



# On traveling wave solutions of the $\theta$ -equation of dispersive type



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## ABSTRACT

Traveling wave solutions to a class of dispersive models,

$$u_t - u_{txx} + uu_x = \theta uu_{xxx} + (1 - \theta)u_x u_{xx},$$

are investigated in terms of the parameter  $\theta$ , including two integrable equations, the Camassa–Holm equation,  $\theta = 1/3$ , and the Degasperis–Procesi equation,  $\theta = 1/4$ , as special models. It was proved in H. Liu and Z. Yin (2011) [39] that when  $1/2 < \theta \leq 1$  smooth solutions persist for all time, and when  $0 \leq \theta \leq 1/2$ , strong solutions of the  $\theta$ -equation may blow up in finite time, yielding rich traveling wave patterns. This work therefore restricts to only the range  $\theta \in [0, 1/2]$ . It is shown that when  $\theta = 0$ , only periodic travel wave is permissible, and when  $\theta = 1/2$  traveling waves may be solitary, periodic or kink-like waves. For  $0 < \theta < 1/2$ , traveling waves such as periodic, solitary, peakon, peaked periodic, cusped periodic, or cusped soliton are all permissible.

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## 1. Introduction

In this work, we investigate traveling wave solutions to a class of dispersive models – the  $\theta$ -equation of the form [37]

$$(1 - \partial_x^2)u_t + (1 - \theta\partial_x^2)\left(\frac{u^2}{2}\right)_x = (1 - 4\theta)\left(\frac{u_x^2}{2}\right)_x, \quad x \in \mathbb{R}, t > 0. \quad (1.1)$$

The equation can be formally rewritten as

$$u_t - u_{txx} + uu_x = \theta uu_{xxx} + (1 - \theta)u_x u_{xx}, \quad (1.2)$$

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which when  $0 < \theta < 1$  involves a convex combination of nonlinear terms  $uu_{xxx}$  and  $u_x u_{xx}$ . In (1.1), two equations are worth of special attention:  $\theta = \frac{1}{3}$  and  $\theta = \frac{1}{4}$ . The  $\theta$ -equation when  $\theta = \frac{1}{3}$  reduces to the Camassa–Holm (CH) equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, in which  $u(x, t)$  denotes the fluid velocity at time  $t$  in the spatial  $x$  direction [3,20,32]. The CH equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [13,15]. Taking  $\theta = \frac{1}{4}$  in (1.1) one finds the Degasperis–Procesi (DP) equation [16]. The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as that for the CH equation [21,22]. This  $\theta$ -class may well have more applications than those mentioned here.

In recent years, nonlocal dispersive models as an extension of the classical KdV equation have been investigated intensively at different levels of treatments: modeling, analysis as well as numerical simulation. The model derives in several ways, for instance, (i) the asymptotic modeling of shallow water waves [45,21,22]; (ii) renormalization of dispersive operators [45,36]; and (iii) model equations of some dispersive schemes [37]. The peculiar feature of nonlocal dispersive models is their capability to capture both smooth long wave propagation and short wave breaking phenomena. The study of traveling wave solutions to dispersive equations proves to be insightful in understanding various wave structures involved in the dispersive wave dynamics, we refer to recent works [19,29,33,34,40,46,51] for such investigations.

### 1.1. $\theta$ -Equations

The class of  $\theta$ -equations was identified by H. Liu [37] in the study of model equations for some dispersive schemes to approximate the Hopf equation

$$u_t + uu_x = 0.$$

With  $\theta = \frac{1}{b+1}$  the model (1.1) under a transformation as shown in [39] links to the  $b$ -model,

$$u_t - \alpha^2 u_{txx} + c_0 u_x + (b+1)uu_x + \Gamma u_{xxx} = \alpha^2 (bu_x u_{xx} + uu_{xxx}),$$

which has been extensively studied in recent years [18,16,24,25,30,31]. Both classes of equations are contained in the more general class introduced in [36] using renormalization of dispersive operators and number of conservation laws, so called the B-equations

$$u_t + uu_x + [Q * B(u, u_x)]_x = 0, \tag{1.3}$$

where  $Q = \frac{1}{2}e^{-|x|}$  and  $B$  is a quadratic function of  $u$  and  $u_x$ . For this class the local well-posedness in  $C([0, T]; H^{3/2+}(\mathbb{R})) \cap C^1([0, T]; H^{1/2+}(\mathbb{R}))$  for (1.3) with initial data  $u_0 \in H^{3/2+}$  is shown in [36]. In fact, up to a scaling of  $t \rightarrow \frac{t}{\theta}$  for  $\theta \neq 0$ , the  $\theta$ -equation can indeed be rewritten as (1.3) with

$$B = \left(\frac{1}{\theta} - 1\right) \frac{u^2}{2} + \left(4 - \frac{1}{\theta}\right) \frac{u_x^2}{2}.$$

In the last decades, a lot of analysis has been given to the CH equation and the DP equation, among other dispersive equations.

The CH equation has a bi-Hamiltonian structure [28,35] and is completely integrable [3,7]. Its solitary waves are smooth if  $c_0 > 0$  and peaked in the limiting case  $c_0 = 0$  [4]. The orbital stability of the peaked solitons is proved in [12], and that of the smooth solitons in [14]. The explicit interaction of the peaked solitons is given in [1]. It has been shown that the Cauchy problem of the CH equation is locally well-posed [8,43] for initial data  $u_0 \in H^{3/2+}(\mathbb{R})$ . Moreover, it has global strong solutions [6,8] and also admits finite time blow-up solutions [6,8,9]. On the other hand, it has global weak solutions in  $H^1(\mathbb{R})$  [2,10,11,47].

The advantage of the CH equation in comparison with the KdV equation,  $u_t + uu_x + \Gamma u_{xxx} = 0$ , lies in the fact that the CH equation has peaked solitons and models the peculiar wave breaking phenomena [4,9].

For the DP equation an inverse scattering approach for computing  $n$ -peakon solutions was presented in [42]. Its traveling wave solutions were investigated in [34,44]. The formal integrability of the DP equation was obtained in [17] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to the CH peakons [17]. Global strong solutions to the Cauchy problem of the DP equation are proved in [26,38,49] and finite time blow-up solutions in [26,38,48,49]. On the other hand, it has global weak solutions in  $H^1(\mathbb{R})$ , see e.g. [26,49] and global entropy weak solutions belonging to the class  $L^1(\mathbb{R}) \cap BV(\mathbb{R})$  and to the class  $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ , cf. [5].

Though both the DP and the CH equation share some nice properties, they differ in that the DP equation has not only peakon solutions [17] and periodic peakon solutions [50], for example,  $u(x, t) = ce^{-|x-ct|}$ , but also shock peakons [41] and periodic shock waves [27], for example

$$u(x, t) = -\frac{1}{t+c} \operatorname{sign}(x)e^{-|x|}, \quad c > 0.$$

The issue of how regularity of solutions changes in terms of the parameter  $\theta$  was studied in [39], where authors prove that as  $\theta$  increases regularity of solutions improves: strong solutions to the  $\theta$ -equation subject to smooth initial data for  $0 < \theta < \frac{1}{2}$  may blow up in finite time, while in the case  $\frac{1}{2} \leq \theta \leq 1$  every strong solution to the  $\theta$ -equation exists globally in time. This presents a clear picture for global regularity and blow-up phenomena of solutions to the  $\theta$ -equation for all  $0 < \theta < 1$ .

The main quest of this paper is to see how the type of traveling wave solutions varies in terms of the parameter  $\theta$  for  $0 \leq \theta \leq 1/2$ . Traveling waves for  $1/2 < \theta \leq 1$  may be studied in a rather similar fashion, but we choose to skip this range since the relevant traveling waves are less interesting.

### 1.2. Main results

Under the traveling wave transformation  $u(x, t) = \varphi(\xi)$ ,  $\xi = x - ct$ ,  $c > 0$ , (1.2) is reduced to the following ordinary differential equation form

$$-c\varphi' + c\varphi''' + \varphi\varphi' = \theta\varphi\varphi''' + (1 - \theta)\varphi'\varphi'' \quad \left( ' = \frac{d}{d\xi} \right). \tag{1.4}$$

After integration we get

$$-c\varphi + c\varphi'' + \frac{1}{2}\varphi^2 = \theta\varphi\varphi'' + \frac{1 - 2\theta}{2}(\varphi')^2 + a, \tag{1.5}$$

where  $a$  is an integral constant. We rewrite (1.5) as

$$\frac{\theta(4\theta - 1)}{2}(\varphi')^2 + \frac{\theta}{2}\varphi^2 - \theta c\varphi = \frac{1}{2}((\theta\varphi - c)^2)'' + a\theta. \tag{1.6}$$

Hence Eq. (1.5) holds in weak sense for all  $\varphi \in H^1_{loc}(\mathbb{R})$ , i.e., it holds

$$\int_{\mathbb{R}} \left[ \frac{\theta(4\theta - 1)}{2}(\varphi')^2 v + \frac{\theta}{2}\varphi^2 v - \theta c\varphi v \right] dy = \int_{\mathbb{R}} \left[ \frac{1}{2}((\theta\varphi - c)^2)v'' + a\theta v \right] dy \tag{1.7}$$

for all  $v \in C^2_0(\mathbb{R})$ .

Our purpose is to find traveling waves of  $\theta$ -equation (1.2) for  $0 \leq \theta \leq \frac{1}{2}$ , and the main results are stated in the following theorems. Different types of solutions are to be clarified in subsequent proofs.

**Theorem 1.1.** For  $0 < \theta < \frac{1}{2}$  and  $a \in \mathbb{R}$ , traveling wave solutions of (1.2) given by (1.5) are as follows:

- (1) If  $a \leq -\frac{c^2}{2}$  or  $a \geq \frac{c^2(1-2\theta)}{2\theta^2}$ , then (1.5) has no bounded solutions.
- (2) If  $-\frac{c^2}{2} < a < 0$ , then (1.5) has smooth periodic solutions and a smooth soliton solution.
- (3) If  $a = 0$ , then (1.5) has smooth periodic solutions and a peakon solution.
- (4) If  $0 < a < \frac{c^2(1-2\theta)}{2\theta^2}$ , then (1.5) has smooth periodic solutions, a peaked periodic solution, cusped periodic solutions and a cusped soliton solution.

**Theorem 1.2.** For  $\theta = 0$  and  $a \in \mathbb{R}$ , traveling wave solutions of (1.2) given by (1.5) are as follows:

- (1) If  $a \leq -\frac{c^2}{2}$ , then (1.5) has no bounded solutions.
- (2) If  $a > -\frac{c^2}{2}$ , then any bounded solution to (1.5) must be periodic.

**Theorem 1.3.** For  $\theta = \frac{1}{2}$  and  $a \in \mathbb{R}$ , traveling wave solutions of (1.2) given by (1.5) are as follows:

- (1) If  $a \leq -\frac{c^2}{2}$ , then (1.5) has no bounded solutions.
- (2) If  $-\frac{c^2}{2} < a < 0$ , then (1.5) admits both solitary wave and periodic solutions.
- (3) If  $a = 0$ , then the only bounded solution to (1.5) is  $\varphi = 2c$ .
- (4) If  $a > 0$ , then (1.5) admits both solitary like waves and kink like wave solutions.

The traveling wave solutions obtained to solve (1.2) contain those for both the CH and the DP equation, and the results generalize some previous traveling wave results [33,34] for these two special models. Our qualitative analysis through some reduced planar systems is elementary, yet we obtain not only the rich phase portraits but also the existence of seven different types of bounded traveling wave solutions: periodic, solitary, peakon, peaked periodic, cusped periodic, cusped soliton and kink-like wave solutions.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 by analyzing traveling wave solutions for  $0 < \theta < 1/2$ . In Section 3, we show how traveling wave solutions are established to complete Theorem 1.2. A detailed account of traveling wave solutions for the case of  $\theta = 1/2$  is presented in Section 4. Some concluding remarks are given in Section 5.

## 2. Traveling waves for $0 < \theta < \frac{1}{2}$ : proof of Theorem 1.1

### 2.1. Traveling wave system and phase portraits

Let  $w = \varphi'$ , then (1.5) can be rewritten as

$$\begin{cases} \frac{d\varphi}{d\xi} = w, \\ \frac{dw}{d\xi} = \frac{\varphi^2 - 2c\varphi - (1-2\theta)w^2 - 2a}{2(\theta\varphi - c)}. \end{cases} \quad (2.1)$$

This is valid only when  $\varphi(\xi) \neq \frac{c}{\theta}$ . For  $\varphi \neq \frac{c}{\theta}$ , we make the transformation  $d\xi = 2(\theta\varphi - c)d\tau$ , then system (2.1) becomes

$$\begin{cases} \frac{d\varphi}{d\tau} = 2(\theta\varphi - c)w, \\ \frac{dw}{d\tau} = \varphi^2 - 2c\varphi - (1-2\theta)w^2 - 2a. \end{cases} \quad (2.2)$$

Note that

$$\frac{dw^2}{d\varphi} = \frac{2ww'}{\varphi'} = \frac{\varphi^2 - 2c\varphi - (1 - 2\theta)w^2 - 2a}{\theta\varphi - c},$$

then systems (2.1) and (2.2) have the same first integral

$$H(\varphi, w) = |c - \theta\varphi|^{\frac{1-2\theta}{\theta}} \left( -w^2 + \varphi^2 - \frac{2a}{1 - 2\theta} \right) = h, \tag{2.3}$$

where  $h$  is an arbitrary constant. Hence both systems (2.1) and (2.2) have the same topological phase portraits except near the line  $\varphi = \frac{c}{\theta}$ .

In order to find out bounded orbits for (2.2), we consider  $-\frac{c^2}{2} < a < \frac{c^2(1-2\theta)}{2\theta^2}$  since in such a case (2.2) has four equilibriums, which are denoted by

$$(\varphi_1^*, 0) = (c - \sqrt{c^2 + 2a}, 0), \quad (\varphi_2^*, 0) = (c + \sqrt{c^2 + 2a}, 0)$$

and

$$\left( \frac{c}{\theta}, w_1^* \right) = \left( \frac{c}{\theta}, -\sqrt{\frac{c^2 - 2c^2\theta - 2a\theta^2}{\theta^2(1 - 2\theta)}} \right), \quad \left( \frac{c}{\theta}, w_2^* \right) = \left( \frac{c}{\theta}, \sqrt{\frac{c^2 - 2c^2\theta - 2a\theta^2}{\theta^2(1 - 2\theta)}} \right).$$

We see that

$$\varphi_1^* < c < \varphi_2^* < \frac{c}{\theta}. \tag{2.4}$$

**Proposition 2.1.** Assume that  $-\frac{c^2}{2} < a < \frac{c^2(1-2\theta)}{2\theta^2}$ .

- (i)  $(\varphi_1^*, 0)$ ,  $(\frac{c}{\theta}, w_1^*)$  and  $(\frac{c}{\theta}, w_2^*)$  are saddle points and  $(\varphi_2^*, 0)$  is a center point.
- (ii) If  $-\frac{c^2}{2} < a < 0$ , then for  $\varphi \in [\varphi_1^*, \frac{c}{\theta})$ , the homoclinic orbit connecting with  $(\varphi_1^*, 0)$  lies in between two separatrices passing  $(\frac{c}{\theta}, w_1^*)$  and  $(\frac{c}{\theta}, w_2^*)$ , respectively.
- (iii) If  $a = 0$ , then separatrices of  $(\varphi_1^*, 0)$  are the same as separatrices of  $(\frac{c}{\theta}, w_1^*)$  and  $(\frac{c}{\theta}, w_2^*)$ , respectively for  $\varphi < \frac{c}{\theta}$ .
- (iv) If  $0 < a < \frac{c^2(1-2\theta)}{2\theta^2}$ , then the connecting orbit between  $(\frac{c}{\theta}, w_1^*)$  and  $(\frac{c}{\theta}, w_2^*)$  lies in between two separatrices passing  $(\varphi_1^*, 0)$  for  $\varphi \in [\varphi_1^*, \frac{c}{\theta})$ .

Phase diagrams for these cases are shown in Fig. 1.

**Proof.** (i) The characteristic equation for (2.2) when linearized at an equilibrium  $(\varphi^*, w^*)$  is

$$D(\lambda) = \begin{vmatrix} 2\theta w^* - \lambda & 2(\theta\varphi^* - c) \\ 2(\varphi^* - c) & -2(1 - 2\theta)w^* - \lambda \end{vmatrix} = \lambda^2 + p\lambda + q = 0,$$

where

$$p = 2(1 - 3\theta)w^* \quad \text{and} \quad q = -4\theta(1 - 2\theta)(w^*)^2 - 4(\varphi^* - c)(\theta\varphi^* - c).$$

The theory of a planar dynamical system (see e.g., [23]) yields that

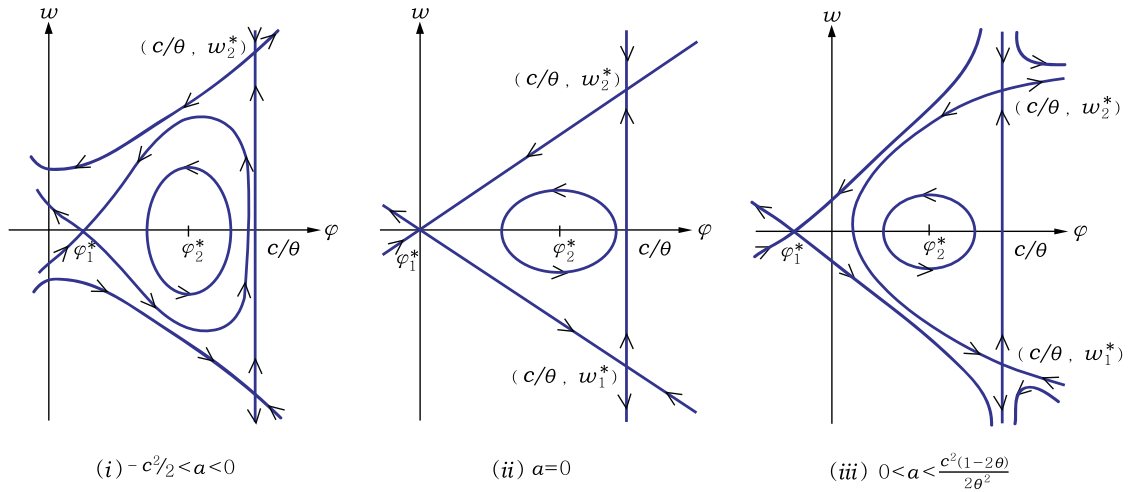


Fig. 1. Phase portraits of systems (2.1) and (2.2).

$(\varphi^*, w^*)$  is a saddle point if  $q < 0$ ,  
 $(\varphi^*, w^*)$  is a center or spiral point if  $q > 0$  and  $p = 0$ .

Since  $H(\varphi, w - w^*) = H(\varphi, w^* - w)$ , in such a case  $(\varphi^*, 0)$  can only be a center.

When  $(\varphi^*, w^*) = (\varphi_1^*, 0)$ , from (2.4) we see that  $q = -4(\varphi_1^* - c)(\theta\varphi_1^* - c) < 0$ . When  $(\varphi^*, w^*) = (\varphi_2^*, 0)$ ,  $p = 0$  and  $q = -4(\varphi_2^* - c)(\theta\varphi_2^* - c) > 0$ . Hence  $(\varphi_1^*, 0)$  is a saddle point and  $(\varphi_2^*, 0)$  is a center point.

When  $\varphi^* = \frac{c}{\theta}$ ,  $q = -4\theta(1 - 2\theta)(w_i^*)^2 < 0$ ,  $i = 1, 2$ . Hence, both  $(\frac{c}{\theta}, w_1^*)$  and  $(\frac{c}{\theta}, w_2^*)$  are saddle points.

(ii) If  $-\frac{c^2}{2} < a < 0$ . The separatrices of  $(\frac{c}{\theta}, w^*)$  are given by

$$-w^2 + \varphi^2 - \frac{2a}{1 - 2\theta} = 0 \quad \text{and} \quad \varphi = \frac{c}{\theta}.$$

These three separatrices together with  $\varphi = \varphi_1^*$  form a closed domain denoted by  $\Omega$ . Note that for any interior point of  $\Omega$ ,  $-w^2 + \varphi^2 - \frac{2a}{1 - 2\theta} > 0$ , hence,  $H(\varphi, w) > 0$ . Moreover,  $H(\varphi_1^*, 0) = (c - \theta\varphi_1^*)^{\frac{1 - 2\theta}{\theta}} ((\varphi_1^*)^2 - \frac{2a}{1 - 2\theta}) > 0$ . Therefore

$$\left\{ (\varphi, w) : H(\varphi, w) = H(\varphi_1^*, 0), \varphi \in \left[ \varphi_1^*, \frac{c}{\theta} \right] \right\} \subset \Omega.$$

This together with the fact that  $(\varphi_1^*, 0)$  is a saddle point ensures that the orbit connecting with  $(\varphi_1^*, 0)$  is homoclinic.

(iii) If  $a = 0$ . The separatrices of  $(\frac{c}{\theta}, w^*)$  are given by

$$w^2 - \varphi^2 = 0 \quad \text{and} \quad \varphi = \frac{c}{\theta}.$$

Since  $H(\varphi_1^*, 0) = 0$ , the orbit passing through  $(\varphi_1^*, 0)$  coincides with the orbit passing through  $(\frac{c}{\theta}, w^*)$ .

By a similar argument as in (ii), we obtain (iv).

According to the above analysis, we draw the phase portraits of (2.1) and (2.2) which are shown in Fig. 1.  $\square$

2.2. Bounded traveling wave solutions

There is a possibility that some interval  $I$  satisfying  $\varphi(\xi) = \frac{c}{\theta}$  on  $\xi \in I$  may exist for  $0 \leq a < \frac{c^2(1-2\theta)}{2\theta^2}$  (see (ii) and (iii) of Fig. 1). This traveling wave is called stumped wave. We now show this is impossible except  $a = \frac{c^2(1-2\theta)}{2\theta^2}$ .

**Lemma 2.1.** *If there exists some interval  $[\xi_1, \xi_2]$  such that  $\varphi(\xi) = \frac{c}{\theta}$  on  $\xi \in [\xi_1, \xi_2]$ , then  $a$  must be  $\frac{c^2(1-2\theta)}{2\theta^2}$ .*

**Proof.** In the weak formulation (1.7), we take  $v \in C_0^2(\mathbb{R})$  but with  $v = 0$  outside  $[\xi_1, \xi_2]$  to obtain

$$\int_{\xi_1}^{\xi_2} \frac{\theta(4\theta - 1)}{2} (\varphi')^2 v + \frac{\theta}{2} \varphi^2 v - \theta c \varphi v dy = \int_{\xi_1}^{\xi_2} \frac{1}{2} ((\theta\varphi - c)^2) v'' + a \theta v dy.$$

From  $\varphi(\xi) = \frac{c}{\theta}$ ,  $\varphi'(\xi) = 0$  on  $\xi \in (\xi_1, \xi_2)$ , it follows that

$$\int_{\xi_1}^{\xi_2} \frac{c^2(1 - 2\theta)}{2\theta} v dy = \int_{\xi_1}^{\xi_2} a \theta v dy.$$

Hence  $a = \frac{c^2(1-2\theta)}{2\theta^2}$ .  $\square$

**Remark 2.1.** By Lemma 2.1, there are no stumped traveling waves for  $-\frac{c^2}{2} < a < \frac{c^2(1-2\theta)}{2\theta^2}$ .

From (2.3) it follows that

$$(\varphi')^2 = F(\varphi), \tag{2.5}$$

where

$$F(\varphi) = \frac{F_0(\varphi) - h}{(c - \theta\varphi)^{\frac{1-2\theta}{\theta}}}, \quad \text{where } F_0(\varphi) = (c - \theta\varphi)^{\frac{1-2\theta}{\theta}} \left( \varphi^2 - \frac{2a}{1 - 2\theta} \right). \tag{2.6}$$

In order to analyze all bounded solutions to (2.5), we investigate the qualitative behavior of solutions to (2.5) near points where  $F$  has a zero or a pole, which corresponds to the set  $\{\varphi' = 0\}$  and  $\{\varphi = \frac{c}{\theta}\}$  in the phase plane  $(\varphi, \varphi')$ . The following lemma describes the behavior of smooth solutions to (2.5) at the point where  $F(\varphi) = 0$ .

**Lemma 2.2.** (See [33].)

- (i) Assume  $F(\varphi)$  has a simple zero at  $\varphi = \ell$ . Then for each  $\xi_0$ , there exists at least one nontrivial solution such that  $\varphi(\xi_0) = \ell$  satisfying

$$\varphi(\xi) = \ell + \frac{1}{4}(\xi - \xi_0)^2 F'(\ell) + O((\xi - \xi_0)^4) \quad \text{as } \xi \rightarrow \xi_0. \tag{2.7}$$

- (ii) If  $F(\varphi)$  has a double zero at  $\varphi = \ell$ , then there exists only one bounded solution  $\varphi = \varphi(\xi)$  such that

$$\varphi(\xi) - \ell \sim \alpha \exp(-|\xi| \sqrt{|F''(\ell)|}) \quad \text{as } |\xi| \rightarrow \infty, \tag{2.8}$$

for some constant  $\alpha$ .

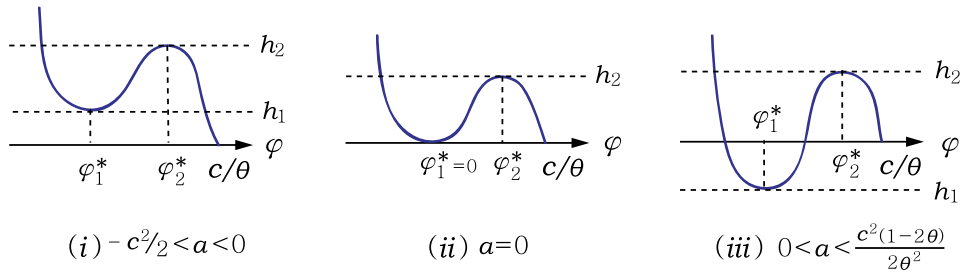


Fig. 2. Graph of  $F_0$ .

We would like to point out that the function  $F$  in [33] corresponds to the case  $\theta = 1/3$ . For a range of parameter  $0 < \theta < 1/2$ ,  $F$  given in (2.6) is well defined, and a very similar proof to that in [33] leads to the above result, as claimed in [33] for the case  $\theta = 1/3$ .

Next, we consider the behavior of solutions to (2.5) near points where  $F$  has a pole. A pole of a function is defined as:

**Definition 2.1.** Let  $r \geq 0$  and  $f$  be a given function. We say that  $f$  has an  $r$ -th pole at  $p$  if

$$\lim_{x \rightarrow p} [f(x)(x - p)^r] \neq 0.$$

**Remark 2.2.** If  $F(\varphi)$  has an  $r$ -th pole at  $\varphi = \ell$ , then no classical solution exists such that  $\varphi(\xi_0) = \ell$ . If there is any weak solution such that  $\varphi(\xi_0) = \ell$ , it must satisfy  $\lim_{\xi \rightarrow \xi_0} \varphi'(\xi) \rightarrow \pm\infty$ .

Using the above as a guide we now discuss different cases for  $F(\varphi)$  defined in (2.6). Note that

$$F'_0(\varphi) = (c - \theta\varphi)^{\frac{1}{\theta}-3}(-\varphi^2 + 2c\varphi + 2a) = -(c - \theta\varphi)^{\frac{1}{\theta}-3}(\varphi - \varphi_1^*)(\varphi - \varphi_2^*). \tag{2.9}$$

Hence,  $\varphi_1^*$  and  $\varphi_2^*$  are critical points of  $F_0(\varphi)$  and  $F_0(\varphi_1^*) = h_1$  and  $F_0(\varphi_2^*) = h_2$  where  $h_1 = H(\varphi_1^*, 0)$ ,  $h_2 = H(\varphi_2^*, 0)$ . Three cases of  $F_0(\varphi)$  are shown in Fig. 2. Once  $a$  is fixed, a change in  $h$  will shift the graph vertically up and down. Hence we can determine which  $h$  yields bounded traveling waves.

*Case (1):*  $-\frac{c^2}{2} < a < 0$ . There are three cases to consider.

(a)  $h_1 < h < h_2$ .

For  $\varphi < \frac{c}{\theta}$ ,  $F$  has three simple zeros at  $z_1, z_2, z_3$  such that  $z_1 < \varphi_1^* < z_2 < \varphi_2^* < z_3 < \frac{c}{\theta}$ . The graph of  $F$  is like (1) of Fig. 3. Since  $F(\varphi) > 0$  for  $z_2 < \varphi < z_3 < \frac{c}{\theta}$  and  $F(z_2) = F(z_3) = 0$ , by Lemma 2.2, for some small  $\epsilon > 0$ ,

$$\lim_{\xi \rightarrow \xi_0 - \epsilon} \varphi'(\xi) < 0, \quad \lim_{\xi \rightarrow \xi_0 + \epsilon} \varphi'(\xi) > 0$$

and

$$\lim_{\xi \rightarrow \xi_1 - \epsilon} \varphi'(\xi) > 0, \quad \lim_{\xi \rightarrow \xi_1 + \epsilon} \varphi'(\xi) < 0,$$

where  $\varphi(\xi_0) = z_2$  and  $\varphi(\xi_1) = z_3$ . Moreover since  $\varphi' \neq 0$  for  $z_2 < \varphi < z_3$ ,  $\varphi$  is strictly decreasing or increasing until one reaches the point  $\varphi = z_2$  or  $z_3$ . This when combined with (ii) of Proposition 2.1 shows that  $\varphi$  is a smooth periodic traveling wave.



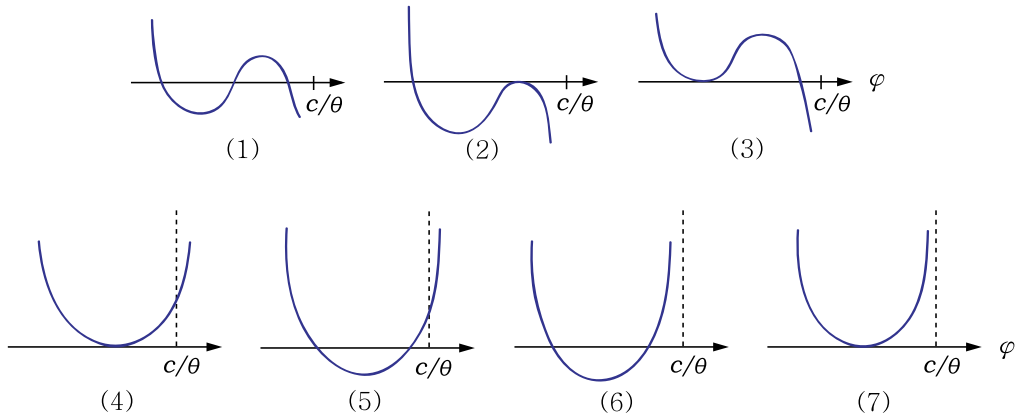


Fig. 3. Types of graph of  $F$ .

(b)  $h = h_2$ .

$F$  has a simple zero at  $z_4$ , double zero at  $\varphi_2^*$  satisfying  $z_4 < \varphi_2^*$ . The graph of  $F$  is like (2) of Fig. 3. Since for  $\varphi < z_4$ ,  $F(\varphi) > 0$  and  $F(z_4) = 0$ ,

$$\lim_{\xi \rightarrow \xi_2 - \epsilon} \varphi'(\xi) > 0 \quad \text{and} \quad \lim_{\xi \rightarrow \xi_2 + \epsilon} \varphi'(\xi) < 0,$$

where  $\varphi(\xi_2) = z_4$ . If there is a bounded solution, then solution must converge to equilibrium point as  $\xi \rightarrow \pm\infty$ . But, equilibrium points  $\varphi_1^*$  and  $\varphi_2^*$  are above  $z_4$ . Hence, there are no bounded solutions.

(c)  $h = h_1$ .

$F$  has a double zero at  $\varphi_1^*$ , a simple zero at  $z_5$  satisfying  $\varphi_1^* < z_5$  and  $F < 0$  for  $z_5 < \varphi < \frac{c}{\theta}$ . The graph of  $F$  is like (3) of Fig. 3. Since  $F(\varphi) > 0$  for  $\varphi_1^* < \varphi < z_5 < \frac{c}{\theta}$  and  $F(z_5) = 0$ , by Lemma 2.2,

$$\lim_{\xi \rightarrow \xi_3 - \epsilon} \varphi'(\xi) > 0 \quad \text{and} \quad \lim_{\xi \rightarrow \xi_3 + \epsilon} \varphi'(\xi) < 0,$$

where  $\varphi(\xi_3) = z_5$ , and  $\varphi \downarrow \varphi_1^*$  as  $\xi \rightarrow \pm\infty$ . Therefore  $\varphi$  is a smooth solitary wave.

Case (2): If  $a = 0$ , we have three cases to consider.

(a)  $0 < h < h_2$ .

For  $\varphi < \frac{c}{\theta}$ ,  $F$  has three simple zeros at  $z_6, z_7, z_8$  such that  $z_6 < \varphi_1^* < z_7 < \varphi_2^* < z_8 < \frac{c}{\theta}$ . Since  $F(\varphi) > 0$  for  $z_7 < \varphi < z_8$  and  $F(z_7) = F(z_8) = 0$ ,  $\varphi$  is strictly monotone until one reaches the point  $\varphi = z_7$  or  $z_8$ . Hence  $\varphi$  is a smooth periodic traveling wave by (iii) of Proposition 2.1.

(b)  $h = h_2$ .

$F$  has a simple zero at  $z_9$ , a double zero at  $\varphi_2^*$  such that  $z_9 < \varphi_1^* < \varphi_2^*$  and  $F > 0$  for  $\varphi < z_9$ . So,  $F$  is well defined on  $\varphi \leq z_9$ . Hence we can consider the area for  $\varphi \leq z_9$ . But, since equilibrium points  $\varphi_1^*$  and  $\varphi_2^*$  are above  $z_9$ , there are no bounded solutions.

(c)  $h = 0$ .

$F$  has a double zero at origin. By Lemma 2.2,  $\varphi \downarrow 0$  as  $\xi \rightarrow \pm\infty$ . The graph of  $F$  is like (4) of Fig. 3. Since  $F(\varphi) > 0$  for  $0 < \varphi < \frac{c}{\theta}$ ,  $\varphi$  is strictly monotone. Moreover  $\varphi$  increases and reaches the point  $\varphi = \frac{c}{\theta}$ . Therefore  $\varphi$  is a peaked solitary wave.

Case (3): If  $0 < a < \frac{c^2(1-2\theta)}{2\theta^2}$ , we have five cases to consider.

(a)  $h = h_2$ .

$F$  has a simple zero at  $z_{10}$ , a double zero at  $\varphi_2^*$  satisfying  $z_{10} < \varphi_1^* < \varphi_2^*$  and  $F(\varphi) > 0$  for  $\varphi < z_{10}$ . Since equilibrium points  $\varphi_1^*$  and  $\varphi_2^*$  are above  $z_{10}$ , there are no bounded solutions.

(b)  $0 < h < h_2$ .

$F$  has three simple zeros at  $z_{11}, z_{12}, z_{13}$  such that  $z_{11} < \varphi_1^* < z_{12} < \varphi_2^* < z_{13} < \frac{c}{\theta}$ . Since  $F(\varphi) > 0$  for  $z_{12} < \varphi < z_{13}$  and  $F(z_{12}) = F(z_{13}) = 0$ ,  $\varphi$  is strictly monotone until one reaches the point  $\varphi = z_{12}$  or  $z_{13}$ . Hence  $\varphi$  is a smooth periodic traveling wave by (iv) of Proposition 2.1.

(c)  $h = 0$ .

$F$  has two simple zeros at  $\pm\sqrt{\frac{2a}{1-2\theta}}$  such that  $-\sqrt{\frac{2a}{1-2\theta}} < \varphi_1^* < \sqrt{\frac{2a}{1-2\theta}} < \varphi_2^* < \frac{c}{\theta}$ . The graph of  $F$  is like (5) of Fig. 2. Since  $F(\varphi) > 0$  for  $\sqrt{\frac{2a}{1-2\theta}} < \varphi < \frac{c}{\theta}$  and  $F(\sqrt{\frac{2a}{1-2\theta}}) = 0$ ,  $\varphi$  is strictly monotone until one reaches the point  $\varphi = \sqrt{\frac{2a}{1-2\theta}}$ . Moreover,  $\varphi$  increases and reaches the point  $\varphi = \frac{c}{\theta}$ . Therefore,  $\varphi$  is a peaked periodic traveling wave by (iv) of Proposition 2.1.

(d)  $h_1 < h < 0$ .

$F$  has two simple zeros at  $z_{14}, z_{15}$  satisfying  $z_{14} < \varphi_1^* < z_{15} < \varphi_2^*$  and  $F$  has the following form

$$F = \frac{(\varphi - z_{14})(\varphi - z_{15})f_1(\varphi)}{(c - \theta\varphi)^{\frac{1-2\theta}{\theta}}}, \quad \text{where } f_1(\varphi) > 0 \text{ for } \varphi < \frac{c}{\theta}.$$

The graph of  $F$  is like (6) of Fig. 3. So for  $z_{15} < \varphi < \frac{c}{\theta}$ ,  $F(\varphi) > 0$  and  $F$  has a  $\frac{1-2\theta}{\theta}$ -th pole at  $\frac{c}{\theta}$ . Hence,  $F(\varphi) \rightarrow \infty$  as  $\varphi \rightarrow \frac{c}{\theta}$ . Thus, by Lemma 2.2,  $\varphi$  is a cusped periodic traveling wave.

(e)  $h = h_1$ .

$F$  has a double zero at  $\varphi_1^*$  and  $F(\varphi) > 0$  for  $\varphi_1^* < \varphi < \frac{c}{\theta}$ . So  $F$  has the following form

$$F = \frac{(\varphi - \varphi_1^*)^2 f_2(\varphi)}{(c - \theta\varphi)^{\frac{1-2\theta}{\theta}}}, \quad \text{where } f_2(\varphi) > 0 \text{ for } \varphi < \frac{c}{\theta}.$$

The graph of  $F$  is like (7) of Fig. 3. Hence  $F(\varphi) > 0$  for  $\varphi_1^* < \varphi < \frac{c}{\theta}$  and  $F$  has a  $\frac{1-2\theta}{\theta}$ -th pole at  $\frac{c}{\theta}$ . Therefore by Lemma 2.2,  $\varphi$  is a cusped solitary wave with  $\varphi \downarrow \varphi_1^*$  as  $\xi \rightarrow \pm\infty$ .

### 3. Traveling waves for $\theta = 0$ : proof of Theorem 1.2

In this section, we study traveling wave solutions to the  $\theta$ -equation with  $\theta = 0$ , i.e.,

$$u_t - u_{txx} + uu_x = u_x u_{xx}. \tag{3.1}$$

Substituting  $\theta = 0$  into (1.5), we have

$$-c\varphi + c\varphi'' + \frac{1}{2}\varphi^2 = \frac{1}{2}(\varphi')^2 + a, \tag{3.2}$$

where  $a$  is an integral constant. Let  $w = \varphi'$  then we have the traveling wave system from (3.2)

$$\begin{cases} \varphi' = w, \\ w' = \varphi - \frac{1}{2c}\varphi^2 + \frac{1}{2c}w^2 + a. \end{cases} \tag{3.3}$$

To find out bounded orbits for (3.3), we consider  $a > -\frac{c^2}{2}$  since in such a case (3.3) has two critical points,  $(\varphi_1^*, w^*) = (c - \sqrt{c^2 + 2a}, 0)$  and  $(\varphi_2^*, w^*) = (c + \sqrt{c^2 + 2a}, 0)$ , respectively. Furthermore, from the theory of planar dynamical system (see e.g., [23]), we know that  $(c - \sqrt{c^2 + 2a}, 0)$  is a saddle point and  $(c + \sqrt{c^2 + 2a}, 0)$  is a center or spiral point.

From (3.3) it follows that

$$\frac{dw}{d\varphi} = \frac{\varphi - \frac{1}{2c}\varphi^2 + \frac{1}{2c}w^2 + a}{w},$$

upon integration we find

$$w^2 = \varphi^2 - 2a + b \exp\left(\frac{1}{c}\varphi\right), \tag{3.4}$$

where  $b$  is an integral constant. This suggests that any orbit must be symmetric about  $\varphi$ -axis.

We first look at the particular orbit passing through the saddle point  $(c - \sqrt{c^2 + 2a}, 0)$ , which corresponds to

$$b = b^* = 2c(\sqrt{c^2 + 2a} - c) \exp\left(\frac{1}{c}\sqrt{c^2 + 2a} - 1\right).$$

Hence two branches of the orbit passing through  $(c - \sqrt{c^2 + 2a}, 0)$  is determined by

$$w^2 = \varphi^2 - 2a + b^* \exp\left(\frac{1}{c}\varphi\right).$$

To proceed we need the following lemma.

**Lemma 3.1.** *For any fixed  $c$ , if  $-\frac{c^2}{2} < a < 0$  or  $a > 0$ , then  $b^* > 2a$ .*

**Proof.** Define  $k(a) = 2c(\sqrt{c^2 + 2a} - c) \exp\left(\frac{1}{c}\sqrt{c^2 + 2a} - 1\right) - 2a$ . We first consider  $a > 0$ . We know that

$$\frac{d}{da}k(a) = 2\left(\exp\left(\frac{1}{c}\sqrt{c^2 + 2a} - 1\right) - 1\right) > 0$$

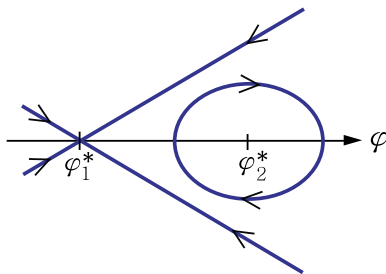


Fig. 4. Phase portrait of (3.3).

and  $k(0) = 0$ . Hence  $k(a)$  is increasing and positive function for  $a > 0$ . Therefore,  $b^* > 2a$  for  $a > 0$ . By a similar argument, we also have  $b^* > 2a$  for  $-\frac{c^2}{2} < a < 0$ .  $\square$

We now examine to see how many times the above mentioned orbit crosses  $\varphi$ -axis. At crossing points  $(\varphi, 0)$ , we have

$$\varphi^2 - 2a + b^* \exp\left(\frac{1}{c}\varphi\right) = 0. \tag{3.5}$$

Note that  $\varphi_1^* = c - \sqrt{c^2 + 2a}$  is a root of (3.5). Define  $g(\varphi) = \varphi^2 - 2a$  and  $h(\varphi) = -b^* \exp(\frac{1}{c}\varphi)$ , then intersections of  $g(\varphi)$  and  $h(\varphi)$  are solutions of (3.5).

We distinguish  $a > 0$  from the case  $-\frac{c^2}{2} < a \leq 0$ .

For  $a > 0$ , we have  $\varphi_1^* < 0$  and  $b^* > 2a$  by Lemma 3.1. Also,  $g(\varphi)$  and  $h(\varphi)$  have following properties:

- (i)  $g(\varphi)$  and  $h(\varphi)$  are monotonically decreasing for  $\varphi \leq 0$ .
- (ii)  $g(\varphi_1^*) = h(\varphi_1^*)$  and  $g'(\varphi_1^*) = h'(\varphi_1^*)$ .

Hence Eq. (3.5) has only one solution,  $\varphi_1^*$ , which implies that the orbit passing through the point  $(\varphi_1^*, 0)$  touches  $\varphi$ -axis only once. Therefore  $(c + \sqrt{c^2 + 2a}, 0)$  must be a center, with a phase portrait shape like Fig. 4. Thus any trajectory issuing from  $(\varphi^*, 0)$  with  $\varphi^* > c - \sqrt{c^2 + 2a}$  must be a closed trajectory, i.e., any bounded traveling wave solution is periodic. The case  $-\frac{c^2}{2} < a \leq 0$  can be treated in similar fashion.

#### 4. Traveling waves for $\theta = \frac{1}{2}$ : proof of Theorem 1.3

In this section, we study traveling wave solutions to the  $\theta$ -equation with  $\theta = \frac{1}{2}$ , i.e.,

$$u_t - u_{txx} + uu_x = \frac{1}{2}uu_{xxx} + \frac{1}{2}u_xu_{xx}. \tag{4.1}$$

Substituting  $\theta = \frac{1}{2}$  into (1.5), we have the following ordinary differential equation

$$-c\varphi + c\varphi'' + \frac{1}{2}\varphi^2 = \frac{1}{2}\varphi\varphi'' + a. \tag{4.2}$$

We first consider the case of  $a = 0$ . (4.2) with  $a = 0$  becomes

$$(\varphi - 2c)(\varphi'' - \varphi) = 0.$$

Hence, the only bounded solution for  $\xi \in \mathbb{R}$  is  $\varphi = 2c$ . Conversely, taking  $\varphi = 2c$  into (4.2), we obtain  $a = 0$ . Therefore,  $\varphi = 2c$  if and only if  $a = 0$ .

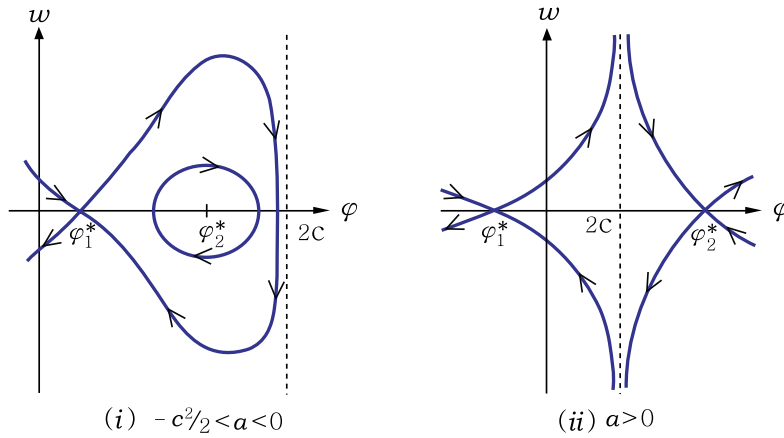


Fig. 5. Phase portraits of system (4.3).

For  $a \neq 0$ , let  $w = \varphi'$ , (4.2) can be rewritten as

$$\begin{cases} \varphi' = w, \\ w' = \frac{\varphi^2 - 2c\varphi - 2a}{\varphi - 2c}, \end{cases} \tag{4.3}$$

when  $\varphi \neq 2c$ . To find out bounded orbits for (4.3), we consider  $a > -\frac{c^2}{2}$  since in such a case (4.3) has two critical points, which are denoted by  $(\varphi_1^*, w^*) = (c - \sqrt{c^2 + 2a}, 0)$  and  $(\varphi_2^*, w^*) = (c + \sqrt{c^2 + 2a}, 0)$ , respectively. We see that for  $-\frac{c^2}{2} < a < 0$ ,  $\varphi_1^* < \varphi_2^* < 2c$  and for  $a > 0$ ,  $\varphi_1^* < 2c < \varphi_2^*$ . System (4.3) has the first integral

$$H(\varphi, w) = w^2 - \varphi^2 + 4a \ln \left| \frac{1}{2}\varphi - c \right| = h, \tag{4.4}$$

where  $h$  is an arbitrary constant. Hence by the qualitative theory of planar dynamical system, we have the following.

- (i) If  $-\frac{c^2}{2} < a < 0$ , then  $(\varphi_1^*, 0)$  is a saddle point, and  $(\varphi_2^*, 0)$  is a center point.
- (ii) If  $a > 0$ , then both  $(\varphi_1^*, 0)$  and  $(\varphi_2^*, 0)$  are saddle points.

According to the above properties, we obtain the phase portraits of system (4.3) (see Fig. 5).

Let

$$G(\varphi) = G_0(\varphi) + h, \quad \text{where } G_0(\varphi) = \varphi^2 - 4a \ln \left| \frac{1}{2}\varphi - c \right|.$$

From (4.4), we have

$$w^2 = G(\varphi). \tag{4.5}$$

Note that

$$G'_0(\varphi) = 2(\varphi - 2c)^{-1}(\varphi - \varphi_1^*)(\varphi - \varphi_2^*).$$

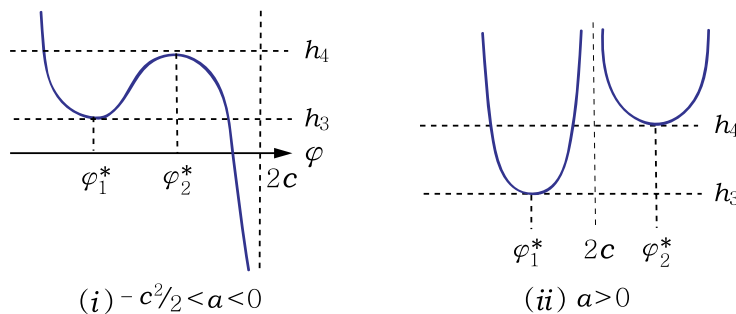


Fig. 6. Graph of  $G_0$ .

Hence,  $\varphi_1^*$  and  $\varphi_2^*$  are critical points of  $G_0(\varphi)$  and  $G_0(\varphi_1^*) = h_3$  and  $G_0(\varphi_2^*) = h_4$  where  $h_3 = H(\varphi_1^*, 0)$ ,  $h_4 = H(\varphi_2^*, 0)$ . Two cases of  $G_0(\varphi)$  are shown in Fig. 6. Once  $a$  is fixed, a change in  $h$  will shift the graph vertically up and down. Hence we can determine which  $h$ 's yield bounded traveling waves.

For  $-\frac{c^2}{2} < a < 0$ , by the same argument as case (1) of Section 2.2, we obtain the result such as (2) of Theorem 1.2.

For  $a > 0$ , we consider following cases.

(a)  $h = -h_3$  or  $-h_4$ .

For  $h = -h_3$ ,  $G$  has a double zero at  $\varphi_1^*$  and  $G(\varphi) > 0$  for  $\varphi_1^* < \varphi < 2c$ . Also  $G(\varphi) \rightarrow +\infty$  as  $\varphi \rightarrow 2c$  but,  $\varphi$  never touches the line  $2c$ . Therefore by Lemma 2.2,  $\varphi$  is a cusped like solitary wave with  $\varphi \downarrow \varphi_1^*$  as  $\xi \rightarrow \pm\infty$ . This wave is similar to shapes connected two kink waves.

Similarly, for  $h = -h_4$ ,  $\varphi$  is cusped like solitary wave with  $\varphi \uparrow \varphi_2^*$  as  $\xi \rightarrow \pm\infty$ .

(b)  $-h_4 < h < -h_3$  or  $h < -h_4$ .

For  $-h_4 < h < -h_3$ ,  $G$  has two simple zeros at  $z_{16}$  and  $z_{17}$  such that  $z_{16} < \varphi_1^* < z_{17} < 2c$  and  $G(\varphi) > 0$  for  $z_{17} < \varphi < 2c$ . Also,  $G(\varphi) \rightarrow +\infty$  as  $\varphi \rightarrow 2c$  but,  $\varphi$  never touches the line  $2c$ . By Lemma 2.2,

$$\lim_{\xi \rightarrow \xi_5 - \epsilon} \varphi'(\xi) < 0 \quad \text{and} \quad \lim_{\xi \rightarrow \xi_5 + \epsilon} \varphi'(\xi) > 0, \quad \text{where } \varphi(\xi_5) = z_{17}.$$

Moreover, since  $\varphi' \neq 0$  for  $z_{17} < \varphi < 2c$ ,  $\varphi$  is strictly decreasing or increasing until it reaches the point  $\varphi = z_{17}$  or  $2c$ . Since  $\varphi$  is not touched at  $2c$  and there is no center point for  $z_{17} < \varphi < 2c$ , implying that this process occurs only once. Therefore,  $\varphi$  is a kink like traveling wave. Similarly, for  $h < -h_4$ ,  $\varphi$  is also a kink like traveling wave.

### 5. Concluding remarks

In this paper, we have investigated traveling wave solutions to the  $\theta$ -equation for  $0 \leq \theta \leq \frac{1}{2}$ , including two well-studied integrable equations, the Camassa–Holm equation,  $\theta = \frac{1}{3}$ , and the Degasperis–Procesi equation,  $\theta = \frac{1}{4}$ . Hence this paper generalizes some known traveling wave results [33,34] for these two special models. As shown in [39], when  $0 \leq \theta \leq \frac{1}{2}$ , strong solutions of the  $\theta$ -equation may blow up in finite time, correspondingly, the traveling waves for  $\theta \in [0, 0.5]$  are very rich. Our study shows that when  $\theta = 0$ , only periodic travel wave is permissible, and when  $\theta = 0.5$  traveling waves may be solitary, periodic or kink-like waves. For  $0 < \theta < 1/2$ , traveling waves such as periodic, solitary, peakon, peaked periodic, cusped periodic, or cusped soliton are all permissible.

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