RESEARCH STATEMENT

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Broadly speaking, my research interests lie in geometric analysis, which combines partial differential equations and the calculus of variations to understand questions in geometry. The projects I work on are often inspired by problems in applied mathematics or elementary questions. As a result, I work on a variety of questions at the intersection of geometry, analysis, and probability.

To provide an overview of my research, Sections 1 - 3 provide a summary of three of my current projects, enumerated below. In Section 4, I will briefly summarize some other work which fall outside of these categories. When relevant, I have interspersed directions for future work throughout the sections.

(1) **Optimal Transport:** I have recently been focused on the regularity theory for the Monge problem. In particular, Jun Zhang and I have found a connection between this regularity theory and Kähler/Hessian geometry. By using this correspondence, we are able to gain insight for both fields as well as address questions in mathematical finance.
(2) **The Spectrum of Drift Laplacians:** I also investigate the spectrum of Laplace-Beltrami operators on Riemannian manifolds with an additional drift term. In particular, I study the principle eigenvalue when minimal assumptions are made about the drift term.
(3) **Geometric Probability and Random Triangles:** I study extremal configurations for several classic questions in geometric probability and their relation to isoperimetric inequalities.

1. Optimal Transport

Optimal transport is a classic field which seeks to find the most cost-efficient way to transport or allocate resources. Given a cost function c and probability measures (X, μ) and (Y, ν), under mild assumptions an optimal *coupling* exists (denoted π), which minimizes the integral

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x,y) d\pi,$$

where $\Pi(\mu, \nu)$ is the set of all couplings between (X, μ) and (Y, ν). Generally, this optimal coupling subdivides mass at a single point and sends it to multiple locations. However, it is often preferable to only consider *deterministic* couplings, in which each point in X is sent to a unique point in Y.

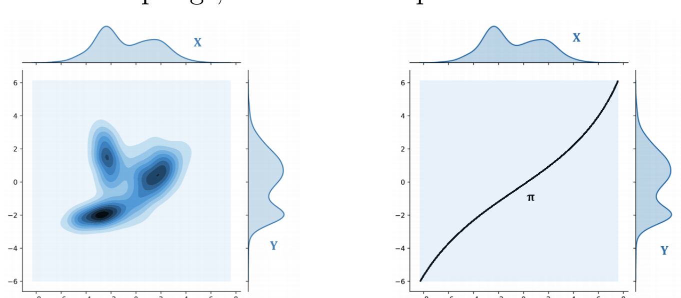

Figure 1: Two examples of couplings, the second of which is deterministic.
(Images from Wikimedia Commons)

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In 1995, building off work by Brenier [8], Gangbo and McCann [11] found conditions on the cost function and probability measures to ensure that the optimal coupling between (X, μ) and (Y, ν) is unique and deterministic. Their work further shows this transport is induced by the c-subdifferential (denoted G_ϕ) of a potential function ϕ, which satisfies the Monge-Ampere equation

$$\det\left(D^2_{xx} c\left(x, G_\phi(x)\right) + D^2\phi\right) = \left|\det D^2_{xy} c\right| \frac{d\mu(x)}{d\nu\left(G_\phi(x)\right)} \qquad \mu\text{-almost everywhere.} \tag{1}$$

For deterministic optimal transport, it is natural to ask whether this transport is continuous, in that nearby points in X are transported to nearby points in Y. From a mathematical perspective, this corresponds to find a priori estimates for solutions to Equation (1).

In general, for smooth measures μ and ν, the potential ϕ will be Lipschitz, but may fail to be C^2, (or even C^1). As a result, the transport will be well-defined (μ-almost everywhere), but need not be continuous. In 2005, a breakthrough paper by Ma, Trudinger, and Wang proved C^2 regularity[1] for solutions to Equation (1), under two additional assumptions, 1) that a certain fourth-order quantity, known as the *MTW tensor*, is non-negative and 2) that the supports of the initial and target measures are *relatively c-convex*. Despite their complexity, these conditions were later shown to be essentially necessary by Loeper [27].

In recent work with Jun Zhang, we show that this regularity problem is intricately linked with Kähler and Hessian geometry.[2]

Theorem 1 (K.-Zhang [24])**.** *For cost functions of the form $c(x, y) = \Psi(x - y)$ with $\Psi : \Omega \to \mathbb{R}$ a convex function, the MTW tensor is proportional to the orthogonal anti-bisectional curvature of the tangent bundle $T\Omega$ with its induced Kähler-Sasaki metric. Furthermore, relative c-convexity can corresponds to geodesic convexity on Ω with respect to a dual affine connection.*

1.1. **Geometric Applications.** By using the interplay between complex geometry and optimal transport, it is possible to gain insight for both fields. For instance, this approach allows us to construct new cost functions which have non-negative MTW tensor, which is important as relatively few examples were known.

In addition to finding examples, a recent preprint with Jun Zhang and Fangyang Zheng explores the geometry in more detail [25]. For instance, we show that structural conditions from optimal transport can be used to find *synthetic* versions of curvature bounds for Kähler manifolds on tube domains. This allows us to *define* "curvature bounds" when the Kähler potential is only C^3. By analogy, this is similar to the Topogonov triangle theorem, which characterizes sectional curvature bounds in a way that can be defined on a length space. In the future, I plan to continue this project, and believe it will yield other geometric insights.

1.2. **The Regularity of Pseudo-Arbitrages.** Apart from geometric implications, this work has direct applications to mathematical finance. In fact, this line of research was primarily motivated by a question in stochastic portfolio theory. In particular, our work provides a regularity theory for *pseudo-arbitrages*, which are investment strategies which beat the market portfolio almost surely in the long run (given two mild and realistic assumptions).

Recently, it was shown by Pal and Wong [29] that pseudo-arbitrages are induced by solutions to an optimal transport problem, where the cost function is given by the so-called *diversification return* (which is closely related to the free energy in statistical mechanics). In [30], Pal and Wong ask for a regularity theory for this transport, which in practical terms is a guarantee that a slight change of the market weights will not trigger a massive sell-off for the portfolio. Using our geometric framework, Jun Zhang and I prove such a regularity result.

[1]For Equation (1), a C^2 estimate ensures that the linearized operator is strongly elliptic, which implies higher order smoothness via elliptic bootstrapping.

[2]There is also a pseudo-Riemannian framework for optimal transport, developed by Kim and McCann [26].

To do so, we use the cost function from [29] to construct an associated Kähler manifold, which turns out to be an (incomplete) Hermitian symmetric space of constant positive holomorphic sectional curvature. Using this observation, we prove the following regularity theorem.

Theorem 2 (K.-Zhang)**.** *Suppose μ and ν are smooth probability measures supported on subsets X and Y of the probability simplex satisfying necessary regularity and convexity assumptions. Suppose $c(p, q)$ be the cost function given by*

$$c(p,q) = \log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{q_i}{p_i}\right) - \frac{1}{n}\sum_{i=1}^{n}\log\frac{q_i}{p_i}.$$

Then the c-optimal map $\mathbb{T}_{\mathrm{U}}$ taking μ to ν is smooth.

One limitation of this result is that it requires that the market weights of each option does not become arbitrarily small (i.e. none of the companies go bankrupt). Pseudo-arbitrages outperform the market almost surely in the long term. While it may be reasonable to assume that large companies do not go bankrupt in the short term, this may not be reasonable over large time scales. In future work, I would like to overcome this limitation by answering the following question:

Question 3. *Is it possible to find a version of Theorem 2 for measures which are supported on the entire probability simplex?*

1.3. **Statistical Mirror Symmetry and Information Geometry.** The construction used to connect Kähler geometry and optimal transport has an interesting duality, which I would like to understand in more detail. To be more precise, our work considers the Sasaki metric, which is an almost Hermitian structure defined on the tangent bundle TM of any Riemannian manifold (M, g) with an affine connection ∇ [9].

By using the metric to dualize the connection, it is possible to construct a *dual* almost Hermitian manifold. In upcoming work with Jun Zhang, we study this phenomena, which we call "statistical mirror symmetry" (due to a formal correspondence to T-duality for semi-flat Calabi-Yau manifolds). This yields many interesting geometric examples, many of which are induced by considering parametrized families of probability distributions. Perhaps surprisingly, this work appears to have connections with number theory. For instance, the Siegel half-space with its Bergman metric (i.e. the complete surface of constant negative holomorphic sectional curvature) is mirror to the half space in $\mathbb{C}^2$ with the Kähler-Berndt metric, which was originally introduced to study automorphic forms on $\mathbb{H} \times \mathbb{C}$ [3]. We are still working to understand this phenomena and this will be an active topic for future research.

2. The spectrum of drift-Laplacians

Moving away from optimal transport, another focus of my research is on spectral geometry. To motivate questions about the spectrum of the Laplacian, it is helpful to consider a hot cup of coffee in a cold room. It is a classic result that the temperature of the coffee will evolve according to the heat equation

$$\frac{\partial}{\partial t}u = \Delta u.$$

The temperature converges at an exponential rate to the temperature of its surroundings. In order to obtain quantitative estimates for how the heat dissipates, we must understand the spectrum of the Laplace operator Δ.[3] If one stirs the coffee, then the temperature instead evolves by the equation

$$\frac{\partial}{\partial t}u = \Delta u + v(\nabla u),$$

[3]This is closely related to the famous question in geometric analysis "Can you hear the shape of a drum?" [16]

where v is some one-form determined by the convection of the liquid. In order to understand how the temperature evolves, the task now becomes to understand the "spectrum" of the operator $\Delta + v(\nabla)$.

Unless the drift v is given by the gradient of a potential, the operator $\Delta + v(\nabla)$ is non-self-adjoint, so its spectrum may be fairly complicated. Nonetheless, on a bounded domain there is a real-valued principle eigenvalue [2], which essentially controls the global behavior of the temperature. On a Riemannian manifold, one can study the same question (where Δ is instead the Laplace-Beltrami operator) and to try to understand the principle eigenvalue with minimal regularity assumptions for the drift term (see [17] for my first paper on the topic). When the fluid is incompressible (i.e. the drift is divergence-free), it can be shown that stirring speeds up the cooling. In general, the drift can actually act to slow the diffusion, but only by so much. To be more precise, my work establishes a lower bound on the principle eigenvalue when the drift is only assumed to be L^∞.

Theorem 4 (K. [20])**.** *Suppose Ω is an open domain of M. Let v be some one form satisfying $\|v\|_{L^\infty} < C$. Suppose that there exists λ real and $u \in W^{2,p}(\Omega)$ satisfying*

$$\begin{cases} \Delta u + v(\nabla u) = \lambda u & x \in \Omega \\ u(x) \equiv 0 & x \in \partial\Omega \end{cases}$$

Then $\log(\lambda) > L$, *where L depends only on $\|Ric_M\|$, n, diam(M), inj(M), and C.*

From the perspective of convection-diffusion equations, this result provides quantitative lower bounds on the exponential convergence to thermal equilibrium. To explain how this improves on existing work, it is worth considering the self-adjoint case, in which $v = df$ for some potential f and $\Delta + v(\nabla)$ is a Witten-Laplacian. In this case, Theorem 4 provides a lower bound on the spectrum using bounds on the Ricci tensor and a Lipschitz estimate on f. Of note is that no assumptions are made on the Bakry-Emery tensor,[4] which is typically needed for spectral estimates.

The proof of Theorem 4 is inspired by a Faber-Krahn inequality for drift-Laplacians in Euclidean domains proven by Hamel, Nadirashvili, and Russ [15]. Crucially, their result uses a geometric argument that only requires L^∞ control on the drift. By adapting this for Riemannian manifolds, the problem reduces to studying the semi-linear elliptic problem

$$\begin{cases} \Delta u - C|\nabla u| = \lambda u & x \in \Omega \\ u(x) \equiv 0 & x \in \partial\Omega \end{cases}.$$

Using this observation and several results on elliptic operators and geometry (e.g. [2], [13]), I was able to modify the Li-Yau estimate [31] to obtain lower bounds on the principle eigenvalue.

In future work, I would like to consider more singular drifts. For L^∞-drifts, by minimizing the eigenvalue, the corresponding eigenfunction satisfied a semi-linear equation. It would be of interest to determine if this also occurs with less initial regularity on the drift terms. Furthermore, it seems likely that this result can be refined to eliminate the upper bound on the Ricci tensor and bound on the injectivity radius. These are necessary for technical steps in the proof, but do not seem necessary to control the spectrum.

3. Geometric Probability and Random Triangles

In a third area of my research, I study the statistics of random geometric objects. In particular, I try to find extremal configurations which maximize or minimize the expected value of a random variables induced from the geometry. Oftentimes, the extremal configurations themselves are quite simple, but proving a given configuration is extremal unveils deep and insightful mathematics.

[4] This roughly corresponds to lower bounds on the Ricci tensor and a C^2 estimate on f.

For example, the following problem initially seemed like an elementary question about triangles. However, as I worked on it, I found connections to general isoperimetric inequalities.

Conjecture (Hall 1982 [14])**.** *Let $S \subset \mathbb{R}^n$ be a bounded convex domain. Consider A_S the event that three points (chosen i.i.d. uniformly from S) form the vertices of an acute triangle. Then the probability of A_S (denoted $P(A_S)$) is maximized when S is the n-ball.*

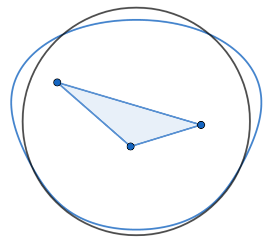

FIGURE 2: A random triangle in a deformed disk

In other words, this question asks us to find the shape of a dartboard for which three randomly thrown darts are most likely to form an acute triangle. In [18], I was able establish the following partial results.

Theorem 5 (K.)**.**

(1) *In dimensions 2 and 3, the ball is a weak local maximum for $P(A_S)$ as a function of S. In other words, it is a local maximum in the C^1 topology (after accounting for similarities).*
(2) *In dimension 2, the disk is a strong local maximum for the probability. More precisely, if S is a unit-area convex domain with $0 < d_{Gromov-Hausdorff}(S, Disk) \ll 1$, then $P(A_S) < P(A_{Disk})$.*
(3) *There is an explicit bound on the isoperimetric ratio of any region which maximizes $P(A_S)$.*

To prove that the disk is a weak local maximum, I used a variational approach. The first variation vanishes, so what was left to show is that the second variation is strongly negative. To show this, I considered the variation in terms of time-dependent measure on the boundary ∂S and decomposed it using Fourier analysis. In three-dimensions, the same method worked but it was necessary to use representation theory on $SO(3)$ to generalize the Fourier analysis. Specifically, it required representation theoretic versions of Plancherel's identity and the auto-correlation theorem. It also required the use of Mathematica to compute the relevant integrals.

To prove that the disk is a local maximum in the C^0-topology, several further ingredients were needed. Firstly, it was necessary to construct a "nice" homotopy from S to the disk. Doing so requires some algebraic topology, in particular that a degree-one self map of a sphere has no retraction. After doing so, it was possible to carefully analyze the geometry to derive a $C^{2,1/2}$ estimate for the probability along this homotopy.

In future work, I would like to find a full proof of Hall's conjecture as well as extend Theorem 5 to higher dimensions. It is worth noting that in dimension two, my results reduce the full conjecture to a finite (though currently intractable) computation.

3.1. **Probabilistic Isoperimetric Inequalities.** At first glance, it might seem surprising that Hall's conjecture requires such sophisticated machinery. However, this problem can be understood as a probabilistic isoperimetric inequality, which helps put it in a broader context. For instance, the following well known inequalities can also be interpreted as probabilistic isoperimetric inequalities.

(1) **The Faber-Krahn Inequality:** By interpreting the principal eigenvalue of a domain in terms of the exit rates of Brownian motion, the Faber-Krahn inequality (see [7] for a sharp

quantitative version) shows that the ball maximizes the expected value of the log of the exit time for Brownian motion.

(2) **Blaschke's Theorem:** In affine geometry, Blaschke [5] proved that the disk minimizes the probability that four points (chosen uniformly at random from a convex region) have a quadrilateral as their convex hull.

Both the Faber-Krahn inequality and Blaschke's theorem can be proven using rearrangement inequalities because they rely on the observation that the mass of a disk concentrates towards its center. In comparison, Hall's conjecture captures roundness in a completely different way and rearrangement techniques do not seem relevant. There are countless variants to Hall's conjecture which can be understood as isoperimetric inequalities. This is an area of research I plan to explore in the future and would be a natural opportunity to collaborate with students.

3.2. **The Curvature of Random Polygons.** The insights from Hall's conjecture can be used to tackle completely different problems in geometric probability. For instance, random polygons in $\mathbb{R}^3$ with fixed edge lengths (henceforth *ideal polygons*) have been widely studied as a simple geometry model for the folding of polymer chains. Because of this connection with materials science, it is of considerable interest to understand the statistical properties of the geometry and topology of such objects. In applications, one important goal is to understand polymers that are tightly confined. A strand of DNA would be several meters long if stretched out, but is confined entirely to the nucleus of a cell. Intuitively, this confinement tangles the polymer, and numerical simulation of confined ideal polygons bears this out. As a quantitative version of "tangling," the expected curvature (i.e. the sum of the angles between edges) of ideal polygons tends to increase as the confinement diameter shrinks.

Theorem 6 (K. [21]). *Confinement increases the expected curvature of ideal quadrilaterals with equal side lengths.*

To prove this result, it was necessary to use the fact that the moduli space of ideal polygons is an (almost toric) symplectic manifold (proven by Kapovich and Millson [26]). By working in Darboux coordinates and using various probabilistic tools, I was able to show the desired monotonicity.

4. Other work

Apart from these three topics, I've worked on several other projects, some of which I'll mention briefly.

4.1. **Non-Kähler Complex Geometry.** In complex geometry, one question of interest is how Riemannian geometry interacts with complex structures. While a Riemannian metric and complex structure are different objects, they interact in subtle ways.

In joint work with Fangyang Zheng and Bo Yang, we studied complex structures which are compatible with a compact Riemannian-flat manifold [23]. For complex three-folds[5], we classify all such complex manifolds. In summary, this shows all of the non-Kähler complex structures on a flat six-dimensional torus are warped products, which are in correspondence with a particular class of elliptic functions.

Theorem 7 (K.-Yang-Zheng [23]). *Let (M^3, g) be a compact Hermitian manifold whose Riemannian curvature tensor is identically zero. Then a finite unbranched cover of M is holomorphically isometric to either a flat complex torus or one of the twisted complex structures studied[6] by Borisov, Salamon and Viaclovsky* [6].

[5]For this problem, three-folds are the first interesting case, as all Riemannian-flat curves and surfaces are simply complex tori.

[6]These spaces were independently discovered by Blanchard [4] and Sommese [32].

In my thesis, I also study a class of complex structures known as k-Gauduchon complex structures (introduced in [10]). Whereas the moduli space of complex structures can be very large (as in the Theorem 7), the moduli of k-Gauduchon complex structures is precompact for certain k. This mirrors the situation for Kähler complex structures, which are compact in the space of orthogonal complex structures.

Theorem 8 (K. [19])**.** *Let (M^n, g) be a compact Riemannian manifold. Suppose that g admits a sequence of complex structure J_i such that the Kähler form is k-Gauduchon for some k with $\frac{n}{2} < k \leq n-1$. Then there exists a subsequence of J_i converging uniformly to a complex structure J compatible with g.*

4.2. **Number Theory.** A lattice point $(u_1, u_2) \in \mathbb{Z}^2$ is said to be visible (from the origin) if $\gcd(u_1, u_2) = 1$ (i.e., the segment from the origin to (u_1, u_2) does not pass through any other lattice point. It is a classic result in number theory that the proportion of visible lattice points in the square $[-r, r]^2$ (denoted N(r)), satisfies the asymptotic $\lim_{r\to\infty} \frac{N(r)}{r^2} = \frac{6}{\pi^2}$. Similar results hold for large convex regions which are not too "thin" [1].

In recent work [22] (joint with Mizan Khan, Joydip Saha, and Peng Zhao), we study the number of visible lattice points within "thin" parallelograms whose edges contain no lattice points. Most of the time, the proportion of visible lattice points is very close to $\frac{6}{\pi^2}$, but some parallelograms have many more or fewer visible points. My main contribution to this project is to find a probabilistic heuristic to explain this phenomena and to characterize which parallelograms are outliers.

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