Riemann-Liouville Fractional Calculus of Blancmange Curve and Cantor Functions

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Outline

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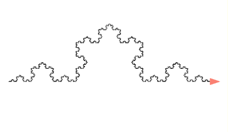
What are Fractals?

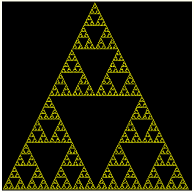
- Fractals occupy important place while studying natural objects structures like leafs, mountains...
- [Mandelbrot B.B., 1977] - coined the term **Fractals**
- He showed many objects in nature were not collection of just smooth components.
- what we see in nature is repetition of patterns at all scales - characteristic of ‘Fractals’.

Examples

- Cantor Sets
- Von-Koch Curve



- Sierpinski Triangle



Outline

Hausdorff Space

- $(\mathcal{X}, d_{\mathcal{X}})$ is a metric space, where $\mathcal{X} \subset \mathbb{R}^n$.
- $\mathcal{H}(\mathcal{X})$ is the class of compact sets of $\mathcal{X}$
- The *Hausdorff distance* $d_{\mathcal{H}}(A, B)$ between two sets A and B in $\mathcal{H}(\mathcal{X})$ is defined by

$$d_{\mathcal{H}}(A, B) = \max(d_{\mathcal{X}}(A, B), d_{\mathcal{X}}(B, A)),$$

where $d_{\mathcal{X}}(A, B) = \max_{x \in A} \min_{y \in B} d_{\mathcal{X}}(x, y)$

Proposition ([Federer H., 1969])

If $(\mathcal{X}, d_{\mathcal{X}})$ is a complete metric space, then $(\mathcal{H}(\mathcal{X}), d_{\mathcal{H}})$ is also a complete metric space and if $\{B_n^*\}_{n=1}^{\infty} \subseteq \mathcal{H}(\mathcal{X})$ is a Cauchy sequence, then $B^* \equiv \lim_{n\to\infty} B_n^* \in \mathcal{H}(\mathcal{X})$ is given by $B^* = \{x \in \mathcal{X}:$ there is a Cauchy sequence $\{x_n : x_n \in B_n^*\}$ that converges to $x\}$.

Hyperbolic Iterated Function System

Definition

Let $\omega_n : \mathcal{X} \to \mathcal{X},\ n = 1, 2, \dots N$ be a finite collection of maps. Then, $\{\mathcal{X}; \omega_n,\ n = 1, 2, \dots, N\}$ is called an *Iterated Function System* (IFS).

Definition

If the maps $\omega_n : \mathcal{X} \to \mathcal{X},\ n = 1, 2, \dots, N$ are *contraction maps*, i.e. $d_{\mathcal{X}}(\omega_n(x), \omega_n(y)) \leq \gamma_n\, d_{\mathcal{X}}(x, y)$, $0 \leq \gamma_n < 1$. Then, the IFS is called hyperbolic IFS. The number $\gamma = \max\{\gamma_n : n = 1, 2, \dots, N\}$ is called *contractivity factor* of the IFS.

Hutchinson Map

- **Hutchinson Map**:

$$W(B) = \bigcup_{n=1}^{N} \omega_n(B), \text{ where } \omega_n(B) = \{\omega_n(x) : \text{ for all } x \in B\}. \qquad (1)$$

- [Hutchinson J.E., 1981] : W is a contraction map on $\mathcal{H}(\mathcal{X})$

- Banach Fixed Point Theorem : $W(A) = A$

Attractor

Definition

Let $\{\mathcal{X}; \omega_n,\ n = 1, 2, \ldots, N\}$ be a hyperbolic IFS. Then, the unique fixed point A of Hutchinson map W, defined by (1), is called **attractor** of the IFS.

Definition

A function $f : E \to \mathcal{X}$, $E \subseteq \mathcal{X}$, is said to be **associated with an IFS** if the graph $\{(x, f(x)) : x \in E\}$ of the function f is the attractor of the IFS.

Outline

Fractal Interpolation Function (FIF)

- Fractal Interpolation Function (FIF) : [Barnsley M.F., 1986] new tool for fitting experimental data
- traditional methods are fitting a straight line, polynomial construction, linear combination of elementary functions
- Advantage of FIF: approximates clouds,mountain ranges,trees,forest....
- **Similarities of FIF and traditional methods**
  - ∗ Geometrical Character - can be plotted on graph
  - ∗ Represented by formulas
- **Difference between FIF and traditional methods**
  - ∗ Fractal Character

Construction of a FIF

- Given interpolation data $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$
- $I = [x_0, x_N],\ I_k = [x_{k-1}, x_k]\ k = 1, 2, \ldots, N$
- $L_k : I \to I_k$

$$L_k(x_0) = a_k x + b_k = \frac{x_k - x_{k-1}}{x_N - x_0}(x - x_0) + x_{k-1} \tag{2}$$

- $F_k : I \times \mathbb{R} \to \mathbb{R}$

$$F_k(x, y) = \gamma_k y + q_k(x) \tag{3}$$

- $|\gamma_k| < 1$, $\quad q_k(x_0) = y_{k-1} - \gamma_k\, y_0, \quad q_k(x_N) = y_k - \gamma_k y_N$

Fractal Interpolation Function (FIF)

- $\omega_k : I \times \mathbb{R} \to I \times \mathbb{R}$ is given by

$$\omega_k(x,y) = (L_k(x), F_k(x,y)), \; k = 1,2,\ldots N \tag{4}$$

- Iterated Function System (IFS) given by

$$\{I \times \mathbb{R}; \omega_k, k = 1,2,\ldots N\} \tag{5}$$

  is hyperbolic with respect to a metric equivalent to Euclidean metric on $\mathbb{R}^2$ [Barnsley M.F., 1986].

- The attractor $G \subseteq \mathbb{R}^2$ such that $G = \bigcup_{k=1}^{N} \omega_k(G)$ of the above IFS is graph of a continuous function $f : I \to \mathbb{R}$ such that $f(x_i) = y_i$ for $i = 0,1,\ldots,N$ [Barnsley M.F., 1986].

Fractal Interpolation Function (FIF)

Definition

The **Fractal Interpolation Function (FIF)** for the given interpolation data $\{(x_i, y_i) : i = 0, 1, \ldots, N\}$ is defined as the continuous function $f : I \to \mathbb{R}$ whose graph is attractor of the above hyperbolic IFS.

- $\mathcal{G} = \{g : \ g : I \to \mathbb{R}$ is continuous, $g(x_0) = y_0$ and $g(x_N) = y_N\}$
- $d_{\mathcal{G}}(g, \hat{g}) = \max_{x \in I} |g(x) - \hat{g}(x)|, \ g, \hat{g} \in \mathcal{G}$
- Read-Bajraktarevic operator
  $T(g)(x) = F_n(L_n^{-1}(x), g(L_n^{-1}(x))), \ x \in I_n$.

Fractal Interpolation Function (FIF)

The FIF f satisfies the recursive equation

$$f(L_k(x)) = \gamma_k f(x) + q_k(x) \quad x \in I \text{ for } k = 1, 2, \ldots, N. \tag{6}$$

Definition

The **Fractal Interpolation Function (FIF)** for the interpolation data $\{(x_i, y_i) : i = 0, 1, \ldots, N\}$ is defined as the continuous function $f : I \to \mathbb{R}$ which satisfies the recursive equation given by (6).

Outline

Riemann-Liouville fractional integral

Definition

Let $-\infty < a < x < b < \infty$. The Riemann-Liouville fractional integral of order $\nu > 0$ with lower limit a is defined for locally integrable functions $f : [a, b] \to \mathbb{R}$ as

$$I_{a+}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x - t)^{\nu-1} f(t) dt \tag{7}$$

for $x > a$. For $\nu = 0$, it is defined that $I_{a+}^{\nu} f(x) = f(x)$.

Riemann-Liouville fractional integral of FIF

Proposition ([Prasad S.A.,2017])

Let f be a FIF passing through the interpolation data given by $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$ constructed using the IFS given by (5). Define

$$q_n^\nu(x) = a_n^\nu \ I_{x_0}^\nu \ q_n(x) + \frac{1}{\Gamma(\nu)} \int_{x_0}^{x_{n-1}} (L_n(x) - t)^{\nu-1} f(t) dt. \tag{8}$$

Then, Riemann-Liouville fractional integral of a FIF of order ν is also a FIF passing through the data $\{(x_i, y_i^\nu) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$, where $y_0^\nu = 0, \ y_N^\nu = \frac{q_N^\nu(x_N)}{1 - a_N^\nu \gamma_N}$, and $y_j^\nu = a_j^\nu \gamma_j y_N^\nu + q_j^\nu(x_N) = q_{j+1}^\nu(x_0)$ for $j = 1, 2, \ldots, N-1$.

Riemann-Liouville fractional derivative

Definition

Let $-\infty < a < x < b < \infty$, $0 < \nu$, $f \in L_1([a,b])$, $D(I^{n-\nu})f \in L_1([a,b])$ and n is the smallest integer greater than ν. The Riemann-Liouville fractional derivative of order ν with lower limit a is defined as

$$(D_{a+}^{\nu} f)(x) = \frac{d^n}{dx^n}(I_{a+}^{n-\nu} f)(x)$$

and $(D_{a+}^{\nu} f)(x) = f(x)$ when $\nu = 0$.

Riemann-Liouville fractional derivative of FIF

Proposition ([Prasad S.A.,2017])

Let f be a FIF passing through the interpolation data given by $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$ and constructed using the IFS given by (5). *Suppose $\gamma_k < a_k^\nu$ for some fixed $\nu > 0$. Then Riemann-Liouville fractional derivative of a FIF of order ν is also a FIF provided*

$$q_k^{d\nu}(x) = a_k^{-\nu} D^\nu q_k(x) + \frac{1}{\Gamma(n-\nu)} \frac{d^n}{d(L_k(x))^n} \left[\int_{x_0}^{x_{k-1}} f(t)(L_k(x) - t)^{n-\nu-1}\, dt \right]$$

$$= a_k^{-\nu} D^\nu q_k(x) + \frac{1}{\Gamma(n-\nu)} \left(\prod_{j=1}^{n} (n-j-\nu) \right) \left[\int_{x_0}^{x_{k-1}} f(t)(L_k(x) - t)^{-\nu-1}\, dt \right] \quad (9)$$

Outline

Blancmange Curve

$$B(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n} \quad x \in [0,1],$$

where, $s(y) = \min_{m \in \mathbb{Z}} |y - m|, y \in \mathbb{R}$.

$B\left(\frac{x+k-1}{2}\right) = \frac{1}{2}B(x) + \frac{k-1+(-1)^{k-1}x}{2} \quad x \in [0,1]$ for $k = 1, 2$.

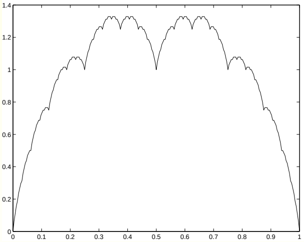


Figure: **Blanchmange Curve**

Blancmange Curve

- $\gamma_k = \frac{1}{2}$ for $k = 1, 2$.
- $L_k(x) = \frac{1}{2}x + \frac{k-1}{2}$
- $q_k(x) = \frac{k-1+(-1)^{k-1}x}{2}$

Theorem

Blancmange Curve is a Fractal Interpolation Function on $[0, 1]$.

Blancmange Curve

Theorem (1)

Riemann-Liouville fractional integral of order ν of Blancmange curve defined on $[0,1]$ is a FIF passing through the data $\{(x_i, y_i^\nu) \in \mathbb{R}^2 : i = 0,1,2\}$, where $y_0^\nu = 0$, $y_1^\nu = \frac{\nu+2}{2^{\nu+1}\Gamma(\nu+2)}$ and $y_2^\nu = \frac{1}{\Gamma(\nu+1)}$.

Theorem (2)

Riemann-Liouville fractional derivative of Blancmange curve defined on $[0,1]$ is not a FIF for any ν.

Sketch of Proof of Theorem 2:

- Since $a_k = \gamma_k = \frac{1}{2}$, the value of ν can lie only between 0 and 1.
- For $0 < \nu < 1$,

$$D^\nu q_k(x) = \frac{d}{dx}\left[\frac{1}{\Gamma(1-\nu)}\int_0^x (x-t)^{-\nu} q_k(t)dt\right]$$

$$= \frac{(-\nu)}{\Gamma(1-\nu)}\int_0^x (x-t)^{-\nu-1}\left[\frac{k-1+(-1)^{k-1}t}{2}\right]dt$$

$$= \frac{1}{2\Gamma(1-\nu)}\left\{(k-1)x^{-\nu} + (-1)^{k-1}\frac{x^{1-\nu}}{(1-\nu)}\right\}$$

- As $x \to 0$, the function $D^\nu q_2(x)$ goes to ∞.

Outline

Cantor Function

$$C(x) = \lim_{n\to\infty} f_n(x) \quad x \in [0,1],$$

where, $f_0(x) = x$, $f_{n+1}(x) = \begin{cases} 0.5 * f_n(3x) & 0 \le x < \frac{1}{3} \\ 0.5 & \frac{1}{3} \le x < \frac{2}{3} \\ 0.5 + 0.5 * f_n(3x-2) & \frac{2}{3} \le x \le 1 \end{cases}$

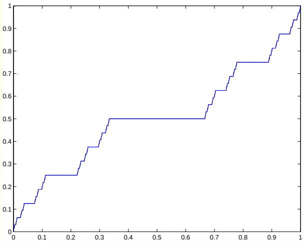


Figure: **Cantor Function**

Cantor Function

- $\Delta = \{(0,0), (\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2}), (1,0)\}$
- $L_k(x) = \frac{x+k-1}{3}$ for $k = 1, 2, 3$.
- $\gamma_1 = \gamma_3 = \frac{1}{2}$ and $\gamma_2 = 0$.
- $q_1(x) = 0$ and $q_2(x) = q_3(x) = \frac{1}{2}$

Theorem

Cantor function is a Fractal Interpolation Function on $[0,1]$.

Cantor Function

Theorem (1)

Riemann-Liouville fractional integral of Cantor function of order ν is a FIF passing through the data $\{(x_i, y_i^\nu) \in \mathbb{R}^2 : i = 0, 1, 2, 3\}$, where $y_0^\nu = 0$, $y_1^\nu = \frac{1}{2.(2.3^\nu - 1)\Gamma(\nu+1)}$, $y_2^\nu = \frac{2^{\nu-1}}{3^\nu \Gamma(\nu+1)}$ and $y_3^\nu = \frac{3^\nu}{(2.3^\nu - 1)\Gamma(\nu+1)}$.

Theorem (2)

Riemann-Liouville fractional derivative of Cantor function defined on $[0, 1]$ is not a FIF for any value of ν.

Cantor Function

Sketch of Proof of Theorem 2:

- Since $a_k = \frac{1}{3}$ for $k = 1, 2, 3$ and $\gamma_1 = \gamma_3 = \frac{1}{2}$, $\gamma_2 = 0$, the value of ν can lie only between 0 and $\frac{\log 2}{\log 3}$.
- For $0 < \nu < \frac{\log 2}{\log 3} < 1$,

$$D^{\nu} q_k(x) = \frac{(-\nu)}{\Gamma(1-\nu)} \int_0^x (x-t)^{-\nu-1} q_k(t) dt$$

- $D^{\nu} q_1(x) = 0 \quad D^{\nu} q_2(x) = \frac{x^{(-\nu)}}{2\Gamma(1-\nu)} = D^{\nu} q_3(x)$
- As $x \to 0$, the function $D^{\nu} q_2(x)$ and $D^{\nu} q_3(x)$ goes to ∞.

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Thank You!