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Fractal measures and polynomial sampling: I.F.S.–Gaussian integration

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Abstract Measures generated by Iterated Function Systems can be used in place of atomic measures in Gaussian integration. A stable algorithm for the numerical solution of the related approximation problem – an inverse problem in fractal construction – is proposed.

Keywords Iterated function systems · Inverse problems · Orthogonal polynomials · Jacobi matrix · Gaussian integration

Mathematics Subject Classifications (2000) 65D32 · 47B36 · 28A80 · 60B10

1 Introduction

Constructing the Jacobi matrix of a system of orthogonal polynomials leads naturally to an approximation of the orthogonality measure by a sum of atomic measures. Despite the fact that the numerical computation of the Jacobi matrix may require considerable theoretical and numerical effort [7] and despite the marvelous convergence properties of Gaussian integration, this approximation is rather crude, especially when the sampled measure is singular continuous

Dedicated to Walter Gautschi.

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[12]. This limitation can be overcome by finding systems of iterated functions with probabilities (I.F.S.) [1, 10] and the associated invariant measures so to integrate exactly polynomials of prescribed order: this defines a generalized I.F.S.–Gaussian integration that will be described in this paper.

I adopt a variant of this approximation problem – working with I.F.S. made of affine maps with equal contraction ratios [2, 4, 9] for which I present here, for the first time to the best of my knowledge, an algorithm that is numerically stable. The theory is accompanied with illustrative examples.

2 Measures, iterated function systems and orthogonal polynomials

Let μ be a normalized positive Borel measure on the unit interval: $\mu \in \mathcal{M}_1([0,1])$. The set of orthogonal polynomials of μ, $\{p_n(\mu;s)\}_{n\in\mathbb{N}}$, is defined by:

$$\int d\mu(s)\ p_n(\mu;s)\ p_m(\mu;s) = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecher's delta. They fulfill a three-terms relation:

$$s\ p_n(\mu;s) = a_n\ p_n(\mu;s) + b_{n+1}\ p_{n+1}(\mu;s) + b_n\ p_{n-1}(\mu;s). \tag{1}$$

The two infinite sequences of coefficients a_n and $b_n \geq 0$ uniquely encode the measure μ, in force of Favard's theorem. The sequence of b_ns is composed of non-negative entries: if a particular n exists such that $b_n = 0$, the sequence terminates at that value and the corresponding measure is discrete, composed of a finite number of atoms. This case is of particular relevance for Gaussian integration. In fact, let the a_ns be the diagonal and b_ns the leading sub-diagonal of the symmetric Jacobi matrix J_μ,

$$J_\mu = \begin{pmatrix} a_0 & b_1 & & & \\ b_1 & a_1 & b_2 & & \\ & b_2 & a_2 & b_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

Let now J_μ^n be the principal minor of rank n of J_μ, that can also be obtained setting $b_n = 0$. It is well known [8] that Gaussian points are the eigenvalues $\{x_k\}_{k=1}^n$ of J_μ^n: $J_\mu^n \mathbf{v}_k = x_k \mathbf{v}_k$ and Gaussian weights, $\{w_k\}_{k=1}^n$, are the squares of the first components of the normalized eigenvectors $\{\mathbf{v}_k\}_{k=1}^n$. Gaussian integration can be seen as an approximation of the measure μ by the discrete, n-point measures μ^n:

$$\mu^n := \sum_{k=1}^{n} w_k \delta_{x_k}, \tag{3}$$

where δ_x is the atomic measure at x. Despite the remarkable convergence properties of Gaussian integration, the approximation obtained from (3) is rather poor, when a characterization of the multifractal properties [16] of the

sampled measure μ is sought. We then select a different family of measures as approximation tool: the balanced measures of Iterated Function Systems [1, 10]. In its simplest form, an I.F.S. is a finite collection of affine, contractive maps of the unit interval into itself:

$$\phi_j(s) = \delta_j(s - \beta_j) + \beta_j, \quad j = 1, \ldots, M. \tag{4}$$

The parameters $\{\delta_j\}_{j=1}^M$, $0 \leq \delta_j < 1$ can be described as contraction ratios, since the unit interval is mapped by ϕ_j into a subinterval of length δ_j. The fixed points of the maps, $\{\beta_j\}_{j=1}^M$, are the second set of parameters specifying an I.F.S. An iterative construction of a fractal set is obtained by mapping $[0, 1]$ by the collection of I.F.S. maps into $\{\phi_j([0, 1])\}_{j=1}^M$ and more generally into

$$\{\phi_{\sigma_k} \circ \cdots \circ \phi_{\sigma_2} \circ \phi_{\sigma_1}([0, 1]),\ \sigma_n \in \{1, \ldots, M\},\ n = 1, \ldots, k.\}$$

One then takes the union of these intervals and lets k go to infinity, thereby approaching (in the Hausdorff metric) the *attractor* of the I.F.S., that is, the unique set $\mathcal{A}$ that satisfies the equation $\mathcal{A} = \bigcup_{j=1}^M \phi_j(\mathcal{A})$.

A family of measures on $\mathcal{A}$ can be constructed in terms of a third set of parameters, the probabilities $\{\pi_j\}_{j=1}^M$, $\pi_j > 0$, $\sum_j \pi_j = 1$. Let the uniform Lebesgue measure be defined on $[0, 1]$, and let us add to the iterative construction of the I.F.S. attractor just described the further specification that the Lebesgue measure on $\phi_{\sigma_k} \circ \cdots \circ \phi_{\sigma_2} \circ \phi_{\sigma_1}([0, 1])$ is multiplied by $\pi_{\sigma_k} \cdots \pi_{\sigma_2} \pi_{\sigma_1}$, and that measures add linearly over overlapping intervals. This construction defines a limit measure μ on $[0, 1]$ that satisfies, and can be equally defined via, the balance equation

$$\int f(s) d\mu(s) = \sum_{j=1}^M \pi_j \int (f \circ \phi_j)(s) d\mu(s), \tag{5}$$

required to hold for any continuous function f. Since I.F.S. measures are supported on a compact set, the associated moment problem admits a unique solution and the orthogonal polynomials $p_n(\mu; s)$ are defined for all orders.

Among all affine I.F.S. we shall consider in this paper a sub-class that is particularly convenient for developing an approximation theory: we suppose that all contraction ratios are equal: $\delta_j = \delta$, $j = 1, \ldots, M$. We shall call this a δ – homogeneous affine I.F.S. (δ-H.I.F.S.). Notice that discrete measures with finitely many atoms are a particular case of this family, that of maps with null contraction ratios: the measure in (3) can be obtained by n maps with $\delta_j = 0$, $\beta_j = x_j$ and $\pi_j = w_j$, for $j = 1$ to n.

3 An I.F.S. approximation problem

We can now introduce our basic approximation problem: *given an arbitrary measure $\mu \in \mathcal{M}_1([0, 1])$, find a δ-H.I.F.S. and its balanced measure μ', so that the Jacobi matrices of μ and μ' coincide up to size N:*

$$J_\mu^N = J_{\mu'}^N. \tag{6}$$

It is clear that (6) is equivalent to say that the first $2N$ moments of μ and μ' are equal. Indeed, moments were employed in the original formulation of this problem. Since the measure μ' integrates exactly all polynomials up to degree $2N-1$ it can be properly termed an *I.F.S.–Gaussian Approximation* of μ.

It is not surprising that this problem may have infinite solutions, or no solution at all. Two main theoretical results describe this situation. The first is Elton and Yan's theorem [4]:

Theorem 1 *Let $\delta > 0$. A solution of the problem (6) exists for all N if and only if the function $\hat{\mu}(y)/\hat{\mu}(\delta y)$ is positive definite, where $\hat{\mu}(y)$ is the Fourier transform of μ.*

A more flexible result is due to Handy and Mantica [9]:

Theorem 2 *Let μ be a measure with an infinite number of points of increase. Then, for all $N > 0$ there exists $\delta_N > 0$ such that for all $\delta \in [0, \delta_N)$ the problem (6) has a unique solution with $M = N$.*

Sketch of the proof of Theorems 1 and 2 Fix the value of δ, so that the remaining free parameters are β_j and π_j, for $j = 1$ to M, where of course we are also free to choose the number of maps, M. Let us now encode these parameters in the discrete measure σ:

$$\sigma = \sum_{j=1}^{M} \pi_j \delta_{\beta_j}. \tag{7}$$

First of all, choosing $f(s) = e^{-iys}$ in (5) it is easy to see that the Fourier transform of σ and μ are related by $\hat{\mu}(y) = \hat{\sigma}(y)\hat{\mu}(\delta y)$, whence Theorem 1 follows.

Successively, choose $f(s) = s^k$ in (5), to link the moments of μ and σ:

$$\mu_k = \sum_j \pi_j \int d\mu(s)(\phi_j(s))^k = \sum_j \pi_j (\delta s + \bar{\delta}\beta_j)^k = \sum_{l=0}^{k} B(k,l)\delta^l \mu_l \bar{\delta}^{k-l} \sigma_{k-l}, \tag{8}$$

where we have put $\bar{\delta} = 1 - \delta$ and where $B(k,l)$ is a binomial coefficient. Equations 8 form a triangular set of relations that permit to derive $\{\sigma_l\}_{l=0}^{k}$ from $\{\mu_l\}_{l=0}^{k}$ and vice versa. Therefore, if the I.F.S. parameters are known, one can easily compute the moments of σ and *en suite* those of μ.

Notice that in the inverse problem, when all moments $\{\mu_l\}_{l=0}^{\infty}$ are supposed to be known, one can in principle compute uniquely the full sequence $\{\sigma_l\}_{l=0}^{\infty}$, for any value of δ. Clearly, given an arbitrary measure μ, this sequence of values is not assured to be the sequence of moments of a positive measure σ, for otherwise one would violate Elton's theorem. Hence, Handy and Mantica considered the truncated, finite sequences $\{\sigma_l\}_{l=0}^{k}$ and imposed Hankel–Hadamard inequalities, that are necessary and sufficient to ensure the

existence of a such a measure. These determinantal inequalities are easily seen to be continuous functions of the contraction rate δ. In addition, for $\delta = 0$ all inequalities are satisfied, since in this case μ and σ coincide: this ensures the existence of the interval $[0, \delta_N)$ in Theorem 2.

The decoding problem of obtaining the map parameters as weights and locations of the atomic measures in (7) from the moments $\{\sigma_l\}_{l=0}^{2M}$ is known to be ill conditioned and the efforts of [4, 9] were bound to a very limited number of maps, as well as those of [2], who employed generalized moments, obtaining only a reduction of the coefficient in the exponential growth of the error. The problem was then basically abandoned at that point.

4 A numerically stable Stieltjes solution

The ill-conditioning of the moment solution of the inverse fractal problem described in the previous section affected another problem: the construction of the Jacobi matrix of an I.F.S. measure. Standard techniques are in fact based on classical polynomial sampling: adapting the analysis of Gautschi [6] it appeared clearly [12] that they cannot reliably sample a measure supported on a fractal set.

A different technique was needed and this was found [5, 12] see also [11] *avoiding* the perilous passage through the moments' mountain range. To further exploit the analogy with the magnificent ambience of this meeting, one could say that the circumventing path was traced in a succession of demanding, yet well conditioned mountain passes, like those encircling the Sella Group in the Dolomites. One of the key points of the technique in [12] is a simple observation: adopt the compact notation $\phi_\beta(s)$ for any of the I.F.S. maps,

$$\phi_\beta(s) := \delta s + \bar{\delta}\beta, \tag{9}$$

where, as before, $\bar{\delta} := 1 - \delta$. Then, $p_n(\mu; \phi_\beta(s))$ is a polynomial of degree n in the variable s, that can be expanded on the first n orthogonal polynomials. In this way, a *direct* technique of computing the Jacobi matrix associated with an I.F.S. measure was found.

In this section we describe a similar algorithm for the basic approximation problem introduced in Section 3, that can be restated as follows: *compute the truncated Jacobi matrix J_σ^N of the measure σ (the distribution of fixed points), taking as input the truncated Jacobi matrix J_μ^N of an arbitrary measure $\mu \in \mathcal{M}_1([0, 1])$ and the value of the real variable δ. Determine the largest value of N for which J_σ^N exists*.

Notice that the symmetrical rôle of β and s in the linear maps ϕ implies that $p_n(\mu; \phi_\beta(s))$ is a homogeneous polynomial of degree n in the variables s and β. This yields the double expansion

$$p_n(\mu; \phi_\beta(s)) = \sum_{0 \le k+r \le n} \Gamma_{k,r}^n p_k(\mu; s) p_r(\sigma; \beta) \tag{10}$$

in terms of the orthogonal polynomials of μ and of those of σ. The algorithm that we describe now is a recursive evaluation of the triangular matrices Γ^n and

of the Jacobi matrix J_σ. For simplicity of notation, in the following we shall let the indices k and r run freely, while assuming that $\Gamma^n_{k,r} = 0$ unless $0 \le k+r \le n$. Of fundamental importance are the following formulæ:

$$\phi_\beta(s)\, p_n(\mu;\phi_\beta(s)) = \sum_{k,r} \Gamma^n_{k,r}\big[\delta\, P_k(\mu;s)\, p_r(\sigma;\beta) + \bar\delta\, P_r(\sigma;\beta)\, p_k(\mu;s)\big], \tag{11}$$

where we have put

$$P_j(\cdot;s) := b_{j+1}(\cdot)\, p_{j+1}(\cdot;s) + a_j(\cdot)\, p_j(\cdot;s) + b_j(\cdot)\, p_{j-1}(\cdot;s) \tag{12}$$

and $\cdot$ can be either μ or σ. To prove (11) observe that

$$\phi_\beta(s)\, p_n(\mu;\phi_\beta(s)) = (\delta s + \bar\delta\beta) \sum_{k,r} \Gamma^n_{k,r}\, p_k(\mu;s)\, p_r(\sigma;\beta)$$

and the products $s p_k(\mu;s)$ and $\beta p_r(\sigma;\beta)$ can be dealt with using (1).

We are now ready to introduce the algorithm, that runs in a cyclic sequence of three steps, described by the Lemmas below. Initialization: $n = 0$. It requires $\Gamma^0_{0,0} = 1$, since $p_0(\mu;s) = 1 = p_0(\sigma;\beta)$ and $b_0(\mu) = b_0(\sigma) = 0$.

Lemma 1 *The matrix element $a_n(\sigma)$ can be computed in terms of Γ^n and linearly in terms of the Jacobi matrix entries $\{a_0(\mu),\dots,a_n(\mu)\}$, $\{b_1(\mu),\dots,b_n(\mu)\}$, $\{a_0(\sigma),\dots,a_{n-1}(\sigma)\}$ and $\{b_1(\sigma),\dots,b_n(\sigma)\}$.*

Proof Equation 1 implies that

$$a_n(\mu) = \int d\mu(s)\, s\, p_n(\mu;s)^2. \tag{13}$$

The balance relation (5), written in terms of the measure σ provides:

$$\begin{aligned} a_n(\mu) &= \int d\sigma(\beta) \int d\mu(s)\, \phi_\beta(s)\, [p_n(\mu;\phi_\beta(s))]^2 = \\ &= \iint d\sigma(\beta) d\mu(s) \sum_{k,r,k',r'} \Gamma^n_{k,r}\Gamma^n_{k',r'}\, p_{k'}(\mu;s)\, p_{r'}(\sigma;\beta) \times \\ &\quad \times\big[\delta\, P_k(\mu;s)\, p_r(\sigma;\beta) + \bar\delta\, P_r(\sigma;\beta)\, p_k(\mu;s)\big], \end{aligned} \tag{14}$$

where we have employed (10), (11). We can compute the integrations explicitly thanks to the orthogonality of the polynomials involved: after some manipulations, one gets

$$a_n(\mu) = \sum_{k,r} \Gamma^n_{k,r} \left(\delta[a_k(\mu)\Gamma^n_{k,r} + 2b_k(\mu)\Gamma^n_{k-1,r}] + \bar\delta[a_r(\sigma)\Gamma^n_{k,r} + 2b_r(\sigma)\Gamma^n_{k,r-1}]\right). \tag{15}$$

Notice that the r.h.s. is a linear combination of the entries of the Jacobi matrices J_μ and J_σ and that the highest index for both k and r is n. Therefore, the term $a_n(\sigma)$ can be easily determined: its linear coefficient is $\bar\delta(\Gamma^n_{0,n})^2$ and this is larger than zero whenever $b_n(\sigma) > 0$ (see (16) below). □

Lemma 2 *The matrix Γ^{n+1} can be computed from Γ^n, Γ^{n-1} and the Jacobi matrix entries $\{a_0(\mu), \ldots, a_n(\mu)\}$, $\{b_1(\mu), \ldots, b_{n+1}(\mu)\}$, $\{a_0(\sigma), \ldots, a_n(\sigma)\}$ and $\{b_1(\sigma), \ldots, b_{n+1}(\sigma)\}$. In addition, the term $b_{n+1}(\sigma)$ only affects the computation of*

$$\Gamma^{n+1}_{0,n+1} = \bar{\delta}\Gamma^n_{0,n} b_{n+1}(\sigma)/b_{n+1}(\mu). \tag{16}$$

Proof We start from the relation

$$b_{n+1}(\mu)p_{n+1}(\mu; s) = (s - a_n(\mu))p_n(\mu; s) - b_n(\mu)p_{n-1}(\mu; s), \tag{17}$$

and map s in $\phi_\beta(s) = \delta s + \bar{\delta}\beta$: define $Q_{n+1}(s) := b_{n+1}(\mu)p_{n+1}(\mu; \phi_\beta(s))$:

$$Q_{n+1}(s) = (\phi_\beta(s) - a_n(\mu))p_n(\mu; \phi_\beta(s)) - b_n(\mu)p_{n-1}(\mu; \phi_\beta(s)). \tag{18}$$

Use now (10) and (11) to express the terms at r.h.s. via the matrices Γ^n and Γ^{n-1}. Then, we multiply both sides of (18) by $p_j(\mu; s)p_l(\sigma; \beta)$ and integrate w.r.t. $d\sigma(\beta)d\mu(s)$, to get

$$\begin{aligned} b_{n+1}(\mu)\Gamma^{n+1}_{j,l} = {} & \delta\left(a_j(\mu)\Gamma^n_{j,l} + b_{j+1}(\mu)\Gamma^n_{j+1,l} + b_j(\mu)\Gamma^n_{j-1,l}\right) + \\ & + \bar{\delta}\left(a_l(\sigma)\Gamma^n_{j,l} + b_{l+1}(\sigma)\Gamma^n_{j,l+1} + b_l(\sigma)\Gamma^n_{j,l-1}\right) \\ & - a_n(\mu)\Gamma^n_{j,l} - b_n(\mu)\Gamma^{n-1}_{j,l}. \end{aligned}$$

The r.h.s. of these equations is a linear combination of the matrix elements of Γ^n and Γ^{n-1}. The coefficients are given by the matrix entries of J_μ and J_σ. Direct inspection, using the triangular nature of the matrices Γ^n and Γ^{n-1}, reveals that the Jacobi entries of highest index appearing are $a_n(\mu)$, $b_{n+1}(\mu)$ and $a_n(\sigma)$, $b_{n+1}(\sigma)$, and that (16) holds. □

Remark that in the previous Lemma the quantity $b_{n+1}(\mu)$ is implicitly assumed to be non-null, for otherwise the measure μ would consist of a finite number of atoms. Finally, we have

Lemma 3 *The matrix elements $b_{n+1}(\sigma)$ and $\Gamma^{n+1}_{0,n+1}$ can be computed in terms of $\Gamma^{n+1}_{k,r}$ (with $_{(k,r)\neq(0,n+1)}$) and of $b_{n+1}(\mu)$, provided $\delta < \delta_{n+1}$, so that a solution of the $n+1$ truncated inverse problem exists.*

Proof Use the balance relation (5) in $\int d\mu(s)p_{n+1}(\mu; s)^2 = 1$, the normalization relation. Working out terms one gets

$$1 = \sum_{k,r}(\Gamma^{n+1}_{k,r})^2 = \sum_{(k,r)\neq(0,n+1)}(\Gamma^{n+1}_{k,r})^2 + \left(\bar{\delta}\Gamma^n_{0,n}\frac{b_{n+1}(\sigma)}{b_{n+1}(\mu)}\right)^2, \tag{19}$$

where the last term of the summation has been singled out using (16), Lemma 2. Equation 19 can then be solved to find uniquely $b_{n+1}(\sigma) \geq 0$, *provided* the difference between one and the first summation at r.h.s. (a positive quantity) is non-negative: this is a consistency condition that tells us when a solution of

the $n+1$-truncated inverse problem exists. Notice that this condition can be computed much more easily than the determinantal inequalities in [9]. □

In conclusion, the algorithm consists of the concatenation of these three lemmas. The sequence of quantities being produced is therefore:

$$\{a_{n-1}(\sigma), b_n(\sigma), \Gamma^n\} \Rightarrow a_n(\sigma) \Rightarrow \Gamma^{n+1}_{r,s} \ {}_{(r,s)\neq(0,n+1)} \Rightarrow \{b_{n+1}(\sigma), \Gamma^{n+1}_{0,n+1}\},$$

for all $n = 0, \ldots$ up to termination, imposed by fixing N or by Lemma 3.

5 Exploring the algorithm: numerical stability and approximation

The first mandatory test of the algorithm presented in the previous section is a check of its numerical stability: I have already remarked that many algorithms dealing with these sorts of problems have been plagued by exponentially growing computational errors. After the Jacobi matrix J^N_σ has been found, we can compute its spectral decomposition in a stable fashion, so obtaining the parameters of the I.F.S.: the fixed points $\{\beta_j\}_{j=1}^N$ and the weights $\{\pi_j\}_{j=1}^N$. We can then use the algorithm of [12] or of [13] as well to *recompute* the Jacobi matrix J_μ and compare the result with the original matrix, input of the first algorithm. As a test case, we take the Jacobi matrix of the (normalised) Chebyshev polynomials, corresponding to the measure $d\mu(s) = \frac{ds}{\pi\sqrt{1-s^2}}$: in Fig. 1 errors in the reconstruction are plotted versus n in doubly logarithmic scale, showing a very mild polynomial increase.

Two comments are in order at this point. First, when δ is small, the Jacobi matrices J^N_σ and J^N_μ are close, and the sensitive part of the computation regards

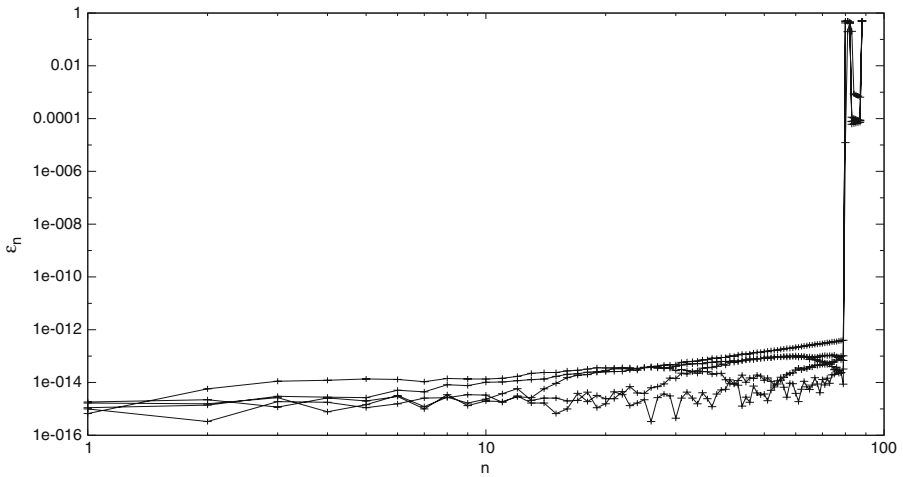

Fig. 1 Absolute differences $\varepsilon_n = |a_n(\mu) - a_n(\mu')| + |b_n(\mu) - b_n(\mu')|$ between μ and the recomputed μ', versus n. *Curves* are drawn for five different values of δ and $N = 80$. μ is the Chebyshev measure. For $n > N$ differences are obviously of the order of unity

only the coefficients with n close to the maximal (where $b_n(\sigma)$ may not exist, see Lemma 3). Therefore, as can be seen in (16), small values of $b_n(\mu)$ do not lead to significant loss of precision. Secondly, the algorithm is a Stieltjes technique that follows the same sequence of operations (Lemmas 1, 2 and 3) as the algorithm in [12] mentioned above. This latter, in turn, has been shown to be equivalent to the $A(1,7)$ version of the Lanczos algorithm discussed in Paige [15]. It is then plausible, although not within the aims of the present paper, that a formal rounding error analysis could be obtained along the same lines.

Having discussed stability, we can now consider the approximation power. Let us compare the approximation of the sampled measure effected by I.F.S. and by Gaussian integration, (3): in Fig. 2 we plot the distribution functions $F(x) = \int^x d\mu(s)$ of these measures and of the original Chebyshev measure. Here, we have chosen a very low value $M = N = 3$: the I.F.S. approximation outperforms Gaussian integration, in particular at the ends of the support and near zero.

Next, we perform the same analysis for the third convolution power of the ternary Cantor measure: this is an example of a fractal measure supported on an interval, $[0, 3]$. It can also be seen as the invariant measure of a homogeneous IFS with four (overlapping) maps and $\delta = 1/3$. Of course, if we set $N = M = 4$ and the correct value of δ, the algorithm retrieves the parameters $\{\pi_j, \beta_j\}_{j=1}^4$ within the numerical precision of the machine. To explore the approximation power of the technique, we have chosen $M = 6$ and varied the parameter δ in order to reproduce exactly one additional term in the

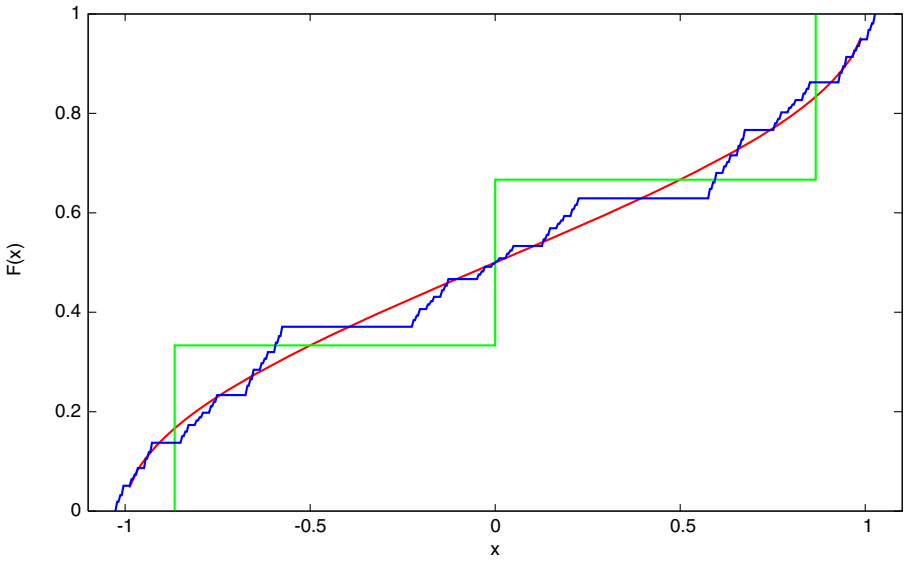

Fig. 2 Distribution functions of the Chebyshev measure (*smooth red curve*), of the discrete, three points Gaussian approximation (*green step function*) and of the I.F.S. approximation with $M = 3$ (*blue fractal curve*)

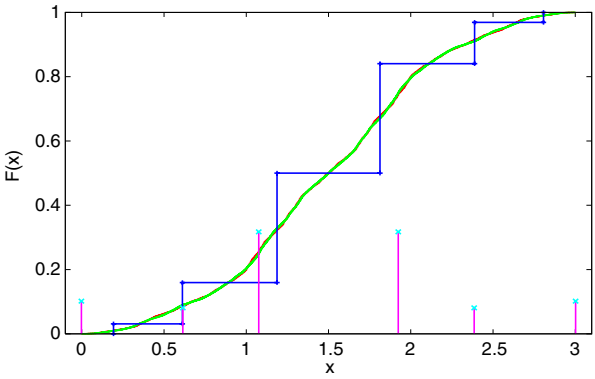

Fig. 3 Distribution functions of the third convolution power of the ternary Cantor measure (*red curve*), of the discrete Gaussian approximation (*blue step function*) and of the I.F.S. approximation with $M = 6$ (*green curve* – almost indistinguishable from the *red*). Also shown (*starred impulses*) are the location and weights of the fixed points of the I.F.S. maps

Jacobi matrix. The distribution functions of the original and reconstructed IFS measures are compared in Fig. 3: only small differences are perceivable. More refined comparisons tests will be described elsewhere.

6 I.F.S. integration turning into Gauss

The algorithm we have developed is ideally suited for enjoying the display of the behaviour of δ-H.I.F.S. parameters in various examples. Let us analyze again the classical Chebyshev measure, to show what happens at fixed N when δ explores the range $[0, \delta_N)$ assured by Theorem 2. Remark that the algorithm of Section 4 can be used to find the value δ_N: this latter is plotted versus N in Fig. 4, displaying an asymptotic power-law decrease.

For $\delta = 0$ we have $J_\sigma = J_\mu$ and the algorithm of Section 4 becomes the identity (although not exactly so, in floating point arithmetics). In the limit of vanishing δ, the fixed points β_j tend to the Gaussian points x_j and the map probabilities π_j to the weights w_j. Figure 5 plots the locations β_j versus δ (green

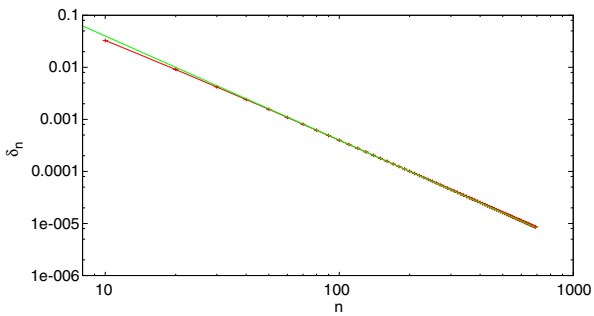

Fig. 4 δ_N versus N for the Chebyshev measure. The fitting power–law has exponent -2

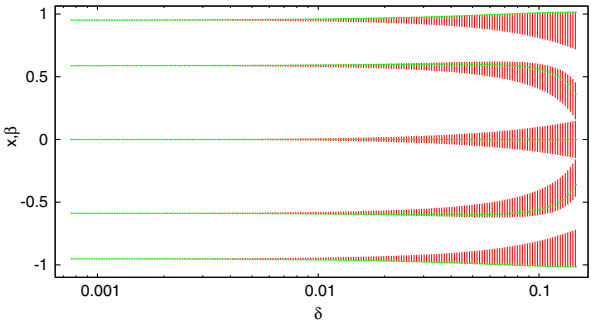

Fig. 5 Fixed points $\{\beta_j\}_{j=1}^5$ versus δ (*green lines*) for the Chebyshev measure, $M = 5$ and their Gauss tongues (*red stripes*). See text for definitions and details

line) together with the first image of the interval $[\beta_1, \beta_5]$ (that is, the convex hull of the support of the reconstructed measure) under the maps ϕ_β. This plot displays a sort of "Gauss tongues" that shrink towards the Gaussian points as δ goes to zero. It should be noticed that the plot of these tongues – when not interrupted at the first iteration – is fractal.

Finally, the uniform Lebesgue measure on [0, 1] can be exactly reproduced by δ-H.I.F.S. with *any* number of maps M with equal weights, equally spaced $\{\beta_j\}_{j=1}^M$ and $\delta = 1/M$. In Fig. 6 we let $M = 5$ and we plot the nonlinear behaviour (versus δ) of the fixed points and probabilities of the I.F.S. maps that join smoothly the Gaussian points and weights obtained for $\delta = 0$ to the exact I.F.S. parameters obtained for $\delta = 1/5$.

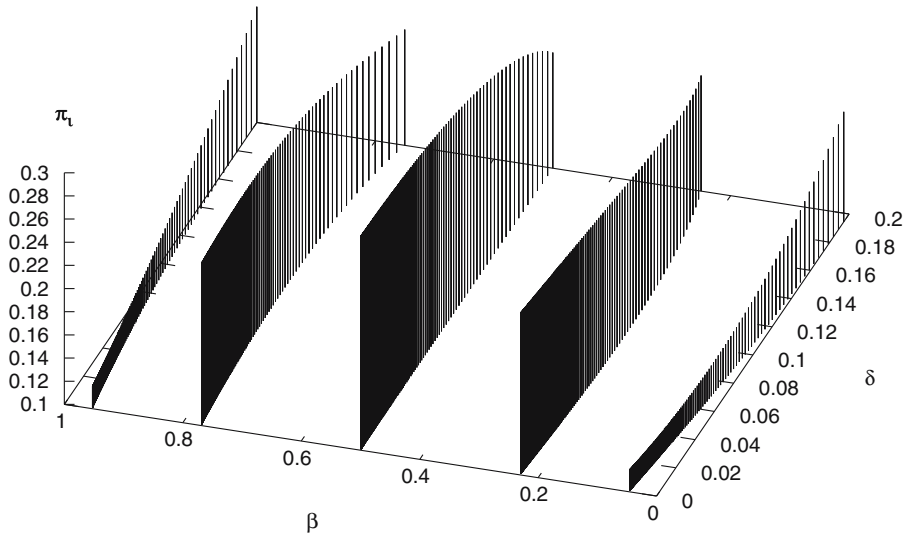

Fig. 6 Fixed points and associated probabilities $\{(\beta_j, \pi_j)\}_{j=1}^5$ for $M = 5$, versus δ in the case of the uniform Lebesque measure

7 Conclusions

We have presented in this paper a new algorithm for the approximation of a target measure by measures in the class of Iterated Function Systems. The algorithm appears to be numerically stable when implemented in floating point; we have also indicated where to look for a formal proof of stability. The algorithm can also be coded symbolically, if square roots are avoided in a trivially modified version that relies on monic, rather than normalized, orthogonal polynomials. It requires the Jacobi matrix of the sampled measure as input: since this is usually needed for conventional Gaussian integration, the new approximation can be performed at no additional sampling cost.

Integration of functions w.r.t. to I.F.S. measures can be efficiently performed and a theory of Richardson extrapolation developed [3]. Therefore, also in the light of previous results [2] indicating a definite advantage of I.F.S. above conventional Gaussian integration, we believe that the combination of these techniques, extended for the first time to large orders thanks to our algorithm, might become interesting in various applications, including the study of the Fourier transforms of singular measures and of their orthogonal polynomials [14].

Finally, the reversible nature of the transformation between J_σ and J_μ prompts for a new algorithm for the computation of Jacobi matrices associated with homogeneous I.F.S. even with uncountably many affine maps, that will be described elsewhere.

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