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FORMATION OF δ-SHOCKS AND VACUUM STATES IN THE VANISHING PRESSURE LIMIT OF SOLUTIONS TO THE EULER EQUATIONS FOR ISENTROPIC FLUIDS*

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Abstract. The phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states in solutions to the Euler equations for isentropic fluids are identified and analyzed as the pressure vanishes. It is shown that, as the pressure vanishes, any two-shock Riemann solution to the Euler equations for isentropic fluids tends to a δ-shock solution to the Euler equations for pressureless fluids, and the intermediate density between the two shocks tends to a weighted δ-measure that forms the δ-shock. By contrast, any two-rarefaction-wave Riemann solution of the Euler equations for isentropic fluids is shown to tend to a two-contact-discontinuity solution to the Euler equations for pressureless fluids, whose intermediate state between the two contact discontinuities is a vacuum state, even when the initial data stays away from the vacuum. Some numerical results exhibiting the formation process of δ-shocks are also presented.

Key words. concentration, cavitation, δ-shocks, vacuum states, Euler equations, vanishing pressure limit, transport equations, measure solutions, isentropic fluids, pressureless fluids, numerical simulations

AMS subject classifications. Primary, 35L65, 35B30, 76E19, 35Q35, 35L67; Secondary, 35B25, 65M06

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1. Introduction.

We are concerned with the phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states in solutions to the Euler equations for compressible fluids as the pressure vanishes. In this paper, we consider the Euler equations of isentropic gas dynamics in Eulerian coordinates,

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.1}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + P) = 0, \tag{1.2}$$

where ρ, P, and $m = \rho v$ represent the density, the scalar pressure, and the momentum, respectively; and ρ and m are in the physical region $\{(\rho, m) \mid \rho \geq 0, |m| \leq V_0 \rho\}$ for some $V_0 > 0$. For $\rho > 0$, $v = m/\rho$ is the velocity with $|v| \leq V_0$. The scalar pressure P is a function of the density ρ and a small parameter $\epsilon > 0$ satisfying

$$\lim_{\epsilon \to 0} P(\rho, \epsilon) = 0.$$

For concreteness, we focus on the prototypical pressure function for polytropic gases:

$$P(\rho, \epsilon) = \epsilon p(\rho), \qquad p(\rho) = \rho^\gamma/\gamma, \quad \gamma > 1. \tag{1.3}$$

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System (1.1)–(1.3) is an example of hyperbolic systems of conservation laws with form

$$\partial_t u + \partial_x f(u, \epsilon) = 0, \tag{1.4}$$

with $u = (\rho, \rho v)$ and $f(u, \epsilon) = (\rho v, \rho v^2 + \epsilon p(\rho))$. Observe that system (1.1)–(1.3) with parameter $\epsilon > 0$ is generic in the sense that such a system can also be obtained under the scaling

$$(x, t) \longrightarrow (\alpha x, \alpha t), \qquad \rho \longrightarrow \alpha\rho,$$

with $\alpha = \epsilon^{-1/(\gamma-1)}$ from system (1.1)–(1.2) with $p = p(\rho)$.

In Chang, Chen, and Yang [4, 5, 6], a phenomenon of concentration in solutions of the two-dimensional Riemann problem was first observed numerically, which led to the occurrence of so-called smoothed δ-shocks for the Euler equations of gas dynamics when the Riemann data produces four initial contact discontinuities with different signs and the initial pressure data is close to zero. One of the main objectives of this paper is to show rigorously that the phenomenon of concentration in solutions, observed numerically in [4, 5, 6], for inviscid compressible fluid flow is fundamental and occurs not only in the multidimensional situations but also naturally in the one-dimensional case.

The limit system as $\epsilon \to 0$ formally becomes the transport equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.5}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2) = 0, \tag{1.6}$$

which are also called the one-dimensional system of pressureless Euler equations, modeling the motion of free particles which stick under collision (see [3, 11, 30]).

The transport equations (1.5)–(1.6) have been analyzed extensively since 1994; for example, see [1, 2, 3, 11, 12, 13, 17, 18, 19, 20, 22, 28, 29] and the references cited therein. In particular, the existence of measure solutions of the Riemann problem was first presented in Bouchut [1], and a connection of (1.5)–(1.6) with adhesion particle dynamics and the behavior of global weak solutions with random initial data were discussed in E, Rykov, and Sinai [11]. Also see [14, 15, 16, 21, 26, 27] for related equations and results. It has been shown that, for the transport equations (1.5)–(1.6), δ-shocks and vacuum states do occur in the Riemann solutions. Since the two eigenvalues of the transport equations coincide, the occurrence of δ-shocks and vacuum states as $t > 0$ can be regarded as a result of resonance between the two characteristic fields.

In this paper, we rigorously analyze the phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states in solutions to the Euler equations for isentropic fluids as the pressure vanishes. The vanishing pressure limit can be regarded as a singular flux-function limit for hyperbolic systems of conservation laws (1.4). We show that such phenomena occur naturally in the one-dimensional case as the pressure vanishes: any two-shock Riemann solution to the Euler equations for isentropic fluids tends to a δ-shock solution to the Euler equations for pressureless fluids, and the intermediate density between the two shocks tends to a weighted δ-measure that forms a δ-shock; by contrast, any two-rarefaction-wave Riemann solution to the Euler equations for isentropic fluids tends to a two-contact-discontinuity solution to the Euler equations for pressureless fluids, whose intermediate state between the two contact discontinuities is a vacuum state even when the initial data stays away from the vacuum. These results show that the δ-shocks for the transport equations result from a phenomenon of concentration, while the vacuum states result from a phenomenon of cavitation in the process of the vanishing pressure limit; both are fundamental and physical in fluid dynamics.

From the point of view of hyperbolic conservation laws, since the limit system loses hyperbolicity, the phenomena of concentration and cavitation in the process of the vanishing pressure limit can be regarded as phenomena of resonance between the two characteristic fields. These phenomena show that the flux-function limit can be very singular: the limit functions of solutions are no longer in the spaces of functions BV or L^∞; and the space of Radon measures, for which the divergences of certain entropy and entropy flux fields are also Radon measures, is a natural space in order to deal with such a limit in general. In this regard, a theory of divergence-measure fields has been established in Chen and Frid [7, 8, 9].

The organization of this paper is as follows. In section 2, we discuss the δ-shocks and vacuum states for the transport equations (1.5)–(1.6) and examine the dependence of the Riemann solutions on the parameter $\epsilon > 0$ for the Euler equations (1.1)–(1.3). In section 3, we analyze the formation of δ-shocks in the Riemann solutions to the Euler equations (1.1)–(1.3) as the pressure vanishes. In section 4, we analyze the formation of vacuum states in the Riemann solutions to (1.1)–(1.3), even when the initial data stays away from the vacuum, as the pressure decreases. In section 5, we present some representative numerical results, produced by using the higher order essentially nonoscillatory (ENO) scheme in [23, 24], to examine the phenomenon of concentration and the formation process of δ-shocks in the level of the Euler dynamics (1.1)–(1.3) as the pressure decreases.

2. δ-shocks, vacuum states, and Riemann solutions. In this section, we first discuss δ-shocks and vacuum states in the Riemann solutions to the transport equations (1.5)–(1.6), and then we examine the dependence of the Riemann solutions on the parameter $\epsilon > 0$ to the Euler equations (1.1)–(1.3).

2.1. δ-shocks and vacuum states for the transport equations. Consider the Riemann problem of the transport equations (1.5)–(1.6) with Riemann initial data

$$(\rho, v)(x, 0) = (\rho_\pm, v_\pm), \qquad \pm x > 0, \tag{2.1}$$

with $\rho_\pm > 0$. Since the equations and the Riemann data are invariant under uniform stretching of coordinates

$$(x, t) \to (\beta x, \beta t), \qquad \beta \text{ constant},$$

we consider the self-similar solutions of (1.5), (1.6), and (2.1):

$$(\rho, v)(x, t) = (\rho, v)(\xi), \quad \xi = x/t.$$

Then the Riemann problem is reduced to a boundary value problem for ordinary differential equations:

$$\begin{aligned}
&-\xi\rho_\xi + (\rho v)_\xi = 0,\\
&-\xi(\rho v)_\xi + (\rho v^2)_\xi = 0,\\
&(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm).
\end{aligned}$$

As shown in [22], in the case in which $v_- < v_+$, we can obtain a solution $(\rho, v)(\xi)$ that consists of two contact discontinuities and a vacuum state which are uniquely determined by the Riemann data $(\rho_\pm, v_\pm)$. That is,

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi \le v_-, \\ (0, \xi), & v_- \le \xi \le v_+, \\ (\rho_+, v_+), & v_+ \le \xi < \infty. \end{cases}$$

In the case in which $v_- > v_+$, a key observation in [22] is that the singularity cannot be a jump with finite amplitude; that is, there is no solution which is piecewise smooth and bounded. Hence a solution containing a weighted δ-measure (i.e., δ-shock) supported on a line was constructed in order to establish the existence in a space of measures from the mathematical point of view (see also [26, 27]).

To define the measure solutions, the weighted δ-measure $w(t)\delta_S$ supported on a smooth curve $S = \{(x(s), t(s)) : a < s < b\}$ can be defined by

$$\langle w(\cdot)\delta_S, \psi(\cdot,\cdot)\rangle = \int_a^b w(t(s))\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}ds$$

for any $\psi \in C_0^\infty((-\infty, \infty) \times [0, \infty))$.

With this definition, a family of δ-measure solutions with parameter σ in the case in which $v_- > v_+$ can be obtained as

$$\rho(x, t) = \rho_0(x, t) + w(t)\delta_S, \qquad v(x, t) = v_0(x, t),$$

where $S = \{(\sigma t, t) : 0 \le t < \infty\}$,

$$\rho_0(x, t) = \rho_- + [\rho]\,\chi(x - \sigma t), \;\; v_0(x, t) = v_- + [v]\,\chi(x - \sigma t), \;\; w(t) = \frac{t}{1 + \sigma^2}(\sigma[\rho] - [\rho v]),$$

in which $[h] := h_+ - h_-$ denotes the jump of function h across the discontinuity, and $\chi(x)$ is the characteristic (or indicator) function that is 0 when $x < 0$ and 1 when $x > 0$.

It was shown in [22] that the δ-measure solutions (ρ, v) constructed above satisfy

$$\langle \rho, \phi_t\rangle + \langle \rho v, \phi_x\rangle = 0, \tag{2.2}$$

$$\langle \rho v, \phi_t\rangle + \langle \rho v^2, \phi_x\rangle = 0 \tag{2.3}$$

for any $\phi \in C_0^\infty((-\infty, \infty) \times (0, \infty))$, where

$$\langle \rho, \phi\rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0\phi\, dxdt + \langle w\delta_S, \phi\rangle$$

and

$$\langle \rho v, \phi\rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 v_0\phi\, dxdt + \langle \sigma w\delta_S, \phi\rangle.$$

A unique solution can be singled out by the so-called δ-Rankine–Hugoniot condition

$$\sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \tag{2.4}$$

that satisfies the δ-entropy condition

$$v_+ < \sigma < v_-. \tag{2.5}$$

The entropy condition (2.5) means that, in the (x, t)-plane, all the characteristic lines on either side of a δ-shock run into the line of δ-shock, which implies that a δ-shock is an overcompressive shock.

2.2. Riemann solutions to the Euler equations for isentropic fluids. The eigenvalues of system (1.1)–(1.3) are

$$\lambda_1 = v - c(\rho, \epsilon), \quad \lambda_2 = v + c(\rho, \epsilon) \qquad \text{for } \rho > 0$$

with

$$c(\rho, \epsilon) = \sqrt{\epsilon\, p'(\rho)} = \sqrt{\epsilon}\, \rho^{\theta}, \qquad \theta = \frac{\gamma - 1}{2}.$$

The Riemann invariants are

$$w = v + \int_0^{\rho} \frac{\sqrt{\epsilon p'(s)}}{s} ds, \qquad z = v - \int_0^{\rho} \frac{\sqrt{\epsilon p'(s)}}{s} ds.$$

Then the Riemann solutions, which are functions of $\xi = x/t$, are solutions of

$$-\xi \rho_\xi + (\rho v)_\xi = 0, \tag{2.6}$$

$$-\xi (\rho v)_\xi + (\rho v^2 + \epsilon\, p(\rho))_\xi = 0, \tag{2.7}$$

$$(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \tag{2.8}$$

Shock curves. The Rankine–Hugoniot conditions for discontinuous solutions of (1.1)–(1.3) are

$$-\sigma[\rho] + [\rho v] = 0, \qquad -\sigma[\rho v] + [\rho v^2 + \epsilon\, p(\rho)] = 0.$$

The Lax entropy inequalities imply

$$\rho_+ > \rho_- \ \text{(one-shock)}, \qquad \rho_+ < \rho_- \ \text{(two-shock)}.$$

Then, given a state $u_- = (\rho_-, \rho_- v_-)$, the shock curves in the phase plane, which are the sets of states that can be connected on the right by a one-shock or a two-shock, are the following.

One-shock curve $S_1(u_-)$:

$$v - v_- = -\sqrt{\frac{\epsilon\,(p(\rho) - p(\rho_-))}{\rho_- \rho (\rho - \rho_-)}}(\rho - \rho_-), \qquad \rho > \rho_-.$$

Two-shock curve $S_2(u_-)$:

$$v - v_- = -\sqrt{\frac{\epsilon\,(p(\rho) - p(\rho_-))}{\rho_- \rho (\rho - \rho_-)}}(\rho - \rho_-), \qquad \rho < \rho_-.$$

Then the shock curves are concave or convex, respectively, with respect to the point $u_- = (\rho_-, \rho_- v_-)$ in the $\rho - m$ plane with $m = \rho v$; that is, the quotient $\frac{m - m_-}{\rho - \rho_-}$ as a function of ρ is monotone.

We now turn to analyzing the Riemann solutions that consist of rarefaction waves and constant states. There are also two families of rarefaction waves, corresponding to characteristic fields λ_1 and λ_2, respectively.

Rarefaction wave curves. A rarefaction wave is a continuous solution of (2.6)–(2.8) of the form $(\rho, \rho v)(\xi)$, $\xi = x/t$, satisfying

$$\xi = v \mp \sqrt{\epsilon\, p'(\rho)}, \qquad -\xi \rho_\xi + (\rho v)_\xi = 0.$$

Then, given a state $u_- = (\rho_-, \rho_- v_-)$, the rarefaction-wave curves in the phase plane, which are the sets of states that can be connected on the right by a one-rarefaction or two-rarefaction wave, are the following.

One-rarefaction wave curve $R_1(u_-)$:

$$v - v_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{\epsilon\, p'(s)}}{s} ds, \qquad \rho < \rho_-.$$

Two-rarefaction wave curve $R_2(u_-)$:

$$v - v_- = \int_{\rho_-}^{\rho} \frac{\sqrt{\epsilon\, p'(s)}}{s} ds, \qquad \rho > \rho_-.$$

The rarefaction wave curves are concave or convex, respectively, in the $\rho - m$ plane.

Given a left state $u_- = (\rho_-, \rho_- v_-)$, the set of states that can be connected on the right by a shock or a rarefaction wave in the phase plane consists of the one-shock curve $S_1(u_-)$, the one-rarefaction curve $R_1(u_-)$, the two-shock curve $S_2(u_-)$, and the two-rarefaction curve $R_2(u_-)$. These curves divide the phase plane into four regions $S_2S_1(u_-)$, $S_2R_1(u_-)$, $R_2S_1(u_-)$, and $R_2R_1(u_-)$; any right state of the Riemann data staying in one of them yields a unique global Riemann solution $R(x/t)$, which contains a one-shock (or a one-rarefaction wave) and/or a two-shock (or a two-rarefaction wave) satisfying

$$w(R(x/t)) \le w(u_+), \;\; z(R(x/t)) \ge z(u_-), \;\; w(R(x/t)) - z(R(x/t)) \ge 0.$$

In particular, when $u_+ \in S_2S_1(u_-)$, $R(x/t)$ contains a one-shock, a two-shock, and a nonvacuum intermediate constant state; and, when $u_+ \in R_2R_1(u_-)$, $R(x/t)$ contains a one-rarefaction wave, a two-rarefaction wave, and an intermediate constant state that may be a vacuum state. Since the other two regions $S_2R_1(u_-)$ and $R_2S_1(u_-)$ have empty interiors when $\epsilon \to 0$, it suffices to analyze the limit process for the two cases $u_+ \in S_2S_1(u_-)$ (in section 3) and $u_+ \in R_2R_1(u_-)$ (in section 4). For more details about Riemann solutions, see [10, 25].

3. Formation of δ-shocks. In this section, we study the formation of δ-shocks in the Riemann solutions to the Euler equations for isentropic fluids in the case $u_+ \in S_2S_1(u_-)$ with $v_- > v_+$ and $\rho_\pm > 0$ as the pressure vanishes.

3.1. Limiting behavior of the Riemann solutions as $\epsilon \to 0$. For fixed $\epsilon > 0$, let $u_*^\epsilon := (\rho_*^\epsilon, \rho_*^\epsilon v_*^\epsilon)$ be the intermediate state in the sense that u_- and u_*^ϵ are connected by one-shock S_1 with speed σ_1^ϵ and that u_*^ϵ and u_+ are connected by two-shock S_2 with speed σ_2^ϵ. Then $(\rho_*^\epsilon, v_*^\epsilon)$ are determined by

$$v_*^\epsilon - v_- = -\sqrt{\frac{\epsilon\,(p(\rho_*^\epsilon) - p(\rho_-))}{\rho_- \rho_*^\epsilon (\rho_*^\epsilon - \rho_-)}}(\rho_*^\epsilon - \rho_-), \qquad \rho_*^\epsilon > \rho_-, \tag{3.1}$$

and

$$v_+ - v_*^\epsilon = -\sqrt{\frac{\epsilon\,(p(\rho_+) - p(\rho_*^\epsilon))}{\rho_+ \rho_*^\epsilon (\rho_+ - \rho_*^\epsilon)}}(\rho_+ - \rho_*^\epsilon), \qquad \rho_*^\epsilon > \rho_+. \tag{3.2}$$

Define $g(s,\tau) = \sqrt{(\frac{1}{s} - \frac{1}{\tau})(p(\tau) - p(s))}$ for $s, \tau > 0$. Thus a combination of the jump conditions (3.1) and (3.2) gives

$$v_- - v_+ = \sqrt{\epsilon}\,(g(\rho_*^\epsilon, \rho_-) + g(\rho_*^\epsilon, \rho_+)) > 0. \tag{3.3}$$

Then one must have $\lim_{\epsilon\to 0} g(\rho_*^\epsilon, \rho_\pm) = \infty$, which yields $\lim_{\epsilon\to 0} \rho_*^\epsilon = \infty$. Letting $\epsilon \to 0$ in (3.3) yields

$$\lim_{\epsilon\to 0} \epsilon(\rho_*^\epsilon)^\gamma = \frac{\sqrt{\rho_-\rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}(v_- - v_+).$$

Therefore, we have the following lemma.

LEMMA 3.1. $\lim_{\epsilon\to 0} \epsilon^{1/\gamma}\rho_*^\epsilon = (\frac{\sqrt{\rho_-\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(v_- - v_+))^{1/\gamma}$.

LEMMA 3.2. *Set* $\sigma = \frac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} \in (v_+, v_-)$. *Then*

$$\lim_{\epsilon\to 0} v_*^\epsilon = \lim_{\epsilon\to 0} \sigma_1^\epsilon = \lim_{\epsilon\to 0} \sigma_2^\epsilon = \sigma$$

and

$$\lim_{\epsilon\to 0}\ \rho_*^\epsilon(\sigma_2^\epsilon - \sigma_1^\epsilon) = \sigma[\rho] - [\rho v].$$

Proof. First, Lemma 3.1 and (3.1)–(3.2) immediately imply that

$$\lim_{\epsilon\to 0} v_*^\epsilon = \sigma.$$

On the other hand, using the Lax entropy inequalities for the shocks, we have

$$v_*^\epsilon - \sqrt{\epsilon}\,(\rho_*^\epsilon)^\theta < \sigma_1^\epsilon < \min(v_*^\epsilon + \sqrt{\epsilon}\,(\rho_*^\epsilon)^\theta, v_- - \sqrt{\epsilon}\,(\rho_-)^\theta) \tag{3.4}$$

and

$$\max(v_*^\epsilon - \sqrt{\epsilon}\,(\rho_*^\epsilon)^\theta, v_+ + \sqrt{\epsilon}\,(\rho_+)^\theta) < \sigma_2^\epsilon < v_*^\epsilon + \sqrt{\epsilon}\,(\rho_*^\epsilon)^\theta. \tag{3.5}$$

Noting that $\sqrt{\epsilon}\,(\rho_*^\epsilon)^\theta = \epsilon^{\frac{1}{2\gamma}}(\epsilon^{1/\gamma}\rho_*^\epsilon)^\theta$, we can see from Lemma 3.1 that, for $\gamma > 1$,

$$\lim_{\epsilon\to 0} \sqrt{\epsilon}\,(\rho_*^\epsilon)^\theta = 0. \tag{3.6}$$

Then (3.4)–(3.6) yield

$$\lim_{\epsilon\to 0} v_*^\epsilon = \lim_{\epsilon\to 0} \sigma_1^\epsilon = \lim_{\epsilon\to 0} \sigma_2^\epsilon = \sigma. \tag{3.7}$$

The Rankine–Hugoniot conditions for (1.1) on the shocks and the results of (3.7) yield

$$\rho_*^\epsilon(\sigma_2^\epsilon - \sigma_1^\epsilon) = \sigma_2^\epsilon\rho_+ - \sigma_1^\epsilon\rho_- - [\rho v] \to \sigma[\rho] - [\rho v] \qquad \text{as} \quad \epsilon \to 0.$$

This completes the proof of Lemma 3.2. $\square$

Remark 3.1. The quantity σ that is the limit of v_*^ϵ, σ_1^ϵ, and σ_2^ϵ uniquely determines the δ-shock solution of (1.5)–(1.6) as the limit of the Riemann solutions when $\epsilon \to 0$ and is consistent with the δ-Rankine–Hugoniot condition (2.4) and the δ-entropy condition (2.5), as proposed for the Riemann solutions for pressureless Euler equations.

3.2. Weighted δ-shocks. We now show the following theorem characterizing the vanishing pressure limit in the case in which $v_- > v_+$.

THEOREM 3.1. *Let $v_- > v_+$. For each fixed $\epsilon > 0$, assume that $(\rho^\epsilon, m^\epsilon) = (\rho^\epsilon, \rho^\epsilon v^\epsilon)$ is a two-shock solution of* (1.1)–(1.3) *with Riemann data $u_\pm = (\rho_\pm, \rho_\pm v_\pm)$, constructed in section* 2.2. *Then, when $\epsilon \to 0$, ρ^ϵ and m^ϵ converge in the sense of distributions, and the limit functions ρ and m are the sums of a step function and a δ-measure with weights*

$$\frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) \quad \textit{and} \quad \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho v] - [\rho v^2]),$$

respectively, which form a δ-shock solution of (1.5)–(1.6) *with the same Riemann data $u_\pm$.*

Proof. 1. Set $\xi = x/t$. Then, for each fixed $\epsilon > 0$, the Riemann solution can be written as

$$\rho^\epsilon(\xi) = \begin{cases} \rho_- & \text{for } \xi < \sigma_1^\epsilon, \\ \rho_*^\epsilon(\xi) & \text{for } \sigma_1^\epsilon < \xi < \sigma_2^\epsilon, \\ \rho_+ & \text{for } \xi > \sigma_2^\epsilon \end{cases}$$

and

$$v^\epsilon(\xi) = \begin{cases} v_- & \text{for } \xi < \sigma_1^\epsilon, \\ v_*^\epsilon(\xi) & \text{for } \sigma_1^\epsilon < \xi < \sigma_2^\epsilon, \\ v_+ & \text{for } \xi > \sigma_2^\epsilon, \end{cases}$$

satisfying the following weak formulations: For any $\psi \in C_0^1(-\infty, \infty)$,

$$-\int_{-\infty}^{\infty} (v^\epsilon(\xi) - \xi)\rho^\epsilon(\xi)\,\psi'(\xi)d\xi + \int_{-\infty}^{\infty} \rho^\epsilon(\xi)\,\psi(\xi)\,d\xi = 0, \tag{3.8}$$

and

$$\begin{aligned} -\int_{-\infty}^{\infty} (v^\epsilon(\xi) - \xi)\rho^\epsilon(\xi)v^\epsilon(\xi)\,\psi'(\xi)d\xi + \int_{-\infty}^{\infty} \rho^\epsilon(\xi)v^\epsilon(\xi)\,\psi(\xi)d\xi \\ = \epsilon \int_{-\infty}^{\infty} p(\rho^\epsilon(\xi))\,\psi'(\xi)d\xi. \end{aligned} \tag{3.9}$$

2. The first integral in (3.8) can be decomposed into

$$-\left\{\int_{-\infty}^{\sigma_1^\epsilon} + \int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon} + \int_{\sigma_2^\epsilon}^{\infty}\right\}(v^\epsilon(\xi) - \xi)\rho^\epsilon(\xi)\psi'(\xi)d\xi. \tag{3.10}$$

The sum of the first and last term of (3.10) is

$$\begin{aligned} &-\int_{-\infty}^{\sigma_1^\epsilon} (v_- - \xi)\rho_-\psi'(\xi)d\xi - \int_{\sigma_2^\epsilon}^{\infty} (v_+ - \xi)\rho_+\psi'(\xi)d\xi \\ &= -\rho_- v_- \psi(\sigma_1^\epsilon) + \rho_+ v_+ \psi(\sigma_2^\epsilon) + \rho_- \sigma_1^\epsilon \psi(\sigma_1^\epsilon) - \rho_+ \sigma_2^\epsilon \psi(\sigma_2^\epsilon) \\ &\quad - \int_{-\infty}^{\sigma_1^\epsilon} \rho_- \psi(\xi)d\xi - \int_{\sigma_2^\epsilon}^{\infty} \rho_+ \psi(\xi)d\xi, \end{aligned}$$

which converges as $\epsilon \to 0$ to

$$([\rho v] - \sigma[\rho])\,\psi(\sigma) - \int_{-\infty}^{\infty} \rho_0(\xi - \sigma)\,\psi(\xi)d\xi$$

with

$$\rho_0(\xi) = \rho_- + [\rho]\,\chi(\xi),$$

where $\chi(\xi)$ is the characteristic function.

For the second term of (3.10),

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)\psi'(\xi)d\xi\\ &\quad= \rho_*^\epsilon(\sigma_2^\epsilon - \sigma_1^\epsilon)\left\{v_*^\epsilon\frac{\psi(\sigma_2^\epsilon)-\psi(\sigma_1^\epsilon)}{\sigma_2^\epsilon-\sigma_1^\epsilon} - \frac{\sigma_2^\epsilon\psi(\sigma_2^\epsilon)-\sigma_1^\epsilon\psi(\sigma_1^\epsilon)}{\sigma_2^\epsilon-\sigma_1^\epsilon} + \frac{1}{\sigma_2^\epsilon-\sigma_1^\epsilon}\int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon}\psi(\xi)d\xi\right\},\end{aligned}$$

which converges as $\epsilon \to 0$ to

$$([\rho v] - \sigma[\rho])\{-\sigma\psi'(\sigma) + \sigma\psi'(\sigma) + \psi(\sigma) - \psi(\sigma)\} = 0$$

since $\psi \in C_0^1(-\infty,\infty)$, $\lim_{\epsilon\to 0} v_*^\epsilon = \sigma$, and $\lim_{\epsilon\to 0}\sigma_j^\epsilon = \sigma$ for $j = 1, 2$.

Then the integral identity (3.8) yields

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\rho^\epsilon(\xi) - \rho_0(\xi-\sigma))\,\psi(\xi)d\xi = (\sigma[\rho] - [\rho v])\,\psi(\sigma)$$

for any function $\psi \in C_0^\infty(-\infty,\infty)$.

3. We now turn to justifying the limit of momentum $m^\epsilon = \rho^\epsilon v^\epsilon$ using the weak formulation (3.9). As done previously, we can obtain the limit for the first term on the left of (3.9) as

$$\begin{aligned}&-\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(v^\epsilon(\xi)-\xi)\,\rho^\epsilon(\xi)v^\epsilon(\xi)\psi'(\xi)d\xi\\ &= \psi(\sigma)\,([\rho v^2] - \sigma[\rho v]) - \int_{-\infty}^{\sigma}\rho_- v_-\,\psi(\xi)d\xi - \int_{\sigma}^{\infty}\rho_+ v_+\,\psi(\xi)d\xi.\end{aligned}$$

The term on the right of (3.9) equals

$$\epsilon\int_{-\infty}^{\infty}p(\rho^\epsilon)\,\psi'(\xi)d\xi = \epsilon\left\{\int_{-\infty}^{\sigma_1^\epsilon} + \int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon} + \int_{\sigma_2^\epsilon}^{\infty}\right\}p(\rho^\epsilon)\,\psi'(\xi)d\xi,$$

which converges to

$$\begin{aligned}&\epsilon\,\{p(\rho_-)\psi(\sigma_1^\epsilon) + p(\rho_*^\epsilon)(\psi(\sigma_2^\epsilon) - \psi(\sigma_1^\epsilon)) - p(\rho_+)\psi(\sigma_2^\epsilon)\}\\ &\quad= o(\epsilon) + \epsilon\,p(\rho_*^\epsilon)(\psi(\sigma_2^\epsilon)-\psi(\sigma_1^\epsilon)) \to 0 \qquad \text{as } \epsilon\to 0,\end{aligned}$$

where we used the fact that $\epsilon p(\rho_*^\epsilon)$ is bounded and $\lim_{\epsilon\to 0}\sigma_j^\epsilon = \sigma$ for $j = 1, 2$.

Returning to the weak formulation (3.9), one has

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}((\rho^\epsilon v^\epsilon)(\xi) - (\rho_0 v_0)(\xi-\sigma))\,\psi(\xi)d\xi = \psi(\sigma)(\sigma[\rho v] - [\rho v^2]).$$

4. Finally, we are in a position to study the limits of ρ^ϵ and $m^\epsilon = \rho^\epsilon v^\epsilon$ by tracking the time-dependence of the weights of the δ-measures as $\epsilon \to 0$.

Let $\phi(x,t) \in C_0^\infty((-\infty,\infty)\times[0,\infty))$ be a smooth test function, and let $\tilde{\phi}(\xi,t) := \phi(\xi t, t)$. Then we have

$$\lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \rho^\epsilon(x/t)\phi(x,t)dxdt = \lim_{\epsilon\to 0}\int_0^\infty t\left(\int_{-\infty}^\infty \rho^\epsilon(\xi)\tilde{\phi}(\xi,t)d\xi\right)dt.$$

On the other hand, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{-\infty}^\infty \rho^\epsilon(\xi)\tilde{\phi}(\xi,t)d\xi &= \int_{-\infty}^\infty \rho_0(\xi-\sigma)\tilde{\phi}(\xi,t)d\xi + (\sigma[\rho]-[\rho v])\tilde{\phi}(\sigma,t)\\ &= t^{-1}\int_{-\infty}^\infty \rho_0(x-\sigma t)\phi(x,t)dx + (\sigma[\rho]-[\rho v])\phi(\sigma t,t).\end{aligned}$$

Combining the two relations above yields

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \rho^\epsilon(x/t)\phi(x,t)dxdt\\ &\quad= \int_0^\infty\int_{-\infty}^\infty \rho_0(x-\sigma t)\phi(x,t)dxdt + \int_0^\infty t([\rho v]-\sigma[\rho])\phi(\sigma t,t)dt.\end{aligned}$$

The last term, by the definition, equals

$$\langle w_1(\cdot)\delta_S, \phi(\cdot,\cdot)\rangle$$

with

$$w_1(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho]-[\rho v]).$$

Similarly, we can show that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty m^\epsilon(x/t)\phi(x,t)dxdt\\ &\quad= \int_0^\infty\int_{-\infty}^\infty (\rho_0 v_0)(x-\sigma t)\phi(x,t)dxdt + \langle w_2(\cdot)\delta_S, \phi(\cdot,\cdot)\rangle\end{aligned}$$

with

$$w_2(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho v]-[\rho v^2]).$$

This completes the proof of Theorem 3.1. $\square$

4. Formation of vacuum states. In this section, we show the formation of vacuum states in the Riemann solutions of (1.1)–(1.3) in the case in which $u_+ \in R_2R_1(u_-)$ with $v_- < v_+$ and $\rho_\pm > 0$ as the pressure decreases.

As stated previously, on the rarefaction waves, the solution satisfies

$$\xi = x/t = v^\epsilon \pm \sqrt{\epsilon\, p'(\rho^\epsilon)} \tag{4.1}$$

for each fixed $\epsilon > 0$. More precisely, we have that, on the one-rarefaction wave,

$$\xi = v^\epsilon - \sqrt{\epsilon\, p'(\rho^\epsilon)}, \qquad v_- - \sqrt{\epsilon\, p'(\rho_-)} < \xi < v_*^\epsilon - \sqrt{\epsilon\, p'(\rho_*^\epsilon)}, \qquad \rho_- > \rho_*^\epsilon,$$

and, on the two-rarefaction wave,

$$\xi = v^\epsilon + \sqrt{\epsilon\, p'(\rho^\epsilon)}, \qquad v_*^\epsilon + \sqrt{\epsilon\, p'(\rho_*^\epsilon)} < \xi < v_+ + \sqrt{\epsilon\, p'(\rho_+)}, \qquad \rho_*^\epsilon < \rho_+,$$

where $(\rho_*^\epsilon, \rho_*^\epsilon v_*^\epsilon)$ is the intermediate state in the Riemann solutions. Since $(\rho_*^\epsilon, \rho_*^\epsilon v_*^\epsilon)$ is on the curve $R_1(u_-)$, we have

$$v_*^\epsilon = v_- - \int_{\rho_-}^{\rho_*^\epsilon} \frac{\sqrt{\epsilon\, p'(s)}}{s} ds \le v_- + \int_0^{\rho_-} \frac{\sqrt{\epsilon\, p'(s)}}{s} ds = v_- + \sqrt{\epsilon}\frac{\rho_-^\theta}{\theta} \equiv A^\epsilon.$$

When $v_- < v_+ < A^\epsilon$, that is,

$$\epsilon > \left(\frac{\theta(v_+ - v_-)}{\rho_-^\theta}\right)^2 \equiv \epsilon_0(u_-, u_+), \tag{4.2}$$

there is no vacuum in the solution. This implies that, for a fluid with strong pressure, no vacuum occurs in the solution generically.

However, when ϵ decreases so that $\epsilon < \epsilon_0(u_-, u_+)$, then $A^\epsilon < v_+$, and the intermediate state $(\rho_*^\epsilon, \rho_*^\epsilon v_*^\epsilon)$ becomes a vacuum state with

$$(\rho_*^\epsilon, v_*^\epsilon)(\xi) = (0, \xi), \qquad v_1^\epsilon \le \xi \le v_2^\epsilon,$$

where

$$v_1^\epsilon = v_- + \int_0^{\rho_-} \frac{\sqrt{\epsilon p'(s)}}{s} ds, \quad v_2^\epsilon = v_+ - \int_0^{\rho_+} \frac{\sqrt{\epsilon p'(s)}}{s} ds.$$

The uniform boundedness of $\rho^\epsilon(\xi)$ with respect to ϵ in this case leads to

$$\lim_{\epsilon \to 0} v_1^\epsilon = v_-, \qquad \lim_{\epsilon \to 0} v_2^\epsilon = v_+,$$

and

$$\lim_{\epsilon \to 0} v^\epsilon(\xi) = \xi \qquad \text{for } \xi \in (v_-, v_+).$$

In summary, the limit function (ρ, v) in this case is

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi \le v_-, \\ (0, \xi), & v_- \le \xi \le v_+, \\ (\rho_+, v_+), & v_+ \le \xi < \infty, \end{cases}$$

which is a solution to the transport equations (1.5)–(1.6) containing a vacuum state that fills up the region formed by the two contact discontinuities $\xi = x/t = v_\pm$.

We can clearly see from the analysis above that, when ϵ decreases, the left boundary of the one-rarefaction wave and the right boundary of the two-rarefaction wave are fixed, the right boundary of the one-rarefaction wave becomes closer to the left boundary of the one-rarefaction wave, and the left boundary of the two-rarefaction wave becomes closer to the right boundary of the two-rarefaction wave; while the state between the right boundary of the one-rarefaction wave and the left boundary of the two-rarefaction wave in the Riemann solution is a vacuum state; and, in the limit, the left boundary of the one-rarefaction wave and the right boundary of the two-rarefaction wave become two contact discontinuities of the transport equations (1.5)–(1.6), and the vacuum state fills up the region between the two contact discontinuities.

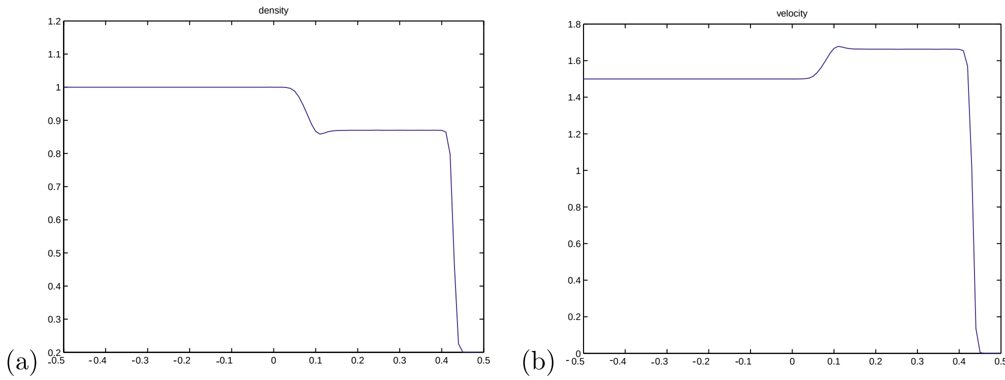

FIG. 5.1. *Density and velocity for* $\epsilon = 1.4$.

5. Formation process of δ-shocks: Numerical simulations. After the cavitation process in the Riemann solutions of (1.1)–(1.3) has been described clearly as the pressure decreases in section 4, understanding the formation process of δ-shocks in the Riemann solutions as the pressure decreases becomes more constructive for comparison. For this purpose, in this section we present a selected group of representative numerical results in the level of Euler dynamics (1.1)–(1.3) starting with Riemann initial data. We have performed many more numerical tests to make sure what we present are not numerical artifacts.

We solve the Riemann problem for (1.1)–(1.3) with $p(\rho) = \rho^\gamma/\gamma$ and $\gamma = 1.4$ for an ideal gas. The Riemann initial data is

$$(\rho, v)(x, 0) = \begin{cases} (1.0, 1.5) & \text{for} \quad x < 0, \\ (0.2, 0.0) & \text{for} \quad x > 0. \end{cases}$$

To discretize the system, we use the higher order ENO scheme to obtain a method-of-line ordinary differential equation in time and then discretize the ordinary differential equation by the classical higher order explicit Runge–Kutta method (see [23, 24]). We calculate by the third-order ENO scheme [24] up to $t = 0.2$ with mesh 100. The numerical simulations with different choices of ϵ are presented in Figures 5.1–5.3. These figures show the formation process of a δ-shock in the two-shock Riemann solutions for the Euler equations (1.1)–(1.3) for isentropic fluids as the pressure decreases. We start with $\epsilon/\gamma = 1.0$ and choose then $\epsilon/\gamma = 0.05$ and finally $\epsilon/\gamma = 0.001$. Figures 5.1a–5.3a show the concentration process of the density yielding a weighted δ-measure in the limit, in which the horizontal axis stands for the space variable x and the vertical axis stands for the density. Figures 5.1b–5.3b show the change of the velocity as ϵ decreases yielding a step function in the limit, in which the horizontal axis stands for the space variable x and the vertical axis stands for the velocity.

We can see clearly from these numerical results that, when ϵ decreases, the locations of the two shocks become closer, and the density of the intermediate state increases dramatically, while the velocity is closer to a step function. In the vanishing pressure limit, the two shocks coincide to form, along with the intermediate state, a δ-shock of the transport equations (1.5)–(1.6), while the velocity is a step function.

We remark that it is delicate to calculate solutions of hyperbolic systems of conservation laws that strict hyperbolicity fails, for which the system of pressureless Euler

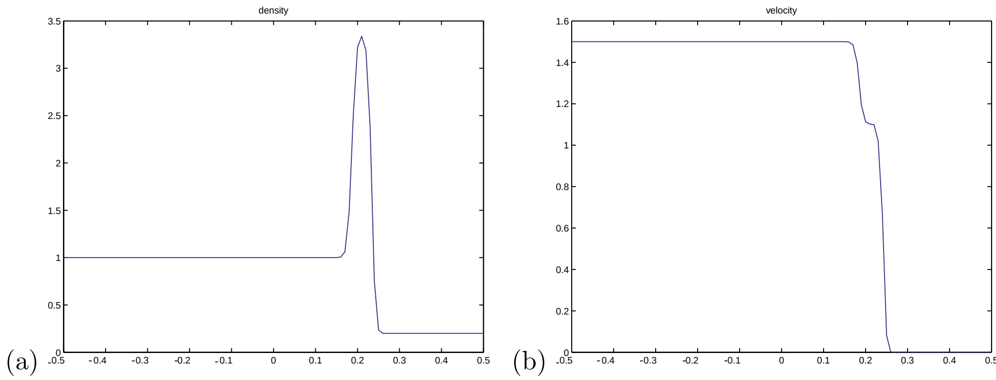

FIG. 5.2. *Density and velocity for* $\epsilon = 0.07$.

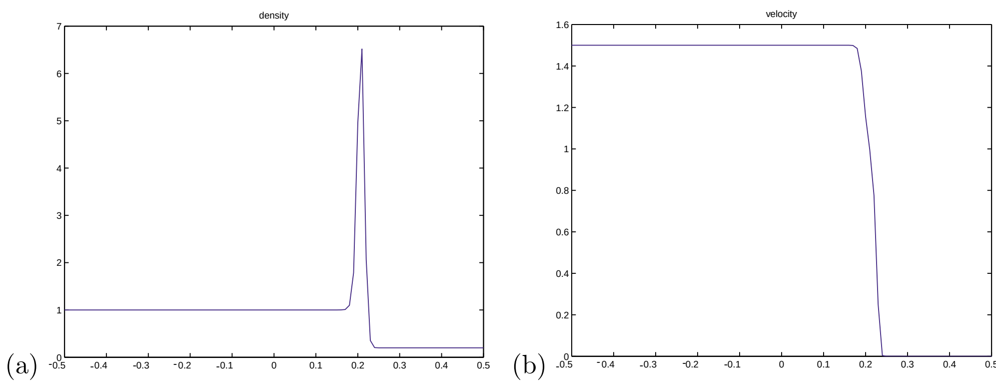

FIG. 5.3. *Density and velocity for* $\epsilon = 0.0014$.

equations for (1.5)–(1.6) is an example. In this section, we have proposed an efficient numerical approach to calculate the solutions containing δ-shocks for (1.5)–(1.6) via the vanishing pressure limit. It would be interesting to apply the approach and ideas set forth here to develop efficient numerical algorithms to calculate solutions for more complex physical models.

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