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Convergence rates to discrete shocks for nonconvex conservation laws

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Summary. This paper is concerned with polynomial decay rates of perturbations to stationary discrete shocks for the Lax-Friedrichs scheme approximating non-convex scalar conservation laws. We assume that the discrete initial data tend to constant states as $j \to \pm\infty$, respectively, and that the Riemann problem for the corresponding hyperbolic equation admits a stationary shock wave. If the summation of the initial perturbation over $(-\infty, j)$ is small and decays with an algebraic rate as $|j| \to \infty$, then the perturbations to discrete shocks are shown to decay with the corresponding rate as $n \to \infty$. The proof is given by applying weighted energy estimates. A discrete weight function, which depends on the space-time variables for the decay rate and the state of the discrete shocks in order to treat the non-convexity, plays a crucial role.

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1 Introduction

Let us consider a function $f \in C^3$ not necessarily convex. For states $u_-,\ u_+ \in \mathrm{IR}$, shock speed $s \in \mathrm{IR}$, space variable $x \in \mathrm{IR}$ and time variable $t \geq 0$, we consider a shock wave solution

$$u(t,x) = \begin{cases} u_-, x - st < 0, \\ u_+, x - st > 0, \end{cases}$$

to the scalar conservation law

$$(1.1) \qquad u_t + f(u)_x = 0.$$

Taking $\mu \in]0,1[,\ \lambda \in \mathrm{IR}$ to be specified later, we consider the modified Lax-Friedrichs (L-F) scheme

$$(1.2) \quad u_j^{n+1} - u_j^n + \frac{\lambda}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) = \frac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

In this paper we study the evolution of perturbations of discrete shock solutions to the L-F scheme and their decay rate. The states $u_\pm$ and related shock speed $s \in \mathrm{IR}$ must satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0.$$

We want to restrict ourselves to the case of stationary shock waves, i.e. the case $s = 0$ and therefore we have

$$(1.3a) \qquad f(u_+) = f(u_-)$$

in this case. Further, we set $Q(u) := f(u) - f(u_\pm)$. Then for any **admissible** stationary shock wave solution the Oleinik entropy condition

$$(1.3b) \quad (u_+ - u_-)Q(u) > 0, \quad \text{for } u \in]\min(u_-, u_+), \max(u_-, u_+)[$$

must hold. This condition plays an important role in proving the existence and monotonicity of discrete shock solutions, cf. Jennings [6], as well as in defining our weight function in a later argument.

It is noted that when $f'(u_\pm) \neq 0$, then (1.3b) implies Lax's shock condition

$$(1.3c) \qquad f'(u_+) < 0 < f'(u_-).$$

For convenience only we shall restrict our considerations to the case $u_+ < u_-$ throughout the paper.

The discrete solution $u^n := (u_j^n)_{j \in \mathbb{Z}}$ should become an approximation of the point values $u(x_j, t_n)$ of an exact solution to the conservation law (1.1) on the grid given by $x_j = j\Delta x$ and $t_n = n\Delta t$, with $\Delta x = r$ and $\Delta t = h$

being the spatial and the temporal mesh lengths. Further, we assume that the mesh ratio $\lambda = \frac{\Delta t}{\Delta x}$ satisfies the Courant-Friedrichs-Lewy (CFL) condition

$$\lambda \max |f'| < \mu < 1. \tag{1.4}$$

Note that by Corollary 2.3 in Tadmor [23] the assumption (1.4) implies the TVD property of the scheme (1.2).

Under the hypotheses (1.3a)-(1.3b) and (1.4), the scheme (1.2) admits a unique **stationary discrete shock solution** $(\phi_j)_{j\in\mathbb{Z}}$ which takes on a given value $u_* \in]u_+, u_-[$ at $j = 0$, i.e., it satisfies the conditions

$$\lambda(f(\phi_{j+1}) - f(\phi_{j-1})) = \mu(\phi_{j+1} - 2\phi_j + \phi_{j-1}), \tag{1.5a}$$

$$\phi_j \to u_\pm, \quad \text{as } j \to \pm\infty, \tag{1.5b}$$

$$\phi_j|_{j=0} = u_*. \tag{1.5c}$$

The existence of this discrete solution and further properties, see Lemma 2.1, have been proved by Jennings [6] provided that $u_\pm$ satisfies (1.3a)-(1.3b). Clearly ϕ_j is a one-parameter family of the discrete shock profile with parameter u^*. As shown in in [12] another equivalent parameter can be taken as the amount of the excess mass from the initial data, that is the parameter u^* can be uniquely determined by the quantity $\sum_j(\phi_j - u_j^0)$ for given data u_j^0.

Let us now define the following weighted l^2 spaces,

$$l_K^2 = \{f = (f_j)_{j\in\mathbb{Z}} : \quad ||f||_{l_K^2} \equiv |f|_K = \Big[\sum_{j\in\mathbb{Z}} |f_j|^2 K_j\Big]^{\frac{1}{2}} < \infty\},$$

where $K = (K_j)_{j\in\mathbb{Z}}$ is any discrete weight function. When for $r = \Delta x$ specifically $K_j = \langle jr\rangle^\alpha := (1 + (jr)^2)^{\frac{\alpha}{2}}$, for some $\alpha \geq 0$, we write $l_K^2 = l_\alpha^2$ and $|\cdot|_K = |\cdot|_\alpha$. We will also consider bounded weight functions $w = (w_j)_{j\in\mathbb{Z}}$ with $C^{-1} \leq w_j \leq C$ for a constant $C > 0$. They are used to define the weights $K_j = \langle jr\rangle^\alpha w_j$. In this case we write $l_K^2 = l_{\alpha,w}^2$ with the norm $|\cdot|_K = |\cdot|_{\alpha,w}$. We note that $l^2 = l_w^2$ with the norm $||\cdot|| \sim |\cdot|_w$ and that $l_{\alpha,w}^2 = l_\alpha^2$ with $|\cdot|_{\alpha,w} \sim |\cdot|_\alpha$. We will denote the difference of a discrete function $(f_j)_{j\in\mathbb{Z}}$ in space by

$$\Delta f := (f_{j+1} - f_j)_{j\in\mathbb{Z}}.$$

Now we state the main theorem in this paper.

Theorem 1. *Suppose that the assumptions (1.3a)-(1.3c) and for any given positive constant $\mu < 1$ the CFL condition (1.4) hold. Further, consider $\lambda >$*

0 suitably small. Let $(\phi_j)_{j\in\mathbb{Z}}$ be the stationary discrete shock profile defined by (1.5a)-(1.5c) connecting u_+ to u_-. Define $v_j^0 = \sum_{k=-\infty}^{j}(u_k^0 - \phi_k)$ assuming that the mass of the perturbation satisfies

$$\sum_{j\in\mathbb{Z}}(u_j^0 - \phi_j) = 0, \tag{1.6}$$

from which u^* in (1.5c) is uniquely determined. Consider that for some $\alpha > 0$ the spatial decay rate

$$|v^0|_\alpha \le \delta_1 \tag{1.7}$$

with some constant $\delta_1 > 0$ is given. Then the unique global solution $(u_j^n)_{j\in\mathbb{Z}}$ to the L-F scheme (1.2) with the initial data $(u_j^0)_{j\in\mathbb{Z}}$ deviates in the maximum norm from the shock profile $(\phi_j)_{j\in\mathbb{Z}}$ by at most

$$\sup_{j\in\mathbb{Z}}|u_j^n - \phi_j| \le C(1+nh)^{-\alpha/2}|v^0|_\alpha, \quad n \ge 0. \tag{1.8}$$

This means that the perturbation decays at the rate $\alpha/2$ in the maximum norm for $n \to \infty$. □

Remark 1. We point that a sufficient condition for (1.7) to hold is that there exists a constant $\kappa > \frac{\alpha}{2} + 1$ such that the estimate

$$\sum_{j\in\mathbb{Z}}(1+j^2)^\kappa|u_j^0 - \phi_j|^2 \le \delta_1 \tag{1.9}$$

holds. This was shown in Liu and Wang [11].

Remark 2. For the stability of a numerical method one always needs the CFL condition. It is a restriction on the product of λ with the maximal wave speed, see (1.4). For large wave speeds this may become a severe restriction on λ. The detailed restriction on λ will be clarified in the course of weighted energy analysis in Sects. 4 and 5.

The study of existence and stability of discrete shocks is important to the understanding of the convergence behavior of numerical shock computations. Jennings [6] proved the existence and the l^1 stability of discrete shocks for general first order monotone schemes approximating scalar conservation laws, see also Engquist-Osher [3] and Osher-Ralston [19]. The existence of discrete shock profiles of finite difference methods for systems of conservation laws was established by Majda and Ralston [15] by means of the center manifold theorem, see also Michelson [17]. Szepessy [22] studied the existence and l^2-stability of stationary discrete shocks for a first order implicit streamline diffusion finite element method for systems of conservation laws. Smyrlis [21] proved stability of a scalar stationary discrete shock wave for the Lax-Wendroff scheme. Tadmor [23] considered the large time behavior of the Lax-Friedrichs scheme approximating scalar genuinely nonlinear conservation laws. The nonlinear stability of discrete shocks to the modified Lax-Friedrichs scheme approximating systems was obtained by Liu and Xin [13,14]. In their study each characteristic field was assumed to be genuinely nonlinear. Recently, Engquist and Yu [4] showed that the stability and existence of discrete shock profiles are closely related to the convergence and stability of the scheme itself.

Discrete shocks for strictly monotone schemes approximating non-convex scalar conservation laws were shown to be stable in the l^1-norm by Jennings [6]. The stability in the l^2-norm of the scheme (1.2) was proved by Liu and Wang [10]. The polynomial convergence rate to discrete shocks for the Lax-Friedrichs scheme (1.2) was first obtained by Liu and Wang [11] for convex flux functions f. In the present paper, we investigate the convergence rate to discrete shocks for (1.2) when f may also be non-convex. We show that polynomial spatial decay yields polynomial temporal decay. Our result suggests that the increase in the spatial decay of initial perturbations leads to an increase in the temporal decay of the corresponding perturbations. Namely, the spatial decay rate is transformed to temporal decay. We would like to point out that the decay result obtained here for non-convex scalar conservation laws involved a much more elaborate analysis than the previous decay results on the convex case and the stability results for the non-convex case mentioned above. The authors are aware of the fact that the restriction of the results presented here to the case of stationary shock, i.e. $s = 0$, is rather severe and unwanted. This choice was made in order to avoid further lengthy technical arguments. We do not have the feeling that it is not possible to extend our results to the case of non-stationary shocks.

Equations of type (1.1) with f non-convex were considered, for example, by Buckley and Leverett [1] as a model for the one-dimensional convection dominated displacement of oil by water in a porous medium. The large time behavior of solutions for non-convex conservation laws exhibits a much richer and more complicated behavior than in the convex case, see Dafermos [2].

Our time-decay estimate is motivated by decay estimates to shock profiles for scalar viscous conservation laws

$$u_t + f(u)_x = \varepsilon u_{xx}, \quad \varepsilon > 0. \tag{1.10}$$

Various time decay rates to viscous shock profiles for (1.10) have been investigated by many authors, see [5,7–9,16,18] and [20] as well as particularly the papers of Liu [9] and Matsumura-Nishihara [16] from which we draw ideas in the present work. For scalar conservation laws with viscosity (1.10)

Il'in and Oleinik [5] showed that if the integral of the initial disturbance over $(-\infty, x]$ decays exponentially for $|x| \to \infty$, then the solution approaches the shock profile solution at an exponential rate as $t \to \infty$. For Burgers' equation, by using the Cole-Hopf transformation, Nishihara [18] showed that if the integral of the initial disturbance over $(-\infty, x]$ has an algebraic order $O(|x|^{-\alpha})(\alpha > 0)$ for $|x| \to \infty$, then the solution converges to the shock profile solution at the same algebraic rate $t^{-\alpha}$ as $t \to \infty$. This fact, that decay rates of the primitive of the perturbation are considered spatially, accounts for the one extra order in the decay rate in assumption (1.9). This makes our results for the discrete case completely analogous. Nishihara [18] also noted that this time decay rate is optimal in general. Therefore, our algebraic decay rate in Theorem 1 seems to be optimal in comparison with the continuous analogue considered by Nishihara. In analogy to the situation to viscous conservation laws (1.10), one expects that exponential spatial decay should yield exponential temporal decay. However, the rigorous justification of exponential decay remains an open problem.

Before concluding this section, we would like point out that in our asymptotic stability analysis in this paper we frequently have to choose constants implying that we have sufficiently small time steps, i.e. a severe restriction of the CFL condition (1.4). Deriving sharper constants would involve more technical analysis or restrictive assumptions on the flux functions. This is not a pleasing situation, but a common occurance in numerical analysis that we have to live with. The asymptotic stability estimates in our paper are in a certain sense similar to error estimates, as for instance in the theory of finite elements. These estimates are also only valid for sufficiently small mesh lengths. In such type of analysis one usually cannot make this very precise. The same holds for truncation error analysis for higher order schemes, which is also only valid for sufficiently small mesh lengths. In practice the situation may be quite different. For a large mesh size a first order scheme may perform better than a second order scheme. In three dimensional unsteady problems this may become quite relevant. Still numerical analysis gives valuable insight into the nature of numerical schemes.

This paper is organized as follows. We reformulate the original problem and restate Theorem 1 in an equivalent form in Sect. 2. There it is shown how Theorem 1 follows from the equivalent Theorem 2.3. Some properties of the weight function are obtained in Sect. 3. In Sect. 4 we give the main part of the proof of Theorem 2.3. We investigate the time decay rate by using a weighted energy method. The key idea is to use a discrete weight function which depends not only on the time-space variable for obtaining the desired rates but also on the discrete shocks for dealing with the non-convexity of the flux function. The proofs of some intermediate technical estimates summarized in Lemmas 4.1-4.2 are relegated to Sect. 5.

2 Reformulation of the problem

Let $(\phi_j)_{j\in\mathbb{Z}}$ be a stationary discrete shock wave for the L-F scheme (1.2). Then $(\phi_j)_{j\in\mathbb{Z}}$ satisfies

$$\lambda(f(\phi_{j+1}) - f(\phi_{j-1})) = \mu(\phi_{j+1} - 2\phi_j + \phi_{j-1}).$$

Summing it over j from $-\infty$ to j yields

$$\mu(\phi_{j+1} - \phi_j) = \lambda(Q_{j+1} + Q_j), \tag{2.1}$$

where $Q_j = Q(\phi_j) = f(\phi_j) - f(u_\pm)$. The equation (2.1) admits an unique solution $(\phi_j)_{j\in\mathbb{Z}}$ satisfying $\phi_{\pm\infty} = u_\pm$ which takes on a given value $u_* \in]u_+, u_-[$ at $j = 0$. Since (1.2) is a first order monotone scheme, Theorem 1 in Jennings [6] implies the following lemma.

Lemma 2.1 *Suppose that (1.3a)-(1.3b) and $u_+ < u_-$ hold. Then for each $u_* \in]u_+, u_-[$, there exists an unique stationary discrete shock profile $(\phi_j)_{j\in\mathbb{Z}}$ to (1.2) satisfying*

$$\phi_0 = u_* \quad \text{as well as} \quad \phi_j > \phi_{j+1}, \quad \text{for} \;\; j \in \mathbb{Z}. \tag{2.2}$$

□

Further, we obtain

Lemma 2.2 *We set $M = \sup_{u\in]u_+,u_-[}\{f'(u)/\mu\}$. For the stationary discrete shock solutions to scheme (1.2), the following estimates hold for any $j \in \mathbb{Z}$,*

$$|\phi_{j+1} - 2\phi_j + \phi_{j-1}| \leq M\lambda|\phi_{j-1} - \phi_{j+1}|, \tag{2.3}$$

$$\phi_j - \phi_{j+1} \leq \frac{1}{2}(1 + M\lambda)(\phi_{j-1} - \phi_{j+1}), \tag{2.4}$$

$$\phi_{j-1} - \phi_j \leq \frac{1}{2}(1 + M\lambda)(\phi_{j-1} - \phi_{j+1}). \tag{2.5}$$

Proof. The estimate (2.3) follows from (1.5a) since $\phi_{j+1} - 2\phi_j + \phi_{j-1} = \frac{\lambda}{\mu} f'(\tilde{\phi}_j)(\phi_{j+1} - \phi_{j-1})$ for some $\tilde{\phi}_j \in]\phi_{j+1}, \phi_{j-1}[$. Furthermore, we use the simple identities

$$\phi_j - \phi_{j+1} = \frac{1}{2}[(\phi_{j-1} - \phi_{j+1}) - (\phi_{j+1} - 2\phi_j + \phi_{j-1})]$$

and

$$\phi_{j-1} - \phi_j = \frac{1}{2}[(\phi_{j-1} - \phi_{j+1}) + (\phi_{j+1} - 2\phi_j + \phi_{j-1})],$$

then (2.4) and (2.5) follow. The proof of Lemma 2.2 is complete. □

To prove Theorem 1, we reformulate the scheme (1.2) by formally introducing

$$v_j^n := \sum_{k=-\infty}^{j} (u_k^n - \phi_k). \tag{2.6}$$

It will be shown below that the summation gives always finite value. Subtracting (1.5a) from (1.2) and summing up the resulting expressions from $-\infty$ to j, one obtains the scheme

$$\begin{aligned} v_j^{n+1} &- v_j^n + \frac{\lambda}{2}\Lambda_{j+1}(v_{j+1}^n - v_j^n) + \frac{\lambda}{2}\Lambda_j(v_j^n - v_{j-1}^n) \\ &- \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) = e_j^n, \end{aligned} \tag{2.7}$$

where

$$\Lambda_j = f'(\phi_j) = Q'(\phi_j) \qquad \text{and} \qquad e_j^n = -\frac{\lambda}{2}(\theta_{j+1}^n + \theta_j^n)$$

with

$$\theta_j^n = f(u_j^n) - f(\phi_j) - f'(\phi_j)(u_j^n - \phi_j).$$

Theorem 1 will be obtained from the following theorem.

Theorem 2.3. *Suppose that the assumptions (1.3a)-(1.3c) and and the CFL condition (1.4) hold,* $(v_j^0)_{j\in\mathbb{Z}} \in l_\alpha^2$ *for some* $\alpha \geq 0$*, and* λ *is suitably small, and that there exists a constant* $\delta_1 > 0$ *(suitably small) such that* $|v^0|_\alpha < \delta_1$*. Then the scheme (2.7) with initial data* $(v_j^0)_{j\in\mathbb{Z}}$ *admits an unique global solution* $(v_j^n)_{j\in\mathbb{Z}}$ *satisfying, for any* $p > 0$*,*

$$\begin{aligned} \sup_{n\in\mathbb{N}_0} &\left[(1+nh)^\alpha ||v^n||^2 + (1+nh)^{-p}\sum_{i<n}(1+ih)^{\alpha+p}||\Delta v^i||^2\right] \\ &\leq C|v^0|_\alpha^2. \end{aligned} \tag{2.8}$$

□

Since the scheme (2.7) is explicit with a given right hand side, we only need an a priori bound in order to guarantee the global existence of the unique discrete solution $(v_j^n)_{j\in\mathbb{Z}}$ for all $n \in \mathbb{N}_0$. Therefore, Theorem 2.3 can be obtained by continuity arguments based on the following proposition.

Proposition 2.4 (A priori estimate) *Let* n_1 *be a natural number. Suppose that the unique solution* $(v_j^n)_{j\in\mathbb{Z}}$ *to the scheme (2.7) with initial data* $(v_j^0)_{j\in\mathbb{Z}}$

defined via (2.6) satisfies $(v_j^n)_{j\in\mathbb{Z}} \in l_\alpha^2$ *for some* $\alpha \geq 0$. *Further, suppose that there exists a constant* $\delta_2 > 0$ *independently of* n_1 *such that* $\sup_{0\leq n\leq n_1} ||v^n|| \leq \delta_2$. *Then the estimate*

(2.9)

$$\sup_{0\leq n\leq n_1}\left[(1+nh)^\alpha||v^n||^2 + (1+nh)^{-p}\sum_{i<n}(1+ih)^{\alpha+p}||\Delta v^i||^2\right] \leq C|v^0|_\alpha^2.$$

holds for a constant $C > 0$ *independently of* n_1. □

The proof of Proposition 2.4 is carried out in the remaining three sections of this paper. The main part of the proof is given in Sect. 4.

Proof of Theorem 1 based on Theorem 2.3: It follows from Theorem 2.3 that v_j^n is well-defined. By (2.6) we have

$$u_j^n = \phi_j + v_j^n - v_{j-1}^n.$$

It follows from (2.7) and (1.5a) that $(u_j^n)_{j\in\mathbb{Z}}$ is the unique solution of the L-F scheme (1.2) with initial data $(u_j^0)_{j\in\mathbb{Z}}$. Moreover, we estimate from above

$$|u_j^n - \phi_j| = |v_j^n - v_{j-1}^n| \leq ||\Delta v^n||.$$

Next we derive the convergence rate of the solution $(u_j^n)_{j\in\mathbb{Z}}$ to (1.2). It follows from the crucial estimate (2.8) that

$$\sum_{i=0}^{n}(1+ih)^{\alpha+p}||\Delta v^i||^2 \leq C(1+nh)^p|v^0|_\alpha^2$$

holds, which implies

$$||\Delta v^n|| \leq C(1+nh)^{-\frac{\alpha}{2}}|v^0|_\alpha.$$

Combining the above facts gives

$$\sup_{j\in\mathbb{Z}}|u_j^n - \phi_j| \leq \|\Delta v^n\| \leq C(1+nh)^{-\frac{\alpha}{2}}|v^0|_\alpha,$$

which yields the estimate (1.8) in Theorem 1. □

3 The weight function

Let the discrete weight $w_j = w(\phi_j)$ be chosen analogously to [9, 16] as

$$w_j = w(\phi_j) = \frac{(\phi_j - u_+)(\phi_j - u_-)}{Q(\phi_j)}. \tag{3.1}$$

The function w will be used to treat the non-convexity of the problem. We introduce with $r = \Delta x$ the abbreviations

$$P_j := \langle jr \rangle^\beta \quad \text{and} \quad H_j := P_j w_j.$$

Next we choose a time-dependent discrete weight function of the form

$$K_j^n = (1 + nh)^\gamma H_j, \qquad j \in \mathbb{Z},$$

which will be used to characterize the decay rate.

The following properties are needed in proofs below.

Lemma 3.1 *For any given flux function $f \in C^3$ and under the assumptions (1.3a)-(1.3c) there exists a positive constant C such that*

$$C^{-1} \le w_j \le C, \tag{3.2}$$

$$|w'(u)|, \quad |w''(u)| \text{ and } |w'''(u)Q(u)| \le C \tag{3.3}$$

for all $u \in [u_+, u_-]$.

Proof. Clearly $C^{-1} \le w_j \le C$ under the shock condition (1.3b)-(1.3c). We only consider the case $u_+ < u_-$. Note that due to the Lax shock condition (1.3c) $Q(u) = f(u) - f(u_\pm)$ has only simple zeros at u_+ and u_-, i.e. $Q'(u_\pm) \neq 0$, and by (1.3b) $Q(u) < 0$ on the interval $]u_+, u_-[$. Setting $h(u) = (u - u_-)(u - u_+)$, we have $w(u) = \frac{h(u)}{Q(u)}$ and

$$w'(u) = \frac{h'(u)Q(u) - h(u)Q'(u)}{Q^2(u)},$$

which takes finite values on the interval $]u_+, u_-[$. At u_+ and u_- the derivatives of numerator and denominator give the quotient

$$\frac{h''(u)Q(u) - h(u)Q''(u)}{2Q(u)Q'(u)}.$$

It is easily seen to be bounded in the limits at $u \to u_\pm$. Here we used the assumption $f \in C^2$. Therefore, by L'Hôpital's rule $w'(u)$ is bounded on the interval $]u_+, u_-[$.

Now note that

$$(w'Q)' = (wQ)'' - (wQ')' = 2 - w'(u)f'(u) - w(u)f''(u).$$

Since the functions w, w', f' and f'' are bounded on the interval $[u_+, u_-]$ we have $(w'Q)'$ bounded.

Similarly we now show that if $f \in C^3[u_+, u_-]$ then w'' and $w'''Q$ are bounded on the interval $[u_+, u_-]$. In fact, as above we have

$$w'' = \frac{M(u)}{Q^3(u)} \quad \text{with} \quad M(u) := 2Q^2 - 2h'QQ' + 2h(Q')^2 - hQQ''$$

which takes finite values on the interval $]u_+, u_-[$. To bound $w''(u)$ at the states $u_\pm$, we have to show that the function M also has zeros of third order at $u_\pm$. We have by $f \in C^3$ and introducing a suitable term $G(u)$

$$M'(u) = 3(hQ' - h'Q)Q'' - hQQ''' = G(u) - hQQ'''$$

which vanishes at $u_\pm$. The last term $-hQQ'''$ in the function M obviously has zeros of second order at $u_\pm$. Now we only have to show that the function G also has double zeros at $u_\pm$. This can be obtained under the assumption $f \in C^3$ giving

$$G'(u) = 3h(Q'Q'')' - 3Q(h'Q'')' = 3h(f'f'')' - 3Q(h'f'')',$$

which vanishes at $u_\pm$. Therefore $M(u)$ has zeros of third order at $u_\pm$. This implies that w'' is bounded on the interval $[u_+, u_-]$.

Note that the identity $(wQ)''' \equiv 0$ enables us to see that $w'''Q$ is bounded on the interval $[u_+, u_-]$. This is due to the fact that we therefore have

$$w'''Q = -3w''Q' - 3w'Q'' - wQ''' = -3w''f' - 3w'f'' - wf'''.$$

This completes the proof of Lemma 3.1. $\square$

We state some basic estimates on the weights $P_j = \langle jr\rangle^\beta = (1 + (jr)^2)^{\beta/2}$.

Lemma 3.2 *For any $j \in \mathbb{Z}$, $\beta \in [0, \alpha]$, there exist constants $\theta \in]0, 1[$, and $c_r > 0$, $C_r > 0$ such that*

$$\theta^{-1}P_j \geq \quad P_{j+1} \quad \geq \theta P_j, \tag{3.4}$$

$$c_r\beta r\langle jr\rangle^{\beta-1} \leq |P_{j+1} - P_j| \leq C_r\beta r\langle jr\rangle^{\beta-1} \tag{3.5}$$

Proof. To prove (3.4) let us consider two cases. Since $P_j = P_{-j}$ the discrete weights are symmetric. They are increasing for $j \geq 0$, decreasing for $j \leq 0$.

Therefore, we could choose any $\theta \in]0,1[$ for the right estimate in case $j \geq 0$, for the left estimate in case $j < 0$. We now consider the right estimate for $j < 0$. Then by the mean value theorem there exists an $\eta_j \in]j, j+1[$ such that

$$(3.6) \qquad P_j - P_{j+1} = |P_{j+1} - P_j| = |\beta \eta_j r \langle \eta_j r \rangle^{\beta-2} r| \leq \beta \langle jr \rangle^{\beta-1} r.$$

This gives the estimates

$$P_{j+1} \geq P_j - \beta \langle jr \rangle^{\beta-1} r = P_j \left(1 - \frac{\beta r}{\langle jr \rangle} \right) \geq \theta P_j,$$

for $\theta = 1 - \frac{\alpha r}{\sqrt{1+r^2}} < 1$, provided that $r = \Delta x$ is suitably small. The left estimate for $j \geq 0$ follows analogously. One has to possibly choose a smaller θ.

The desired estimate (3.5) follows from (3.6) by simply defining

$$(C_r) c_r = \left(\sup_{j \in \mathbb{Z}} \right) \inf_{j \in \mathbb{Z}} \frac{|\eta_j r| \langle \eta_j r \rangle^{\beta-2}}{\langle jr \rangle^{\beta-1}},$$

which exist and are positive. They may depend on r. The technical derivation of the fact that these bounds $c_r, \quad C_r$ exist has been omitted. This proves the lemma. □

4 Energy estimates

Throughout this section we suppose that the scheme (2.7) with $(v_j^0)_{j \in \mathbb{Z}}$ as initial data admits a solution $(v_j^n)_{j \in \mathbb{Z}} \in l_\alpha^2$ for some $\alpha \geq 0$ and $n = 0, 1, \ldots, n_1$. We write C as a generic positive constant which may depend on $(\phi_j)_{j \in \mathbb{Z}}$ and λ, but is independent of n for $0 \leq n \leq n_1$, and of $(v_j^n)_{j \in \mathbb{Z}}$.

To get the desired estimate, we use the weighted energy method. Taking a time-dependent discrete weight function

$$(4.1) \qquad K_j^n = (1 + nh)^\gamma H_j, \qquad j \in \mathbb{Z},$$

as chosen in the Sect. 3.

Now we begin with the energy estimates. Multiplying (2.7) by $2 v_j^n K_j^n$ and summing the resulting expressions over j, one obtains

$$\underbrace{\sum_{j \in \mathbb{Z}} 2(v_j^{n+1} - v_j^n) v_j^n K_j^n}_{I_1}$$

$$+\underbrace{\lambda\left[\sum_{j\in\mathbb{Z}}\Lambda_{j+1}v_j^nK_j^n(v_{j+1}^n-v_j^n)+\sum_{j\in\mathbb{Z}}\Lambda_jv_j^nK_j^n(v_j^n-v_{j-1}^n)\right]}_{I_2}$$

$$+\underbrace{\mu\sum_{j\in\mathbb{Z}}v_j^nK_j^n(2v_j^n-v_{j+1}^n-v_{j-1}^n)}_{I_3}=2\sum_{j\in\mathbb{Z}}v_j^nK_j^ne_j^n. \tag{4.2}$$

We now estimate each term denoted by I_i for $i = 1, 2, 3$ on the left hand side of (4.2). We rewrite the first term as

$$\begin{aligned}I_1&=\sum_{j\in\mathbb{Z}}\left[(v_j^{n+1})^2-(v_j^{n+1}-v_j^n)^2-(v_j^n)^2\right]K_j^n\\&=\sum_{j\in\mathbb{Z}}(v_j^{n+1})^2K_j^{n+1}-\sum_{j\in\mathbb{Z}}(v_j^n)^2K_j^n\\&\quad-\sum_{j\in\mathbb{Z}}(v_j^{n+1}-v_j^n)^2K_j^n-\sum_{j\in\mathbb{Z}}(v_j^{n+1})^2(K_j^{n+1}-K_j^n).\end{aligned}$$

Using an index shift for the second term we get

$$\begin{aligned}I_2&=\lambda\left[\sum_{j\in\mathbb{Z}}\Lambda_{j+1}K_j^nv_j^nv_{j+1}^n-\sum_{j\in\mathbb{Z}}\Lambda_{j+1}K_j^n(v_j^n)^2\right.\\&\qquad\left.+\sum_{j\in\mathbb{Z}}\Lambda_jK_j^n(v_j^n)^2-\sum_{j\in\mathbb{Z}}\Lambda_jK_j^nv_j^nv_{j-1}^n\right]\\&=\lambda\left[-\sum_{j\in\mathbb{Z}}(\Lambda_{j+1}K_{j+1}^n-\Lambda_jK_j^n)(v_j^n)^2\right.\\&\qquad\left.+\sum_{j\in\mathbb{Z}}\Lambda_{j+1}(K_j^n-K_{j+1}^n)v_j^n(v_{j+1}^n-v_j^n)\right]\end{aligned}$$

and the third is rewritten as

$$\begin{aligned}I_3&=\mu(1+nh)^\gamma\left[\sum_{j\in\mathbb{Z}}v_j^nH_j(v_j^n-v_{j+1}^n)-\sum_{j\in\mathbb{Z}}v_j^nH_j(v_{j-1}^n-v_j^n)\right]\\&=\mu(1+nh)^\gamma\left[\sum_{j\in\mathbb{Z}}\frac{H_j}{2}\left[(v_j^n-v_{j+1}^n)^2+(v_j^n)^2-(v_{j+1}^n)^2\right]\right.\end{aligned}$$

$$
\begin{aligned}
&\left.+\sum_{j\in\mathbb{Z}}\frac{H_{j+1}}{2}[(v_j^n - v_{j+1})^2 + (v_{j+1}^n)^2 - (v_j^n)^2]\right]\\
&= \mu(1+nh)^\gamma\left[\sum_{j\in\mathbb{Z}}(v_j^n - v_{j+1}^n)^2\frac{H_j + H_{j+1}}{2}\right.\\
&\left.+\sum_{j\in\mathbb{Z}}(H_{j+1}-H_j)\frac{v_{j+1}^n + v_j^n}{2}(v_{j+1}^n - v_j^n)\right].
\end{aligned}
$$

Furthermore for the very last term in I_3 we obtain

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}(H_{j+1}-H_j)\frac{v_{j+1}^n + v_j^n}{2}(v_{j+1}^n - v_j^n)\\
&= \sum_{j\in\mathbb{Z}}\left[(H_{j+1} - w_jP_{j+1}) - (H_j - w_jP_{j+1})\right]\frac{v_j^n + v_{j+1}^n}{2}(v_{j+1}^n - v_j^n)\\
&= \sum_{j\in\mathbb{Z}}P_{j+1}(w_{j+1}-w_j)\frac{{v_{j+1}^n}^2 - v_j^{n2}}{2}\\
&\quad + \sum_{j\in\mathbb{Z}}\left[(P_{j+1}-P_j)w_j\frac{v_j^n + v_{j+1}^n}{2}\right](v_{j+1}^n - v_j^n)\\
&= -\sum_{j\in\mathbb{Z}}\left[P_j\frac{w_j - w_{j-1}}{2} - P_{j+1}\frac{w_{j+1}-w_j}{2}\right](v_j^n)^2\\
&\quad + \sum_{j\in\mathbb{Z}}\left[(P_{j+1}-P_j)w_j\frac{v_j^n + v_{j+1}^n}{2}\right](v_{j+1}^n - v_j^n).
\end{aligned}
$$

Then, we finally have for the third term

$$
\begin{aligned}
I_3 = \mu(1+nh)^\gamma&\left(\sum_{j\in\mathbb{Z}}(v_j^n - v_{j+1}^n)^2\frac{H_j + H_{j+1}}{2}\right.\\
&-\sum_{j\in\mathbb{Z}}\left[P_j\frac{w_j - w_{j-1}}{2} - P_{j+1}\frac{w_{j+1}-w_j}{2}\right](v_j^n)^2\\
&\left.+\sum_{j\in\mathbb{Z}}\left[(P_{j+1}-P_j)w_j\frac{v_j^n + v_{j+1}^n}{2}\right](v_{j+1}^n - v_j^n)\right).
\end{aligned}
$$

We introduce the abbreviations

$$
A_j = -\lambda(\Lambda_{j+1}H_{j+1} - \Lambda_j H_j)
$$

$$-\mu\left[P_{j+1}\frac{w_{j+1}-w_j}{2}-P_j\frac{w_j-w_{j-1}}{2}\right], \tag{4.3a}$$

$$B_j^n=\left[-\lambda\Lambda_{j+1}v_j^n(H_j-H_{j+1})-\mu(P_{j+1}-P_j)w_j\frac{v_j^n+v_{j+1}^n}{2}\right] \times(v_{j+1}^n-v_j^n). \tag{4.3b}$$

Inserting the new terms obtained for I_i with $i=1,2,3$ into (4.2) and rearranging these terms in a suitable way, we get

$$\begin{aligned}\sum_{j\in\mathbb{Z}}(v_j^{n+1})^2K_j^{n+1}&-\sum_{j\in\mathbb{Z}}(v_j^n)^2K_j^n+(1+nh)^\gamma\sum_{j\in\mathbb{Z}}A_j(v_j^n)^2\\ &+\mu(1+nh)^\gamma\sum_{j\in\mathbb{Z}}\frac{H_j+H_{j+1}}{2}|v_{j+1}^n-v_j^n|^2\\ &=(1+nh)^\gamma\sum_{j\in\mathbb{Z}}(v_j^{n+1}-v_j^n)^2H_j-(1+nh)^\gamma\sum_{j\in\mathbb{Z}}B_j^n\\ &+\sum_{j\in\mathbb{Z}}(K_j^{n+1}-K_j^n)(v_j^{n+1})^2+2(1+nh)^\gamma\sum_{j\in\mathbb{Z}}v_j^nH_je_j^n,\end{aligned} \tag{4.4}$$

We use the discrete weighted norms $|v^n|_\beta$ and $|v^n|_{\beta,w}$ defined in the introduction for $\beta=\alpha$. They satisfy the relations

$$\sum_{j\in\mathbb{Z}}H_j|v_j^n|^2=|v^n|_{\beta,w}^2 \quad\text{and}\quad \sum_{j\in\mathbb{Z}}K_j^n|v_j^n|^2=(1+nh)^\gamma|v^n|_{\beta,w}^2. \tag{4.5}$$

Clearly, for a suitable constant $C>0$ and using the Mean Value Theorem the estimate

$$\begin{aligned}K_j^{n+1}-K_j^n&=H_j\Big[(1+(n+1)h)^\gamma-(1+nh)^\gamma\Big]\\ &\le\gamma(1+Ch)H_j(1+nh)^{\gamma-1}h\end{aligned}$$

holds. Using this estimate and the discrete weighted norms introduced above we have

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}(K_j^{n+1}-K_j^n)(v_j^{n+1})^2\\ &\le 2\sum_{j\in\mathbb{Z}}\Big[(v_j^{n+1}-v_j^n)^2+(v_j^n)^2\Big](K_j^{n+1}-K_j^n)\\ &=2\gamma(1+Ch)(1+nh)^{\gamma-1}\left[\sum_{j\in\mathbb{Z}}H_j|v_j^{n+1}-v_j^n|^2+\sum_{j\in\mathbb{Z}}H_j|v_j^n|^2\right]h\end{aligned}$$

$$= 2\gamma(1+Ch)(1+nh)^{\gamma-1}\Big[|v^{n+1}-v^n|^2_{\beta,w}+|v^n|^2_{\beta,w}\Big]h$$
$$\le C\gamma(1+nh)^{\gamma-1}|v^n|^2_{\beta,w}h+\frac{2\gamma(1+Ch)}{1+nh}((1+nh)^{\gamma}|v^{n+1}-v^n|^2_{\beta,w}h$$
$$\le C\Big[\gamma(1+nh)^{\gamma-1}|v^n|^2_{\beta,w}h+(1+nh)^{\gamma}|v^{n+1}-v^n|^2_{\beta,w}h\Big].$$

(4.6)

Now we take (4.4) and use (4.6) as well as (4.5) in order to obtain

$$(1+(n+1)h)^{\gamma}|v^{n+1}|^2_{\beta,w}-(1+nh)^{\gamma}|v^n|^2_{\beta,w}+(1+nh)^{\gamma}\sum_{j\in\mathbb{Z}}A_j(v_j^n)^2$$
$$+\mu(1+nh)^{\gamma}\sum_{j\in\mathbb{Z}}|v_{j+1}^n-v_j^n|^2\frac{H_j+H_{j+1}}{2}$$
$$\le(1+Ch)(1+nh)^{\gamma}|v^{n+1}-v^n|^2_{\beta,w}+(1+nh)^{\gamma}\sum_{j\in\mathbb{Z}}|B_j^n|$$
$$+C\gamma(1+nh)^{\gamma-1}h|v^n|^2_{\beta,w}+2(1+nh)^{\gamma}\sum_{j\in\mathbb{Z}}v_j^nH_je_j^n. \tag{4.7}$$

Next we estimate the terms on the right hand side of (4.7). We set

$$N(n_1)=\sup_{n\le n_1}(\sum_{j\in\mathbb{Z}}|v_j^n|^2)^{1/2}, \tag{4.8}$$

and assume a priori that $N(n_1)$ is suitably small. Obviously we have the bound

$$\sup_{n\le n_1,j}|v_j^n|\le N(n_1). \tag{4.9}$$

The scheme (2.7) gives

$$v_j^{n+1}-v_j^n=\Big(\frac{\mu}{2}-\frac{\lambda}{2}\Lambda_{j+1}\Big)(v_{j+1}^n-v_j^n)-\Big(\frac{\mu}{2}+\frac{\lambda}{2}\Lambda_j\Big)(v_j^n-v_{j-1}^n)+e_j^n.$$

From (2.6) $u_j^n-\phi_j=v_j^n-v_{j-1}^n$, we have by using the remainder term in the Taylor expansion of f

$$|\theta_j^n|\le C|v_j^n-v_{j-1}^n|^2.$$

Note that

$$\sup_{n\le n_1,j}|v_{j+1}^n-v_j^n|\le 2N(n_1).$$

Combining these gives for $e_j^n=-\frac{\lambda}{2}(\theta_{j+1}^n+\theta_j^n)$ the estimate

$$|e_j^n|\le CN(n_1)\Big[|v_j^n-v_{j-1}^n|+|v_{j+1}^n-v_j^n|\Big]. \tag{4.10}$$

Now note that for any $a, b, c \in \mathrm{IR}$ and any $\delta > 0$, the inequality

$$(a+b+c)^2 \leq 2(1+\delta)a^2 + 2(1+\delta)b^2 + (1+\frac{1}{\delta})c^2$$

holds. This leads to the bound

$$\begin{aligned} &|v_j^{n+1} - v_j^n|^2 \\ &\quad \leq \frac{1+\delta}{2}\left[(\mu - \lambda\Lambda_{j+1})^2|v_{j+1}^n - v_j^n|^2 + (\mu+\lambda\Lambda_j)^2|v_j^n - v_{j-1}^n|^2\right] \\ &\quad +C(1/\delta)(N(n_1))^2\left[|v_{j+1}^n - v_j^n|^2 + |v_j^n - v_{j-1}^n|^2\right], \end{aligned} \tag{4.11}$$

where $C(1/\delta)$ depends on $\frac{1}{\delta}$.

Considering the identity

$$\sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 H_j + \sum_{j\in\mathbb{Z}} |v_j^n - v_{j-1}^n|^2 H_j = \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 (H_j + H_{j+1})$$

and multiplying (4.11) by H_j as well as summation over $j \in \mathbb{Z}$ gives

$$\begin{aligned} |v^{n+1} - v^n|_{\beta,w}^2 \leq (1+\delta)&\left[(\mu + \lambda \max|f'|)^2 + C(N(n_1))^2\right] \\ &\times \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n| \frac{H_j + H_{j+1}}{2}. \end{aligned} \tag{4.12}$$

Next, using (4.9) and (4.10), one obtains

$$\begin{aligned} 2\sum_{j\in\mathbb{Z}} |v_j^n H_j e_j^n| &\leq CN(n_1) \sum_{j\in\mathbb{Z}} H_j \left[|v_{j+1}^n - v_j^n|^2 + |v_j^n - v_{j-1}^n|^2\right] \\ &\leq CN(n_1) \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 \frac{H_j + H_{j+1}}{2}. \end{aligned} \tag{4.13}$$

Substituting the estimates (4.11)-(4.13) into (4.7) yields

$$\begin{aligned} &(1+(n+1)h)^\gamma |v^{n+1}|_{\beta,w}^2 - (1+nh)^\gamma |v^n|_{\beta,w}^2 + (1+nh)^\gamma \sum_{j\in\mathbb{Z}} A_j (v_j^n)^2 \\ &\quad +(1+nh)^\gamma \left[\mu - (1+Ch)(1+\delta)[(\mu+\lambda\max|f'|)^2 \right. \\ &\quad \left. +CN^2(n_1)] - CN(n_1)\right] \cdot \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 \frac{H_j + H_{j+1}}{2} \\ &\leq C\gamma(1+nh)^{\gamma-1} h |v^n|_{\beta,w}^2 + (1+nh)^\gamma \sum_{j\in\mathbb{Z}} |B_j^n|. \end{aligned} \tag{4.14}$$

To get the desired final estimate, one has to bound the terms $\sum_{j\in\mathbb{Z}} A_j^n (v_j^n)^2$ and $\sum_{j\in\mathbb{Z}} |B_j^n|$ respectively. This is done in Lemma 4.1 and 4.2 which will be proved in Sect. 5.

Lemma 4.1. *Consider the assumptions made at the beginning of this section. For any $\beta \in [0,\alpha]$, there are constants $c_0 > 0$ and $\theta \in]0,1[$ independent of β such that*

$$A_j \geq \theta\lambda\langle jr\rangle^\beta(\phi_j - \phi_{j+1}) + c_0\beta\langle jr\rangle^{\beta-1}h, \tag{4.15}$$

for all $j \in \mathbb{Z}$, provided that $\lambda = \frac{\Delta t}{\Delta x}$ is suitably small.

Lemma 4.2. *Consider the assumptions made at the beginning of this section. For any $\beta \in [0,\alpha]$, and any given constants $\varepsilon > 0, J > 0$ there exists a constant $C > 0$, independently of n_1, such that the estimate*

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |B_j^n| \leq \varepsilon \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 \frac{H_j + H_{j+1}}{2} + \frac{C}{J}|\Delta v^n|^2_{\beta,w} \\ + C\beta||\Delta v^n||^2 + \frac{\beta c_0}{2} h|v^n|^2_{\beta-1,w} \\ + \frac{\lambda^2 C}{\varepsilon}\sum_{j\in\mathbb{Z}}(\phi_j - \phi_{j+1})|v_j^n|^2\langle jr\rangle^\beta\end{aligned} \tag{4.16}$$

holds for all $n \leq n_1$ provided $\lambda = \frac{\Delta t}{\Delta x}$ is suitably small.

Equipped with these lemmas, we turn to the proof of the following estimate.

Proposition 4.3 *Consider the assumptions made at the beginning of this section. Let $(v_j^n)_{j\in\mathbb{Z}}$ be a solution of (2.7) for $n \leq n_1$. Then there exists a positive constant C independently of n_1 such that for all $n \leq n_1$*

$$\begin{aligned}(1+nh)^\gamma|v^n|^2_\beta + \beta\sum_{i<n}(1+ih)^\gamma|v^i|^2_{\beta-1}h + \sum_{i<n}(1+ih)^\gamma|\Delta v^i|^2_\beta \\ \leq C\left[|v^0|^2_\beta + \gamma\sum_{i<n}(1+ih)^{\gamma-1}|v^i|^2_\beta h + \beta\sum_{i<n}(1+nh)^\gamma||\Delta v^i||^2\right],\end{aligned} \tag{4.17}$$

provided that $\lambda = \frac{\Delta t}{\Delta x}$, $r = \Delta x$ and $N(n_1)$ are suitably small.

Proof. Substituting the estimates of $\sum_{j\in\mathbb{Z}} A_j$ and $\sum_{j\in\mathbb{Z}} |B_j^n|$ into inequality (4.14) yields

$$(1+(n+1)h)^\gamma|v^{n+1}|^2_{\beta,w} - (1+nh)^\gamma|v^n|^2_{\beta,w}$$

$$
\begin{aligned}
&+(1+nh)^\gamma \left[\mu-\varepsilon-(1+Ch)(1+\delta)[(\mu+\lambda\max|f'|)^2\right.\\
&\left.+CN^2(n_1)]-CN(n_1)\right]\sum_{j\in\mathbb{Z}}|v^n_{j+1}-v^n_j|^2\frac{H_j+H_{j+1}}{2}\\
&-(1+nh)^\gamma\frac{C}{J}|\Delta v^n|^2_{\beta,w}+\frac{c_0\beta h}{2}(1+nh)^\gamma|v^n|^2_{\beta-1,w}\\
&+(1+nh)^\gamma\left(\theta-\frac{\lambda C}{\varepsilon}\right)\lambda\sum_{j\in\mathbb{Z}}(\phi_j-\phi_{j+1})\langle jr\rangle^\beta|v^n_j|^2\\
\le\ & C\left[\gamma(1+nh)^{\gamma-1}h|v^n|^2_{\beta,w}+\beta(1+nh)^\gamma||\Delta v^n||^2\right].
\end{aligned}
$$

Noting that $H_{j+1}>0$, we have by (4.5) and dropping H_{j+1}

$$
\sum_{j\in\mathbb{Z}}|v^n_{j+1}-v^n_j|^2\frac{H_j+H_{j+1}}{2}>\frac{1}{2}|\Delta v^n|^2_{\beta,w}.
$$

On the other hand, since $\mu<1$, we take $r=\Delta x$ and $\lambda=\frac{\Delta t}{\Delta x}$ suitably small, take J suitably large, $\varepsilon<\mu$ and δ suitably small, then

$$
\frac{1}{2}\Big[\mu-\varepsilon-(1+Ch)(1+\delta)[(\mu+\lambda\max|f'|)^2+CN^2(n_1)]
$$
$$
-CN(n_1)\Big]-\frac{C}{J}\ge\nu>0,
$$

provided that $N(n_1)$ is suitably small. Here we see that we may choose a suitably small δ_2 independent of n_1 such that $N(n_1)\le\delta_2$, as given in the statement of Proposition 2.4. Fix the chosen ϵ, let $\lambda=\frac{\Delta t}{\Delta x}$ be suitably small such that also

$$
\theta-\frac{\lambda C}{\varepsilon}>0,
$$

here C depends only on $u_\pm$, θ and $f(u)$, see the proof of Lemma 4.2.

Combining these facts we obtain

$$
\begin{aligned}
&(1+(n+1)h)^\gamma|v^{n+1}|^2_{\beta,w}-(1+nh)^\gamma|v^n|^2_{\beta,w}+\nu(1+nh)^\gamma|\Delta v^n|^2_{\beta,w}\\
&\quad+\frac{c_0\beta h}{2}(1+nh)^\gamma|v^n|^2_{\beta-1}\\
&\le C\left[\gamma(1+nh)^{\gamma-1}h|v^n|^2_{\beta,w}+\beta(1+nh)^\gamma||\Delta v^n||^2\right]. \qquad (4.18)
\end{aligned}
$$

Finally, summing up (4.18) with respect to n from 0 to $n-1$, we have

$$
(1+nh)^\gamma|v^n|^2_{\beta,w}+\nu\sum_{i<n}(1+ih)^\gamma|\Delta v^i|^2_{\beta,w}
$$

$$+\frac{c_0\beta}{2}\sum_{i<n}(1+ih)^\gamma|v^i|^2_{\beta-1}h$$

$$\leq C\left[|v^0|^2_{\beta,w}+\beta\sum_{i<n}(1+ih)^\gamma||\Delta v^i||^2\right.$$

$$\left.+\gamma\sum_{i<n}(1+ih)^{\gamma-1}|v^i|^2_{\beta,w}h\right]. \tag{4.19}$$

Noting that $C^{-1}\leq w_j\leq C$ by (3.2), the desired estimate (4.17) follows. □

Next we proceed to estimate the solution of the scheme (2.7) with an argument analogously to Liu and Wang [11].

First, taking $\beta=\gamma=0$ in (4.17), we get the following lemma.

Lemma 4.4. *Under the assumptions made at the beginning of this section there exists a constant $C>0$ independently of n_1, such that for any $n\leq n_1$*

$$||v^n||^2+\sum_{i<n}||\Delta v^i||^2\leq C||v^0||^2 \tag{4.20}$$

holds, provided that $N(n_1)$ and λ are suitably small. □

Applying induction to (4.17) as in Liu and Wang [11], one gets

Lemma 4.5. *Under the assumptions made at the beginning of this section. Let $\gamma\in[0,\alpha]$ be an integer, then the following estimate holds for any $n\leq n_1$*

$$(1+nh)^\gamma|v^n|^2_{\alpha-\gamma}+(\alpha-\gamma)\sum_{i<n}(1+ih)^\gamma|v^i|^2_{\alpha-\gamma-1}h$$

$$+\sum_{i<n}(1+ih)^\gamma|\Delta v^i|^2_{\alpha-\gamma}\leq C|v^0|^2_\alpha. \tag{4.21}$$

Consequently, if α is an integer, then for $0\leq\gamma\leq\alpha$ we obtain the bound

$$(1+nh)^\gamma||v^n||^2+\sum_{i<n}(1+ih)^\gamma||\Delta v^i||^2\leq C|v^0|^2_\alpha \tag{4.22}$$

for any $n\leq n_1$ and a constant $C>0$ independent of n_1. □

From Lemma 4.5, if α is an integer, then

$$(1+nh)^\alpha||v^n||^2+\sum_{i<n}(1+ih)^\alpha||\Delta v^i||^2\leq C|v^0|^2_\alpha$$

which obviously implies (2.9).

We show a sharper estimate when α is not an integer. Taking $\beta = 0$ in (4.17) gives

$$(1+nh)^{\gamma}|v^n|_0^2 + \sum_{i<n}(1+ih)^{\gamma}|\Delta v^i|_0^2 \le C\left[|v^0|_0^2 + \gamma\sum_{i<n}(1+ih)^{\gamma-1}|v^i|_0^2 h\right]. \tag{4.23}$$

Using (4.21) with $\gamma = [\alpha]$, one gets

$$(1+nh)^{[\alpha]}|v^n|^2_{\alpha-[\alpha]} + (\alpha-[\alpha])\sum_{i<n}(1+ih)^{[\alpha]}|v^i|^2_{\alpha-[\alpha]-1}h + \sum_{i<n}(1+ih)^{[\alpha]}|\Delta v^i|^2_{\alpha-[\alpha]} \le C|v^0|^2_{\alpha}. \tag{4.24}$$

We estimate the final term in (4.23) as follows

$$\begin{aligned}
&\sum_{i<n}(1+ih)^{\gamma-1}|v^i|_0^2 h \\
&\quad = \sum_{i<n}(1+ih)^{\gamma-1}\sum_{j\in\mathbb{Z}}\langle jr\rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha)-(\alpha-[\alpha])([\alpha]+1-\alpha)} \\
&\qquad \times(|v_j^i|^2)^{([\alpha]+1-\alpha)+(\alpha-[\alpha])}h \\
&\quad \le \sum_{i<n}(1+ih)^{\gamma-1}\left(\sum_{j\in\mathbb{Z}}\langle jr\rangle^{\alpha-[\alpha]}|v_j^i|^2\right)^{[\alpha]+1-\alpha} \\
&\qquad \times\left(\sum_{j\in\mathbb{Z}}\langle jr\rangle^{-([\alpha]+1-\alpha)}|v_j^i|^2\right)^{\alpha-[\alpha]} h \\
&\quad = \sum_{i<n}(1+ih)^{-([\alpha]+1-\gamma)}\left((1+ih)^{[\alpha]}|v^i|^2_{\alpha-[\alpha]}\right)^{[\alpha]+1-\alpha} \\
&\qquad \times\left((1+ih)^{[\alpha]}|v^i|^2_{\alpha-[\alpha]-1}\right)^{\alpha-[\alpha]} h,
\end{aligned}$$

where we have used the Hölder inequality

$$\sum ab \le \left(\sum a^p\right)^{1/p}\left(\sum b^{p'}\right)^{1/p'},$$

with $p = \frac{1}{[\alpha]+1-\alpha}$ and $p' = \frac{1}{\alpha-[\alpha]}$. Further, again using the Hölder inequality and (4.24) one obtains

$$\sum_{i<n}(1+ih)^{\gamma-1}|v^i|_0^2 h$$

$$
\begin{aligned}
&\le C|v^0|_\alpha^{2([\alpha]+1-\alpha)} \sum_{i<n} (1+ih)^{-([\alpha]+1-\gamma)} \\
&\quad \times ((1+ih)^{[\alpha]} |v^i|^2_{\alpha-[\alpha]-1})^{\alpha-[\alpha]} h \\
&\le C|v^0|_\alpha^{2([\alpha]+1-\alpha)} \left[\sum_{i<n} (1+ih)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}} \right]^{[\alpha]+1-\alpha} \\
&\quad \times \left[\sum_{i<n} (1+ih)^{[\alpha]} |v^i|^2_{\alpha-[\alpha]-1}) \right]^{\alpha-[\alpha]} h \\
&\le C(1+nh)^p |v^0|^2_\alpha h,
\end{aligned}
$$

where we take $\gamma = \alpha + p$ for any $p > 0$ instead of $[\alpha] \le \gamma < \alpha$ in Liu and Wang [11]. Thus we obtain the following from (4.23).

Lemma 4.6 *Under the assumptions made at the beginning of this section there exists a constant $C > 0$ independent of n_1 such that the estimate*

$$
(4.25) \quad (1+nh)^{\alpha+p} ||v^n||^2 + \sum_{i<n} (1+ih)^{\alpha+p} ||\Delta v^i||^2 \le C(1+nh)^p |v^0|^2_\alpha
$$

holds for any $p > 0$ and any $n \le n_1$.

Combining the latter part of Lemma 4.5 with Lemma 4.6, we have completed the proof of Proposition 2.4.

5 Estimates of terms involving A_j^n, B_j^n

Proof of Lemma 4.1. For this estimate, we need some properties of stationary discrete shocks. Since the discrete shock profile $(\phi_j)_{j\in\mathbb{Z}}$ is strictly decreasing in $j \in \mathbb{Z}$ stated in Lemma 2.1, there exists a unique integer j_0 such that $\phi_{j_0} \le \bar{u} < \phi_{j_0-1}$, where $\bar{u} = \frac{u_+ + u_-}{2} \in]u_+, u_-[$. Without loss of generality one can assume $j_0 = 0$, thus $\phi_{-1} > \bar{u} \ge \phi_0 > \phi_1$. Otherwise we would have to consider the weight $\langle (j-j_0)r \rangle^\beta$ instead of $\langle jr \rangle^\beta$.

We have for the discrete shock solution by definition

$$
(5.1) \qquad \lambda(f(\phi_{j+1}) - f(\phi_{j-1})) = \mu(\phi_{j+1} - 2\phi_j + \phi_{j-1}),
$$

which implies by the mean value theorem for an appropriate ξ_j

$$
(5.2) \quad \phi_{j-1} - \phi_{j+1} = \frac{2\mu}{\mu - \lambda f'(\xi_j)} (\phi_j - \phi_{j+1}) \ge (\phi_j - \phi_{j+1})
$$

for λ satisfying (1.4).

By the definition of A_j and equation (2.1) we have

$$
\begin{aligned}
A_j &= \mu\left[\frac{w_j - w_{j+1}}{2}P_{j+1} - \frac{w_{j-1} - w_j}{2}P_j\right] - \lambda(\Lambda_{j+1}H_{j+1} - \Lambda_j H_j) \\
&= \left[\frac{w_j - w_{j+1}}{\phi_j - \phi_{j+1}}\mu\frac{\phi_j - \phi_{j+1}}{2}P_{j+1} - \frac{w_{j-1} - w_j}{\phi_{j-1} - \phi_j}\mu\frac{\phi_{j-1} - \phi_j}{2}P_j\right] \\
&\quad -\lambda(Q'_{j+1}H_{j+1} - Q'_j H_j) \\
&= -\lambda\Bigg[\frac{w_j - w_{j+1}}{\phi_j - \phi_{j+1}} \cdot \frac{Q_{j+1} + Q_j}{2}P_{j+1} - \frac{w_{j-1} - w_j}{\phi_{j-1} - \phi_j}\frac{Q_j + Q_{j-1}}{2}P_j \\
&\quad +Q'_{j+1}H_{j+1} - Q'_j H_j\Bigg]. \qquad (5.3)
\end{aligned}
$$

By using the Taylor expansion we easily get

$$\frac{w_j - w_{j+1}}{\phi_j - \phi_{j+1}} = w'_{j+1} + \frac{1}{2}w''_{j+1}(\phi_j - \phi_{j+1}) + \frac{w'''(\xi_j)}{6}(\phi_j - \phi_{j+1})^2$$

and

$$\frac{Q_{j+1} + Q_j}{2} = Q_{j+1} + \frac{Q'_{j+1}}{2}(\phi_j - \phi_{j+1}) + \frac{Q''(\eta_j)}{4}(\phi_j - \phi_{j+1})^2.$$

where $\xi_j,\ \ \eta_j$ are values in the interval $]\phi_{j+1}, \phi_j[$. When taking the product of the two terms note that using (2.1)

$$
\begin{aligned}
&\left|\frac{1}{8}w''_{j+1}Q''(\eta_j)(\phi_j - \phi_{j+1})^3\right| \\
&= \left|\frac{1}{8}w''_{j+1}Q''(\eta_j)\frac{\lambda}{\mu}(\phi_j - \phi_{j+1})^2(Q_{j+1} + Q_j)\right| \\
&\leq C(\phi_j - \phi_{j+1})^2.
\end{aligned}
$$

Also due to the boundness of $w'''Q$ shown in Lemma 3.1, we have

$$\left|\frac{Q_{j+1} + Q_j}{2}\frac{w'''(\xi_j)}{6}(\phi_j - \phi_{j+1})^2\right| \leq C(\phi_j - \phi_{j+1})^2.$$

Substituting these into (5.3) and suitably regrouping terms, and using the fact that w', w'' and $w'''Q$ are bounded by suitable constants C one gets the lower estimate

$$
\begin{aligned}
A_j \geq -\lambda\Bigg[&P_{j+1}[(wQ)'_{j+1} + \frac{(w'Q)'_{j+1}}{2}(\phi_j - \phi_{j+1}) + C(\phi_j - \phi_{j+1})^2] \\
&-P_j[(wQ)'_j + \frac{(w'Q)'_j}{2}(\phi_{j-1} - \phi_j) - C(\phi_{j-1} - \phi_j)^2]\Bigg]
\end{aligned}
$$

$$\geq -\lambda\Big[P_{j+1}[(wQ)'_{j+1} - (wQ)'_j] + (wQ)'_j(P_{j+1} - P_j)$$
$$+\frac{P_{j+1}}{2}\Big[(w'Q)'_{j+1}(\phi_j - \phi_{j+1}) - (w'Q)'_j(\phi_{j-1} - \phi_j)\Big]$$
$$+\frac{P_{j+1} - P_j}{2}(w'Q)'_j(\phi_{j-1} - \phi_j) + CP_{j+1}(\phi_j - \phi_{j+1})^2$$
$$-CP_j(\phi_{j-1} - \phi_j)^2\Big].$$

Further, by $(wQ)'_j = 2(\phi_j - \bar{u})$ we obtain

$$A_j \geq -\lambda\Big[-2P_{j+1}(\phi_j - \phi_{j+1}) + (wQ)'_j(P_{j+1} - P_j)$$
$$+\frac{P_{j+1}}{2}\Big[((w'Q)'_{j+1} - (w'Q)'_j)(\phi_j - \phi_{j+1})$$
$$-(w'Q)'_j(\phi_{j+1} - 2\phi_j + \phi_{j-1})\Big] + \frac{P_{j+1} - P_j}{2}(w'Q)'_j(\phi_{j-1} - \phi_j)$$
$$+CP_{j+1}(\phi_j - \phi_{j+1})^2 - CP_j(\phi_{j-1} - \phi_j)^2\Big].$$

From the mean value theorem with $(w'Q)'' = w'''Q + 2w''Q' + w'Q''$, which is bounded as we have seen in Lemma 3.1, and (2.3) we obtain

$$A_j \geq \lambda P_{j+1}(\phi_j - \phi_{j+1}) - \lambda\Big[P_{j+1}(\phi_{j+1} - \phi_j) + 2(\phi_j - \bar{u})(P_{j+1} - P_j)$$
$$+\frac{P_{j+1}}{2}[C(\phi_j - \phi_{j+1})^2 + C\lambda(\phi_{j-1} - \phi_{j+1})] - CP_j(\phi_{j-1} - \phi_j)^2$$
$$+\frac{P_{j+1} - P_j}{2}(w'Q)'_j(\phi_{j-1} - \phi_j)\Big]. \tag{5.4}$$

Applying (5.2), Lemma 3.2 and Lemma 2.2 we proceed as follows

$$A_j = \lambda P_{j+1}(\phi_j - \phi_{j+1}) - \lambda\Big[P_{j+1}(\phi_{j+1} - \phi_j) + 2(\phi_j - \bar{u})(P_{j+1} - P_j)$$
$$+\frac{P_{j+1}}{2}\left[C(\phi_j - \phi_{j+1})^2 + C\lambda(\phi_{j-1} - \phi_{j+1})\right] - CP_j(\phi_{j-1} - \phi_j)^2$$
$$+\frac{P_{j+1} - P_j}{2}(w'Q)'_j(\phi_{j-1} - \phi_j)\Big].$$
$$\geq \theta\lambda P_j(\phi_j - \phi_{j+1}) + \lambda\Big[P_{j+1}(\phi_j - \phi_{j+1})\Big[1 - C\lambda - C(\phi_j - \phi_{j+1})\Big]$$
$$-2(P_{j+1} - P_j)(\phi_j - \bar{u}) - CP_j(\phi_{j-1} - \phi_j)^2$$
$$-C|P_{j+1} - P_j|(\phi_{j-1} - \phi_j)\Big]. \tag{5.5}$$

Note that $|P_{j+1} - P_j| \le \beta C_r r \langle jr\rangle^{\beta-1}$ for some positive constant C_r by (3.5). Therefore, we get

$$
\begin{aligned}
A_j &\ge \theta\lambda P_j(\phi_j - \phi_{j+1}) + \lambda\Big[\theta P_j(\phi_j - \phi_{j+1})\left[1 - \lambda C - C(\phi_j - \phi_{j+1})\right] \\
&\quad -2(P_{j+1} - P_j)(\phi_j - \bar{u}) - CP_j(\phi_j - \phi_{j+1})^2 \\
&\quad -\beta C_r \langle jr\rangle^{\beta-1} r C(\phi_{j-1} - \phi_j)\Big] \\
&\ge \theta\lambda P_j(\phi_j - \phi_{j+1}) + \lambda\Big(\theta P_j(\phi_j - \phi_{j+1}) \\
&\quad \times\Big[1 - \lambda C - C(\phi_j - \phi_{j+1}) - C\frac{r}{\langle jr\rangle}\Big] - 2(P_{j+1} - P_j)(\phi_j - \bar{u})\Big) \\
&= \theta\lambda\langle jr\rangle^{\beta}(\phi_j - \phi_{j+1}) + \lambda\Big(\theta\langle jr\rangle(\phi_j - \phi_{j+1}) \\
&\quad \times\Big[1 - \lambda C - C(\phi_j - \phi_{j+1}) \\
&\quad -C\frac{r}{\langle jr\rangle}\Big] - 2\frac{(P_{j+1} - P_j)}{\langle jr\rangle^{\beta-1}}(\phi_j - \bar{u})\Big)\langle jr\rangle^{\beta-1}.
\end{aligned}
\tag{5.6}
$$

We want to estimate the second summand further from below. Noting that $(\phi_j - \phi_{j+1}) \le C\lambda$ we may find a constant $\nu \in]0,1[$ for $r = \Delta x,\ \ \lambda = \frac{\Delta t}{\Delta x}$ taken suitably small such that the lower bound

$$
1 - C\lambda - C(\phi_j - \phi_{j+1}) - C\frac{r}{\langle jr\rangle} \ge \nu
$$

holds independently of j. From the equations (2.1), (5.1) as well as the choice of the weights w_j, it is clear that the constants C involved in the above estimates depend only on $u_\pm$, $M = \max f'(u)/\mu$ and the bound encountered in Lemma 3.1.

First we consider the case $j = 0$. In this case $\phi_0 - \bar{u}$ may vanish. But we have $-2(P_1 - P_0)(\phi_0 - \bar{u}) \ge 0$ due to $\phi_0 \le \bar{u}$. So with $c_1 = \frac{\theta(\phi_0 - \phi_1)\nu}{\alpha r}$ we obtain for any $\beta \in [0, \alpha]$ the lower estimate

$$
A_0 \ge \theta\lambda(\phi_0 - \phi_1) + \lambda\theta(\phi_0 - \phi_1)\nu \ge \theta\lambda(\phi_0 - \phi_1) + c_1\beta h.
$$

Now we consider the case $j \ne 0$. We set $c_2 := 2c_r \min_{j\ne 0}|\phi_j - \bar{u}| > 0$. By Lemma 3.2 we get the inequality

$$
-2(\phi_j - \bar{u})\frac{(P_{j+1} - P_j)}{\langle jr\rangle^{\beta-1}} = 2|\phi_j - \bar{u}|\frac{|P_{j+1} - P_j|}{\langle jr\rangle^{\beta-1}} \ge c_2\beta r.
$$

Since

$$
\theta\langle jr\rangle(\phi_j - \phi_{j+1})\nu > 0,
$$

we may neglect this term in the estimate from below. Therefore we obtain with $\lambda r = h$

$$\begin{aligned} A_j &\geq \theta\lambda\langle jr\rangle^\beta(\phi_j - \phi_{j+1}) + \lambda c_2\beta r\langle jr\rangle^{\beta-1} \\ &\geq \theta\lambda\langle jr\rangle^\beta(\phi_j - \phi_{j+1}) + c_2\beta\langle jr\rangle^{\beta-1}h. \end{aligned}$$

Now taking $c_0 := \min\{c_1, c_2\}$ we have for any $j \in \mathbb{Z}$ the final estimate

$$A_j \geq \theta\lambda\langle jr\rangle^\beta(\phi_j - \phi_{j+1}) + c_0\beta\langle jr\rangle^{\beta-1}h,$$

which proves the Lemma 4.1. □

Proof of Lemma 4.2. Rearranging the terms in B_j^n we obtain

$$\begin{aligned} B_j^n &= \left[-\lambda\Lambda_{j+1}v_j^n(H_j - H_{j+1}) - \mu(P_{j+1} - P_j)w_j\frac{v_j^n + v_{j+1}^n}{2}\right] \\ &\quad\times(v_{j+1}^n - v_j^n) \\ &= \lambda\Lambda_{j+1}v_j^n(H_{j+1} - H_j)(v_{j+1}^n - v_j^n) \\ &\quad-\mu(P_{j+1} - P_j)w_j\left(v_j^n + \frac{v_{j+1}^n - v_j^n}{2}\right)(v_{j+1}^n - v_j^n) \\ &= I_1 + I_2. \end{aligned} \tag{5.7}$$

Since $H_j > 0$ for all $j \in \mathbb{Z}$ one has $|H_{j+1} - H_j| \leq H_{j+1} + H_j$. By using the Cauchy-Schwarz inequality we estimate I_1 as follows

$$\begin{aligned} |I_1| &= |\lambda\Lambda_{j+1}v_j^n(H_{j+1} - H_j)(v_{j+1}^n - v_j^n)| \\ &\leq \sqrt{H_j + H_{j+1}}|v_{j+1}^n - v_j^n|\lambda\sqrt{|H_{j+1} - H_j|}|\Lambda_{j+1}v_j^n| \\ &\leq \varepsilon(v_{j+1}^n - v_j^n)^2\frac{H_j + H_{j+1}}{2} + \frac{(\lambda\Lambda_{j+1})^2}{2\varepsilon}|H_{j+1} - H_j|(v_j^n)^2. \end{aligned} \tag{5.8}$$

We set

$$\tilde{C} = \max\left\{\sup_{j\in\mathbb{Z}}\frac{w_{j+1}}{w_j}, \sup_{j\in\mathbb{Z}}\left|\frac{w_{j+1} - w_j}{\phi_{j+1} - \phi_j}\right|\right\}.$$

This quantity exists due to the fact that $C^{-1} \leq w_j \leq C$ by (3.2) for some suitable constant $C > 0$ and the fact that the first derivative of w is bounded, see (3.3).

Using the identity

$$H_{j+1} - H_j = w_{j+1}(P_{j+1} - P_j) + P_j(w_{j+1} - w_j),$$

and Lemma 3.2 one obtains

$$\begin{aligned} |H_{j+1} - H_j| &\leq \beta\tilde{C}C_r w_j\langle jr\rangle^{\beta-1}r + \left|\frac{w_{j+1} - w_j}{\phi_{j+1} - \phi_j}\right| \cdot |\phi_{j+1} - \phi_j|\langle jr\rangle^\beta \\ &\leq \beta\tilde{C}C_r\langle jr\rangle^{\beta-1}w_j r + \tilde{C}|\phi_{j+1} - \phi_j|\langle jr\rangle^\beta. \end{aligned}$$

By $\lambda r = h$ and setting $C = \max\{\tilde{C}, \tilde{C} C_r \max |f'|^2\}$ we have

$$\sum_{j\in\mathbb{Z}} |I_1| \le \varepsilon \sum_{j\in\mathbb{Z}} (v_{j+1}^n - v_j^n)^2 \frac{H_j + H_{j+1}}{2}$$
$$+ \frac{\lambda C}{2\varepsilon}\left[\beta h |v_j^n|_{\beta-1,w}^2 + \lambda \sum_{j\in\mathbb{Z}} (\phi_j - \phi_{j+1}) |v_j^n|^2 \langle jr\rangle^\beta\right]. \tag{5.9}$$

On the other hand

$$I_2 \le \mu |P_{j+1} - P_j| w_j \left[|v_j^n||v_{j+1}^n - v_j^n| + \frac{(v_{j+1}^n - v_j^n)^2}{2}\right]$$
$$\le \mu\beta C_r r \langle jr\rangle^{\beta-1}\left[\varepsilon_1 |v_j^n|^2 + \frac{1}{4\varepsilon_1}|v_{j+1}^n - v_j^n|^2 + \frac{1}{2}|v_{j+1}^n - v_j^n|^2\right] w_j,$$

which, after summing up over j from $-\infty$ to $+\infty$, becomes

$$\sum_{j\in\mathbb{Z}} |I_2| \le \mu\beta C_r r \left[\varepsilon_1 |v^n|_{\beta-1,w}^2 + \left(\frac{1}{4\varepsilon_1} + \frac{1}{2}\right) |\Delta v^n|_{\beta-1,w}^2\right]. \tag{5.10}$$

Noting that

$$|\Delta v^n|_{\beta-1,w}^2 = \sum_{|j|\le J} \langle jr\rangle^{\beta-1} |v_{j+1}^n - v_j^n|^2 w_j + \sum_{|j|\ge J} \frac{\langle jr\rangle^\beta}{\langle jr\rangle} |v_{j+1}^n - v_j^n|^2 w_j$$
$$\le C(J) \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 + \frac{1}{Jr}\sum_{j\in\mathbb{Z}} \langle jr\rangle^\beta |v_{j+1}^n - v_j^n|^2 w_j$$
$$= C(J)\|\Delta v^n\|^2 + \frac{1}{Jr}|\Delta v^n|_{\beta,w}^2,$$

for some large fixed number $J > 0$, we get

$$\sum_{j\in\mathbb{Z}} |B_j^n| \le \varepsilon \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 \frac{H_j + H_{j+1}}{2} + \frac{\mu\beta C_r(\frac{1}{2\varepsilon_1} + 1)}{2J} |\Delta v^n|_{\beta,w}^2$$
$$+ \overline{C}\beta ||\Delta v^n||^2 r + \left[\varepsilon_1 \mu\beta C_r r + \frac{\lambda C\beta}{2\varepsilon} h\right] |v^n|_{\beta-1,w}^2$$
$$+ \frac{\lambda^2 C}{2\varepsilon} \sum_{j\in\mathbb{Z}} (\phi_j - \phi_{j+1}) |v_j^n|^2 \langle jr\rangle^\beta$$
$$\le \varepsilon \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n|^2 \frac{H_j + H_{j+1}}{2} + \frac{\mu\beta C_r(\frac{1}{2\varepsilon_1} + \mu)}{2J} |\Delta v^n|_{\beta,w}^2$$
$$+ \overline{C}\beta ||\Delta v^n||^2 r + \frac{\beta c_0}{2} h |v^n|_{\beta-1,w}^2$$

$$+\frac{\lambda^2 C}{2\varepsilon}\sum_{j\in\mathbb{Z}}(\phi_j-\phi_{j+1})|v_j^n|^2\langle jr\rangle^\beta,$$

where we have chosen

$$\varepsilon_1=\frac{[c_0-\frac{\lambda C}{\varepsilon}]\lambda}{2\mu c_r}$$

and $\overline{C} = C(J)\left(\frac{1}{4\varepsilon_1}+\frac{1}{2}\right)\mu c_r$. We see that $\varepsilon_1 > 0$ provided that λ is suitably small. This completes the proof of Lemma 4.2. □

References

1. S. E. Buckley, M. C. Leverett: Mechanism of fluid displacement in sands. A.I.M. E. **146**, 107–116 (1942)
2. C. M. Dafermos: Regularity and large time behavior of solutions of a conservation law without convexity. Proc. R. Soc. Edinb. **99A**, 201–239 (1985)
3. B. Engquist, S. Osher: One-sided difference approximations for nonlinear conservation laws. Math. Comp. **36**, 321–351 (1981)
4. B. Engquist, Shih-Hsien Yu: Convergence of finite difference schemes for piecewise smooth solutions with shocks (manuscript)
5. A. M. Il'in, O. A. Oleinik: Asymptotic behaviour of solutions of the Cauchy problem for certain quasi-linear equations for large time. Mat. Sbornik **51**, 191–216 (1960)
6. G. Jennings: Discrete shocks. Comm. Pure Appl. Math. **27**, 25–37 (1974)
7. C. Jones, R. Gardner, T. Kapitula: Stability of travelling waves for non-convex scalar viscous conservation laws. Comm. Pure Appl. Math. **46**, 505–526 (1993)
8. S. Kawashima, A. Matsumura: Asymptotic stability of travelling wave solutions of system for one-dimensional gas motion. Comm. Math. Phys. **101**, 97–127 (1985)
9. H. L. Liu: Asymptotic stability of shock waves for non-convex Convection-Diffusion equation. Appl. Math. Lett. **10**, 129–134 (1997)
10. H. L. Liu, J. H. Wang: Nonlinear stability of stationary discrete shocks for non-convex scalar conservation laws. Math. Comp. **65**, 1137–1153 (1996)
11. H. L. Liu, J. H. Wang: Decay rate for perturbations of stationary discrete shocks for convex scalar conservation laws. Math. Comp. **66**, 69–84 (1997)
12. H. L. Liu, J. H. Wang: Asymptotic stability of stationary discrete shocks of Lax-Friderichs scheme for non-convex conservation laws. Jn. J. Indust. Appl. Math. **15**, 147–162 (1998)
13. J. G. Liu, Z. P. Xin: Nonlinear stability of discrete shocks for systems of conservation laws. Arch. Rat. Mech. Anal. **125**, 217–256 (1993)
14. J. G. Liu, Z. P. Xin: L^1 stability of stationary discrete shocks. Math. Comp. **60**, 233–244 (1993)
15. A. Majda, J. Ralston: Discrete shock profiles for systems of conservation laws. Comm. Pure Appl. Math. **32**, 445–483 (1979)
16. A. Matsumura, K. Nishihara: Asymptotic stability of travelling waves for scalar viscous conservation laws with non-convex nonlinearity. Comm. Math. Phys. **165**, 83–96 (1994)
17. D. Michelson: Discrete shocks for difference approximations to systems of conservation laws. Adv. Appl. Math. **5**, 433–469 (1984)

18. K. Nishihara: A note on the stability of travelling solutions of Burgers equation. Jn. J. Appl. Math. **2**, 27–35 (1985)
19. S. Osher, J. Ralston: L^1 stability of travelling waves with applications to convective porous medium flow. Comm. Pure Appl. Math. **35**, 737–749 (1982)
20. D. Sattinger: On the stability of waves of nonlinear parabolic systems. Adv. Math. **22**, 312–355 (1976)
21. Y.S. Smyrlis: Existence and stability of stationary profiles of the LW scheme. Comm. Pure Appl. Math. **43**, 508–545 (1990)
22. A. Szepessy: On the stability of finite element methods for shock waves. Comm. Pure Appl. Math. **45**, 923–946 (1992)
23. E. Tadmor: The large-time behavior of the scalar genuinely nonlinear Lax-Friedrichs scheme. Math. Comp. **43**, 353–368 (1984)