A Diffusive Subcharacteristic Condition for Hyperbolic Systems with Diffusive Relaxation

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Abstract

In this paper, we study a diffusive limit of a semilinear hyperbolic system with relaxation. For a Cauchy problem with initial data prescribed around a traveling wave solution, we rigorously justify this limit using the energy method under a rather weak characteristic condition, called the diffusive subcharacteristic condition.

1 Introduction

In [5], we studied a diffusive scaling for the semilinear hyperbolic system with relaxation

$$(E_\epsilon) \qquad \begin{cases} u_t^\epsilon + v_x^\epsilon = 0, \\ \epsilon^2 v_t^\epsilon + a u_x^\epsilon = f(u^\epsilon) - v^\epsilon, \end{cases}$$

with the initial data

$$(u^\epsilon, v^\epsilon)(x,0) = (u_0^\epsilon, v_0^\epsilon)(x) \overset{x\to\pm\infty}{\longrightarrow} (u_\pm, v_\pm), \quad v_\pm = f(u_\pm). \tag{1}$$

Here $\epsilon > 0$ is the relaxation time, the constant $a > 0$ will be specified later. It is easy to see that, as $\epsilon \to 0$, (E_ϵ) asymptotically reduces to the following problem

$$(E_0) \qquad \begin{cases} u_t + f(u)_x = a u_{xx}\,, \\ v = f(u) - a u_x\,, \end{cases}$$

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with initial data

$$(u,v)(x,0) = (u_0,v_0)(x).$$

In [5], for $\epsilon < 1$, under the *subcharacteristic condition*, see [10], [16]

$$a > f'(u)^2\,, \tag{2}$$

we rigorously justified this asymptotic limit with the initial data prescribed around a traveling wave solution $(U^\epsilon, V^\epsilon)(x-st)$ of (E_ϵ).

In order to understand condition (2), we briefly review the study on the semilinear hyperbolic system with relaxation under the Euler scaling:

$$\begin{cases} u^\epsilon_t + v^\epsilon_x = 0, \\ v^\epsilon_t + a u^\epsilon_x = -\frac{1}{\epsilon}\left[v^\epsilon - f(u^\epsilon)\right]. \end{cases} \tag{3}$$

This system was introduced by Jin and Xin [6] as a new way of regularizing the hyperbolic conservation law

$$u_t + f(u)_x = 0. \tag{4}$$

The necessity of the subcharacteristic condition (2) may be seen from the Chapman-Enskog expansion on (3), which yields

$$u_t + f(x)_x = \epsilon[(a - f'(u)^2)u_x]_x\,. \tag{5}$$

Clearly, (2) guarantees the dissipative nature of this convection-diffusion equation, and the stability of (3). Indeed, under this condition, the rigorous passage from (3) to (4) has been justified. See [4], [7], [13], [14].

Now back to the diffusive scaling (E_ϵ). Using the Chapman-Enskog expansion on (E_ϵ), one obtains

$$u_t + f(u)_x = [(a - \epsilon^2 f'^2)u_x]_x + a\epsilon^2[f(u)_{xx} + f'(u)u_{xx}]_x - a^2\epsilon^2 u_{xxxx}. \tag{6}$$

Obviously, to ensure the dissipative nature of (6), one just needs the following weakened characteristic condition

$$a > \epsilon^2 f'^2\,. \tag{7}$$

We call this condition the *diffusive subcharacteristic condition.* With the diffusive scaling in (E_ϵ), this is clearly the physically more natural condition than (2). The purpose of this paper is to justify that (E_0) is the limit of (E_ϵ)

as $\epsilon \to 0$ under the diffusive subcharacteristic condition. Due to technical reason, this analysis works only for the traveling waves with suitable strength i.e., $|u_+ - u_-| \le \beta$ with β defined in (26).

The diffusive scaling introduced in (E_ϵ) is very typical in many important physical problems, for example, in transport equation in diffusive regime [8], [2], [15], in kinetic equations near incompressible Navier-Stokes regimes [1], [3], and in hyperbolic balance laws [11]. In particular, similar diffusive limits have been studied for rather general hyperbolic systems with relaxation and discrete-velocity kinetic models by using L^1 compactness tools (compensated-compactness or BV estimates)[12], [9], which are in general not working for the system case (i.e. when the reduced equation is not a scalar one). Although the results stated in this article could also be established under possibly weaker assumptions by these approaches, nevertheless, due to the restriction of these delicated techniques, it is always worthwhile to explore the applicability of the more general energy method, which is capable of generalizations to system case. Moreover, our arguments provide a clear picture of the large time behavior of solutions for both original system and the reduced equation.

2 Review and Main Results

We first decompose the solution of (E_ϵ) and (1) into

$$(u^\epsilon, v^\epsilon) = (U^\epsilon, V^\epsilon)(z) + (\bar{u}^\epsilon, \bar{v}^\epsilon)(z,t)\,,$$

where $(U^\epsilon(z), V^\epsilon(z))$ is the traveling wave solution of (E_ϵ) in a moving coordinate $z = x - st + x_0$ with possible space shift x_0, and x_0 can be uniquely determined by the relation

$$x_0 = \frac{\int_{\mathbf{R}} (u_0^\epsilon - U^\epsilon)(y)dy}{u_+ - u_-}.$$

Without loss of generality, we may take $x_0 = 0$ hereafter. As shown in [5], this traveling wave exists as long as Rankine-Hugoniot condition

$$H(u_+) = 0, \quad \text{where} \quad H(u) \equiv -s(u - u_-) + f(u) - f(u_-) \tag{8}$$

and the Lax shock condition

$$f'(u_+) < s < f'(u_-) \tag{9}$$

are satisfied. In fact, the $U-$component of this traveling wave satisfies the following equation

$$(I_\epsilon) \qquad U_z^\epsilon = \frac{H(U^\epsilon)}{a - \epsilon^2 s^2}\,, \qquad z = x - st\,, \tag{10}$$

and decays exponentially to $u_\pm$ at far fields $x = \pm\infty$. The traveling wave U^0 of the relaxed equation (E_0) solves the above equation with $\epsilon = 0$. Moreover, one can establish the following result:

$$||U^\epsilon - U^0||_{H^1} \to 0\,, \qquad \text{as } \epsilon \to 0\,, \tag{11}$$

provided $\int (U^\epsilon - U^0)dx = 0$. Hereafter H^m denotes the usual Sobolev space on $\mathbb{R}$ of order m with norm $\|\cdot\|_{H^m}$. With this, one may naturally expect the solution of (E_0) to be of the form

$$(u, v)(x, t) = (U^0, V^0)(z) + (\bar{u}, \bar{v})(z, t), \quad z = x - st.$$

With (11), the investigation of limiting behavior of (u^ϵ, v^ϵ) can be reduced to the study of the convergence of the family $\{\bar{u}^\epsilon, \bar{v}^\epsilon\}$, as $\epsilon \downarrow 0$, in L^2.

To obtain such a result, we make the following assumptions on initial data:

(H_1) $\quad u_\pm$ generate a traveling wave (U^ϵ, V^ϵ) of (E_ϵ),

(H_2) $\quad \| \int_{-\infty}^x (u^\epsilon - U^\epsilon)(y)dy\|_{H^3} + \|\epsilon(v_0^\epsilon - V^\epsilon)(x)\|_{H^2} \le c_1$,

(H_3) $\quad \|(v_0^\epsilon - V^\epsilon - s(u_0^\epsilon - U^\epsilon)\|_{H^2} \le c_2$ and

$$\frac{1}{\epsilon}\|v_0^\epsilon - f(u_0^\epsilon) + a\partial_x u_0^\epsilon\|_{H^1} \le c_3,$$

(H_4) $\quad \int_R (u_0^\epsilon - u_0)(x)dx = 0$ and, as $\epsilon \to 0$,

$$u_0^\epsilon - U^\epsilon \to u_0 - U^0\,, \qquad v_0^\epsilon - V^\epsilon \to v_0 - V^0\,, \qquad \text{in} \quad H^2 \quad \text{as} \quad \epsilon \to 0.$$

In these assumptions, c_1, c_2, c_3 are positive constants independent of ϵ.

Under these conditions, we proved [5]

Theorem 1 (Jin-Liu) *Assume* $0 < \epsilon \le 1$, *f is a smooth convex function satisfying*

$$a > f'^2 \quad \text{for } u \in (u_+, u_-)\,, \tag{12}$$

and $(H_1) - (H_4)$ *hold, then there exists an* $u \in C^1([0,T] \times \mathbb{R})$ *for any* $T \geq 0$ *such that*

$$u^\epsilon - U^\epsilon \to u(x,t) - U^0 \quad in \quad C^0([0,T]; H_{loc}^{2-\delta_1}) \quad for\ any \quad \delta_1 > 0,$$

$$v^\epsilon - V^\epsilon \to v(x,t) - V^0 \quad in \quad C^0([0,T]; H_{loc}^{2-\delta_1}),$$

and the limit function (u,v) *solves the relaxed parabolic problem:*

$$\begin{cases} u_t + f(u)_x = a u_{xx}, \\ u|_{t=0} = u_0(x), \end{cases}$$

and

$$v = f(u) - a u_x.$$

As mentioned in the introduction section, our goal of the paper is to establish the same conclusion as in Theorem 1, but with the diffusive subcharacteristic condition (7). Our new result is stated in the following theorem.

Theorem 2 *Assume that there exists a constant* $\epsilon_0 > 0$ *such that*

$$0 < \epsilon \leq \epsilon_0, \qquad a > \epsilon^2 f'^2 \quad \text{for} \quad u \in (u_+, u_-), \tag{13}$$

and $(H_1) - (H_4)$ *hold, then the convergence result in Theorem 1 still holds if* $|u_+ - u_-| \leq \beta$ *with* β *defined in (26).*

Note that the stability condition $a > \epsilon^2 f'^2$ is not restricted and is automatically satisfied for $\epsilon \leq \epsilon_0$ with suitable small ϵ_0. This reflects the case in the diffusion limit.

The proof of Theorem 1 in [5] consists of three steps. First, under the subcharacteristic condition (12) and the assumption (H_2), we established the global existence and uniform estimates for (E_ϵ). Secondly, the regularity in time was established for (u^ϵ, v^ϵ). Finally, under assumption (H_3), we proved the convergence to the local equilibrium and furthermore, under assumption (H_4), the compactness of the solution sequence was established. Since the stability condition (12) was essentially used only in constructing the energy function in the first step, to prove Theorem 2, it suffices to show that one can establish similar global existence results under the mild diffusive subcharacteristic condition (13). In the next section, we introduce a new energy function that allows us to establish the global existence under (13). Since the other two steps do not involve explicitly the characteristic condition, they still hold in the present setting. Interested readers may consult [5] for details.

3 A New Energy Function and the Global Existence

In this section, under the diffusive subcharacteristic condition (13), we shall establish the global existence for the solutions (u^ϵ, v^ϵ) of the Cauchy problem (E_ϵ) with the initial data (1). The main result is included in the following theorem.

Theorem 3 *Assume that there exists* $\epsilon_0 < 1$ *and* $c_1 > 0$ *independent of* ϵ *such that (13) and* $(H_1) - (H_2)$ *hold. Let* $(U^\epsilon, V^\epsilon)(x - st)$, $f'(u_+) < s < f'(u_-)$, *be a traveling wave solution generated by* $u_\pm$ *with* $u_0^\epsilon - U^\epsilon$ *being integrable on* $\mathbb{R}$. *Then there exists a unique global solution* (u^ϵ, v^ϵ) *to* (E_ϵ), *with initial condition (1) where* $|u_+ - u_-|$ *is small, such that*

$$(u^\epsilon - U^\epsilon, v^\epsilon - V^\epsilon) \in C^0([0,\infty); H^2) \cap L^2([0,\infty); H^2).$$

Moreover,

$$\|(u^\epsilon - U^\epsilon, v^\epsilon - V^\epsilon)\|_{L^2([0,\infty);H^2)} \le C,$$
$$\|(u^\epsilon - U^\epsilon, \epsilon(v^\epsilon - V^\epsilon))\|_{C^0([0,\infty);H^2)} \le C.$$

Let

$$(u^\epsilon, v^\epsilon)(x,t) = (U^\epsilon, V^\epsilon)(z) + (\bar{u}^\epsilon(z,t), \bar{v}^\epsilon(z,t))\,, \tag{14}$$

where (U^ϵ, V^ϵ) is the solution of traveling wave equation

$$\begin{cases} -sU_z^\epsilon + V_z^\epsilon = 0, \\ -s\epsilon^2 V_z^\epsilon + aU_z^\epsilon = f(U^\epsilon) - V^\epsilon. \end{cases} \tag{15}$$

Combination of equation (E_ϵ) and this equation (dropping the index ϵ for notational convenience) gives

$$\begin{cases} \bar{u}_t + \bar{v}_x &= 0, \\ \epsilon^2 \bar{v}_t + a\bar{u}_x &= f(U^\epsilon + \bar{u}) - f(U^\epsilon) - \bar{v}. \end{cases} \tag{16}$$

By the assumption (H_2) and the conservative form of the first equation in (16), for $t \ge 0$,

$$\int_{-\infty}^{\infty} \bar{u}(z,t)dz = 0, \qquad \bar{v}(\pm\infty, t) = 0.$$

Let

$$\phi(z,t) = \int_{-\infty}^{z} \bar{u}(y,t)dy, \qquad \psi(z,t) = -\bar{v}(z,t)\,.$$

Applying these variables in (16) and combining the resulting two equations, one gets

$$L(\phi, \psi) = \epsilon^2 \{\psi_t - s\psi_z\} - a\phi_{zz} + \phi_t - s\phi_z + \lambda\phi_z = F \tag{17}$$

with $\psi = \phi_t - s\phi_z$, $\lambda = f'(U^\epsilon)$ and

$$F = -\{f(U^\epsilon + \phi_z) - f(U^\epsilon) - f'(U^\epsilon)\phi_z\} = O(1)\phi_z^2 \tag{18}$$

being the higher order term.

We introduce the solution space of the problem (17) with initial data

$$(\phi, \psi)(z, 0) = (\phi_0, \psi_0)(z) := (\int_{-\infty}^{z} (u_0^\epsilon - U^\epsilon)(y)dy, V^\epsilon(z) - v_0^\epsilon(z)) \tag{19}$$

as follows

$$\begin{aligned} X(0,T) = \{(\phi, \psi) \ \ : \ \ & (\phi, \psi) \in C^0([0,T); H^3) \times C^0([0,T); H^2), \\ & (\phi_z, \psi) \in L^2([0,T); H^2)^2\}, \qquad 0 < T < +\infty. \end{aligned}$$

One can conclude Theorem 3 from the following proposition.

Proposition 1 *Let $N(t) = \sup_{0 \leq \tau \leq t}\{\|\phi(\tau)\|_3 + \|\epsilon\psi(\tau)\|_2\}$. Under the assumptions in Theorem 3, the problem (17), (19) admits a unique solution $(\phi, \psi) \in X(0, +\infty)$ satisfying*

$$\|\phi(t)\|_{H^3(R)} + \|\epsilon\psi(t)\|_{H^2(R)} + \left[\int_0^t \|(\psi, \phi_z)\|_{H^2}^2 d\tau\right]^{1/2} \leq CN(0), \tag{20}$$

where C is a positive constant independent of ϵ.

To prove this proposition, we need to establish the following key a priori estimate.

Lemma 1 *If the assumptions of Theorem 3 hold, and $N(T) \leq \delta$ for some $\delta > 0$, then*

$$\begin{aligned} \|\phi(t)\|_1^2 \ \ + \ \ & \epsilon^2\|\psi(t)\|^2 + \int_0^t \|(\psi, \phi_z)(\tau)\|^2 d\tau + \int_0^t \int_R |U_z^\epsilon|\phi^2 dz\, d\tau \\ \leq \ \ & C\{\|\phi_0\|_1^2 + \epsilon^2\|\psi_0\|^2\} \end{aligned} \tag{21}$$

for $t \in [0, T]$.

Proof. As was done in [5], we multiply (17) by $\phi + 2\psi$ to get

$$\tilde{E}_1(\phi, \psi, \phi_z)_t + \tilde{E}_2(\phi, \psi, \phi_z) + \{\cdots\}_z = F\{\phi + 2\psi\}, \tag{22}$$

where

$$\begin{aligned}
\tilde{E}_1(\phi, \psi, \phi_z) &= \epsilon^2\psi^2 + \epsilon^2\phi\psi + \frac{1}{2}\phi^2 + a\phi_z^2, \\
\tilde{E}_2(\phi, \psi, \phi_z) &= (2 - \epsilon^2)\psi^2 + 2\lambda\phi_z\psi + a\phi_z^2 - \lambda_z\left(\frac{\phi^2}{2}\right),
\end{aligned}$$

and $\{\cdots\}_z$ denotes the term which will disappear after integration over $\mathbb{R}$.

It is easy to see that, to ensure the positivity of the energy functions $\tilde{E}_1$ and $\tilde{E}_2$, one needs

$$\epsilon^2 < 2, \quad f'^2 < (2 - \epsilon^2)a$$

which is ensured by the condition (12). This was the condition used in [5].

To relax this stability condition, we need to introduce a new energy function. For this purpose, we use a new multiplier of the form $w(z)\phi_z$, with $w(z)$ a smooth function to be determined.

Multiplying the equation (17) by $w(z)\phi_z$, one gets

$$\begin{aligned}
w\phi_z L(\phi, \psi) = \quad & \epsilon^2\left[(w\psi\phi_z)_t + \frac{w'}{2}\phi_t^2 - s^2 w'\frac{\phi_z^2}{2}\right] \\
& + \frac{a}{2}w'\phi_z^2 + w\psi\phi_z + \lambda w\phi_z^2 + \{\cdots\}_z \\
= \quad & Fw(z)\phi_z\,,
\end{aligned}$$

where $\lambda = f'(U^\epsilon)$. Addition of this to the identity (22) gives us the following new energy relation

$$E_1(\phi, \psi, \phi_z)_t + E_2(\phi, \psi, \phi_z) + \{\cdots\}_z = F\{\phi + 2\psi + w\phi_z\}, \tag{23}$$

where

$$\begin{aligned}
E_1(\phi, \psi, \phi_z) &= \epsilon^2\psi^2 + \epsilon^2\phi\psi + \frac{1}{2}\phi^2 + a\phi_z^2 + \epsilon^2 w\phi_z\psi \\
&= (\phi, \psi, \phi_z)^T A_1(\phi, \psi, \phi_z), \\
E_2(\phi, \psi, \phi_z) &= (2 - \epsilon^2 + \epsilon^2 w')\psi^2 + [2\lambda + 2w + 2\epsilon^2 w' s]\psi\phi_z \\
&\quad + [a + 2w\lambda + aw]\phi_z^2 - \lambda_z\left(\frac{\phi^2}{2}\right) \\
&= -\lambda_z\left(\frac{\phi^2}{2}\right) + (\psi, \phi_z)^T A_2(\psi, \phi_z),
\end{aligned}$$

and $A_j (j = 1, 2)$ are the matrices defined as

$$A_1 = \begin{bmatrix} 1/2 & \epsilon^2/2 & 0 \\ \epsilon^2/2 & \epsilon^2 & \epsilon^2 w/2 \\ 0 & \epsilon^2 w/2 & a \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 2 - \epsilon^2 + \epsilon^2 w' & \lambda + w + s\epsilon^2 w' \\ \lambda + w + s\epsilon^2 w' & a + 2\lambda w + a w' \end{bmatrix}.$$

Since the convexity of f and the monotonicity of $U^\epsilon(z)$ imply that there exists a constant $c > 0$ such that

$$E_2(\phi, \psi, \phi_y) \geq c|U_z|\phi^2 + (\psi, \phi_z)^T A_2 (\psi, \phi_z)\,,$$

it remains to choose a suitable w such that the matrices $A_j (j = 1, 2)$ are both positive definite.

It is easy to see that A_1 is positive definite if and only if

$$\epsilon^2 < 2, \quad w^2\epsilon^2 < 4a \quad \text{and} \quad \epsilon^2 w^2 < a(2 - 2\epsilon^2).$$

This is possible if we choose w such that $\epsilon \leq \epsilon_0 = 1/\sqrt{2}$ and

$$\epsilon^2 w^2 < a \quad . \tag{24}$$

To ensure the positive definiteness of A_2, one needs to choose w such that

$$2 - \epsilon^2 + \epsilon^2 w' > 0, \quad a + 2\lambda w + a w' > 0$$

and

$$D := (\lambda + w + s\epsilon^2 w')^2 - (2 - \epsilon^2 + \epsilon^2 w')(a + 2\lambda w + a w') < 0\,.$$

Further calculation shows that

$$D = (\lambda - w)^2 + 2s\epsilon^2 w'(w + \lambda) + \epsilon^4 s^2 w'^2 + 2\epsilon^2(1 - w')w\lambda - a(1 + w')(2 - \epsilon^2 + \epsilon^2 w').$$

For ϵ sufficiently small, the requirement that $D < 0$ may be replaced by

$$(\lambda - w)^2 < a(1 + w')(2 - \epsilon^2) \quad \text{and} \quad 1 + w' > 0\,. \tag{25}$$

Next, we show the existence of w satisfying the above requirements. If the strength of traveling wave is suitably bounded, say $|u_+ - u_-| < \beta$ with

$$\beta = \frac{a}{2\max_{u\in(u_+,u_-)}|f'(u) - s||f''(u)|}, \tag{26}$$

we can take

$$w = \lambda\,.$$

Then (25) is satisfied using (10) ,

$$1 + w' = 1 + f''U_z^\epsilon = 1 + f''H(U^\epsilon)/(a - \epsilon s^2) \geq 1 - \beta^{-1}|u_+ - u_-| > 0$$

for $\epsilon \leq \epsilon_0$ with $\epsilon_0 = \min\left\{1/\sqrt{2}, a/2s^2\right\}$. We may take smaller ϵ_0 such that $D < 0$ is also satisfied for $\epsilon \leq \epsilon_0$. From (24) the new stability condition is

$$\epsilon^2\lambda^2 < a \quad \text{for } \epsilon \leq \epsilon_0\,.$$

After the integration with respect to t and z, (23) gives

$$\begin{aligned} \|\phi(t)\|_1^2 \quad &+ \quad \epsilon^2\|\psi(t)\|^2 + \int_0^t \|(\psi,\phi_z)(\tau)\|^2 d\tau + \int_0^t \int_R |U_z^\epsilon|\phi^2 dz\, d\tau \\ &\leq \quad C\{\|\phi_0\|_1^2 + \epsilon^2\|\psi_0\|^2 + \int_0^t \int_R |F|(|\phi| + |\psi|) dz\, d\tau\} \end{aligned} \tag{27}$$

for $t \in [0, T]$. By recalling the definition of F in (18), we have

$$|F| = O(1)|\phi_z|^2.$$

Since $\sup_z\{|\phi|, |\phi_z|, |\psi|\} \leq CN(T)$, the integral on the right hand side of (27) is dominated by

$$CN(T)\int_0^t \|(\psi,\phi_z)\|^2 d\tau.$$

Therefore, if $N(T) < \frac{1}{2C}$, we obtain the desired estimate (21) for $t \in [0, T]$. $\square$

Based on this basic estimate we can get the higher order estimates in the same way as in [5] to close the energy estimate (20). The details are omitted here.

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