# Adversarial Projections for Inverse Problems

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October 2, 2020

# Agenda

1. Background
2. Adversarial Projections
3. Numerical Experiments
4. Conclusions

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# Background

Forward problem is given by

$$b = A(u^\star) + \epsilon,$$

where

- $b \in \mathbb{R}^m$ = measurement data
- $A \colon \mathbb{R}^n \to \mathbb{R}^m$ = forward operator
- $\epsilon \in \mathbb{R}^m$ = measurement noise
- $u^\star \in \mathbb{R}^n$ = true signal

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- $u^\star \in \mathbb{R}^n$ = true signal

**Inverse Problem:** recover signal $u^\star$ given noisy measurements $b$

## Inverse Problem

$$\tilde{u} \in \arg\min_{u} \quad \ell(A(u), b) + J(u),$$

where

- $\ell \colon \mathbb{R}^m \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is a fidelity term (e.g., Least Squares)
- $J \colon \mathbb{R}^m \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is a regularizer (e.g., TV, L1, Learned)

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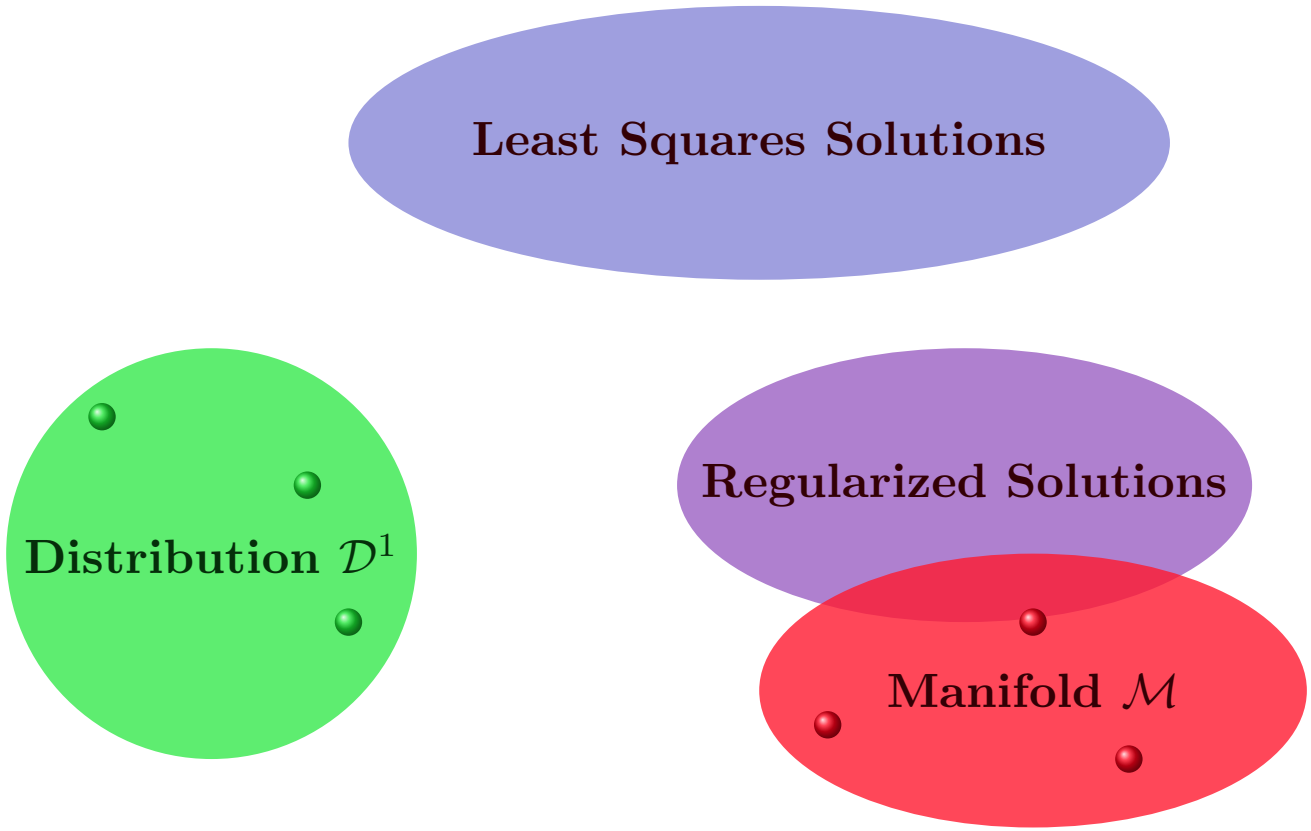
**Remark:** regularizers often picked heuristically.

# Distribution Illustration



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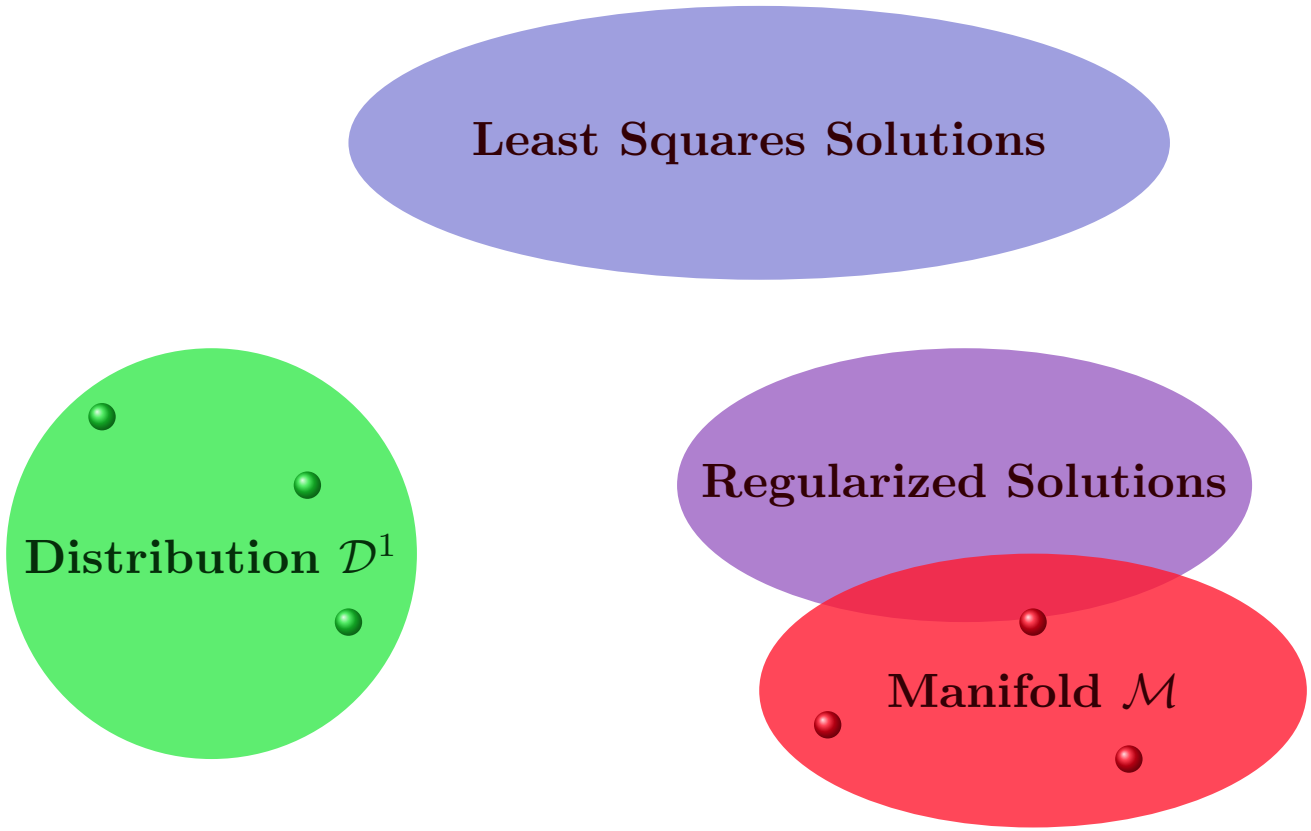


**Key question:** how can we guarantee reconstructed signal estimates are in the manifold of true signals?

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# Adversarial Projections

**Recurring theme:** signals represented in high dimensional spaces often exhibit redundancies, i.e., they lie on some low-dimensional manifold.

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**Our Task:** learn an operator that projects onto manifold of true signals in unsupervised manner.

# Manifold Assumptions

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The projection operator $P_{\mathcal{M}} : \mathbb{R}^n \to \mathbb{R}^n$ gives the closest point in $\mathcal{M}$ to the point $u$, i.e.,

$$P_{\mathcal{M}}(u) := \arg\min_{v \in \mathcal{M}} \|v - u\|.$$

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The distance function is given by

$$d_{\mathcal{M}}(u) := \inf_{v \in \mathcal{M}} \|v - u\| = \|P_{\mathcal{M}}(u) - u\|.$$

# Manifold Assumptions

## Assumption 2

The initial distribution $\mathcal{D}^1$ is uniformly bounded.

## Assumption 3

The push forward of the projection operation $P_{\mathcal{M}}$ applied to $\mathcal{D}^1$ recovers the true distribution $\mathcal{D}_{\text{true}}$ up to a set of measure zero, i.e.,

$$\mathcal{D}_{\text{true}} = (P_{\mathcal{M}})_{\#}(\mathcal{D}^1) := \{P_{\mathcal{M}}(u) : u \in \mathcal{D}^1\}$$

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**Remark:** We do not assume each signal can be recovered, but instead the overall distribution is not "too noisy."

## Distance Function Theorem

Under the previous assumptions,

$$d_{\mathcal{M}} = \arg\min_{J \in \mathcal{I}} \mathbb{E}_{u \sim \mathcal{D}_{\text{true}}} [J(u)] - \mathbb{E}_{u \sim \tilde{\mathcal{D}}} [J(u)] + \tau \cdot \mathbb{E}_{u \sim \mathcal{D}_{\text{true}}} [(J(u))^p],$$

where $\tau \in (0, \infty)$, $p > 1$, $\tilde{\mathcal{D}}$ is the approximate distribution, and $\mathcal{I}$ is the set of all nonnegative 1-Lipschitz functions.

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- we will parameterize $J$ by a neural network

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# Distribution Updates

For step size $\lambda_k$, define the operator

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To update an individual signal, we use a Halpern update

$$u^{k+1} = \gamma_k u^1 + (1 - \gamma_k) g_k(u^k),$$

where $\gamma_k \in (0, 1]$, $\gamma_k \to 0$, and $\sum_{k \in \mathbb{N}} \gamma_k = \infty$.

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How to choose step size $\lambda_k$?

# Choice of Step Size

Denote the mean distance by

$$\beta_k = \mathbb{E}_{\omega \sim \Omega} \left[ d_{\mathcal{M}}(u^k(\omega)) \right] - \mathbb{E}_{u \sim \mathcal{D}_{\text{true}}} \left[ d_{\mathcal{M}}(u) \right].$$

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Three options for choosing step size $\lambda_k$:

- $\lambda_k(u) = \mu d_{\mathcal{M}}(u^k)$ individual distance to manifold
- $\lambda_k(u) = \mu\beta_k$ mean distance to manifold, or
- $\lambda_k(u) = \mu(\beta_k + d_{\mathcal{M}}(u^k))$ combined distance to manifold

for relaxation parameter $\mu \in (0, 1)$

# Projection Illustration

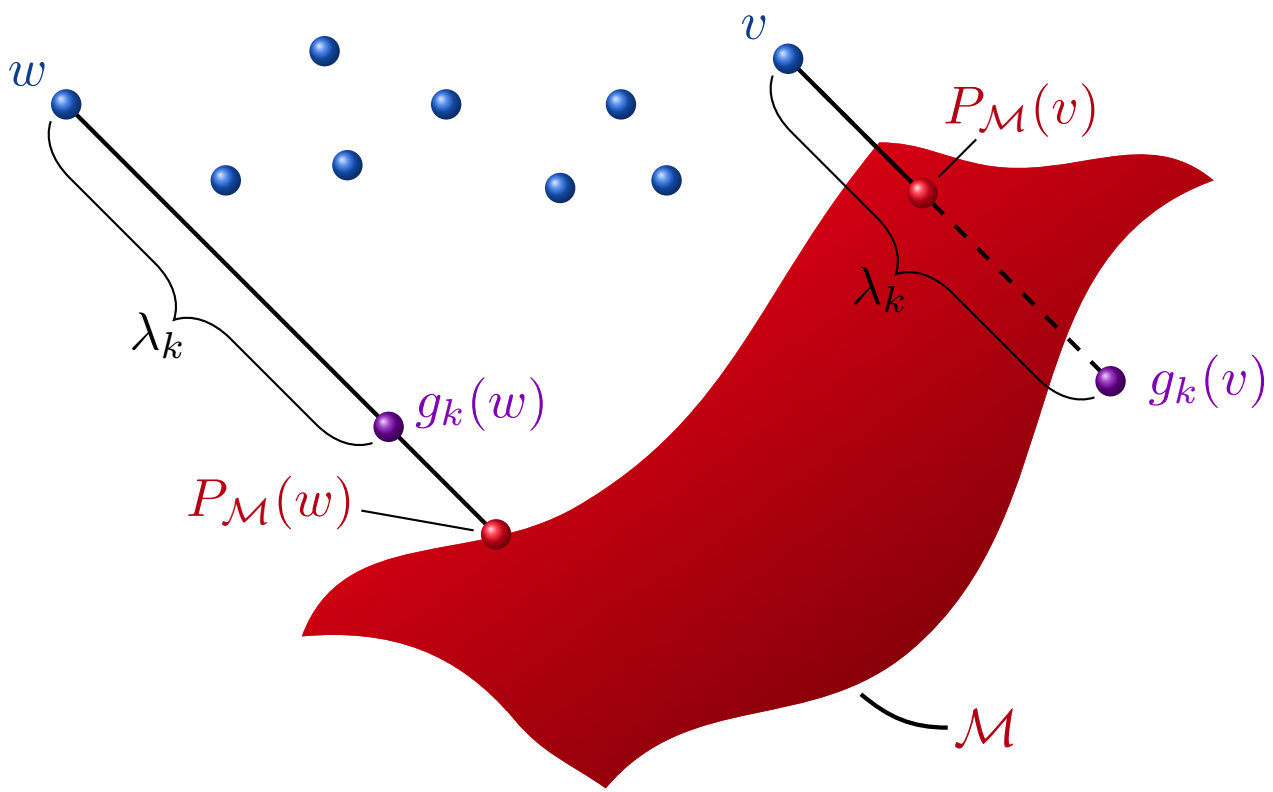


**Figure 1:** Blue points are samples drawn from distribution $\mathcal{D}^k$, which are updated to the relaxed projection $g_k$ of the projection $P_{\mathcal{M}}$ manifold $\mathcal{M}$.

# Learned Projection Algorithm

**Algorithm 1:** Training to generate parameters for distribution sequence $\{\mathcal{D}^k\}$

Choose relaxation parameter $\alpha \in (0, 1)$

Choose anchoring sequence $\{\gamma_k\} \subset (0, 1]$

Choose function parameterization $\mathcal{I}$

Choose initial distribution $\mathcal{D}^1 = \{u^1(\omega) : \omega \in \Omega\}$ ◁ Initial signal estimates

**for** $k = 1, 2, \ldots$ **do**

  $\theta^k \leftarrow \underset{\theta \in \mathcal{I}}{\operatorname{argmin}} \; \mathbb{E}_{u \sim \mathbb{P}_{\mathcal{U}}} \left[ J_\theta(u) + \tau J_\theta(u)^p \right] - \mathbb{E}_{u \sim \mathbb{P}^k} \left[ J_\theta(u^k) \right]$ ◁ Train to find weights

  $\beta_k \leftarrow \mathbb{E}_{u^k \sim \mathbb{P}^k} \left[ J_{\theta^k}(u^k) \right] - \mathbb{E}_{u \sim \mathbb{P}_{\mathcal{U}}} \left[ J_{\theta^k}(u) \right]$ ◁ Assign mean distance

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Return $\{\mu, \beta_k, \gamma_k, \theta^k\}$

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## Remarks:

- $J$ is parameterized by a neural network, where $\theta$ are the weights
- we learn a sequence of relaxed projections which amount to $P_{\mathcal{M}}$
- encode information on NN by forcing $J(u) \geq 0 \ \ \forall u \in \mathbb{R}^n$

# Convergence

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**end**

Return $\{\mu, \beta_k, \gamma_k, \theta^k\}$

## Convergence Theorem

Under previously stated assumptions, the sequence $\{u^k\}$ converges to $P_{\mathcal{M}}(u^1)$ in probability.

# Related Works

Three key connections:

- **Wasserstein GANS**[1]: Our learned distance function $J$ can be interpreted as a discriminator function in W-GANs. $J$ discriminates between "true" and approximate signals.

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- **Adversarial Regularizers**[2]: Our work extends the work from adversarial regularizers and provides convergence guarantees.
- **Expert Regularizers**[3]: $J$ can be interpreted as an expert regularizer, which attain small values at signals similar to the distribution of true signals and larger values at signals drawn elsewhere.

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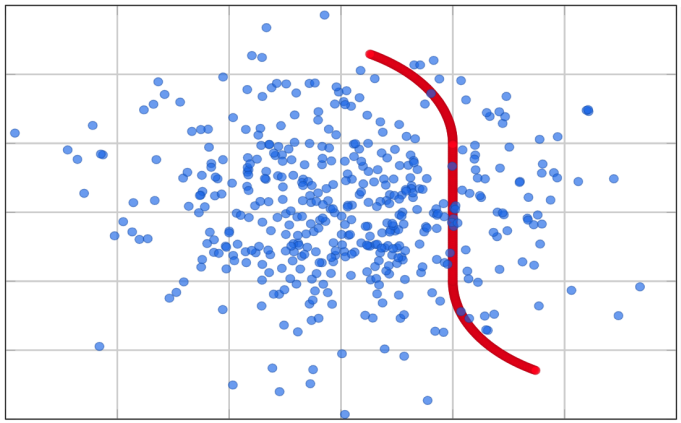
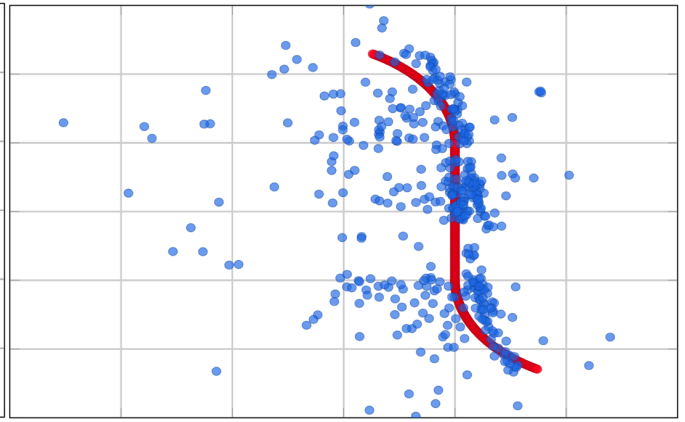
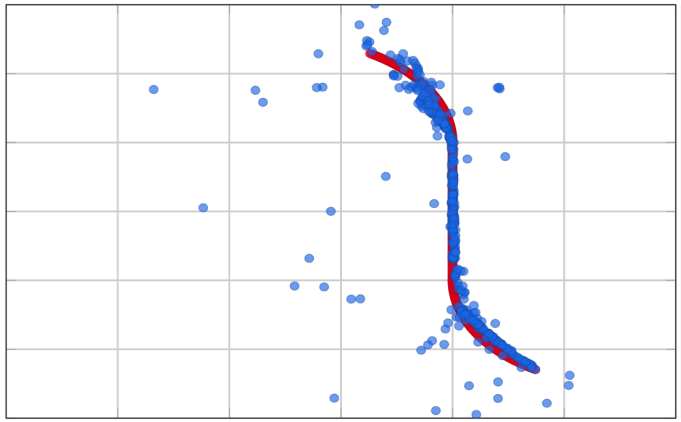
# Toy Example



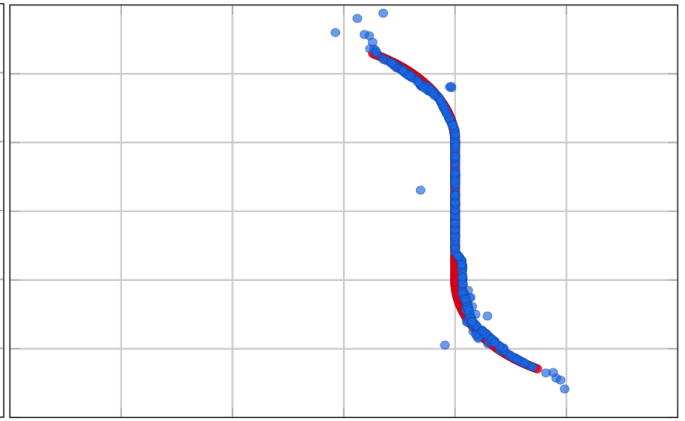
**(a)** $\mathcal{D}^1$



**(b)** $\mathcal{D}^5$

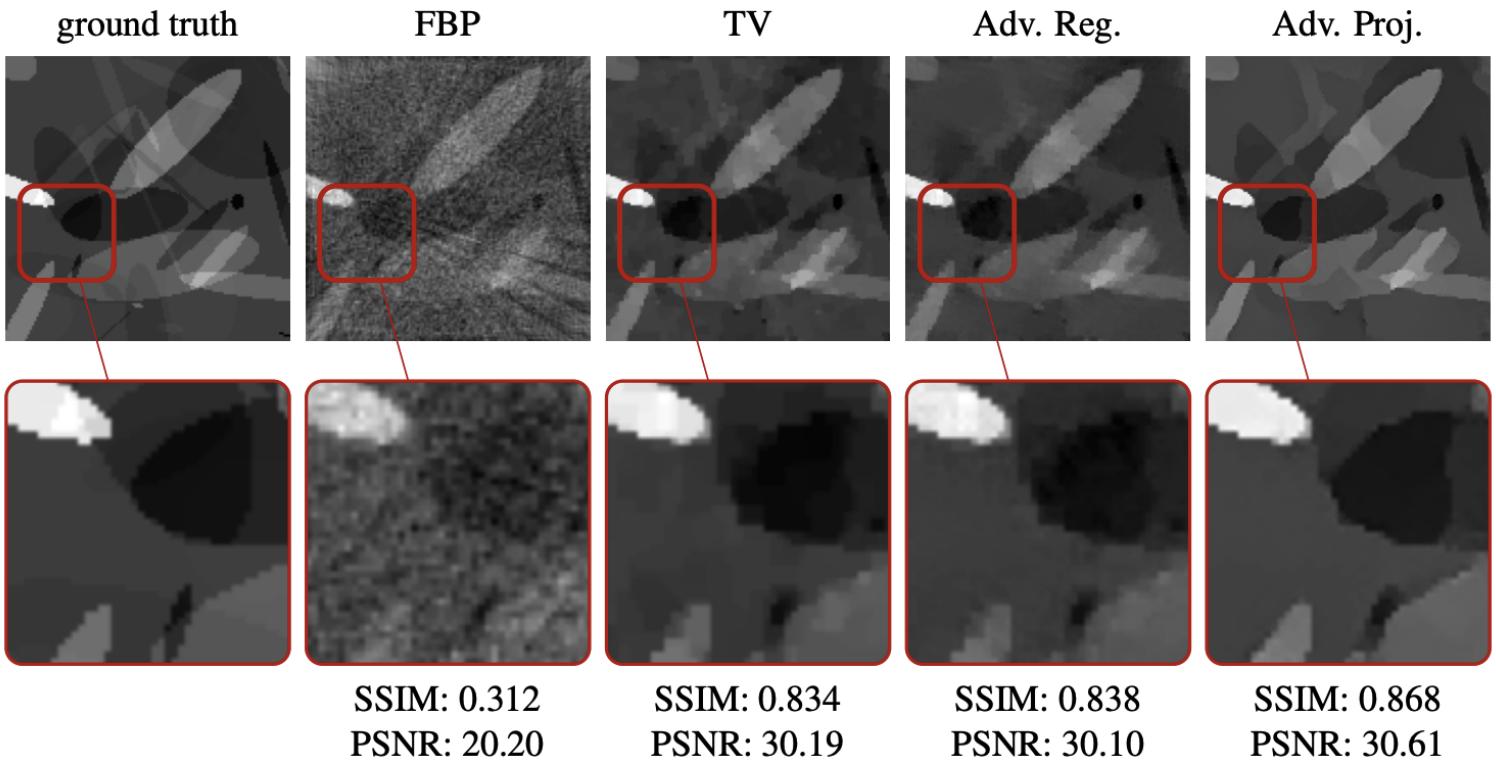


**(c)** $\mathcal{D}^{25}$



**(d)** $\mathcal{D}^{300}$

# Low-Dose CT: Ellipses



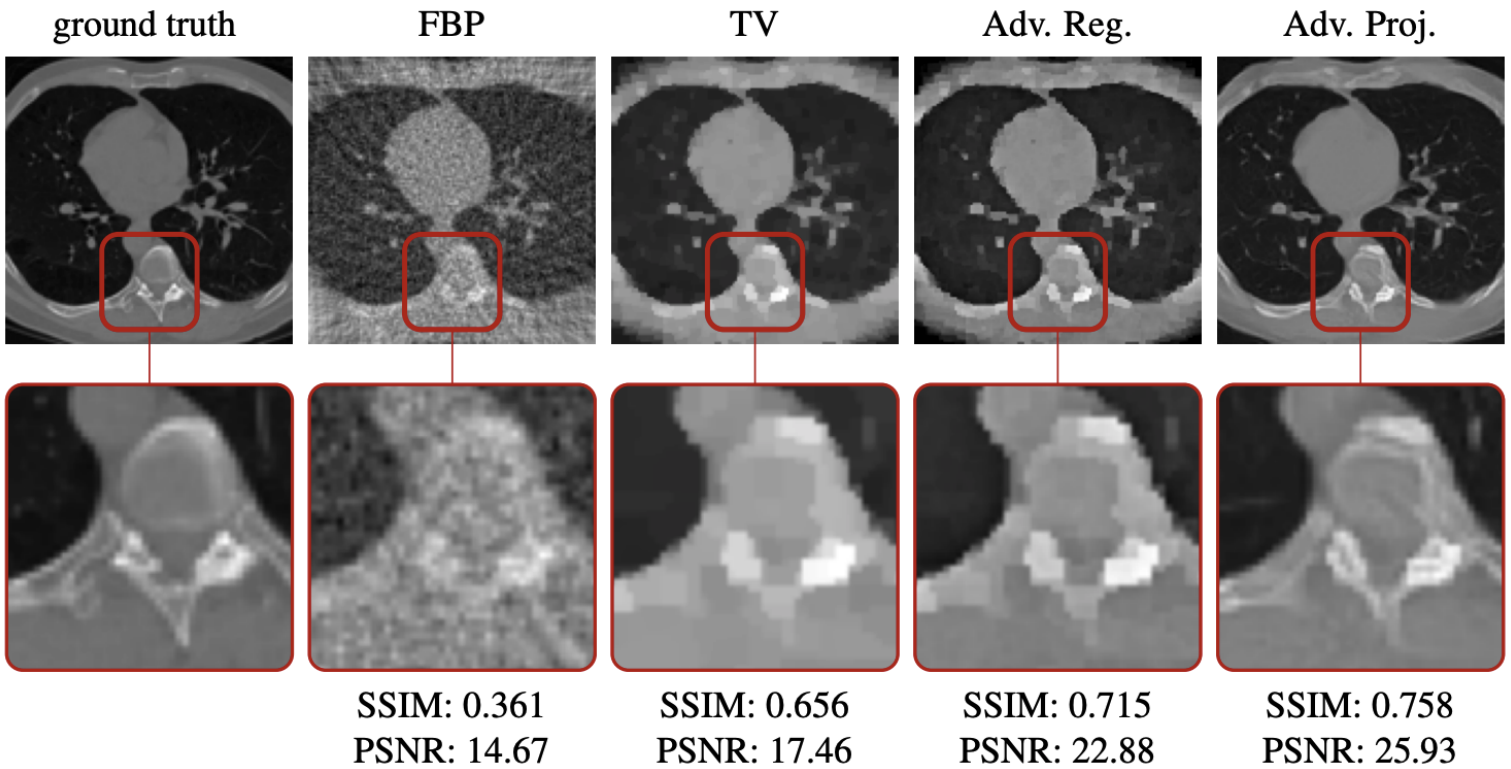
# Low-Dose CT: Ellipses

CT Results on Ellipses Dataset

| Method | Avg. PSNR (dB) | Avg. SSIM |
|---|---|---|
| Filtered Backprojection | 16.53 | 0.179 |
| Total Variation | 26.43 | 0.624 |
| Adversarial Regularizers | 26.95 | 0.680 |
| Adversarial Projections (ours) | **28.11** | **0.783** |

**Table 1:** Average PSNR and SSIM on a validation dataset consisting 1,000 images of random ellipses.

# Low-Dose CT: Human Phantoms



CT Results on Human Phantoms

| Method | Avg. PSNR (dB) | Avg. SSIM |
|---|---|---|
| Filtered Backprojection | 14.77 | 0.314 |
| Total Variation | 19.72 | 0.672 |
| Adversarial Regularizers | 22.58 | 0.691 |
| Adversarial Projections (ours) | **26.07** | **0.750** |

**Table 2:** Average PSNR and SSIM on a validation dataset consisting 2,000 images of human phantoms.

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# Conclusions

**Summary:** We present a new mechanism for solving inverse problems

- Use projections to find closest "true" point to measurements
- Leverage ideas from Wasserstein GANs and Expert Regularizers
- Focus on distribution, not individual signals

Preprint will be updated soon!

# Team and Acknowledgments

Howard Heaton

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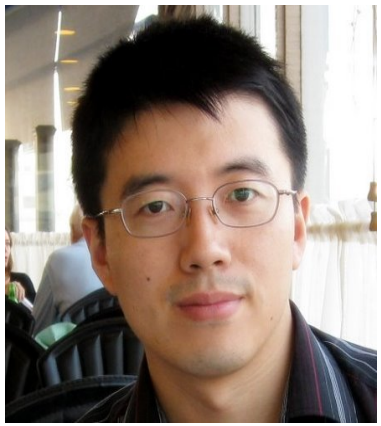

**Funding:** AFOSR MURI FA9550-18-1-0502, AFOSRFA9550-18-1-0167, ONR N00014-18-1-2527, NSF DGE-1650604