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Threshold for shock formation in the hyperbolic Keller–Segel model

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ABSTRACT

We identify a sub-threshold for finite time shock formation in solutions to the one-dimensional hyperbolic Keller–Segel model. The main result states that under some assumptions on the initial potential, if the slope of the initial density is above a threshold at even one location, the solution must become discontinuous in finite time.

1. Introduction

In this work we investigate the hyperbolic Keller–Segel model with logistic sensitivity of the form

$$\begin{cases} \partial_t u + \partial_x (f(u)\partial_x S) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(t=0) = u_0, & 0 \le u_0 \le 1, \\ -\partial_x^2 S + S = u. \end{cases} \tag{1.1}$$

This system of equations models the cell motion with a collective chemotactic attraction through the chemical potential S governed by the Poisson equation. The unknown is the cell density $u = u(t,x)$, with initial data $u_0(x)$. The function $f(u)$ given by

$$f(u) = u(1-u),$$

ensures the a priori bounds for the cell density, $0 \le u(t,x) \le 1$, therefore we restrict ourselves to solutions within $[0,1]$.

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This model was proposed in [1] to highlight the biological observations as wave propagation, which is in contrast to the parabolic version of the Keller–Segel system [2]. The logistic sensitivity $(1-u)$ takes into account the so called volume filling effect, which prevents overcrowding [3,4].

This model falls within a general class of non-local conservation laws with the nonlinear advection coupling both local and non-local mechanism, i.e.,

$$\begin{cases} \partial_t u + \partial_x F(u, \bar{u}) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Here, u is the unknown, F is a given smooth function, and $\bar{u}$ is given by

$$\bar{u}(t,x) = (K * u)(t,x) = \int_{\mathbb{R}} K(x-y)u(t,y)\,dy. \tag{1.3}$$

This class of conservation laws, identified in [5], appears in several applications including traffic flows [6,7], the collective motion of biological cells [1,8,9], dispersive water waves [10–13], the radiating gas motion [14–16], high-frequency waves in relaxing medium [17–19] and the kinematic sedimentation model [20–22].

One of aspects of our interest in this class of non-local conservation laws is the critical threshold phenomena, as in many other hyperbolic balance laws. As is known that the typical well-posedness result asserts that either a solution of a time-dependent PDE exists for all time or else there is a finite time such that some norm of the solution becomes unbounded as the life span is approached. The natural question is whether there is a critical threshold for the initial data such that the persistence of the C^1 solution regularity depends only on crossing such a critical threshold. This concept of critical threshold (CT) and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [23–25] for a class of Euler–Poisson equations. Following their CT concept, the authors [5] identified sub-thresholds for finite time shock formation in a class of nonlocal conservation laws, which we summarize here. Under the following two assumptions:

(H_1) $F \in C^3(\mathbb{R}, \mathbb{R})$, and the kernel $K(r) \in W^{1,1}$ satisfying

$$K(r) = \begin{cases} Nondecreasing, & r \le 0, \\ 0, & r > 0. \end{cases}$$

(H_2) $F(0,\cdot) = F(m,\cdot) = 0$ and

$$F_{uu} < 0, \quad F_{\bar{u}\bar{u}} > 0, \quad F_{\bar{u}} < 0 \quad \text{for } u \in [0,m],$$

we have the following result.

Theorem 1.1 *([5]). Consider (1.2) with (1.3) under assumptions (H_1)–(H_2). If $u_0 \in H^2$ and $0 \le u_0(x) \le m$ for all $x \in \mathbb{R}$, then there exists a non-increasing function $\lambda(\cdot)$ such that if*

$$\sup_{x\in\mathbb{R}}[u_0'(x)] > \lambda(\inf_{x\in\mathbb{R}}[u_0'(x)]),$$

then u_x must blow up at some finite time.

There is some distinguished special case of (1.2):

• Traffic flow model with Arrhenius look-ahead dynamics proposed by [7],

$$\partial_t u + \partial_x(u(1-u)e^{-K*u}) = 0, \tag{1.4}$$

where $u(t,x)$ represents a vehicle density normalized in the interval $[0,1]$ and the relaxation kernel

$$K(r) = \begin{cases} \dfrac{K_0}{\gamma}, & \text{if } -\gamma \le r \le 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

is the constant interaction potential, where γ is a positive constant proportional to the look-ahead distance and K_0 is a positive interaction strength. This model corresponds to (1.2) with $F(u,\bar{u}) = u(1-u)e^{-\bar{u}}$. In a recent work, D. Li and T. Li [26] present several finite time shock formation scenarios for solutions to (1.4) with (1.5). An improved linear interaction potential for (1.4) is introduced in [6]. The improved interaction potential is intended to take into account the fact that a vehicle's speed is affected more by nearby vehicles than distant ones. Sub-thresholds for finite time shock formation in the traffic flow models with constant interaction potential and the improved linear interaction potential are identified in [5].

Note that (1.1) corresponds to (1.2) with

$$F(u,\bar{u}) = u(1-u)\bar{u} \quad \text{and} \quad K(r) = \partial_r\left(\frac{e^{-|r|}}{2}\right). \tag{1.6}$$

However, this kernel does not satisfy condition (H_1), hence the result in Theorem 1.1 does not apply to (1.1) directly. Our goal of this work is to prove the following result.

Theorem 1.2. *Consider the system (1.1). Suppose that $u_0 \in H^2$ and $0 \le u_0(x) \le 1$ for all $x \in \mathbb{R}$. If there exist an x and a constant μ such that $0 < \mu < \partial_x S(0,x)$ and*

$$\partial_x u_0(x) > \frac{1}{\mu}\left(\frac{4+3\sqrt{2}}{8}\right)\left(1-\exp\left(-\frac{4+3\sqrt{2}}{8}(\partial_x S(0,x)-\mu)\right)\right)^{-1},$$

then u_x must blow up at some finite time.

Remark 1.3. Note that $\partial_x S(0,x)$, though not explicitly given, can be evaluated from the initial data $u_0(x)$ as follows

$$\partial_x S(0,x) = \frac{1}{2}\int_0^\infty e^{-y}(u_0(x+y)-u_0(x-y))dy,$$

which lies between $[-1/2,1/2]$ for $0 \le u_0 \le 1$ (see Lemma 2.1). The existence of μ is possible unless $u_0(x)$ is strictly decreasing.

The threshold analysis to be carried out is the a priori estimate on smooth solutions as long as they exist. For the hyperbolic Keller–Segel problem (1.1), local existence of smooth solutions was established in [1]; see also [27] for the multi-dimensional case. For the general class of nonlocal conservation laws (1.2), C^1 solution regularity was established in [5] by some standard contraction argument, which we summarize here.

Theorem 1.4 (*Local existence [5]*)**.** *Suppose $F \in C^3(\mathbb{R},\mathbb{R})$ and $K \in W^{1,1}$. If $u_0 \in H^2$, or $u_0 \in L^\infty$ and $u_{0x} \in H^1$, then there exists $T > 0$, depending on the initial data, such that (1.2) admits a unique solution $u \in C^1([0,T)\times\mathbb{R})$. Moreover, if the maximum life span $T^* < \infty$, then*

$$\lim_{t\to T^*-} \|\partial_x u(t,\cdot)\|_{L^\infty} = \infty.$$

One can verify that the flux function and the kernel (1.6) in the hyperbolic Keller–Segel model (1.1) satisfy the assumptions of Theorem 1.4, hence this C^1 solution regularity result applies to (1.1). In case there is a finite time singularity formation, blow up of solutions to (1.1) must first occur in the first derivative, as stated in Theorem 1.4.

The details of the proof of Theorem 1.2 is carried out in the rest of the paper.

2. Proof of Theorem 1.2

By solving the third equation of (1.1), we obtain

$$S(t,x) = \frac{1}{2}\int_{-\infty}^\infty e^{-|x-y|}u(t,y)\,dy,$$

with which the first equation in (1.1) may be written as

$$\partial_t u + \partial_x(u(1-u)\bar{u}) = 0, \tag{2.1}$$

where

$$\begin{aligned}\bar{u}(t,x) = \partial_x S(t,x) &= \frac{1}{2}\left(-\int_{-\infty}^{x} e^{-x+y}u(t,y)\,dy + \int_{x}^{\infty} e^{x-y}u(t,y)\,dy\right)\\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|z|}\partial_x u(t,x+z)\,dz.\end{aligned} \tag{2.2}$$

Let $d := \partial_x u$ and apply ∂_x to (2.1), then we obtain

$$\begin{aligned}\dot{d} := (\partial_t + (1-2u)\bar{u}\partial_x)d &= 2\bar{u}d^2 - 2(1-2u)\partial_x\bar{u}d - (u-u^2)\partial_x^2\bar{u}\\ &= 2\bar{u}d^2 + \{-2(1-2u)\partial_x\bar{u} + (u-u^2)\}d - (u-u^2)\bar{u}.\end{aligned} \tag{2.3}$$

We first estimate the a priori bounds of $\bar{u}$ and $\partial_x\bar{u}$.

Lemma 2.1. *For any $t \geq 0$,*

$$-\frac{1}{2} \leq \bar{u}(t,x) \leq \frac{1}{2} \quad \text{and} \quad -1 \leq \partial_x\bar{u}(t,x) \leq 1.$$

Proof. We use the *a priori* bound $0 \leq u(t,x) \leq 1$. From (2.2),

$$\begin{aligned}\bar{u}(t,x) &= \frac{1}{2}\left(-\int_{-\infty}^{x} e^{-x+y}u(t,y)\,dy + \int_{x}^{\infty} e^{x-y}u(t,y)\,dy\right)\\ &\leq \frac{1}{2}\left(-\int_{-\infty}^{x} e^{-x+y}\cdot 0\,dy + \int_{x}^{\infty} e^{x-y}\cdot 1\,dy\right)\\ &= \frac{1}{2}.\end{aligned}$$

The lower bound of $\bar{u}$ is obtained similarly.

For the bounds of $\partial_x\bar{u}$, consider

$$\begin{aligned}\partial_x\bar{u}(t,x) &= \frac{1}{2}\left(\int_{-\infty}^{x} e^{-x+y}u(t,y)\,dy + \int_{x}^{\infty} e^{x-y}u(t,y)\,dy - 2u(t,x)\right)\\ &\leq \frac{1}{2}\left(\int_{-\infty}^{x} e^{-x+y}\cdot 1\,dy + \int_{x}^{\infty} e^{x-y}\cdot 1\,dy - 2\cdot 0\right)\\ &\leq 1.\end{aligned}$$

The lower bound of $\partial_x\bar{u}$ is obtained similarly. □

We then trace the evolution of $\bar{u}$ along the same characteristic. That is, we estimate $\dot{\bar{u}} = \partial_t\bar{u} + (1-2u)\bar{u}\partial_x\bar{u}$.

Lemma 2.2. *For $t > 0$, it holds*

$$\dot{\bar{u}}(t,x) \geq -\frac{3}{4}. \tag{2.4}$$

Proof. We first estimate $\partial_t\bar{u}$.

$$\begin{aligned}\partial_t\bar{u} &= \frac{1}{2}\partial_t\left(-\int_{-\infty}^{x} e^{-x+y}u(t,y)\,dy + \int_{x}^{\infty} e^{x-y}u(t,y)\,dy\right)\\ &= \frac{1}{2}\left(-\int_{-\infty}^{x} e^{-x+y}\partial_t u(t,y)\,dy + \int_{x}^{\infty} e^{x-y}\partial_t u(t,y)\,dy\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\int_{-\infty}^{x} e^{-x+y}\partial_y\big(f(u)S_y\big)\,dy - \int_{x}^{\infty} e^{x-y}\partial_y\big(f(u)S_y\big)\,dy\right)\\
&= \frac{1}{2}\bigg(e^{-x}\big[e^{y}f(u)\bar{u}\big]_{-\infty}^{x} - e^{-x}\int_{-\infty}^{x} e^{y}f(u)\bar{u}\,dy\\
&\quad - e^{x}\big[e^{-y}f(u)\bar{u}\big]_{x}^{\infty} + e^{x}\int_{x}^{\infty} -e^{-y}f(u)\bar{u}\,dy\bigg)\\
&= \frac{1}{2}\left(f(u)\bar{u} - e^{-x}\int_{-\infty}^{x} e^{y}f(u)\bar{u}\,dy + f(u)\bar{u} - e^{x}\int_{x}^{\infty} e^{-y}f(u)\bar{u}\,dy\right)\\
&\geq \frac{1}{2}\left(\frac{1}{4}\cdot\left(-\frac{1}{2}\right) - e^{-x}\cdot\frac{1}{4}\cdot\frac{1}{2}\int_{-\infty}^{x} e^{y}\,dy + \frac{1}{4}\cdot\left(-\frac{1}{2}\right) - e^{x}\cdot\frac{1}{4}\cdot\frac{1}{2}\int_{x}^{\infty} e^{-y}\,dy\right)\\
&= -\frac{1}{4}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot{\bar{u}} &= \partial_t\bar{u} + (1-2u)\bar{u}\partial_x\bar{u}\\
&\geq -\frac{1}{4} + (1-2u)\bar{u}\partial_x\bar{u}\\
&\geq -\frac{1}{4} + (-1)\left(-\frac{1}{2}\right)(-1) = -\frac{3}{4}.
\end{aligned}
$$

Here in the last inequality, the bounds in Lemma 2.1 are used. This proves the lemma. □

From (2.3), i.e.,

$$\dot{d} = 2\bar{u}d^2 + \{-2(1-2u)\partial_x\bar{u} + (u-u^2)\}d - (u-u^2)\bar{u}, \tag{2.5}$$

it follows that

$$\dot{d} = 2\bar{u}\big(d(t) - R_1(t)\big)\big(d(t) - R_2(t)\big), \tag{2.6}$$

where $R_2(t)(\geq R_1(t))$ is given by

$$R_2 = \frac{-\{-2(1-2u)\partial_x\bar{u} + (u-u^2)\} + \sqrt{\{-2(1-2u)\partial_x\bar{u} + (u-u^2)\}^2 + 8(u-u^2)\bar{u}^2}}{4\bar{u}}. \tag{2.7}$$

From (2.5), if $\bar{u} > 0$, then R_2 is non-negative and R_1 is non-positive because $-(u-u^2)\bar{u} \leq 0$. Integration over $[0,t]$ of (2.4) gives

$$\bar{u}(t) \geq -\frac{3}{4}t + \bar{u}_0, \quad t > 0.$$

For $\mu \in (0, \bar{u}_0)$, and

$$t^* = \frac{4}{3}(\bar{u}_0 - \mu),$$

we see that μ serves as a short time lower bound of $\bar{u}$, since

$$\bar{u}(t) \geq \mu > 0, \quad \text{for } t \in [0, t^*].$$

We claim that $R_2(t)$ has a uniform upper bound, for $t \in [0,t^*]$,

$$R_2(t) \leq \frac{1}{\mu}\cdot\left(\frac{4+3\sqrt{2}}{8}\right) =: R_M. \tag{2.8}$$

This can be derived from directly estimating terms in (2.7),

$$R_2(t) \leq \frac{1}{4\mu}\left(2 + \sqrt{2^2 + 8\cdot\frac{1}{4}\cdot(1/2)^2}\right) = R_M.$$

Here, we use the bounds in Lemma 2.1 and the fact $\max\{-2(1-2u)\bar{u}_x + (u-u^2)\} = 2$, which can be verified easily since the underlying function is linear in $\bar{u}_x$ and quadratic in u.

Finding the blow-up condition of system (2.6) is carried out by comparison with the following equation:

$$\dot{\beta} = 2\mu(\beta(t) - R_M)\beta(t). \tag{2.9}$$

Lemma 2.3.

$$R_M < \beta(0) < d(0) \quad \textit{implies that} \quad R_M < \beta(t) < d(t),$$

$\forall t \in [0, t^*)$.

Proof. The monotonicity relation $R_M < \beta(t)$ is straightforward from (2.9). The inequality $\beta(t) < d(t)$, $t \in [0, t^*)$ can be proved by contradiction. Suppose that $t_1 < t^*$ is the earliest time when the above lemma is violated. Then $\beta(t_1) = d(t_1)$. Consider

$$(\dot{d} - \dot{\beta})\Big|_{t=t_1} = 2\bar{u}(t_1)\big(d(t_1) - R_1(t_1)\big)\big(d(t_1) - R_2(t_1)\big) - 2\mu\big(\beta(t_1) - R_M\big)\beta(t_1). \tag{2.10}$$

Since $d(t) - \beta(t) > 0$ for $t < t_1$ and $d(t_1) - \beta(t_1) = 0$, at $t = t_1$ we have

$$(\dot{d} - \dot{\beta})\Big|_{t=t_1} \leq 0.$$

But $\bar{u}(t_1) > \mu$,

$$d(t_1) - R_1(t_1) = \beta(t_1) - R_1(t_1) \geq \beta(t_1)$$

and

$$d(t_1) - R_2(t_1) = \beta(t_1) - R_2(t_1) \geq \beta(t_1) - R_M.$$

These three inequalities and the fact that $\beta(t_1) > 0$ and $\beta(t_1) - R_M > 0$ lead to that

$$\textit{the right hand side of } (2.10) \geq 2(\bar{u}(t_1) - \mu)(\beta(t_1) - R_M)\beta(t_1) > 0.$$

So the right-hand side of (2.10) is positive, which leads to the contradiction. □

We first discuss finite-time blow up of (2.9):

Lemma 2.4. *If* $\beta(0) > R_M$, *then*

$$\beta(t) \to \infty \quad \textit{as } t \to \frac{1}{2\mu R_M} \ln\left(\frac{\beta(0)}{\beta(0) - R_M}\right).$$

Proof. By solving (2.9) directly, we obtain

$$\beta(t) = \frac{R_M \beta(0)}{\beta(0) + (R_M - \beta(0))e^{2\mu R_M t}},$$

and the desired result follows immediately. □

The last step of proving Theorem 1.2 is to combine the comparison principle in Lemma 2.3 with Lemma 2.4. For any given initial data

$$d(0) > \lambda(\bar{u}_0, \mu),$$

where

$$\lambda(\bar{u}_0, \mu) = \frac{R_M e^{\frac{4+3\sqrt{2}}{3}(\bar{u}_0 - \mu)}}{e^{\frac{4+3\sqrt{2}}{3}(\bar{u}_0 - \mu)} - 1}$$

for system (2.6), we can always find the initial data $\beta(0)$ for (2.9) such that

$$\beta(0) \in [\lambda(\bar{u}_0, \mu), d(0)].$$

Since $\beta(0) \geq \lambda(\bar{u}_0, \mu) > R_M$, by Lemma 2.4, $\beta(0)$ will lead to finite-time blow up of $\beta(t)$. We notice that the finite time blow-up occurs before t^*. Indeed, the blow-up time in Lemma 2.4, can be estimated as follows

$$\begin{aligned} \frac{1}{2\mu R_M} \ln\left(\frac{\beta(0)}{\beta(0) - R_M}\right) &\leq \frac{1}{2\mu R_M} \ln\left(\frac{\lambda(\bar{u}_0, \mu)}{\lambda(\bar{u}_0, \mu) - R_M}\right) \\ &= \frac{4}{3}(\bar{u}_0 - \mu) = t^*. \end{aligned}$$

We now have $\beta(t) \to \infty$ before t^*, which in turn by Lemma 2.3, implies that $d(t) \to \infty$ in finite time.

Therefore, we complete the proof of Theorem 1.2.

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