m^{-1}V(r^2 l)\|_2)\|m\hat{u}\|_2 \\ &\leq C r^{1/2} \|u\|_{W_2^{2,1}}.\end{aligned}$$

This proves the embedding $W_2^{2,1}(\mathbb{R}^2) \to C^{1/2,1/4}(\mathbb{R}^2)$.

(ii) For $2 \leq s < 6$ we have $s' = \frac{s}{s-1} \leq 2$ and $r > 3$, where

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{s}.$$

We may then use the Hausdorff–Young inequality (A.2) and Lemma 31 to obtain

$$\|\partial_x u\|_s \leq C_s \|\xi\hat{u}\|_{s'} \leq \|m\hat{u}\|_2 \|A_{1,1}\|_r, \leq C\|u\|_{W_2^{2,1}}.$$ ∎

Proof of Theorem 27. Let D be the given closed rectangle in $\mathbb{R}^2$. For functions u defined a.e. on D, we extend u from D to a larger rectangle $\hat{D}$ containing D in its interior, in two steps, using linear combinations of dilated reflections as shown in Adams [1, Theorem 4.26]. The extension is to be made so that the weak derivatives are preserved across ∂D.

For instance, we reflect across the faces of D sequentially. First, from x faces $\{a, b\}$ with $c \le t \le d$, writing $\hat{a} = a - (b - a)$, $\hat{b} = b + (b - a)$, let

$$E_x u(x,t) = \begin{cases} -3u(2a - x, t) + 4u(-x/2 + 3a/2), & \hat{a} \le x \le a, \\ u(x,t), & a \le x \le b, \\ -3u(2b - x, t) + 4u(-x/2 + 3b/2), & b \le x \le \hat{b}. \end{cases}$$

and then from the t faces $\{c, d\}$ in an entirely similar manner, such that

$$\tilde{u} = E_t E_x u(x,t)$$

is an C^1 extension when crossing ∂D and well defined in $\hat{D} = [\hat{a}, \hat{b}] \times [\hat{c}, \hat{d}]$. Then multiply by a fixed smooth cut-off function $\phi(x,t)$ that is 1 on D and 0 near the boundary of $\hat{D}$ to obtain

$$Eu = \phi(x,t) E_t E_x u(x,t).$$

In this way, given u such that $\partial_t u$ and $\partial_x^j u$ are in $L^2(D)$ for $j = 0, 1, 2$, we obtain Eu such that $\partial_t Eu$ and $\partial_x^j Eu$ are in $L^2(\mathbb{R}^2)$ for $j = 0, 1, 2$. The extension E is thus a bounded linear operator from $\in W_2^{2,1}(D)$ to $W_2^{2,1}(\mathbb{R}^2)$. Moreover,

$$Eu = u \quad \text{a.e.} \quad \text{in} \quad D,$$

Eu has support in $\hat{D}$, and

$$\|Eu\|_{W_2^{2,1}(\mathbb{R}^2)} \le C \|u\|_{W_2^{2,1}(D)}.$$

This combined with Theorem 29 when applied to Eu proves Theorem 27. ◻

Appendix B. Existence for the truncated problem. In this appendix, we establish the existence of a classical solution to the truncated problem (4.6). We aim to prove Proposition 13. This global existence result does not appear to follow easily from stated results in standard parabolic theories, due to the fact that the boundary condition $J = 0$ at $x = 1$ is nonlinear. For the convenience of the reader, we indicate how to establish Theorem 13 by use of an approximation method that involves cutting off the nonlinear term in the flux J together with interior regularity theory. This will result in a problem with standard linear Robin-type boundary conditions that still respects a maximum principle which keeps the solution uniformly bounded.

B.1. Approximation by flux cut-off. We consider, then, the following problem. Let $\chi(x)$ be a smooth, nondecreasing function with $\chi(x) = 0$ for $x < -1$ and $\chi(x) = 1$ for $x > 1$ as in (4.4). For small $h > 0$ define $\chi_h(x) = \chi(1 + (x-1)/h)$, so that

$$\chi_h(x) = \begin{cases} 0\,, & x < 1 - 2h\,, \\ 1\,, & x = 1\,. \end{cases} \tag{B.1}$$

Writing

$$J_h = x^2 \partial_x n_h - 2x n_h + n_h^2 + (3n_h - n_h^2)\chi_h\ , \tag{B.2}$$

we consider the problem

$$\partial_t n_h = \partial_x J_h\,, \qquad x \in (\epsilon, 1),\ t \in (0,\infty), \tag{B.3a}$$

$$n_h = n_h^{\rm in}, \qquad x \in (\epsilon, 1),\ t = 0, \tag{B.3b}$$

$$0 = J_h\,, \qquad x = 1,\ t \in [0,\infty), \tag{B.3c}$$

$$0 = \epsilon^2 \partial_x n_h - 2\epsilon n_h, \qquad x = \epsilon,\ t \in [0,\infty)\,. \tag{B.3d}$$

We construct the initial data $n_h^{\rm in}$ from the given $n_\kappa^{\rm in}$ so that at $t = 0$, the cut-off flux J_h is the original J_ϵ. Namely, we require that at $t = 0$,

$$J_h = x^2 \partial_x n_\kappa^{\rm in} - 2x n_\kappa^{\rm in} + (n_\kappa^{\rm in})^2\ . \tag{B.4}$$

We make $n_h^{\rm in}(x) = n_\kappa^{\rm in}(x)$ for $x < 1 - 2h$ and use (B.4) to determine $n_h^{\rm in}(x)$ for $x \in [1 - 2h, 1]$. These initial data are compatible with the boundary conditions (B.3c)–(B.3d). Clearly in the limit $h \downarrow 0$, we have $n_h^{\rm in} \to n_\kappa^{\rm in}$ uniformly on $[\epsilon, 1]$.

B.2. Uniform bounds on the cut-off problem. Because $\chi_h(1) = 1$, the boundary condition in (B.3c) is linear in n_h, taking the form

$$\partial_x n_h = -n_h, \qquad x = 1,\ t \in [0,\infty)\ . \tag{B.5}$$

Moreover, note that (B.3a) takes the explicit form

$$\partial_t n_h = x^2 \partial_x^2 n_h - 2n_h + (\partial_x n_h)(2n_h + (3 - 2n_h)\chi_h) + (3n_h - n_h^2)\chi_h'\ . \tag{B.6}$$

For this problem, comparison principles hold, whence we obtain positivity and uniform sup-norm bounds on solutions.

LEMMA 33. *Suppose* $\min_{[\epsilon,1]} n_h^{\rm in} > 0$ *and* $\max_{[\epsilon,1]} n_h^{\rm in} < M_1$*, where* $M_1 \ge 3$. *Suppose* n_h *is a classical solution of* (B.3) *in* $[\epsilon, 1] \times (0, T]$ *with* n_h *continuous on* $[\epsilon, 1] \times [0, T]$*. Then* $0 < n_h(x, t) < M_1$ *for all* $(x, t) \in [\epsilon, 1] \times [0, T]$.

Proof. The proof of strict positivity is similar to Lemma 14. To prove the upper bound, suppose $n_h(X^*) = M$ with $X^* = (x^*, t^*)$, where $t^* > 0$ is minimal. Because $\partial_x n_h = -n_h < 0$ holds at $x = 1$, and (B.3d) holds at $x = \epsilon$, x^* must lie strictly between ϵ and 1. But because (B.6) holds and $\chi_h' \ge 0$, this is impossible. □

We may obtain global existence of a classical solution to problem (B.3) with cut-off flux from the proof of Proposition 7.3.6 of [24], due to the time-uniform bounds on n_h in this lemma and the fact that the nonlinear terms in (B.3a) appear in the divergence form $N_h(n_h) := \partial_x(n_h^2(1 - \chi_h))$, which enjoys a local Lipschitz bound in the L^∞ norm of the form

$$\|N_h(u) - N_h(v)\|_\infty \le K\Big(\|u - v\|_\infty \|u\|_{C^1} + \|v\|_\infty \|u - v\|_{C^1}\Big) \tag{B.7}$$

with $K = 1 + \|\chi_h'\|_\infty$.

From the proof of [24, Proposition 7.3.6], this solution n_h is continuous on $[\epsilon, 1] \times [0,\infty) = \bar{Q}^\epsilon$, and the quantities $\partial_x n_h$, $\partial_t n_h$, and $\partial_x^2 n_h$ are continuous on $[\epsilon, 1] \times (0,\infty)$. However, these quantities are actually all continuous on $\bar{Q}^\epsilon$ by the local-time existence theorem 8.5.4 of [24], due to the fact that the initial data are C^3 and satisfy the compatibility conditions. (A simple energy estimate for the difference, along the lines of step 1 in subsection B.3 below, shows that the local solution given by this theorem agrees with that given by Proposition 7.3.6.)

Additionally, these quantities are also locally Hölder-continuous on $[\epsilon,1]\times(0,\infty)$, due to the regularity results stated in [24, Proposition 7.3.3(iii)]. From standard interior regularity theory for parabolic problems (e.g., based on Theorem 8.12.1 of [21] and bootstrapping), we infer that n_h is smooth in Q^ϵ. In particular, the flux J_h is a classical solution in Q^ϵ of the equation

$$\partial_t J_h = x^2\partial_x^2 J_h + (\partial_x J_h)(-2x + 2n_h + (3-2n_h)\chi_h)\,. \tag{B.8}$$

Since J_h is continuous on $\bar{Q}^\epsilon$, by the maximum principle it is bounded in terms of its initial and boundary values—recall $J_h = n_h^2$ at $x=\epsilon$. From this and the sup-norm bound in the previous lemma, we obtain (ϵ-dependent) uniform bounds on $\partial_x n_h$.

LEMMA 34. *There is a constant M_2 depending on n_κ^{in} and independent of h and ϵ such that $|J_h| + \epsilon^2|\partial_x n_h| \le M_2$ for all $(x,t)\in\bar{Q}^\epsilon$,*

B.3. Energy estimates. These are simpler than the corresponding ones in section 5, because here $\epsilon>0$ is fixed, and the initial data is smooth.

1. The basic energy estimate is (using that n_h is positive and bounded)

$$\begin{aligned}\frac{d}{dt}\int_\epsilon^1 \frac12 n_h^2\,dx &= \int_\epsilon^1 n_h\,\partial_x J_h\,dx = n_h J_h\Big|_\epsilon^1 - \int_\epsilon^1 (\partial_x n_h) J_h\,dx\\ &= -n_h(\epsilon,t)^3 - \int_\epsilon^1 (\partial_x n_h)(x^2\partial_x n_h - 2xn_h + n_h^2 + (3n_h - n_h^2)\chi_h)\,dx\\ &\le -\frac{\epsilon^2}{2}\int_\epsilon^1 (\partial_x n_h)^2\,dx + C\int_\epsilon^1 n_h^2\,dx.\end{aligned}$$

Here C is independent of h and t, and after integration we conclude that $\partial_x n_h$ (and also J_h) is uniformly bounded independent of h in L^2 on $[\epsilon,1]\times[0,T]$ for any T.

2. For $(x,t)\in Q^\epsilon$, the flux J_h satisfies (B.8), and we find

$$\begin{aligned}\frac{d}{dt}\int_\epsilon^1 \frac12 J_h^2\,dx &= J_h(x^2\partial_x J_h)\Big|_\epsilon^1 - \int_\epsilon^1 (x\partial_x J_h)^2\,dx\\ &\quad + \int_\epsilon^1 J_h(\partial_x J_h)(-4x + 2n_h + (3-2n_h)\chi_h)\\ &\le -\frac{\epsilon^2}{3}\partial_t(n_h(\epsilon,t)^3) - \frac{\epsilon^2}{2}\int_\epsilon^1 (\partial_x J_h)^2\,dx + C\int_\epsilon^1 J_h^2\,dx\,.\end{aligned}$$

Upon integration in time, we conclude $\partial_x J_h = \partial_t n_h$ is uniformly bounded independent of h in L^2 on $[\epsilon,1]\times[0,T]$ for any given T. And further, using (B.2) for $x<1-2h$, we deduce that $\partial_x^2 n_h$ is uniformly bounded independent of h in L^2 on any compact set

$$[\epsilon, 1-\hat\epsilon]\times[0,T]\subset[\epsilon,1)\times[0,\infty) \tag{B.9}$$

fixed independent of h. (This does not work for $\hat\epsilon = 0$ because χ_h' is not uniformly bounded.)

3. Next, we have

$$\begin{aligned}\frac{d}{dt}\int_\epsilon^1 \frac12(\partial_x J_h)^2\,dx &= (\partial_x J_h)(\partial_t J_h)\Big|_\epsilon^1 - \int_\epsilon^1 (\partial_x^2 J_h)(\partial_t J_h)\,dx\\ &\le -2n_h(\epsilon,t)(\partial_t n_h(\epsilon,t))^2 - \frac{\epsilon^2}{2}\int_\epsilon^1 (\partial_x^2 J_h)^2\,dx + C\int_\epsilon^1 (\partial_x J_h)^2\,dx\,.\end{aligned}$$

Because $\partial_x J_h$ is continuous on $\bar{Q}^\epsilon$, we may integrate this inequality over $t \in [0,T]$, and use the bound on $\partial_x J_h$ from the previous step, to conclude that $\partial_x^2 J_h$ and $\partial_t J_h$ are uniformly bounded independent of h in L^2 on $[\epsilon,1]\times[0,T]$. By the anisotropic Sobolev estimates in Appendix A, we deduce that $\partial_x J_h$ is uniformly bounded independent of h in L^4 on $[\epsilon,1]\times[0,T]$, as well.

4. Last we derive an interior estimate on $\partial_x^3 J_h$. We define

$$\beta(x,t) = -2x + 2n_h + (3-2n_h)\chi_h\,, \quad \text{so} \quad \partial_t\beta = 2(1-\chi_h)\partial_x J_h\,.$$

Then

$$\partial_t^2 J_h = x^2\partial_x^2\partial_t J_h + \beta\partial_x\partial_t J_h + 2(1-\chi_h)(\partial_x J_h)^2, \tag{B.10}$$

and we let $\eta(x) = x-\epsilon$ so that $\eta(\epsilon)=0$ and $\eta'=1$,

$$\begin{aligned}
\frac{d}{dt}\int_\epsilon^1 \frac{1}{2}\eta^2(\partial_t J_h)^2\,dx &= \int_\epsilon^1 \eta^2(\partial_t J_h)(\partial_t^2 J_h)\,dx\\
&= \int_\epsilon^1 \eta^2(\partial_t J_h)(\beta\partial_x\partial_t J_h + 2(1-\chi_h)(\partial_x J_h)^2)\,dx\\
&\quad - \int_\epsilon^1 \eta^2x^2(\partial_x\partial_t J_h)^2\,dx - \int_\epsilon^1 (\partial_t J_h)(\partial_x\partial_t J_h)\partial_x(\eta^2x^2)\,dx\\
&\le -\frac{\epsilon^2}{2}\int_\epsilon^1 \eta^2(\partial_x\partial_t J_h)^2\,dx + C\int_\epsilon^1 (\partial_t J_h)^2 + (\partial_x J_h)^4\,dx.
\end{aligned}$$

Because we only know $\partial_t J_h$ is continuous for $t>0$, we integrate this over $t\in[s,T]$, then over $s\in[0,\tau]$, and use the bounds from the previous step. We infer that $\eta\partial_x\partial_t J_h$ is uniformly bounded independent of h in L^2 on $[\epsilon,1]\times[\tau,T]$. Due to (B.8) and (B.1), we infer that $\partial_t(\partial_x J_h)$ and $\partial_x^2(\partial_x J_h)$ are uniformly bounded independent of h in $[\epsilon+\hat\epsilon, 1-\hat\epsilon]\times[\tau,T]$ for any small fixed $\hat\epsilon>0$ and compact $[\tau,T]\subset(0,\infty)$.

B.4. Compactness argument. By the anisotropic Sobolev estimates in Appendix A, from the bounds on $\partial_t J_h$ and $\partial_x^2 J_h$ in step 3 above, we have that J_h is uniformly Hölder-continuous (independent of h) on any compact set

$$[\epsilon,1]\times[0,T]\subset[\epsilon,1]\times[0,\infty) = \bar{Q}^\epsilon\,. \tag{B.11}$$

Also, by the bounds on $\partial_t n_h$ and $\partial_x^2 n_h$ in step 2, n_h is uniformly Hölder-continuous on any compact set of the form in (B.9). From this we infer by (B.2) for $x<1-2h$ the same for $\partial_x n_h$. By step 4, $\partial_t n_h = \partial_x J_h$ and $\partial_x^2 n_h$ are uniformly Hölder-continuous on any compact set

$$[\epsilon+\hat\epsilon, 1-\hat\epsilon]\times[\tau,T]\subset(\epsilon,1)\times(0,\infty)\,. \tag{B.12}$$

From the Arzela–Ascoli theorem and a diagonalization argument, along a subsequence of $h\to 0$ we get uniform convergence of J_h to J_ϵ in sets of form (B.11), n_h and $\partial_x n_h$ to respective limits n_ϵ and $\partial_x n_\epsilon$ in sets of form (B.9), and $\partial_t n_h$ to $\partial_t n_\epsilon$ and $\partial_x^2 n_h$ to $\partial_x^2 n_\epsilon$ in sets of form (B.12), with all limits Hölder-continuous on the indicated sets.

In the limit, the PDE $\partial_t n_\epsilon = \partial_x J_\epsilon$ holds for $(x,t)\in Q^\epsilon$, and

$$J_\epsilon = x^2\partial_x n_\epsilon - 2xn_\epsilon + n_\epsilon^2\,, \quad (x,t)\in[\epsilon,1)\times[0,\infty)\,. \tag{B.13}$$

Because of the continuity of J_ϵ on the sets in (B.11) and n_ϵ on the sets in (B.9), by regarding (B.13) as an ODE for n_ϵ we deduce that n_ϵ and $\partial_x n_\epsilon$ are continuous on the sets in (B.11) also (i.e., up to the boundary $x = 1$), and both boundary conditions in (4.6c)–(4.6d) hold.

From standard parabolic theory as before, we find that n_ϵ is smooth in Q^ϵ. This concludes the proof of Proposition 13.

Appendix C. Regularity away from the origin. What we seek to do in this section is to prove Theorem 5, providing sufficient local regularity in the domain $Q = (0, 1] \times (0, \infty)$ to infer that the solutions n in Theorem 4 are classical, with at least the regularity needed for the strict contraction estimate in Lemma 11. For higher regularity in the interior of Q we will rely on standard parabolic theory via bootstrap arguments.

The idea is to obtain uniform local bounds on L^2 norms of the solutions n_h of the flux-cut-off problem, the associated fluxes J_h in (B.2), and certain space-time derivatives $\partial^\alpha n_h$, $\partial^\beta J_h$. These bounds will be independent of h, ϵ and the smoothing parameter κ. The local L^2 bounds on these derivatives are inherited by $\partial^\alpha n_\epsilon$, $\partial^\beta J_\epsilon$ in the limit $h \to 0$, then by $\partial^\alpha n_\kappa$, $\partial^\beta J_\kappa$ after taking $\epsilon \to 0$, and then by $\partial^\alpha n$, $\partial^\beta J$ after taking $\kappa \to 0$. Local Hölder-norm bounds for each quantity $v \in \{n, \partial_x n, \partial_t n, \partial_x^2 n\}$ in Q will follow from the local L^2 bounds on $\partial_t v$ and $\partial_x^2 v$, due to Theorem 27.

In order to achieve this, we proceed to first obtain the needed estimates for n_h and J_h, independent of h, ϵ, and κ, and then pass to the limits. Select a smooth function $\bar\eta\colon \mathbb{R} \to [0, \infty)$, convex and nondecreasing with $0 = \bar\eta(0) < \bar\eta(x) \le x$ for $x > 0$. Weighted energy estimates with weight $\eta(x) = \bar\eta(x - ma)$ will yield uniform estimates in $L^2(W_m)$, where the sets $W_m \subset [\epsilon, 1] \times [s, T]$ have the form

$$W_m = [(m + 1)a, 1] \times [ms, T], \qquad m = 1, 2, \dots, \tag{C.1}$$

for $a, s > 0$ arbitrary but fixed independent of h, ϵ, and κ.

0. As a preliminary step, we seek a uniform pointwise bound on n_h independent of h, ϵ, and κ, in domains of the form

$$[\epsilon, 1] \times [\tau, \infty), \qquad \tau > 0. \tag{C.2}$$

From the form of J_h and J_ϵ it follows that

$$\begin{aligned} n_h(x,t) - n_\epsilon(x,t) &= n_h(1/2,t) - n_\epsilon(1/2,t) + \int_{1/2}^x \frac{1}{y^2}(J_h - J_\epsilon)\,dy \\ &\quad + \int_{1/2}^x \left[\frac{2}{y}(n_h - n_\epsilon) - \frac{1}{y^2}(n_h^2 - n_\epsilon^2)\right] dy \quad + \int_{1-2h}^x \frac{1}{y^2}(n_h^2 - 3n_h)\chi_h\,dy. \end{aligned}$$

Using the uniform convergence of n_h to n_ϵ in $[\epsilon, 1 - \hat\epsilon] \times [0, T]$ (proven previously) and of J_h to J_ϵ in $[\epsilon, 1] \times [0, T]$, as well as the bounds on n_h in Lemma 33 and on $n_\epsilon \le S(x, t)$ in Lemma 15, we obtain the uniform convergence of n_h toward n_ϵ as $h \to 0$. Therefore, we get the following uniform pointwise bound independent of h, ϵ, and κ: For any $\tau > 0$, for sufficiently small $h > 0$ we have

$$0 < n_h(x,t) \le M_\tau = \max_{x\in[0,1]} S(x,\tau) + 1, \qquad (x,t) \in [\epsilon, 1] \times [\tau, \infty). \tag{C.3}$$

(Here and below, the required smallness of h depends on κ, because the bound in Lemma 15 depends on κ. But we will not mention this further.)

1. Next we proceed to obtain bounds using weighted energy estimates. The weighted energy estimate with $\eta(x) = \bar{\eta}(x-a)$ is

$$\begin{aligned}\frac{d}{dt}\int_\epsilon^1 \frac{1}{2}\eta^2 n_h^2\,dx &= \int_\epsilon^1 \eta^2 n_h\,\partial_x J_h\,dx = -\int_\epsilon^1 (\eta^2\partial_x n_h + 2\eta\eta' n_h)J_h\,dx \\ &= -\int_a^1 (\eta^2\partial_x n_h + 2\eta\eta' n_h)(x^2\partial_x n_h - 2xn_h + n_h^2 + (3n_h - n_h^2)\chi_h)\,dx \\ &\le -\frac{1}{2}\int_a^1 (x\eta\,\partial_x n_h)^2\,dx + C\int_a^1 (n_h + n_h^2)^2\,dx.\end{aligned}$$

By integration over $t \in [s,T]$ and using (C.3), we infer that

$$\begin{aligned}&\int_s^T\int_{2a}^1 (x\eta\,\partial_x n_h)^2 dx\,dt \\ &\qquad \le \int_\epsilon^1 \eta^2 n_h^2(x,s)\,dx + C\int_s^T\int_a^1 (n_h + n_h^2)^2\,dx\,dt \\ &\qquad \le C_s\,, \end{aligned} \tag{C.4}$$

where C_s may depend on s (and T, but we suppress this dependence) but is independent of h, ϵ, κ. Because $\eta(2a) > 0$, we conclude that $\partial_x n_h$, hence also J_h, is uniformly bounded independent of h, ϵ and κ in $L^2(W_1)$ (with a bound that depends on a and s).

2. The cut-off flux J_h satisfies

$$\partial_t J_h = x^2\partial_x^2 J_h + \beta\partial_x J_h\,, \tag{C.5}$$

with boundary condition $J_h(1,t) = 0$ for $t > 0$, where

$$\beta(x,t) = -2x + 2n_h + (3 - 2n_h)\chi_h\,.$$

Multiply by $\eta^2 J_h$ with $\eta(x) = \bar{\eta}(x-2a)$, integrate by parts, and use the inequality $uv \le \frac{1}{4}u^2 + v^2$ to obtain

$$\begin{aligned}&\frac{d}{dt}\int_\epsilon^1 \frac{1}{2}\eta^2 J_h^2\,dx \\ &\quad = -\int_\epsilon^1 (x\eta\,\partial_x J_h)^2\,dx + \int_\epsilon^1 J_h(\partial_x J_h)(\eta^2\beta - \partial_x(x^2\eta^2))\,dx \\ &\quad \le -\int_\epsilon^1 (x\eta\,\partial_x J_h)^2\,dx + \int_\epsilon^1 |J_h\partial_x J_h|\cdot 2x\eta C_s\,dx \\ &\quad \le -\frac{1}{2}\int_\epsilon^1 (x\eta\,\partial_x J_h)^2\,dx + C_s\int_{2a}^1 J_h^2\,dx\,.\end{aligned}$$

Integrating over $t \in [\tau, T]$ first, then averaging over $\tau \in [s, 2s]$, we obtain

$$\begin{aligned}&\int_{2s}^T\int_{3a}^1 (x\eta\,\partial_x J_h)^2 dx\,dt \\ &\qquad \le \frac{1}{s}\int_s^{2s}\int_{2a}^1 \eta^2 J_h^2(x,\tau)dx\,d\tau + C_s\int_s^T\int_{2a}^1 J_h^2\,dx\,dt \\ &\qquad \le C(a,s). \end{aligned} \tag{C.6}$$

Here we have used $|J_h|^2 \le C(x^2|\partial_x n_h|^2 + n_h^2 + n_h^4)$, (C.4), and the upper bound on n_h in (C.3). We conclude that $\partial_x J_h$ is uniformly bounded in $L^2(W_2)$, independent of h, ϵ, κ. Thus $\partial_t n_h$ (but not $\partial_x^2 n_h$) is uniformly bounded in the same L^2 sense.

3. Let us write $n_1 = \partial_t n_h = \partial_x J_h$. Then for $t > 0$,

$$\partial_t n_1 = \partial_x J_1, \qquad J_1(1,t) = 0, \tag{C.7}$$

where

$$J_1 = \partial_t J_h = x^2 \partial_x n_1 + \beta n_1. \tag{C.8}$$

Note that the validity of the zero-flux condition $J_1(1,t) = 0$ is implied by the Hölder continuity of J_1. To see this is valid, set $v = J_h - n_h^2(1-\chi_h)$. From (B.8) it follows that v solves

$$\partial_t v - x^2 \partial_x^2 v = F$$

subject to homogeneous boundary conditions, where the source term

$$F = \partial_x J_h(-2x + 3\chi_h) + x^2 \partial_x^2 (n_h^2(1-\chi_h)).$$

From the results in Appendix B, F is locally Hölder-continuous on $[\epsilon,1] \times (0,\infty)$. Hence, $J_1 = \partial_t v + 2n_h \partial_t n_h (1-\chi_h)$ is continuous up to $x = 1$.

Multiply (C.7) by $\eta^2 n_1$ with $\eta(x) = \bar\eta(x - 3a)$, and integrate in x over $[\epsilon, 1]$ to obtain

$$\begin{aligned}
\frac{d}{dt}\int_\epsilon^1 \frac{1}{2}\eta^2 n_1^2\,dx &= -\int_\epsilon^1 (\eta^2\partial_x n_1 + 2\eta\eta' n_1)(x^2\partial_x n_1 + \beta n_1)\,dx \\
&\le -\int_\epsilon^1 (x\eta\,\partial_x n_1)^2\,dx + \int_\epsilon^1 \Big((2x^2\eta\eta' + |\beta|\eta^2)|n_1\partial_x n_1| + 2\eta\eta'|\beta|n_1^2\Big)\,dx \\
&\le -\frac{1}{2}\int_\epsilon^1 (x\eta\,\partial_x n_1)^2\,dx + C_s\int_{3a}^1 n_1^2\,dx\,.
\end{aligned}$$

Integrating over $t \in [\tau, T]$ first, then over $\tau \in [2s, 3s]$, we obtain

$$\begin{aligned}
&\int_{3s}^T \int_{4a}^1 (x\eta\,\partial_x n_1)^2 dx\,d\tau \\
&\qquad \le \frac{1}{s}\int_{2s}^{3s}\int_{3a}^1 \eta^2 n_1^2(x,\tau)\,dx\,d\tau + C_s\int_{2s}^T\int_{3a}^1 n_1^2\,dx\,d\tau \\
&\qquad \le C(a,s),
\end{aligned} \tag{C.9}$$

where the bound on $n_1 = \partial_x J_h$ in (C.6) from step 2 has been used. We conclude that $\partial_x^2 J_h$ and $\partial_t J_h$ (by (C.5)) are uniformly bounded independent of h, ϵ, and κ in L^2 on W_3, hence J_h is uniformly Hölder continuous on W_3.

4. Next we compute $\partial_t J_1$ to complete the estimates for classical solutions. Differentiating (C.8) with respect to t we find that for $t > 0$,

$$\partial_t J_1 = x^2 \partial_x^2 J_1 + \beta\partial_x J_1 + \partial_t\beta\, n_1, \quad J_1(1,t) = 0. \tag{C.10}$$

Recall that $|\beta| \le C_s$ and note $\partial_t \beta = 2(1-\chi_h)\partial_x J_h$, hence $|\partial_t\beta| \le 2|n_1|$. Multiply by $\eta^2 J_1$ with $\eta(x) = \bar\eta(x - 4a)$, and integrate by parts to find

$$
\begin{aligned}
\frac{d}{dt}\int_\epsilon^1 \frac{1}{2}\eta^2 J_1^2\,dx &+ \int_\epsilon^1 (x\eta\,\partial_x J_1)^2\,dx \\
&= \int_\epsilon^1 (\beta\eta^2 - \partial_x(x^2\eta^2))J_1(\partial_x J_1) + \eta^2 J_1 \partial_t \beta\, n_1\,dx \\
&\le \frac{1}{2}\int_\epsilon^1 (x\eta\,\partial_x J_1)^2\,dx + C_s\left(\int_{4a}^1 J_1^2\,dx + \int_{4a}^1 n_1^4\,dx\right).
\end{aligned}
$$

Integrating over $t \in [\tau, T]$ first, then averaging over $\tau \in [3s, 4s]$, we obtain

$$
\begin{aligned}
\int_{4s}^T \int_{5a}^1 (x\eta\,\partial_x J_1)^2 dx\,dt &\le \frac{1}{s}\int_{3s}^{4s}\int_{4a}^1 \eta^2 J_1^2(x,\tau)\,dx\,d\tau \\
&\quad + C_s \int_{3s}^T \left(\int_{4a}^1 J_1^2\,dx + \int_{4a}^1 n_1^4\,dx\right) dt \\
&\le C_s \int_{3s}^T \int_{4a}^1 (|\partial_x n_1|^2 + |\partial_x J_h|^2 + |n_1|^4)\,dx\,dt.
\end{aligned}
$$

The first two terms are bounded using the bounds from the previous steps, (C.6) and (C.9). Note also that $n_1 = \partial_x J_h$ is in $L^4(W_3)$ due to an anisotropic embedding theorem. Hence

$$
\int_{4s}^T \int_{5a}^1 (x\eta\,\partial_x J_1)^2 dx\,d\tau \le C(a,s). \tag{C.11}
$$

We can conclude that $\partial_x J_1$ $(= \partial_x\partial_t J_h = \partial_t \partial_x J_h = \partial_t^2 n_h)$ is bounded in $L^2(W_4)$ independent of h, ϵ, and κ.

5. After taking the limit $h \to 0$ along a suitable subsequence, we conclude from steps 1 and 2 that $\partial_t n_\epsilon = \partial_x J_\epsilon$ is uniformly bounded in $L^2(W_2)$, hence the same is true of $\partial_x^2 n_\epsilon$ due to the form of J_ϵ in (4.5). By Theorem 27, n_ϵ is uniformly Hölder-continuous on W_2, independent of ϵ and κ.

Next we conclude from step 3 that J_ϵ is uniformly Hölder-continuous on W_3, and the same is true of $\partial_x n_\epsilon$ by (4.5).

From step 4 we then conclude $\partial_t \partial_x J_\epsilon$ is uniformly bounded in $L^2(W_4)$ and by differentiating (4.5) we conclude the same for $\partial_x^3 J_\epsilon$. Therefore $\partial_x J_\epsilon = \partial_t n_\epsilon$ is uniformly Hölder-continuous on W_4, and the same holds for $\partial_x^2 n_\epsilon$.

After taking the limits $\epsilon \to 0$, and finally $\kappa \to 0$, these estimates ensure that the weak solution n of Theorem 4 is a classical solution in $Q = (0,1] \times (0,\infty)$, with the local Hölder regularity indicated in Theorem 5.

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REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Pure Appl. Math. 65, Academic Press New York, 1975.

[2] O. V. Besov, V. P. Il'in, and S. M. Nikol'skiĭ, *Integral Representations of Functions and Imbedding Theorems*, V. H. Winston & Sons, Washington, D.C., 1978.

[3] R. E. Caflisch and C. D. Levermore, *Equilibrium for radiation in a homogeneous plasma*, Phys. Fluids, 29 (1986), pp. 748–752, http://dx.doi.org/10.1063/1.865928.

[4] J. A. Carrillo, M. Di Francesco, and G. Toscani, *Condensation Phenomena in Nonlinear Drift Equations*, arXiv:1307.2275, 2013.

[5] G. Cooper, *Compton Fokker-Planck equation for hot plasmas*, Phys. Rev. D, 3 (1971), http://dx.doi.org/10.1103/PhysRevD.3.2312.

[6] M. Escobedo, M. A. Herrero, and J. J. L. Velazquez, *A nonlinear Fokker-Planck equation modelling the approach to thermal equilibrium in a homogeneous plasma*, Trans. Amer. Math. Soc., 350 (1998), pp. 3837–3901, http://dx.doi.org/10.1090/S0002-9947-98-02279-X.

[7] M. Escobedo and S. Mischler, *On a quantum Boltzmann equation for a gas of photons*, J. Math. Pures Appl. (9), 80 (2001), pp. 471–515, http://dx.doi.org/10.1016/S0021-7824(00)01201-0.

[8] M. Escobedo, S. Mischler, and M. A. Valle, *Homogeneous Boltzmann Equation in Quantum Relativistic Kinetic Theory*, Electron. J. Differ. Equ. Monogr. 4, Southwest Texas State University, San Marcos, TX, 2003.

[9] M. Escobedo, S. Mischler, and J. J. L. Velazquez, *Asymptotic description of Dirac mass formation in kinetic equations for quantum particles*, J. Differential Equations, 202 (2004), pp. 208–230, http://dx.doi.org/10.1016/j.jde.2004.03.031.

[10] M. Escobedo and J. J. L. Velázquez, *Finite time blow-up and condensation for the bosonic Nordheim equation*, Invent. Math., 200 (2015), pp. 761–847, http://dx.doi.org/10.1007/s00222-014-0539-7.

[11] E. Ferrari and A. Nouri, *On the Cauchy problem for a quantum kinetic equation linked to the Compton effect*, Math. Comput. Model., 43 (2006), pp. 838–853, http://dx.doi.org/10.1016/j.mcm.2005.09.034.

[12] C. Josserand, Y. Pomeau, and S. Rica, *Self-similar singularities in the kinetics of condensation*, J. Low Temp. Phys., 145 (2006), pp. 231–265, http://dx.doi.org/10.1007/s10909-006-9232-6.

[13] A. Jüngel and M. Winkler, *A degenerate fourth-order parabolic equation modeling Bose-Einstein condensation. Part* I*: Local existence of solutions*, Arch. Ration. Mech. Anal., 217 (2015), pp. 935–973, http://dx.doi.org/10.1007/s00205-015-0847-0.

[14] A. Jüngel and M. Winkler, *A degenerate fourth-order parabolic equation modeling Bose-Einstein condensation Part* II*: Finite-time blow-up*, Comm. Partial Differential Equations, 40 (2015), pp. 1748–1786, http://dx.doi.org/10.1080/03605302.2015.1043558.

[15] G. Kaniadakis and P. Quarati, *Kinetic equation for classical particles obeying an exclusion principle*, Phys. Rev. E, 48 (1993), pp. 4263–4270, http://dx.doi.org/10.1103/PhysRevE.48.4263.

[16] G. Kaniadakis and P. Quarati, *Classical model of bosons and fermions*, Phys. Rev. E, 49 (1994), pp. 5103–5110, http://dx.doi.org/10.1103/PhysRevE.49.5103.

[17] O. Kavian, *Remarks on the Kompaneets equation, a simplified model of the Fokker-Planck equation*, in Non-linear Partial Differential Equations and their Applications, Collége de France Seminar, Stud. Math. Appl. 31, D. Cioranescu and J. L. Lions, eds, North-Holland Elsevier, 2002, Vol. 14, pp. 467–487.

[18] O. Kavian and C. D. Levermore, *The Kompaneets equation: A semi-linear parabolic equation with blow-up*, unpublished.

[19] J. Klaers, J. Schmitt, F. Vewinger, and M. Weitz, *Bose-Einstein condensation of photons in an optical microcavity*, Nature, 468 (2010), pp. 545–548, http://dx.doi.org/10.1038/nature09567.

[20] A. S. Kompaneets, *The establishment of the thermal equilibrium between quanta and electrons*, J. Exp. Theor. Phys., 4 (1957), pp. 730–737.

[21] N. V. Krylov, *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*, Grad. Stud. Math. 12, AMS, Providence, RI, 1996, http://dx.doi.org/10.1090/gsm/012.

[22] R. Lacaze, P. Lallemand, Y. Pomeau, and S. Rica, *Dynamical formation of a Bose-Einstein condensate*, Phys. D, 152/153 (2001), pp. 779–786, http://dx.doi.org/10.1016/S0167-2789(01)00211-1.

[23] X. Lu, *The Boltzmann equation for Bose-Einstein particles: Condensation in finite time*, J. Stat. Phys., 150 (2013), pp. 1138–1176, http://dx.doi.org/10.1007/s10955-013-0725-9.

[24] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Prog. Nonlinear Differential Equations Appl. 16, Birkhäuser Verlag, Basel, 1995, http://dx.doi.org/10.1007/978-3-0348-9234-6.

[25] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984, http://dx.doi.org/10.1007/978-1-4612-5282-5.

[26] G. Rybicki, *A new kinetic equation for Compton scattering*, Astrophys. J., 584 (2003), pp. 528–540, http://dx.doi.org/10.1086/345683.

[27] D. Semikoz and I. Tkachev, *Condensation of bosons in the kinetic regime*, Phys. Rev. D, 55 (1997), pp. 489–502, http://dx.doi.org/10.1103/PhysRevD.55.489.

[28] D. V. Semikoz and I. I. Tkachev, *Kinetics of Bose-condensation*, Phys. Rev. Lett., 74 (1995), pp. 3093–3097, http://dx.doi.org/10.1103/PhysRevLett.74.3093.

[29] H. Spohn, *Kinetics of the Bose-Einstein condensation*, Phys. D, 239 (2010), pp. 627–634, http://dx.doi.org/10.1016/j.physd.2010.01.018.

[30] G. Toscani, *Finite time blow up in Kaniadakis-Quarati model of Bose-Einstein particles*, Comm. Partial Differential Equations, 37 (2012), pp. 77–87, http://dx.doi.org/10.1080/03605302.2011.592236.

[31] Y. B. Zel'Dovic and E. V. Levich, *Bose condensation and shock waves in photon spectra*, J. Exp. Theor. Phys., 28 (1969), pp. 1287–1290.