Graph Laplace and Markov operators on a measure space

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Outline

- Motivation
- Weighted (electrical) networks
- Discrete Laplace and Markov operators
- Discrete vs measurable networks
- Symmetric measures and reversible Markov processes
- Graph Laplace operator in $L^2(\mu)$- and energy spaces
- Harmonic functions

Weighted Networks

$G = (V, E) =$ a countably infinite locally finite connected graph
$V =$ the vertex set
$E =$ the edge set (with no loops)
$E(x) := \{y \in V : y \sim x\} =$ all neighbors of $x \in V$, $|E(x)| < \infty$

Definition

A weighted (electrical) network (G, c) is a graph G with a symmetric conductance function $c : V \times V \to [0, \infty)$, i.e., $c_{xy} = c_{yx}$ for any edge $(xy) \in E$. Moreover, $c_{xy} > 0$ if and only if $(xy) \in E$, and

$$c(x) := \sum_{y \sim x} c_{xy}$$

is called the total conductance at x.

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Hilbert spaces and operators related to networks

For a weighted network $(G, c) = (V, E, c)$, associate the Hilbert spaces:

$$l^2(V) := \{u : V \to \mathbb{R} : ||u||_{l^2}^2 = \sum_{x \in V} u(x)^2 < \infty\},$$

$$l^2(V, c) := \{u : V \to \mathbb{R} : ||u||_{l^2(V,c)}^2 = \sum_{x \in V} c(x)u(x)^2 < \infty\},$$

$\mathcal{H}_E$ = equivalence classes of functions on V ($u_1 \sim u_2$ if $u_1 - u_2 = \text{constant}$)

$$||u||_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{(xy) \in E} c_{xy}(u(x) - u(y))^2 < \infty$$

The Hilbert space $\mathcal{H}_E$ is called the finite energy space

Hilbert spaces and operators (cont'd)

Given (G, c), define the Markov kernel $P = (p(x, y))_{x,y \in V}$ where $p(x, y) = \frac{c_{xy}}{c(x)}$. Let P be the Markov operator

$$P(f)(x) = \sum_{y \sim x} p(x, y) f(y), \quad x \in V.$$

For any $x, y \in V$, the Markov process defined by P is reversible, i.e, $p(x, y)c(x) = p(y, x)c(y)$.

Definition

The Laplacian on (G, c):

$$(\Delta f)(x) := \sum_{y \sim x} c_{xy}(f(x) - f(y)), \quad f : V \to \mathbb{R}.$$

$f : V \to \mathbb{R}$ is called harmonic if $\Delta f(x) = 0$ for every $x \in V$ (equivalently, $Pf = f$).

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Properties of Δ and P for discrete networks

(1) Given (G, c),

$$\Delta(f)(x) = c(x)(f(x) - P(f)(x))$$

or $\Delta = c(I - P)$;

(2) Δ is an **essentially self-adjoint, positive definite**, generally **unbounded operator** with **dense domain** in $l^2(V)$;

(3) Δ is an **unbounded, positive definite, closed, and symmetric operator with dense domain** in $\mathcal{H}_E$; in general, Δ is not a self-adjoint operator in $\mathcal{H}_E$;

(4) P is a **positive normalized** ($P(\mathbb{1}) = \mathbb{1}$) operator which is **bounded and self-adjoint** on $l^2(V, c)$, and its spectrum is in $[-1, 1]$.

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Discrete vs measurable settings

(1) **Space**

$$(V, |\cdot|) \rightsquigarrow (V, \mathcal{B}, \mu)$$

where $|\cdot|$ is the counting measure, μ is σ-finite.

(2) **Network**

$$G = (V, E, c) \rightsquigarrow (V \times V, \mathcal{B} \times \mathcal{B}, \rho)$$

where ρ is a σ-finite measures on a Borel set $E_\rho \subset V \times V$ and $\rho = \int_V \rho_x d\mu(x)$.

(3) **Symmetry**

$$c : E \to \mathbb{R},\ c_{xy} = c_{yx} \rightsquigarrow \rho(A \times B) = \rho(B \times A)$$

(4) **Local finiteness**

$$c(x) = \sum_{y \sim x} c_{xy} < \infty \rightsquigarrow c(x) = \rho_x(V) \in L^1_{\text{loc}}(\mu)$$

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Discrete vs measurable settings (cont'd)

(5) **Laplacian**

$$\Delta(f)(x) = \sum_{y \sim x} c_{xy}(f(x) - f(y)) \quad \rightsquigarrow \quad \Delta(f)(x) = \int_V (f(x) - f(y))\, d\rho_x(y).$$

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$$p(x,y) = \frac{1}{c(x)} c_{xy} \quad \rightsquigarrow \quad P(x,A) = \int_V \chi_A(y) \frac{1}{c(x)}\, d\rho_x(y),$$

$$P(x,A) = P(\chi_A), \quad A \in \mathcal{B}.$$

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Discrete vs measurable settings (cont'd)

(8) L^2**-Hilbert spaces**

$$l^2(V),\ l^2(V,c) \ \rightsquigarrow\ L^2(\mu),\ L^2(c\mu) = L^2(\nu).$$

(9) **Energy space**

$$||f||^2_{\mathcal{H}_E} = \frac{1}{2}\sum_{x,y} c_{xy}(f(x) - f(y))^2 \ \rightsquigarrow\ ||f||^2_{\mathcal{H}_E} = \frac{1}{2}\iint_{V\times V}(f(x) - f(y))^2\, d\rho(x,y)$$

(10) **Finitely supported functions**

$$\langle \delta_x, f\rangle_{\mathcal{H}_E} = \Delta f(x) \ \rightsquigarrow\ \langle \chi_A, f\rangle_{\mathcal{H}_E} = \int_A \Delta f\, d\mu,\ \mu(A) < \infty.$$

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Useful facts

(1) Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space and ρ a σ-finite measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $\rho \circ \pi_1^{-1} \ll \mu$. Then there exists a **unique system of conditional σ-finite measures** (ρ_x) such that

$$\rho(f) = \int_V \rho_x(f)\, d\mu(x).$$

(2) Every symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ defines **transition probabilities** (Markov kernel)

$$P(x, dy) := \frac{1}{c(x)} d\rho_x(y).$$

(3) For a symmetric measure ρ and any Borel function f on $V \times V$

$$\iint_{V\times V} f(x,y)\, d\rho(x,y) = \iint_{V\times V} f(y,x)\, d\rho(x,y).$$

(4) $x \mapsto P(x, \cdot)$ is called **reversible** if $\forall A, B \in \mathcal{B}$

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Revirsible Markov process

Theorem

The following are equivalent:
(i) $x \mapsto P(x,\cdot)$ *is* ***reversible;***
(ii) the Markov operator P *defined by* $x \to P(x,\cdot)$ *is* ***self-adjoint*** *on* $L^2(\nu)$ *and* $\nu P = \nu$ *where* $d\nu(x) = c(x)d\mu(x)$*;*
(iii)

$$c(x)P(x,dy)d\mu(x) = c(y)P(y,dx)d\mu(y);$$

(iv) the operator R *defined by the relation* $R(f)(x) = c(x)P(f)(x)$ *is* ***symmetric:***

$$\int_V fR(g)\,d\mu = \int_V R(f)g\,d\mu;$$

(v) the measure ρ *on* $(V \times V, \mathcal{B} \times \mathcal{B})$ *defined by*

$$\rho(A \times B) = \int_V \chi_A R(\chi_B)\,d\mu = \langle \chi_A, P(\chi_B)\rangle_{L^2(\nu)}$$

is ***symmetric.***

Spectral properties of Δ and P

Theorem

*(1) The Markov operator $P : L^2(\nu) \to L^2(\nu)$ is **self-adjoint**. Moreover, $\nu P = \nu$ where $d\nu(x) = c(x)d\mu(x)$.*
*(2) The Markov operator P considered in the spaces $L^2(\nu)$ and $L^1(\nu)$ is **contractive**, i.e.,*

$$||P(f)||_{L^2(\nu)} \leq ||f||_{L^2(\nu)}, \qquad ||P(f)||_{L^1(\nu)} \leq ||f||_{L^1(\nu)}.$$

*(3) **Spectrum** of P in $L^2(\nu)$ is a subset of $[-1, 1]$.*
*(4) The graph Laplace operator Δ is a **positive definite essentially self-adjoint operator** acting in $L^2(\mu)$.*
*(5) The graph Laplace operator $\Delta : \mathcal{H}_E \to \mathcal{H}_E$ is **positive definite and symmetric**, but not self-adjoint, in general.*

Functions from the energy space

Theorem

Let $\mathcal{H}_E = \mathcal{H}_E(\rho)$ be the energy space defined by a symmetric measure ρ on $V \times V$. Then
(1)

$$\|f\|^2_{\mathcal{H}_E} = \frac{1}{2}\left(\int_V (P(f^2) - P(f)^2)\, d\nu + \|f - P(f)\|^2_{L^2(\nu)}\right).$$

(2) Suppose the functions f and $\Delta(f)$ belong also to $L^2(\mu)$. Then

$$\|f\|^2_{\mathcal{H}_E} = \int_V f\Delta(f)\, d\mu.$$

(3) For $\mathcal{D}_{\text{fin}} := \text{span}\{\chi_A : A \in \mathcal{B}, \mu(A) < \infty\}$, one has $\overline{\mathcal{D}_{\text{fin}}} \oplus \mathcal{H}arm = \mathcal{H}_E$ and

$$\int_A c(x)\, d\mu(x) = \|\chi_A\|^2_{\mathcal{H}_E} + \rho(A \times A), \quad \mu(A) < \infty.$$

Corollaries on harmonic functions

Corollary

(1) If $f \in \mathcal{H}_E$, then $f - P(f) \in L^2(\nu)$ and $P(f^2) - P(f)^2 \in L^1(\nu)$. The operator

$$I - P : f \mapsto f - P(f) : \mathcal{H}_E \to L^2(\nu)$$

*is **contractive**, i.e., $\|I - P\|_{\mathcal{H}_E \to L^2(\nu)} \leq 1$.*
(2)

$$\|f\|_{\mathcal{H}_E} = 0 \iff \begin{cases} P(f^2) = P(f)^2 \\ P(f) = f \end{cases} \qquad \nu - \text{a.e.}$$

$$\iff \text{both } f \text{ and } f^2 \text{ are harmonic functions.}$$

(3)

$$f \in \mathcal{H}arm_E \iff \|f\|^2_{\mathcal{H}_E} = \frac{1}{2} \int_V (P(f^2)(x) - (Pf)^2(x)) \, d\nu(x)$$

Harmonic functions

Theorem

Suppose P is a Markov operator on $L^2(\nu)$ such that $\nu P = \nu$. Then

$$L^2(\nu) \cap \mathcal{H}arm(P) = \begin{cases} 0, & \nu(V) = \infty \\ \mathbb{R}\mathbb{1}, & \nu(V) < \infty. \end{cases}$$

Theorem

Assume that, for every $A \in \mathcal{B}$, the ***Green's function***

$$G(x, A) := \sum_{n \in \mathbb{N}_0} P_n(x, A)$$

is finite a.e. Then every function $f \in \mathcal{H}_E$ admits a unique decomposition $f = G(\varphi) + h$ where $\varphi \in L^2(\nu)$ and $h \in \mathcal{H}arm_E$, and

$$\|f\|^2_{\mathcal{H}_E} = \frac{1}{2}\left(\|\varphi\|^2_{L^2(\nu)} + \int_V (P(h^2) - h^2)\, d\nu\right).$$

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Thank you!