

"Harmonic Analysis: Smooth and Non-smooth."

Introduction to the Conference: Historical context, Reading list, Basic definitions, some relevant results, and an Outline of the Ten Lectures by Jorgensen.

CBMS conference to be held at Iowa State University,
June 4–8, 2018.

Speaker: Palle Jorgensen, Professor, University of Iowa.
Organizers: Professors John Herr, Justin Peters, and Eric Weber.

- Lecture 1.** Harmonic Analysis of Measures: Analysis on Fractals
- Lecture 2.** Spectra of measures, tilings, and wandering vectors
- Lecture 3.** The universal tiling conjecture in dimension one and operator fractals
- Lecture 4.** Representations of Cuntz algebras associated to quasi-stationary Markov measures
- Lecture 5.** The Cuntz relations and kernel decompositions
- Lecture 6.** Harmonic analysis of wavelet filters: input-output and state-space models
- Lecture 7.** Spectral theory for Gaussian processes: reproducing kernels, boundaries, and L^2 -wavelet generators with fractional scales
- Lecture 8.** Reproducing Kernel Hilbert Spaces arising from groups
- Lecture 9.** Extensions of positive definite functions
- Lecture 10.** Reflection positive stochastic processes indexed by Lie groups

Background There is a recent and increasing interest in understanding the harmonic analysis of non-smooth geometries, typically fractal like. They are unlike the familiar smooth Euclidean geometry. In the non-smooth case, nearby points are not locally connected to each other. Real-world examples where these types of geometry appear include large computer networks, relationships in datasets, and fractal structures such as those found in crystalline substances, light scattering, and other natural phenomena where dynamical systems are present.

Details and Format A series of ten lectures by Professor Palle Jorgensen from the University of Iowa, a leader in the fields of smooth and non-smooth harmonic analysis. The conference aims to demonstrate surprising connections between the two domains of geometry and Fourier spectra. In addition to the 10 lectures by Jorgensen, there will be other invited speakers.

The conference aims to bring both experienced and new researchers together to stimulate collaboration on this timely topic. It also aims to advance representation and participation of underrepresented minorities within mathematics, and the development of a globally competitive STEM workforce.

There is NSF support, including to current graduate students. The conference will contribute to those graduate students' educational and professional development and hence prepare the nation's next generation of researchers to engage this increasingly important subject area.

Jorgensen and Pedersen showed in 1998 that there exists a Cantor-like set with the property that the uniform measure supported on that set is spectral, meaning that there exists a sequence of frequencies for which the corresponding exponential functions form an orthonormal basis in the Hilbert space of square-integrable functions with respect to that measure. Research that has been inspired by this stunning result includes: fractal Fourier analyses, spectral theory of Ruelle transfer operators, representation theory of Cuntz algebras, convergence of the cascade algorithm in wavelet theory, reproducing kernels and their boundary representations, Bernoulli convolutions, and Markov processes. The remarkable feature of this array of subjects is that they straddle both the smooth and non-smooth settings. The lectures presented by Professor Jorgensen will unify these far-reaching research areas at the interface of smooth and non-smooth harmonic analysis.

Preview

Smooth harmonic analysis refers to harmonic analysis over a connected or locally connected domain—typically Euclidean space or locally connected subsets of Euclidean space. The classical example of this is the existence of Fourier series expansions for square integrable functions on the unit interval. Non-smooth harmonic analysis then refers to harmonic analysis on discrete or disconnected domains—typical examples of this setting are Cantor like subsets of the real line and analogous fractals in higher dimensions. In 1998, Jorgensen and Steen Pedersen proved a remarkable result: there exists a Cantor like set (of Hausdorff dimension $1/2$) with the property that the uniform measure supported on that set is spectral, meaning that there exists a sequence of frequencies for which the exponentials form an orthonormal basis in the Hilbert space of square integrable functions with respect to that measure. This surprising result, together with results of Robert Strichartz, has led to a plethora of new research directions in non-smooth harmonic analysis.

Research that has been inspired by this surprising result includes: fractal Fourier analyses (fractals in the large), spectral theory of Ruelle operators; representation theory of Cuntz algebras; convergence of the cascade algorithm in wavelet theory; reproducing kernels and their

boundary representations; Bernoulli convolutions and Markov processes. The remarkable aspect of these broad connections is that they often straddle both the smooth and non-smooth domains. This is particularly evident in Jorgensen’s research on the cascade algorithm, as wavelets already possess a “dual” existence in the continuous and discrete worlds, and also his research on the boundary representations of reproducing kernels, as the non-smooth domains appear as boundaries of smooth domains. In work with Dorin Dutkay, Jorgensen showed that the general affine IFS-systems, even if not amenable to Fourier analysis, in fact do admit wavelet bases, and so in particular can be analyzed with the use of multiresolutions. In recent work with Herr and Weber, Jorgensen has shown that fractals that are not spectral (and so do not admit an orthogonal Fourier analysis) still admits a harmonic analysis as boundary values for certain subspaces of the Hardy space of the disc and the corresponding reproducing kernels within them.

The lectures to be given by Jorgensen will cover the following overarching themes: the harmonic analysis of Cantor spaces (and measures) arising as fractals (including fractal dust) and iterated function systems (IFSs), as well as the methods used to study their harmonic analyses that span both the smooth and non-smooth domains. A consequence of the fact that these methods form a bridge between the smooth and non-smooth domain is that the topics to be discussed— while on the surface seem largely unrelated—actually are closely related and together form a tightly focused theme. The breadth of topics will attract a broader audience of established researchers, while the interconnectedness and sharply focused nature of these topics will prove beneficial to beginning researchers in non-smooth harmonic analysis.

Introduction

One of the most fruitful achievements of mathematics in the past five hundred years has been the development of Fourier series. Such a series may be thought of as the decomposition of a periodic function into sinusoid waves of varying frequencies. Application of such decompositions are naturally abundant, with waves occurring in all manner of physics, and uses for periodic functions being present in other areas such as economics and signal processing, just to name a few. The importance of Fourier series is well-known and incontestable.

Historical context While to many non-mathematicians and undergraduate math majors, a Fourier series is regarded as a breakdown into sine and cosine waves, the experienced analyst will usually think of it (equivalently), as a decomposition into sums of complex exponentials. For instance, in the classical setting of the unit interval $[0, 1)$, a Lebesgue integrable function $f : [0, 1) \rightarrow \mathbb{C}$ will induce a Fourier series

$$f(x) \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{i2\pi nx} \tag{1}$$

where

$$\hat{f}(n) := \int_0^1 f(x) e^{-i2\pi nx} dx. \tag{2}$$

Because the Fourier series is intended to represent the function $f(x)$, it is only natural to ask in what senses, if any, the sum above converges to $f(x)$. One can ask important questions about pointwise convergence, but it is more relevant for our purposes to restrict attention to various normed spaces of functions or, as we will be most concerned with hereafter, a Hilbert space consisting of square-integrable functions, and then ask about norm convergence. In our

present context, if we let $L^2([0, 1])$ denote the Hilbert space of (equivalence classes of) functions $f : [0, 1] \rightarrow \mathbb{C}$ satisfying

$$\|f\|^2 := \int_0^1 |f(x)|^2 dx < \infty \quad (3)$$

and equipped with the inner product

$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx, \quad (4)$$

then if $f \in L^2([0, 1])$, the convergence in (1) will occur in the norm of $L^2([0, 1])$. It is also easy to see that in $L^2([0, 1])$,

$$\langle e^{i2\pi mx}, e^{i2\pi nx} \rangle = \int_0^1 e^{i2\pi mx} e^{-i2\pi nx} dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

That is, the set of complex exponentials $\{e^{i2\pi nx}\}_{n \in \mathbb{Z}}$ is orthogonal in $L^2([0, 1])$. Since every function in $L^2([0, 1])$ can be written in terms of these exponentials, $\{e^{i2\pi nx}\}_{n \in \mathbb{Z}}$ is in fact an orthonormal basis of $L^2([0, 1])$.

Because there exists a countable set of complex exponential functions that form an orthogonal basis of $L^2([0, 1])$, we say that the set $[0, 1]$ is spectral. The set of frequencies of such an orthogonal basis of exponentials, which in this case is \mathbb{Z} , is called a spectrum.

Like most areas of analysis, the historical and most common contexts for Fourier series are also the most mundane: The functions they decompose are defined on \mathbb{R} , the unit interval $[0, 1]$, or sometimes a discrete set. The underlying measure used for integration is Lebesgue measure. It is thanks to the work of many individuals, including Palle Jorgensen, that modern Fourier analysis has been able to aspire beyond these historical paradigms.

The first paradigm break is to consider a wider variety of domains in a wider variety of dimensions. In general, if C is a compact subset of \mathbb{R}^n of nonzero Lebesgue measure, then we say that C is spectral if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that $\{e^{i2\pi \lambda \cdot \vec{x}}\}_{\lambda \in \Lambda}$ is an orthogonal basis of $L^2(C)$, where

$$L^2(C) := \left\{ f : C \rightarrow \mathbb{C} \mid \int_C |f(\vec{x})|^2 d\lambda^n(\vec{x}) < \infty \right\}. \quad (6)$$

Here λ^n is Lebesgue measure in \mathbb{R}^n .

The famous Fuglede Conjecture surmised that C would be spectral if and only if it would tessellate by translation to cover \mathbb{R}^n . Iosevich, Katz, and Tao proved in 2001 that the conjecture holds for convex planar domains [IKT03]. In the same year, they also proved that a smooth, symmetric, convex body with at least one point of nonvanishing Gaussian curvature cannot be spectral [IKT01]. However, in 2003 Tao devised counterexamples to the Fuglede Conjecture in \mathbb{R}^5 and \mathbb{R}^{11} [Tao04]. The conjecture remains open in low dimensions.

The second paradigm break is to substitute a different Borel measure in place of Lebesgue measure. For example, if μ is any Borel measure on $[0, 1]$, one can form the Hilbert space

$$L^2(\mu) = \left\{ f : [0, 1] \rightarrow \mathbb{C} \mid \int_0^1 |f(x)|^2 d\mu(x) < \infty \right\} \quad (7)$$

with inner product

$$\langle f, g \rangle_\mu = \int_0^1 f(x) \overline{g(x)} d\mu(x). \quad (8)$$

Comparing equations (7) and (8) with equations (3) and (4), respectively, we see that we can then regard spectrality as a property of measures rather than of sets: The measure μ is spectral if there exists a countable index set Λ such that the set of complex exponentials $\{e^{i2\pi\lambda x}\}_{\lambda \in \Lambda}$ is an orthogonal basis of $L^2(\mu)$. The index set Λ is then called a spectrum of μ .

Iterated function systems (IFS)

There do, of course, exist some measures that are not spectral. Of great interest to Jorgensen are measures that arise naturally from affine iterated function systems. An iterated function system (IFS) is a finite set of contraction operators $\tau_0, \tau_1, \dots, \tau_n$ on a complete metric space S . As a consequence of Hutchinson's Theorem [Hut81], for an IFS on \mathbb{R}^n , there exists a unique compact set $X \subset \mathbb{R}^n$ left invariant by system in the sense that $X = \cup_{j=0}^n \tau_j(X)$. There will then exist a unique Borel measure μ on X such that

$$\int_X f(x) d\mu(x) = \frac{1}{n+1} \sum_{j=0}^n \int_X f(\tau_j(x)) d\mu(x) \quad (9)$$

for all continuous f .

In many cases of interest, X is a fractal set. In particular, if we take the iterated function system

$$\tau_0(x) = \frac{x}{3}, \quad \tau_1(x) = \frac{x+2}{3}$$

on \mathbb{R} , then the attractor is the ternary Cantor set C_3 . The set C_3 has another construction: One starts with the interval $[0, 1]$ and removes the middle third, leaving only the intervals $[0, 1/3]$ and $[2/3, 1]$, and then successively continues to remove the middle third of each remaining interval. Intersecting the sets remaining at each step yields C_3 . The ternary Cantor measure μ_3 is then the measure induced in (9). Alternatively, μ_3 is the Hausdorff measure of dimension $\frac{\ln 2}{\ln 3}$ restricted to C_3 .

In [JP98] Jorgensen and Pedersen used the zero set of the Fourier-Stieltjes transform of μ_3 to show that μ_3 is not spectral. Equally remarkably, they showed that the quaternary (4-ary) Cantor set, which is the measure induced in (9) under the IFS

$$\tau_0(x) = \frac{x}{4}, \quad \tau_1(x) = \frac{x+2}{4} \quad (10)$$

is spectral by using Hadamard matrices and a completeness argument based on the Ruelle transfer operator. The attractor set for this IFS can be described in a manner similar to the ternary Cantor set: The 4-ary set case is as follows,

$$C_4 = \left\{ x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{a_k}{4^k}, a_k \in \{0, 2\} \right\},$$

and the invariant measure is denoted by μ_4 . Jorgensen and Pedersen prove that

$$\begin{aligned} \Gamma_4 &= \left\{ \sum_{n=0}^N l_n 4^n : l_n \in \{0, 1\}, N \in \mathbb{N} \right\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \dots\} \end{aligned}$$

is a spectrum for μ_4 , though there are many spectra [DHS09, DHL13]. The proof that this is a spectrum is a two step process: first the orthogonality of the exponentials with frequencies in Γ_4 is verified, and second the completeness of those exponentials are verified.

The orthogonality of the exponentials can be checked in several ways:

1. checking the zeroes of the Fourier-Stieltjes transform of μ_4 ;
2. using the representation of a particular Cuntz algebra on $L^2(\mu_4)$;
3. generating Γ_4 as the invariant set for a second IFS that is “dual” in a sense to the IFS in (10) (“fractals in the large”).

While these three methods are distinct, they all rely on the fact that a certain matrix associated to the IFSs is a (complex) Hadamard matrix. All three of these methods are, more or less, contained in the original paper [JP98].

As a Borel probability measure, μ_4 is determined uniquely by the following IFS-fixed-point property:

$$\mu_4 = \frac{1}{2} (\mu_4 \circ \tau_0^{-1} + \mu_4 \circ \tau_1^{-1}),$$

see (10) for the affine maps τ_i , $i = 0, 1$; and one checks that the support of μ_4 is the 4-ary Cantor set C_4 .

By contrast, when this is modified to (μ_3, C_3) , the middle-third Cantor, Jorgensen and Pedersen proved that then there cannot be more than two orthogonal Fourier functions $e_\lambda(x) = e^{i2\pi\lambda x}$, for any choices of points λ in \mathbb{R} .

The completeness of the exponentials (for the cases when the specified Cantor measure is spectral) can be shown in several ways as well, though the completeness is more subtle. The original argument for completeness given in [JP98] uses a delicate analysis of the spectral theory of a Ruelle transfer operator. Jorgensen and Pedersen construct an operator on $C(\mathbb{R})$ using filters associated to the IFS in (10), which they term a Ruelle transfer operator. The argument then is to check that the eigenvalue 1 for this operator is a simple eigenvalue. An alternative argument for completeness given by Strichartz in [Str98] uses the convergence of the cascade algorithm from wavelet theory [Mal89, Dau88, Law91]. Later arguments for completeness were developed in [DJ09, DJ12b] again using the representation theory of Cuntz algebras.

The Cuntz algebra \mathcal{O}_N for $N \geq 2$ is the universal C^* -algebra generated by a family $\{S_0, \dots, S_{N-1}\}$ of N isometries satisfying the relation

$$\sum_{j=0}^{N-1} S_j S_j^* = I, \quad \text{and} \quad S_i^* S_j = \delta_{ij} I.$$

There are many ways to generate such families. For example, consider the isometries S_0, S_1 on $L^2[0, 1]$ given by defining their adjoints

$$(S_0^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right) \quad \text{and} \quad (S_1^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+1}{2}\right),$$

$f \in L^2[0, 1]$, $x \in [0, 1]$. One can check that the range isometries $S_0 S_0^* = \chi_{[0, 1/2]}$ and $S_1 S_1^* = \chi_{[1/2, 1]}$, so that the Cuntz relations are satisfied.

Developing this example a bit further, we can see a relationship between Cuntz isometries and iterated function systems. Let C be the standard Cantor set in $[0, 1]$, consisting of those real numbers whose ternary expansions are of the form $x = \sum_{k=1}^{\infty} \frac{x_k}{3^k}$ where $x_k \in \{0, 2\}$ for all k . Let

$$\varphi : C \rightarrow [0, 1], \quad \varphi\left(\sum_{k=1}^{\infty} \frac{x_k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{x_k}{2^{k+1}}.$$

Let m be Lebesgue measure on $[0, 1]$, and define the Cantor measure μ on C by $\mu(\varphi^{-1}(B)) = m(B)$ if $B \subset [0, 1]$ is Lebesgue measurable. This is well defined since φ is bijective except at countably many points.

Now define isometries R_0, R_1 on $(L^2(C), \mu)$ by defining their adjoints:

$$R_0^*(f) = S_0^*(f \circ \varphi) \quad \text{and} \quad R_1^*(f) = S_1^*(f \circ \varphi), \quad f \in (L^2(C), \mu).$$

Then

$$R_0^*(f)(x) = \frac{1}{\sqrt{2}}f\left(\frac{x}{3}\right) \quad \text{and} \quad R_1^*(f)(x) = \frac{1}{\sqrt{2}}f\left(\frac{x+2}{3}\right),$$

$f \in (L^2(C), \mu)$, $x \in C$. Thus we see the iterated function system for the Cantor set $\tau_0(x) = x/3$, $\tau_1(x) = (x+2)/3$ arising in the definition of Cuntz isometries on the Cantor set.

The Cuntz relations can be represented in many different ways. In their paper [DJ15a], Dutkay and Jorgensen look at finite Markov processes, and the infinite product of the state space is a compact set on which different measures can be defined, and these form the setting of representations of the Cuntz relations.

To construct a Fourier basis for a spectral measure arising from an iterated function system generated by contractions $\{\tau_0, \dots, \tau_{N-1}\}$, Jorgensen (and others, [JP98, DPS14, DJ15a, PW17]) choose filters m_0, \dots, m_{N-1} and define Cuntz isometries S_0, \dots, S_{N-1} on $L^2(\mu)$ by

$$S_j f = \sqrt{N} m_j f \circ R,$$

where R is the common left inverse of the τ 's. The filters, functions defined on the attractor set of the iterated function system, are typically chosen to be continuous, and are required to satisfy the relation $\sum_{j=0}^{N-1} |m_j|^2 = 1$. The Cuntz relations are satisfied by the S_j 's provided the filters satisfy the orthogonality condition

$$\mathcal{M}^* \mathcal{M} = I, \quad (M)_{jk} = m_j(\tau_k(\cdot)). \quad (11)$$

To obtain Fourier bases, the filters m_j are chosen specifically to be exponential functions when possible. This is not possible in general, however, and is not possible in the case of the middle-third Cantor set and its corresponding measure μ_3 .

The fact that some measures, such as μ_3 , are not spectral leaves us with a conundrum: We still desire Fourier-type expansions of functions in $L^2(\mu)$, that is, a representation as a series of complex exponential functions, but we cannot get such an expansion from an orthogonal basis of exponentials in the case of a non-spectral measure. For this reason, we turn to another type of sequence called a frame, which has the same ability to produce series representations that an orthogonal basis does, but has redundancy that orthogonal bases lack and has no orthogonality requirement. Frames for Hilbert spaces were introduced by Dun and Schaeer [DS52] in their study of non-harmonic Fourier series. The idea then lay essentially dormant until Daubechies, Grossman, and Meyer reintroduced frames in [DGM86]. Frames are now pervasive in mathematics and engineering.

	Scaling factor	Number of affine maps τ_i	Ambient dimension	Hausdorff dimension
Middle-third C_3	3	2	1	$\log_3 2 = \frac{\ln 2}{\ln 3}$
The 4-ary C_4	4	2	1	$\frac{1}{2}$
Sierpinski triangle	2	3	2	$\log_2 3 = \frac{\ln 3}{\ln 2}$

Table 1: Some popular affine IFSs

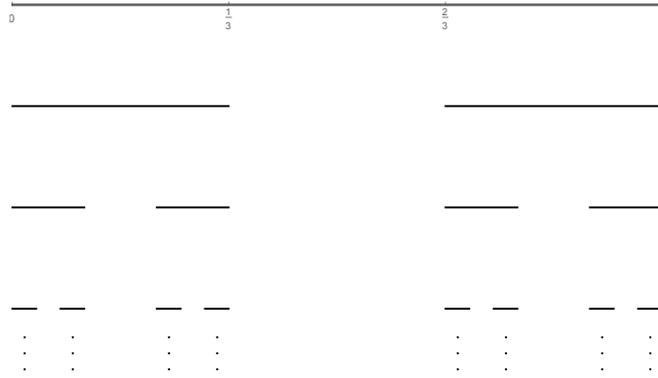


Figure 1: Middle-third Cantor C_3

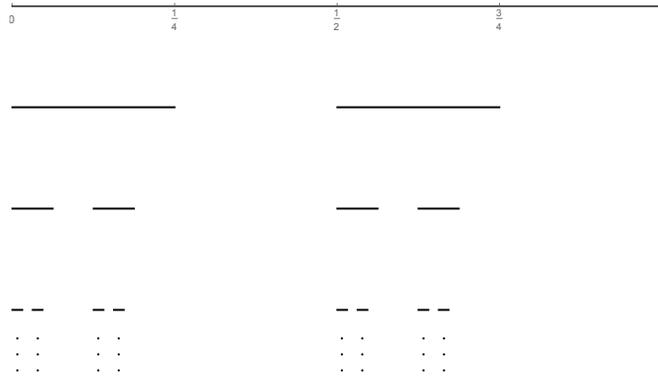


Figure 2: The 4-ary Cantor C_4

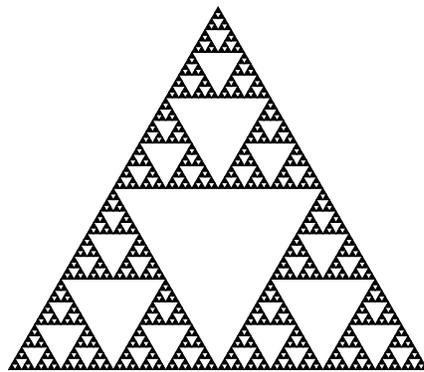


Figure 3: Sierpinski triangle

Frequency bands, filters, and representations of the Cuntz-algebras

Our analysis of the Cuntz relations here in the form $\{S_i\}_{i=0}^{N-1}$ turns out to be a modern version of the rule from signal-processing engineering (SPEE): When complex frequency response functions are introduced, the (SPEE) version of the Cuntz relations $S_i^* S_j = \delta_{ij} I_{\mathcal{H}}$, $\sum_{i=0}^{N-1} S_i S_i^* = I_{\mathcal{H}}$, where \mathcal{H} is a Hilbert space of time/frequency signals, and where the N isometris S_i are expressed in the following form:

$$(S_i f)(z) = m_i(z) f(z^N), \quad f \in \mathcal{H}, \quad z \in \mathbb{C};$$

and where $\{m_i\}_{i=0}^{N-1}$ is a system of bandpass-filters, m_0 accounting for the low band, and the filters $m_i(z)$, $i > 0$, accounting for the remaining bands in the subdivision into a total of N bands. The diagram form (SPEE) is then as in Figure 4.

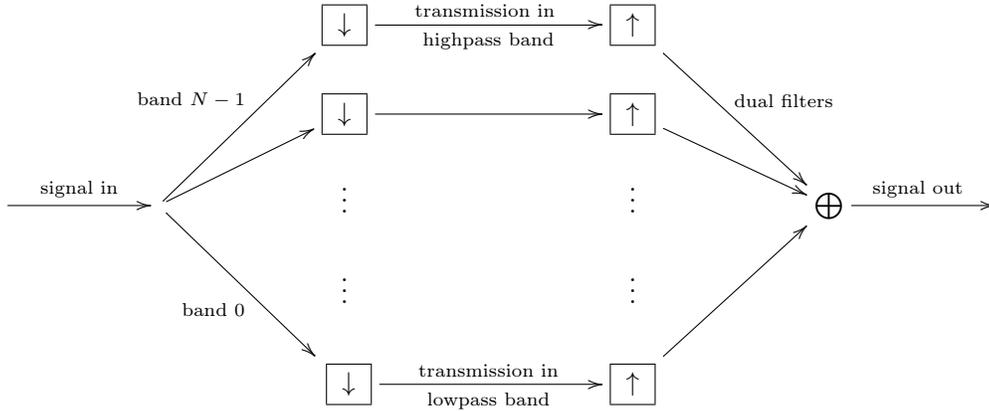


Figure 4: The picture is a modern math version of one I (PJ) remember from my early childhood: In our living room, my dad was putting together some of the early versions of low-pass/high-pass frequency band filters for transmitting speech signals over what was then long distance. One of the EE journals had a picture which is much like the one I reproduce here; after hazy memory. Strangely, the same multi-band constructions are still in use for modern wireless transmission, both speech and images. The down/up arrows in the figure stand for down-sampling, up-sampling, respectively. Both operations have easy expressions in the complex frequency domain. For example up-sampling becomes substitution of z^N where N is the fixed total number of bands.

Frames

Let \mathcal{H} be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let \mathbb{J} be a countable index set. A frame for \mathcal{H} is a sequence $\{x_j\}_{j \in \mathbb{J}} \subset \mathcal{H}$ such that there exist constants $0 < C_1 \leq C_2 < \infty$ such that for all $v \in \mathcal{H}$,

$$C_1 \|v\|^2 \leq \sum |\langle v, x_j \rangle|^2 \leq C_2 \|v\|^2.$$

If C_1 and C_2 can be chosen so that $C_1 = C_2 = 1$, we say that $\{x_j\}$ is a *Parseval frame*.

If $\mathbb{X} \subset \mathcal{H}$ is a frame, then any other frame $\tilde{\mathbb{X}} := \{\tilde{x}_j\} \subset \mathcal{H}$ that satisfies

$$\sum \langle v, \tilde{x}_j \rangle x_j = v, \tag{12}$$

for all $v \in \mathcal{H}$ is called a *dual frame* for \mathbb{X} . Every frame possesses a dual frame, and in general, dual frames are not unique. A Parseval frame is self-dual, that is, $v = \sum \langle v, x_j \rangle x_j$.

Returning to our current interest, we say that a measure μ is frame-spectral if there exists a countable set $\Lambda \subset \mathbb{R}$ such that $\{e^{i2\pi\lambda x}\}_{\lambda \in \Lambda}$ is a frame in $L^2(\mu)$. In general, for a compact subset C of \mathbb{R}^d with nonzero measure, Lebesgue measure restricted to that set is not spectral, but it will always be frame spectral. In general, a singular measure will not be frame spectral [DHSW11, DL14], but many singular measures are frame-spectral [EKW16, PW17]. It is currently unknown whether or not μ_3 is frame-spectral.

The redundancy of frames makes them more immune to error in transmission: Multiple frame elements will capture the same dimensions of information, and so if one series coefficient in the frame expansion of a function is transmitted incorrectly, the adverse effect on the reconstructed function will be minimal. However, expansions in terms of a given frame are in general not unique, and this can be a desirable or undesirable quality depending on the application. If we want the best of both worlds — a frame with redundancy but with a unique expansion for each function — then we must turn to the realm of Riesz bases.

A Riesz basis in a Hilbert space \mathcal{H} is a sequence $\{x_j\}_{j=1}^\infty$ which has dense span in \mathcal{H} and is such that there exist $0 < A \leq B$ such that for any finite sequence of scalars c_1, c_2, \dots, c_N , we have

$$A \sum_{j=1}^N |c_j|^2 \leq \left\| \sum_{j=1}^N c_j x_j \right\|^2 \leq B \sum_{j=1}^N |c_j|^2. \quad (13)$$

A Riesz basis is a frame that has only one dual frame. Equivalently, $\{x_j\}_{j=1}^\infty$ is a Riesz basis if and only if there is a topological isomorphism $T : \mathcal{H} \rightarrow \mathcal{H}$ such that $\{Tx_j\}_{j=1}^\infty$ is an orthonormal basis of \mathcal{H} .

The unit disk \mathbb{D} , for example, as a convex planar body has no orthogonal basis of complex exponential functions, but it does possess a frame of complex exponential functions. However, it is still an open problem whether it possesses a Riesz basis of complex exponential functions.

Description of Lectures

The lectures will cover the following overarching themes: the harmonic analysis of Cantor spaces (and measures) arising as fractals (including fractal dust) and iterated function systems (IFSs), as well as the methods used to study their harmonic analyses that span both the smooth and non-smooth domains. The topics that arise from these overarching themes include fractal Fourier analyses (fractals in the large), spectral theory of Ruelle operators; representation theory of Cuntz algebras; convergence of the cascade algorithm in wavelet theory; reproducing kernels and their boundary representations; Bernoulli convolutions and Markov processes. The remarkable aspect of these broad connections is that they often straddle both the smooth and non-smooth domains. This is particularly evident in Jorgensen’s research on the cascade algorithm, as wavelets already possess a “dual” existence in the continuous and discrete worlds, and also his research on the boundary representations of reproducing kernels, as the non-smooth domains appear as boundaries of smooth domains. Another remarkable aspect is that these broad connections weave together to form a sharply focused theme.

The logical flow and interconnectedness of these topics are illustrated in Figure 5. The numbers in the figure correspond to the lecture numbers.

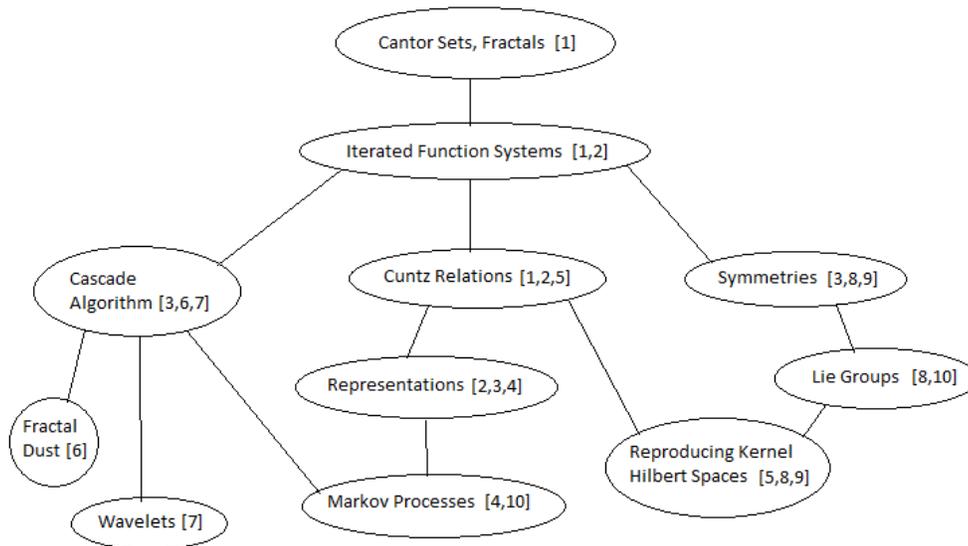


Figure 5: Flow and Connections of Topics.

Lecture 1. Harmonic Analysis of Measures: Analysis on Fractals

Beginning with the foundational results in “*Dense analytic subspaces in fractal L^2 -spaces*” [JP98], the first talk will cover the construction of spectral measures, the constructions of various spectra, characterizations and invariance of spectra for spectral measures. The talk will include initial connections to representation theory of Cuntz algebras, spectra and tiling properties in \mathbb{R}^d , the Fuglede conjecture, and Reproducing Kernel Hilbert spaces.

The existence of orthogonal Fourier bases for classes of fractals came as somewhat of a surprise, referring to the 1998 Jorgensen-Pedersen paper. There are several reasons for why existence of orthogonal Fourier bases might have been unexpected: For one, existence of orthogonal Fourier bases, as in the classical case of Fourier, tends to imply a certain amount of “smoothness” which seems inconsistent with fractal geometries, and fractal dimension. Nonetheless, when feasible, such an orthogonal Fourier analysis holds out promise for applications to large chaotic systems, or to analysis of noisy signals; areas that had previously resisted analysis by Fourier tools.

When Fourier duality holds, it further yields a duality of scale, fractal scales in the small, and for the dual frequency domain, fractals in the large.

While the original framework for the Jorgensen-Pedersen fractals, and associated L^2 -spaces, was a rather limited family, this original fractal framework for orthogonal Fourier bases has since been greatly expanded. While the original setting was restricted to that of affine selfsimilarity, determined by certain iterated affine function systems (IFSs) in one and higher dimension, this has now been broadened to the setting of say conformal selfsimilar IFS systems, and to associated maximal entropy measures. And even when the strict requirements entailed by orthogonal Fourier bases is suitably relaxed, there are computational Fourier expansions (Herr-Jorgensen-Weber) which lend themselves to analysis/synthesis for most singular measures.

Inherent in the study of fractal scales is the notion of multi-resolution analyses, in many ways parallel to the more familiar Daubechies wavelet multi-resolutions. Moreover, Strichartz proved that when an orthogonal Fourier expansions exist, they have localization properties

which parallel the kind of localization which has made wavelet multi-resolutions so useful. The presence of multi-resolutions further implies powerful algorithms, and it makes connections to representation theory and to signal/image processing; subjects of the later lectures. Dutkay-Jorgensen proved that all affine IFS fractals have wavelet bases.

References and reading list. [JP98, JKS14a, JKS14c, Jor12, DJS12, DJ07]

Lecture 2. Spectra of measures, tilings, and wandering vectors

The second talk will build on the themes from the first talk, detailing the constructions of spectra arising from Cuntz algebras, characterizations of spectra using the spectral theory of Ruelle operators, connections between tilings, and wandering vectors for unitary groups and unitary systems.

There is an intimate relations between systems of tiling by translations on the one hand, and orthogonal Fourier bases on the other. Representation theory makes a link between the two, but the tile-spectral question is deep and difficult; so far only partially resolved. One tool of inquiry is that of “wandering vectors” or wandering subspaces. The term “wandering” has its origin in the study of systems of isometries in Hilbert space. It has come to refer to certain actions in a Hilbert space which carries representations: When the action generates orthogonal vectors, we refer to them as wandering vectors; similarly for closed subspaces. In the case of representations of groups, this has proved a useful way of generating orthogonal Fourier bases; — when they exist. In the case of representations of the Cuntz algebras, the “wandering” idea has become a tool for generating nested and orthogonal subspaces. The latter includes multiresolution subspaces for wavelet systems and for signal/image processing algorithms.

References and reading list. [Kad16, IK13, DJ15b, DJ15c, DJ11, She15]

Lecture 3. The universal tiling conjecture in dimension one and operator fractals

The third talk will focus on the tiling properties arising from the study of spectral measures, specifically in dimension one; advances in the Fuglede conjecture in dimension one, non-commutative fractal analogues in infinite dimensions.

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum (has an orthogonal Fourier basis) if and only if it is possible to tile \mathbb{R}^d by a set of translates of Ω [Fug74]. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of \mathbb{R}^d and Iosevich et al. [IKT01] proved that no smooth symmetric convex body Ω with at least one point of nonvanishing Gaussian curvature can admit an orthogonal basis of exponentials.

Using complex Hadamard matrices of orders 6 and 12, Tao [Tao04] constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in \mathbb{R}^5 or \mathbb{R}^{11} . Tao’s results were extended to lower dimensions, down to $d = 3$, but the problem is still open for $d = 1$ and $d = 2$.

Summary of some affirmative recent results: The conjecture has been proved in a great number of special cases (e.g., all convex planar bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling \mathbb{Z} indeed implies one side of Fuglede’s conjecture [CM99]. Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group [Ami05].

Ironically, despite a large number of great advances in the area, Fuglede’s original question is still unsolved in the planar case. In the planar case, the Question is: Let Ω be a bounded

open and connected subset of \mathbb{R}^2 , does it follow that $L^2(\Omega)$ with respect to planar Lebesgue measure has an orthogonal Fourier basis if and only if Ω tiles \mathbb{R}^2 with translations by some set of vectors from \mathbb{R}^2 . Of course, if Ω is a fundamental domain for some rank-2 lattice, the answer is affirmative on account of early work.

Another direction is to restrict the class of sets Ω in \mathbb{R}^3 to be studied. One such recent direction is the following affirmative theorem for the case when Ω is assumed to be a convex polytope: Nir Lev et al [GL17] proved that a spectral convex polytope (i.e., having a Fourier basis) must tile by translations. This implies in particular that Fuglede's conjecture holds true for convex polytopes in \mathbb{R}^3 .

References and reading list. [JKS14b, DJ13a, DJ13b, DHJP13]

Lecture 4. Representations of Cuntz algebras associated to quasi-stationary Markov measures

The fourth talk concerns representations of Cuntz algebras that arise from the action of stochastic matrices on sequences from \mathbb{Z}_n . This action gives rise to an invariant measure, which depending on the choice of stochastic matrices, may satisfy a finite tracial condition. If so, the measure is ergodic under the action of the shift on the sequence space, and thus yields a representation of a Cuntz algebra. The measure provides spectral information about the representation in that equivalent representations of the Cuntz algebras for different choices of stochastic matrices occur precisely when the measures satisfy a certain equivalence condition.

Recursive multiresolutions and basis constructions in Hilbert spaces are key tools in analysis of fractals and of iterated function systems in dynamics: Use of multiresolutions, selfsimilarity, and locality, yield much better pointwise approximations than is possible with traditional Fourier bases. The approach here will be via representations of the Cuntz algebras. It is motivated by applications to an analysis of frequency sub-bands in signal or image-processing, and associated multi-band filters: With the representations, one builds recursive subdivisions of signals into frequency bands.

Concrete realizations are presented of a class of explicit representations. Starting with Hilbert spaces \mathcal{H} , the representations produce recursive families of closed subspaces (projections) in \mathcal{H} , in such a way that "non-overlapping, or uncorrelated, frequency bands" correspond to orthogonal subspaces in \mathcal{H} . Since different frequency bands must exhaust the range for signals in the entire system, one looks for orthogonal projections which add to the identity operator in \mathcal{H} . Representations of Cuntz algebras achieve precisely this: From representations we obtain classification of families of multi-band filters; and representations allow us to deal with non-commutativity as it appears in both time/frequency analysis, and in scale-similarity. The representations further offer canonical selections of special families of commuting orthogonal projections.

References and reading list. [DJ15a, DHJ15]

Lecture 5. The Cuntz relations and kernel decompositions

The fifth talk concerns representations of Cuntz algebras and their relationship to harmonic analysis of measures, particularly singular measures. As will be demonstrated in earlier talks, the constructions of spectral measures often utilize "Cuntz isometries", namely isometries that satisfy the Cuntz relations. This talk will discuss how understanding specific representations of the Cuntz algebras yields information concerning other spectra for a spectral measure. Conversely,

beginning with a representation of a Cuntz algebra, a Markov measure can be associated to the representation which gives spectral information about the representation.

References and reading list. [[DJ14](#), [DJ12b](#), [DJ12a](#)]

Lecture 6. Harmonic analysis of wavelet filters: input-output and state-space models

The sixth talk will focus on the connections between harmonic analysis on fractals and the cascade algorithm from wavelet theory. Wavelets have a dual existence between the discrete and continuous realms manifested in the discrete and continuous wavelet transforms. Wavelet filters give another bridge between the smooth and non-smooth domains in that the convergence of the cascade algorithm yields wavelets and wavelet transforms in a smooth setting, i.e. \mathbb{R}^d , and also the non-smooth setting such as the Cantor dust, depending on the parameters embedded in the choice of wavelet filters.

References and reading list. [[AJL16](#), [AJL15](#), [AJLM13](#), [BJMP05](#)]

Lecture 7. Spectral theory for Gaussian processes: reproducing kernels, boundaries, and L^2 -wavelet generators with fractional scales

The seventh talk concerns Gaussian processes for whose spectral (meaning generating) measure is spectral (meaning possesses orthogonal Fourier bases). These Gaussian processes admit an Itô-like stochastic integration as well as harmonic and wavelet analyses of related Reproducing Kernel Hilbert Spaces.

References and reading list. [[AJK15](#), [AJ12](#), [AJL11](#), [JS09](#), [DJ06a](#)]

Lecture 8. Reproducing Kernel Hilbert Spaces arising from groups

The eighth talk concerns Reproducing Kernel Hilbert Spaces that appear in the study of spectral measures. Spectral measures give rise to positive definite functions via the Fourier transform. Reversing this process will be the focus of the ninth talk. This talk will set the stage by discussing Reproducing Kernel Hilbert Spaces that appear in the context of positive definite functions, and the harmonic analysis of those Reproducing Kernel Hilbert Spaces.

References and reading list. [[JPT16](#), [DJ09](#), [DJ08](#), [DJ06b](#)]

Lecture 9. Extensions of positive definite functions

The ninth talk will consider the question of spectral measures from the perspective of positive definite functions. Since the measures are spectral, the corresponding positive definite functions have special properties in terms of their zero sets. This correspondence leads to the natural question of whether this process can be reversed. Bochner's theorem implies that positive definite functions are the Fourier transform of measures, but whether those measures are spectral becomes a subtle problem. Thus, by considering certain functions on appropriate subsets, the question of spectrality can be formulated as whether the function can be extended to a positive definite function. The answer is sometimes yes, using the harmonic analysis of Reproducing Kernel Hilbert Spaces.

References and reading list. [[JT15](#), [JP10](#), [DJ10](#), [CBG16](#)]

Lecture 10. Reflection positive stochastic processes indexed by Lie groups

The tenth talk will focus on stochastic processes that appear in the representation theory of Lie groups. Motivated by reflection symmetries in Lie groups, this talk will consider representation theoretic aspects of reflection positivity by discussing reflection positive Markov processes indexed by Lie groups, measures on path spaces, and invariant Gaussian measures in spaces of distribution vectors. This provides new constructions of reflection positive unitary representations.

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of reflection positivity has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

The topics will be outlined in lecture 10, and there has been recent Oberwolfach conferences in the field, where Jorgensen has played a leading role.

References and reading List. [JNO16, She15]

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Lecture 1. Harmonic Analysis of Measures: Analysis on Fractals

By Palle Jorgensen

Abstract

The existence of orthogonal Fourier bases for classes of fractals came as somewhat of a surprise, referring to the 1998 Jorgensen-Pedersen paper. There are several reasons for why existence of orthogonal Fourier bases might have been unexpected: For one, existence of orthogonal Fourier bases, as in the classical case of Fourier, tends to imply a certain amount of “smoothness” which seems inconsistent with fractal geometries, and fractal dimension. Nonetheless, when feasible, such a orthogonal Fourier analysis holds out promise for applications to large chaotic systems, or to analysis of noisy signals; areas that had previously resisted analysis by Fourier tools.

When Fourier duality holds, it further yields a duality of scale, fractal scales in the small, and for the dual frequency domain, fractals in the large.

1. Tiling in \mathbb{R}^N , Fuglede, Poisson summation, and Shannon interpolation

Setting

- ▶ \mathbb{R}^N , $N \in \mathbb{N}$ fixed;
- ▶ lattice Γ , dual lattice $\Gamma^0 := \Lambda$.
- ▶ Fourier exponentials

$$e_\lambda(x) := e^{i2\pi\lambda \cdot x}, \quad \lambda \cdot x = \sum_{j=1}^N \lambda_j x_j. \quad (1.1)$$

Assume $\Omega \subset \mathbb{R}^N$ is a fundamental domain for Γ , i.e.,

$$\Omega \dot{+} \Gamma = \mathbb{R}^N. \quad (1.2)$$

(“ $\dot{+}$ ” denotes zero overlap in measure.)

Consider functions F on \mathbb{R}^N with suitable restrictions, for example, s.t.

$$f(x) := \sum_{\gamma \in \Gamma} F(x + \gamma) \quad (1.3)$$

is well defined.

Poisson (normalized version).

$$\begin{aligned} \sum_{\gamma \in \Gamma} F(x + \gamma) &= \sum_{\lambda \in \Lambda} \left(\int_{\Omega} f(y) \overline{e_{\lambda}(y)} dy \right) e_{\lambda}(x) \\ &= \sum_{\lambda \in \Lambda} \left(\int_{\mathbb{R}^N} F(y) \overline{e_{\lambda}(y)} dy \right) e_{\lambda}(x) \\ &= \sum_{\lambda \in \Lambda} \widehat{F}(\lambda) e_{\lambda}(x). \end{aligned} \quad (1.4)$$

Shannon.

Assumption as above on Γ , Λ , and F , but now assume also that \widehat{F} is supported in Ω , and $\widehat{F} \in L^2(\Omega)$. Then

$$\begin{aligned} F(x) &= \sum_{\lambda \in \Lambda} \left(\int_{\Omega} f \bar{e}_{\lambda} \right) e_{\lambda}(x) \\ &\stackrel{\text{by (1.3)\&(1.4)}}{=} \sum_{\lambda \in \Lambda} F(\lambda) \int_{\Omega} e_y(x - \lambda) dy \\ &= \sum_{\lambda \in \Lambda} F(\lambda) \widehat{\chi}_{\Omega}(x - \lambda). \end{aligned}$$

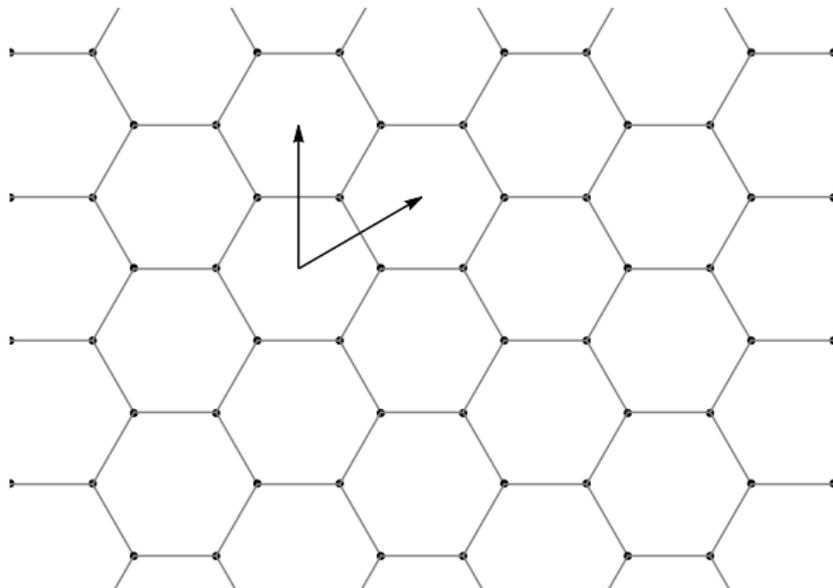
2. Spectral Pairs

Definition. Let μ be a positive measure support in \mathbb{R}^N , and let Λ be a (discrete) subset of \mathbb{R}^N ; we say that (μ, Λ) is a *spectral pair* iff (D) $\{e_\lambda ; \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\mu)$. The case when $d\mu(x) = \chi_\Omega(x) (dx)^N$ is of special interest; we shall say that (Ω, Λ) is a spectral pair.

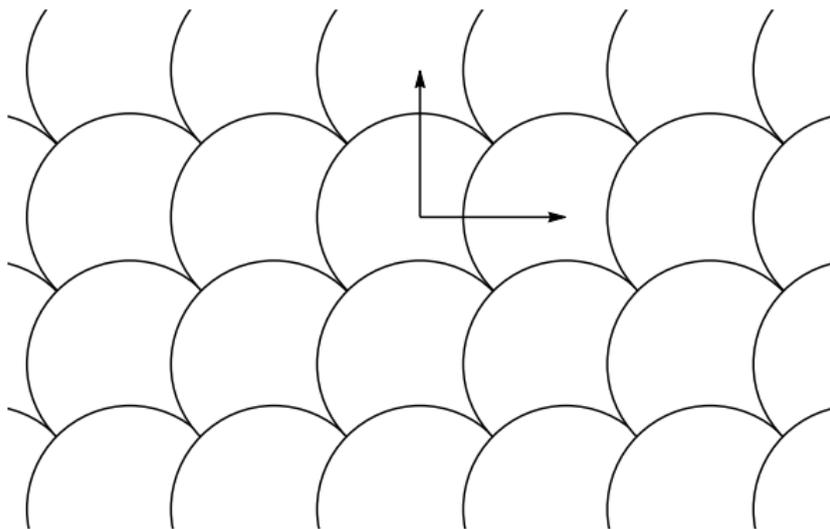
Example 2.1. $N = 1, \Omega = (0, 1) \cup (2, 3)$.



Example 2.2. $N = 2$



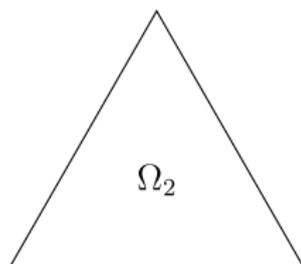
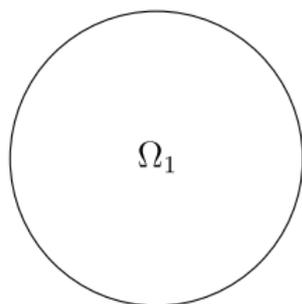
Example 2.3. $N = 2$



The case of $N = 2$ continued

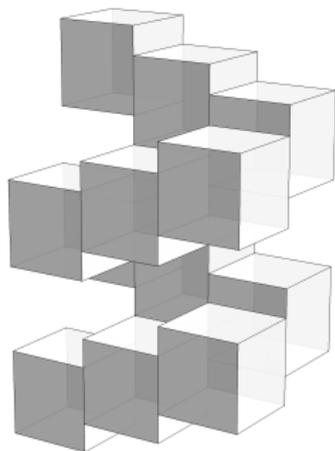
A subset $\Omega \subset \mathbb{R}^2$ is said to be admissible iff (D) there is an $\Lambda \subset \mathbb{R}^2$ such that (Ω, Λ) is a spectral pair w.r.t. Lebesgue measure.

Here are two examples of non-admissible sets Ω :



- ▶ Ω_1 : \exists at most a finite number of orthogonal e_λ functions in $L^2(\Omega_1)$.
- ▶ Ω_2 : many choices of infinite sets Λ s.t. $\{e_\lambda \mid \lambda \in \Lambda\}$ is orthogonal in $L^2(\Omega_2)$ but non is total.

Example 2.4. $N = 3$.



$\Omega = \bigcup_{i=1}^{12} Q_i$,
connected and open.

- ▶ First 3 cubes in front, moving from left to right; then the next 3 move back by one unit, moving now from right to left; and the next 3 then return to the front, moving from left to right; and finally the last 3 move back again, then moving back to the left.
- ▶ Front vs back is indicated with a y -coordinate.
- ▶ In each move from cube 1 through cube 12, the vertical z -coordinate increases by $1/3$. This ensures that the union of the 12 is a connected open set.

- ▶ The idea is to get the union Ω of the 12 cubes to be connected. We will thus get a connected set $\Omega \subset \mathbb{R}^3$, which has a spectrum that is not a (rank-3) lattice. None of the sets which serve as spectrum for this set Ω can be chosen to be a lattice.
- ▶ There is no example known in the plane: open, connected, and having a spectrum which is not a rank-2 lattice.

3. Dense analytic subspaces in fractal L^2 -spaces

The first example of a fractal which admits an orthogonal Fourier expansion

Example 3.1. Let C_4 be the 4-ary Cantor set, and $\mu = \mu_4 =$ fractal measure, determined by

$$\int f d\mu = \frac{1}{2} \left(\int f \left(\frac{x}{4} \right) d\mu(x) + \int f \left(\frac{x}{4} + \frac{1}{2} \right) d\mu(x) \right)$$

for all continuous f . Hausdorff dimension: $d_H = \frac{\ln 2}{\ln 4} = \frac{1}{2}$.

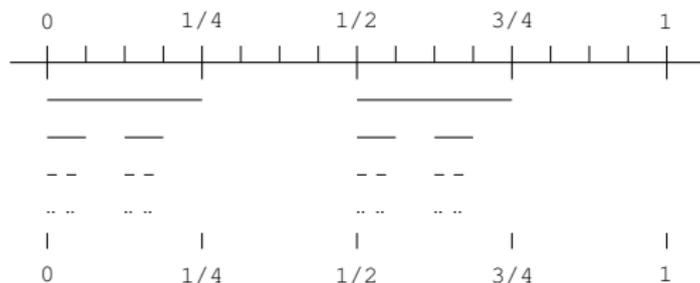


Figure 3.1: Support of μ

A more general IFS-formula: Some spectral, and some not.

General construction in ν dimensions ($\nu \geq 1$)

- ▶ $B \subset \mathbb{R}^\nu$: finite subset
- ▶ R : real matrix, has eigenvalues ξ_i , satisfying

$$|\xi_i| > 1. \quad (3.1)$$

- ▶ Introduce

$$\sigma_b x = R^{-1}x + b, \quad x \in \mathbb{R}^\nu. \quad (3.2)$$

It is assumed that there is a nonempty, bounded open set V such that

$$\bigcup_{b \in B} \sigma_b V \subset V \quad (3.3)$$

with the union disjoint corresponding to distinct points in B .

Uniform distribution affine IFS

- ▶ If $N = \#(B)$, then the corresponding measure μ on \mathbb{R}^ν (depending on R and B) has compact support, and satisfies

$$\int f d\mu = \frac{1}{N} \sum_{b \in B} \int f(\sigma_b(x)) d\mu(x) \quad (3.4)$$

for all continuous f .

A class of Hadamard matrices

Orthogonal Frequencies and Fractal Hardy Spaces

Assume that the matrix R in (3.2) has integral entries, and that

$$RB \subset \mathbb{Z}^\nu, \quad 0 \in B, \quad (3.5)$$

but that none of the differences $b - b'$ is in \mathbb{Z}^ν when $b, b' \in B$ are different. Furthermore, assume that some subset $L \subset \mathbb{Z}^\nu$ satisfies $0 \in L$, $\#(L) = N (= \#(B))$, and the matrix

$$H_{BL} := N^{-\frac{1}{2}} \left(e^{i2\pi b \cdot l} \right) \text{ is unitary as an } N \times N \text{ complex matrix,}$$

i.e., $H_{BL}^* H_{BL} = I_N$ (* = transposed conjugate). (3.6)

Hardy space H_2 and fractals

Theorem 3.2 (Jor-Pedersen). Let $H_2(P, \mu)$ be the closed span in $L^2(\mu)$ of the functions

$$\{e^{i2\pi nx} : n = 0, 1, 4, 5, 16, 17, 20, 21, \dots\},$$

i.e., $P = \{l_0 + 4l_1 + 4^2l_2 + \dots : l_i \in \{0, 1\}, \text{ finite sums}\}$. Then

$$H_2(P, \mu) = L^2(\mu). \quad (3.7)$$

A multi-resolution in H_2

Corollary 3.3. There is a canonical isometric embedding Φ of $L^2(\mu)$ into the subspace $H_2(z^4) + zH_2(z^4)$ of H_2 where $H_2(z^4) := \{f(z^4) : f \in H_2\}$; and it is given by

$$\Phi\left(\sum_{\lambda \in P} c_\lambda e_\lambda\right) = \sum_{n \in P} c_{4n} z^{4n} + z \sum_{n \in P} c_{4n+1} z^{4n}. \quad (3.8)$$

Fractal Hardy spaces

- ▶ For each $n \in \mathbb{N}$, there is a natural *isometric* embedding Φ_n of $L^2(\mu)$ into the subspace of H_2 characterized as n increases by:

$$H_2(z^{4^n}) + zH_2(z^{4^n}) + z^4H_2(z^{4^n}) + z^5H_2(z^{4^n}) \\ + z^{16}H_2(z^{4^n}) + z^{17}H_2(z^{4^n}) \cdots + z^{\frac{4^n-1}{3}}H_2(z^{4^n}).$$

- ▶ Specifically, let $n \in \mathbb{N}$ be fixed, and let $P_n = \{l_0 + 4l_1 + \cdots + 4^{n-1}l_{n-1} : l_i \in \{0, 1\}\}$. Then the functions in $\Phi_n(L^2(\mu))$ ($\subset H_2$) have the following characteristic module representation:

$$\left\{ \sum_{p \in P_n} z^p f_p(z^{4^n}) : f_p \in H_2 \right\}.$$

For each n , Φ_n maps *into* this space, and not onto.

Spectral pairs

Definition 3.4. Consider subsets Ω and Λ in \mathbb{R}^ν , with Ω of finite positive Lebesgue measure, and let $L^2(\Omega)$ be the corresponding Hilbert space from the Ω -restricted and normalized ν -dimensional Lebesgue measure. Let $e_\lambda(x) := e^{i2\pi\lambda \cdot x}$ on Ω .

► We say that (Ω, Λ) is a *spectral pair* if

$$\{e_\lambda : \lambda \in \Lambda\}$$

is an orthonormal basis in $L^2(\Omega)$.

Generalized spectral pair

Definition 3.5. Let G be a locally compact abelian group with dual group Γ . Let μ be a Borel measure on G , and ρ one on Γ . For f of compact support and continuous, introduce

$$F_{\mu}f(\xi) = \int_G \overline{\langle \xi, x \rangle} f(x) d\mu(x)$$

where $\langle \xi, x \rangle$ denotes the pairing between points ξ in Γ and x in G .

- ▶ If $f \mapsto F_{\mu}f$ extends to an isomorphic isometry (i.e., unitary) of $L^2(\mu)$ onto $L^2(\rho)$, then we say that (μ, ρ) is a *spectral pair*.

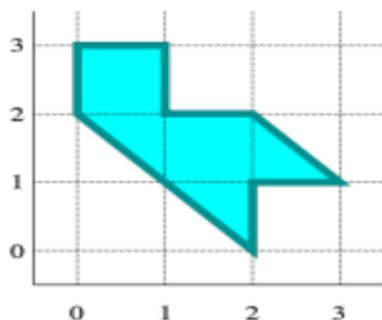
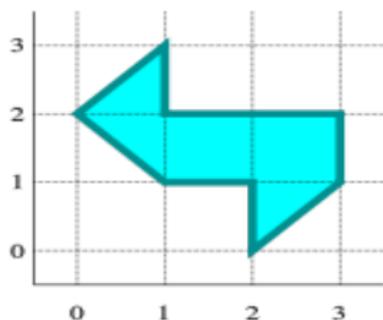
Fractals in the large: lacunary expansions

Remark 3.6. It is not immediate that there are examples (μ, ρ) of the new spectral pair type which cannot be reduced to the old one. Theorem 3.2 shows that this is indeed the case: Let $G = \mathbb{R}$, and let μ be the fractal measure. Let

$$\begin{aligned} P &= \{l_0 + 4l_1 + 4^2l_2 + \cdots : l_i \in \{0, 1\}, \text{ finite sums}\} \\ &= \{0, 1, 4, 5, 16, 17, \cdots\}, \end{aligned}$$

and let $\rho = \rho_P$ be the counting measure of P . By Theorem 3.2, (μ, ρ_P) is a spectral pair.

4. Spectral pairs in Cartesian coordinates



Extensions of symmetric operators

Definition 4.1. If $\Omega \subset \mathbb{R}^d$ is open, then we consider $\frac{\partial}{\partial x_j}$, $j = 1, \dots, d$, defined on $C_c^\infty(\Omega)$ as unbounded skew-symmetric operators in $\mathcal{L}^2(\Omega)$. We say that Ω has the *extension property* if there are commuting self-adjoint extension operators H_j , i.e.,

$$\frac{1}{i} \frac{\partial}{\partial x_j} \subset H_j, \quad j = 1, \dots, d. \quad (4.1)$$

Theorem 4.2 (Fuglede, Jorgensen, Pedersen). Let $\Omega \subset \mathbb{R}^d$ be open and connected with finite and positive Lebesgue measure. Then Ω has the extension property if and only if it is a spectral set. If Ω is only assumed open, then the spectral-set property implies the extension property, but not conversely.

Cubes in dimension $d > 1$

Conjecture. Let $L \subset \mathbb{R}^d$. Then (I^d, L) is a spectral pair if and only if (I^d, L) is a tiling pair. (I^d = the d -dimensional unit cube.)

Dimensions Two and Three

Proposition 4.3. The only subsets $\Lambda \subset \mathbb{R}$ such that (I, Λ) is a spectral pair are the translates

$$\Lambda_\alpha := \alpha + \mathbb{Z} = \{\alpha + n : n \in \mathbb{Z}\} \quad (4.2)$$

where α is some fixed real number.

Theorem 4.4 (Jor-Pedersen). The only subsets $\Lambda \subset \mathbb{R}^2$ such that (I^2, Λ) is a spectral pair must belong to either one or the other of the two classes, indexed by a number α , and a sequence $\{\beta_m \in [0, 1) : m \in \mathbb{Z}\}$, where

$$\Lambda = \left\{ \begin{pmatrix} \alpha + m \\ \beta_m + n \end{pmatrix} : m, n \in \mathbb{Z} \right\}, \text{ or} \quad (4.3)$$

$$\Lambda = \left\{ \begin{pmatrix} \beta_n + m \\ \alpha + n \end{pmatrix} : m, n \in \mathbb{Z} \right\}. \quad (4.4)$$

Each of the two types occurs as the spectrum of a pair for the cube I^2 , and each of the sets Λ as specified is a tiling set for the cube I^2 .

Theorem 4.5 (Jor-Pedersen). (I^3, Λ) is a spectral pair iff after a possible translation by a single vector and a possible permutation of the coordinates (x_1, x_2, x_3) , Λ can be brought into the following form: there is a partition of \mathbb{Z} into disjoint subsets A, B (one possibly empty) with associated functions

$$\begin{aligned} \alpha_0: A &\longrightarrow [0, 1), & \beta_0: B &\longrightarrow [0, 1), \\ \alpha_1: A \times \mathbb{Z} &\longrightarrow [0, 1), & \beta_1: B \times \mathbb{Z} &\longrightarrow [0, 1) \end{aligned}$$

such that Λ is the (disjoint) union of

$$\left(\begin{array}{c} a \\ \alpha_0(a) + k \\ \alpha_1(a, k) + l \end{array} \right) \quad \text{and} \quad \left(\begin{array}{c} b \\ \beta_1(b, n) + m \\ \beta_0(b) + n \end{array} \right) \quad (4.5)$$

as $a \in A, b \in B$, and $k, l, m, n \in \mathbb{Z}$.

5. Orthogonal harmonic analysis and scaling of fractal measures

Pairs of measures

Consider pairs of measures (μ, ν) on \mathbb{R}^d such that the following generalized transform,

$$F_\mu f: \lambda \mapsto \int e^{-i2\pi\lambda \cdot x} f(x) d\mu(x), \quad (5.1)$$

induces an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$, specifically making precise the following unitarity:

$$\int |f(x)|^2 d\mu(x) = \int |(F_\mu f)(\lambda)|^2 d\nu(\lambda).$$

New Pairs from Old Pairs

Definition 5.1. Let μ and ν be Borel measures on \mathbb{R}^d . We say that (μ, ν) is a spectral pair if the map F_μ from (5.1) above, defined for $f \in L^1 \cap L^2(\mu)$, extends by continuity to an isometric isomorphism mapping $L^2(\mu)$ onto $L^2(\nu)$.

Remark.

1. It was shown that if μ is the restriction of Lebesgue measure to a connected set of infinite measure, then the result on extensions of the directional derivatives, described above, remains valid.
2. Our work on generalized spectral pairs is motivated by M.N. Kolountzakis and J.C. Lagarias who discuss related tilings of the real line \mathbb{R}^1 by a function.

An uncertainty relation by Heisenberg

Theorem 5.2 (Jor-Pedersen). Suppose (μ, ν) is a spectral pair, $f \in L^2(\mu)$, $f \neq 0$, and $A, B \subset \mathbb{R}^d$. If $\|f - \chi_A f\|_\mu \leq \varepsilon$ and $\|Ff - \chi_B Ff\|_\nu \leq \delta$, then $(1 - \varepsilon - \delta)^2 \leq \mu(A)\nu(B)$.

Theorem 5.3 (Jor-Pedersen). Suppose (μ, ν) is a spectral pair, and $t \in \mathbb{R}^d$. If \mathcal{O} and $\mathcal{O} + t$ are subsets of the support of μ , then $\mu(\mathcal{O}) = \mu(\mathcal{O} + t)$.

Theorem 5.4 (Jor-Pedersen). Suppose (μ, ν) is a spectral pair. If $\mu(\mathbb{R}^d) < \infty$, then ν must be a counting measure with uniformly discrete support.

6. The role of transfer operators and shifts in the study of fractals

A transfer operator R

Definition 6.1. Let B be a compact Hausdorff space, and \mathcal{B} be a σ -algebra of subsets in B . Consider

$$R : C(B) \rightarrow C(B) \text{ or } R : M(B) \rightarrow M(B), \quad (6.1)$$

where $M(B)$ is the set of all measurable functions on B . We say that R is *positive* iff

$$\varphi(x) \geq 0 \text{ for all } x \in B \text{ implies } (R\varphi)(x) \geq 0, \text{ for all } x \in B. \quad (6.2)$$

Theorem 6.2 (Dutkay-Jor). Let R be a positive operator as in (6.1), with $R1 = 1$. Then for each $x \in B$ there exists a unique Borel probability measure \mathbb{P}_x on $B^{\mathbb{N}}$ such that

$$\int_{B^{\mathbb{N}}} \varphi_1 \otimes \cdots \otimes \varphi_n d\mathbb{P}_x = (M_{\varphi_1} R M_{\varphi_2} \cdots R M_{\varphi_n} 1)(x), \quad (6.3)$$

where $\varphi_i \in C(B)$, $n \in \mathbb{N}$.

Corollary 6.3. Let B and $R : C(B) \rightarrow C(B)$ be as in Theorem 6.2, and let $\mu \in M_1(B)$ be given. Let $\Sigma = \Sigma^{(\mu)}$ be the measure on $\Omega = B^{\mathbb{N}}$ given by

$$\int f d\Sigma := \int_B \int_{\pi_1^{-1}(x)} f d\mathbb{P}_x d\mu(x). \quad (6.4)$$

Then

1. $V_1 : L^2(B, \mu) \rightarrow L^2(\Omega, \Sigma)$ given by $V_1\varphi := \varphi \circ \pi_1$ is isometric.
2. For its adjoint operator V_1^* , we have $V_1^* : L^2(\Omega, \Sigma) \rightarrow L^2(B, \mu)$ with

$$(V_1^* f)(x) = \int_{\pi_1^{-1}(x)} f d\mathbb{P}_x. \quad (6.5)$$

Theorem 6.4 (Dutkay-Jor). With $B, R, \mu, \Sigma = \Sigma^{(\mu)}$ and V_n specified as above, we have the following formulas:

1. $V_1^* V_{n+1} = R^n$ on $L^2(B, \mu)$, $n = 0, 1, 2, \dots$;
2. $Q_1 := V_1 V_1^*$ is a conditional expectation onto

$$\mathcal{A}_1 := \{\varphi \circ \pi_1 : \varphi \in L^\infty(B, \mu)\}$$

$$Q_1((\varphi \circ \pi_1)f) = (\varphi \circ \pi_1)Q_1(f) \quad (6.6)$$

for all $\varphi \in L^\infty(B, \mu)$, $f \in L^\infty(\Omega, \Sigma)$.

3. $Q_1(\varphi \circ \pi_{n+1}) = (R^n \varphi) \circ \pi_1$, for all $\varphi \in C(B)$, $n = 0, 1, 2, \dots$

7. Isospectral measures

Theorem 7.1 (Dutkay-Jor). Let μ be a Borel probability measure on \mathbb{R}^d . The following statements are equivalent:

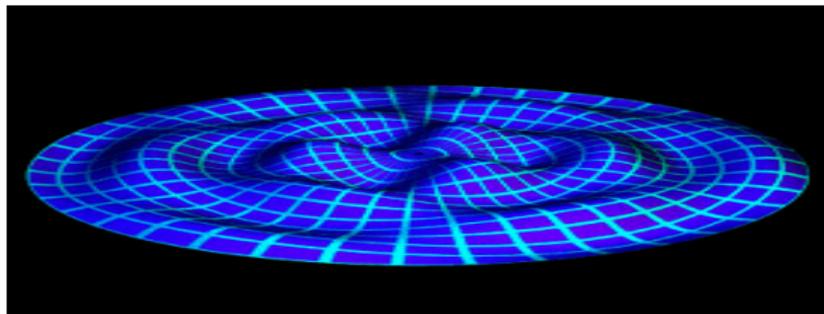
1. The set $\{e_n : n \in \mathbb{Z}^d\}$ forms an orthonormal set in $L^2(\mu)$.
2. There exists a bounded measurable function $\varphi \geq 0$ that satisfies

$$\sum_{k \in \mathbb{Z}^d} \varphi(x + k) = 1, \text{ for Lebesgue a.e. } x \in \mathbb{R}^d, \quad (7.1)$$

such that $d\mu = \varphi dx$.

Theorem 7.2 (Dutkay-Jor). Let μ be a Borel probability measure on \mathbb{R}^d . Then μ has spectrum \mathbb{Z}^d iff μ is the Lebesgue measure restricted to a set E which is translation congruent to $Q := [0, 1)^d$.

8. Affine fractals as boundaries and their harmonic analysis



Affine IFSs

Definition 8.1. Let R be a $d \times d$ expansive real matrix, i.e., all its eigenvalues have absolute value strictly bigger than one. Let B be a finite subset of \mathbb{R}^d . We define the *affine iterated function system (IFS)* denoted (R, B) :

$$\tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d, b \in B) \quad (8.1)$$

The unique Borel probability measure μ_B with the property that

$$\mu_B(E) = \frac{1}{\#B} \sum_{b \in B} \mu_B(\tau_b^{-1}(E)), \quad (8.2)$$

for all Borel sets in \mathbb{R}^d is called *the invariant measure* for the affine IFS (R, B) for details.

Hadamard system

Definition 8.2. Let R be a $d \times d$ matrix, and B, L two finite subsets of \mathbb{R}^d . We call (R, B, L) a *Hadamard system* if $\#B = \#L$ and the matrix

$$\frac{1}{\sqrt{\#B}} \left(e^{2\pi i R^{-1}b \cdot l} \right)_{b \in B, l \in L} \quad (8.3)$$

is unitary.

Theorem 8.3 (Dutkay-Jor). Let μ be a probability measure on \mathbb{R} and assume $\Gamma \subset \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is a spectrum for μ . Then

1. The map $J : L^2(\mu) \rightarrow H^2$

$$J e_\gamma = z^\gamma, \quad (\gamma \in \Gamma) \quad (8.4)$$

extends to an isometric embedding of $L^2(\mu)$ into H^2 .

2. Define the map G on $\mathbb{D} \times \mathbb{R}$

$$G(z, x) := \sum_{\gamma \in \Gamma} \bar{z}^\gamma e_\gamma(x), \quad (z \in \mathbb{D}, x \in \mathbb{R}) \quad (8.5)$$

Then

$$(Jf)(z) = \int f(x) \overline{G(z, x)} d\mu(x) = \langle G(z, \cdot), f \rangle_{L^2(\mu)}, \quad (z \in \mathbb{D}) \quad (8.6)$$

Theorem 8.3 cont.

3. Assume in addition that $\Gamma = R\Gamma + L$ for some $R \in \mathbb{N}$, $R \geq 2$ and some finite set $L \subset \mathbb{N}_0$ such that no two elements in L are congruent modulo R . Then

$$\overline{G}(z, x) = \prod_{n=0}^{\infty} \left(\sum_{l \in L} z^{R^n l} \overline{e}_l(R^n x) \right), \quad (z \in \mathbb{D}, x \in \mathbb{R}). \quad (8.7)$$

The infinite product is uniformly convergent for z in a compact subsets of \mathbb{D} and $x \in \mathbb{R}$.

Theorem 8.4 (Dutkay-Jor). Suppose A is a subset of \mathbb{N}_0 such that there exists $A' \subset \mathbb{N}_0$ and $R \in \mathbb{N}, R \geq 2$ such that $A \oplus A' = \{0, \dots, R - 1\}$ and $A, A' \neq \{0\}$. Then

1. There exists finite subsets $L, L' \subset \{0, \dots, R - 1\}$ such that $L \oplus L' = \{0, \dots, R - 1\}$ and with the property that (R, A, L) and (R, A', L') are Hadamard systems. Also $\gcd(A)$ divides R . The set L can be picked such that $\gcd(A) \cdot \max(L) < R$. Similarly for L' . Here $\gcd(A)$ represents the greatest common divisor of A .

Theorem 8.4 cont.

2. Let μ_A be the invariant measure associated to the IFS (R, A) and similarly for $\mu_{A'}$. Then the convolution $\mu_A * \mu_{A'} = \lambda|_{[0,1]}$ = the Lebesgue measure restricted to $[0, 1]$.
3. μ_A is spectral with spectrum $\Gamma(L) = \{\sum_{k=0}^n R^k l_k : l_k \in L, n \in \mathbb{N}_0\}$ and similarly $\mu_{A'}$ is spectral with spectrum $\Gamma(L')$.
4. The kernels satisfy the following relation

$$G_{\Gamma(L)} G_{\Gamma(L')} = k, \quad (8.8)$$

where k is the classical Szegő kernel.

9. Additive spectra of the $1/4$ Cantor measure

Definition 9.1.

1. $L^2(\mu_{\frac{1}{4}})$: $\mu_{\frac{1}{4}}$ = the $\frac{1}{4}$ -Bernoulli convolution measure
2. Γ : spectrum of $\mu_{\frac{1}{4}}$
3. $e_t(\cdot) := e^{2\pi it(\cdot)}$
4. $E(\Gamma) = \{e_\gamma : \gamma \in \Gamma\}$

Note. $E(\Gamma)$ is an ONB for $L^2(\mu_{\frac{1}{4}})$ exactly when the function

$$c_\Gamma(t) := \sum_{\gamma \in \Gamma} |\langle e_t, e_\gamma \rangle|^2 = \sum_{\gamma \in \Gamma} \prod_{k=1}^{\infty} \cos^2\left(\frac{2\pi(t - \gamma)}{4^k}\right) \quad (9.1)$$

is the constant function 1. We call the function c_Γ the *spectral function* for the set Γ .

Additive Spectrum

A connection between the scaled spectrum $p\Gamma$ and what we call an *additive spectrum* $E(4\Gamma) \cup E(4\Gamma + p)$ is as follows:

Theorem 9.2 (Jorgensen, Kornelson, Shuman). Given any odd natural number p , if $E(p\Gamma)$ is an ONB then $E(4\Gamma) \cup E(4\Gamma + p)$ is also an ONB.

Theorem 9.3. Given any odd integer p , the set $E[(4\Gamma) \cup (4\Gamma + p)]$ is an ONB for $L^2(\mu)$.

10. Positive Matrices in the Hardy Space
with Prescribed Boundary
Representations via the Kaczmarz
Algorithm

Expansions for general singular measures μ in an ambient dimension $d = 1$

- ▶ Let $e_\lambda(x) := e^{2\pi i \lambda x}$. It is known that if μ is a singular probability measure, then the sequence $\{g_n\}_{n=0}^\infty$ defined by
- $$(10.1)$$

is a Parseval frame in $L^2(\mu)$ satisfying

$$\sum_{n=0}^{\infty} \langle f, g_n \rangle e_n = f \quad (10.2)$$

in norm for all $f \in L^2(\mu)$.

- ▶ Equations (10.1) and (10.2) are referred to as the Kaczmarz algorithm. (10.2) can be interpreted as a Fourier expansion of $f \in L^2(\mu)$.

Kernels and frames

Theorem 10.1 (Herr, Jorgensen, Weber). Let μ be a singular probability measure with corresponding inner function b and associated sequence $\{g_n\}_{n=0}^{\infty} \subset L^2(\mu)$ defined by (10.1). Then

$$k_z^b(w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle g_n, g_m \rangle_{\mu} \bar{z}^n w^m. \quad (10.3)$$

Theorem 10.2 (Herr, Jorgensen, Weber). Let μ be a finite singular measure on $[0, 1)$, and let b be the inner function corresponding to μ via the Herglotz representation. Then for any $f \in H^2$, there exists a unique sequence of functions $\{\phi_n\}_{n=0}^\infty \subset \mathcal{H}(b)$ such that

$$f = \sum_{n=0}^{\infty} \phi_n \cdot b^n.$$

11. A matrix characterization of boundary representations of positive matrices in the Hardy space

Kernels and Hardy-space

Definition 11.1. Let μ be a Borel measure on $[0, 1)$. We define $\mathcal{K}(\mu)$ to be the set of positive matrices K on \mathbb{D} such that for each fixed $z \in \mathbb{D}$, $K(w, \cdot)$ possesses an $L^2(\mu)$ -boundary $K^*(w, \cdot)$, and $K(w, z)$ reproduces itself with respect to integration of these $L^2(\mu)$ -boundaries, i.e.

$$K(w, z) = \int_0^1 K^*(w, x) \overline{K^*(z, x)} d\mu(x) \quad (11.1)$$

for all $z, w \in \mathbb{D}$.

Definition 11.2. Let K be a positive matrix on \mathbb{D} . We define $\mathcal{M}(K)$ to be the set of nonnegative Borel measures μ on $[0, 1)$ such that for each fixed $z \in \mathbb{D}$, (11.1) holds.

Theorem 11.3 (Herr, Jorgensen, Weber). Suppose $C = (c_{mn})$ defines a bounded, positive, self-adjoint operator on $\ell^2(\mathbb{N}_0)$. The following are equivalent:

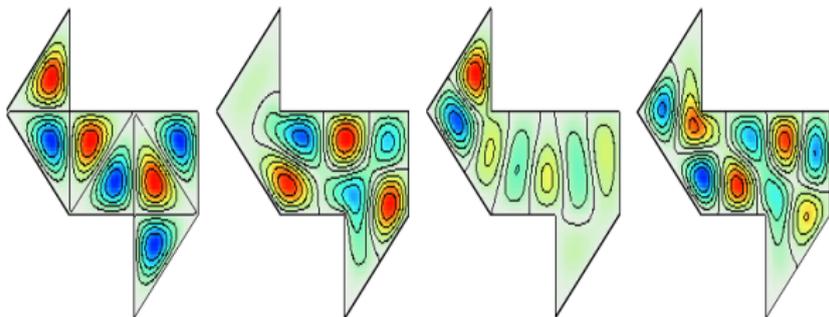
1. $\lambda \in \mathcal{M}(K_C)$;
2. the coefficient matrix C is a projection;
3. the norm induced by K_C is equal to the Hardy space norm in the following sense: for all $\xi_1, \dots, \xi_N \in \mathbb{C}$ and $w_1, \dots, w_N \in \mathbb{D}$,

$$\left\| \sum_{j=1}^N \xi_j K_C(w_j, \cdot) \right\|_{K_C} = \left\| \sum_{j=1}^N \xi_j K_C(w_j, \cdot) \right\|_{H^2} ;$$

Theorem 11.3 cont.

4. there exists a subspace M of the Hardy space such that the Parseval frame $g_n = P_M z^n$ is such that $c_{mn} = \langle g_n, g_m \rangle$;
5. there exists a subspace M of the Hardy space such that the projection of the Szegő kernel onto M is K_C .

12. Scaling by 5 on a $1/4$ -Cantor Measure



Scaling by 5

We consider here a particular additional symmetry relation for the subclass of Cantor-Bernoulli measures that form spectral pairs.

- ▶ Starting with a spectral pair (μ, Γ) , we consider an action which scales the set Γ . In the special case of $\mu_{\frac{1}{4}}$, we scale Γ by 5.
- ▶ Scaling by 5 induces a natural unitary operator U in $L^2(\mu_{\frac{1}{4}})$, and we study the spectral-theoretic properties of U .

Definition 12.1. Set

$$\begin{aligned}\Gamma &= \left\{ \sum_{i=0}^m a_i 4^i : a_i \in \{0, 1\}, m \text{ finite} \right\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \dots\}.\end{aligned}\tag{12.1}$$

(Jorgensen and Pedersen showed that Γ is a spectrum for $\mu := \mu_{\frac{1}{4}}$.) Consider

$$5\Gamma = \{0, 5, 20, 25, 80, 85, 100, 105, 320, \dots\}.$$

Definition 12.2. The 5–scaling property for the ONB (12.1) induces a unitary operator U in $L^2(\mu)$, acting on the orthonormal basis $E(\Gamma)$ by

$$U(e_\gamma) := e_{5\gamma}. \quad (12.2)$$

Theorem 12.3 (Jorgensen, Kornelson, Shuman). If $Uv = v$, with $\|v\| = 1$, then $v = \alpha e_0$ for some $\alpha \in \mathbb{T}$, i.e. U is an ergodic operator.

Remark. The theorem states that the only functions which are fixed by U are the constant functions—in other words, U is an ergodic operator in the sense of Halmos.

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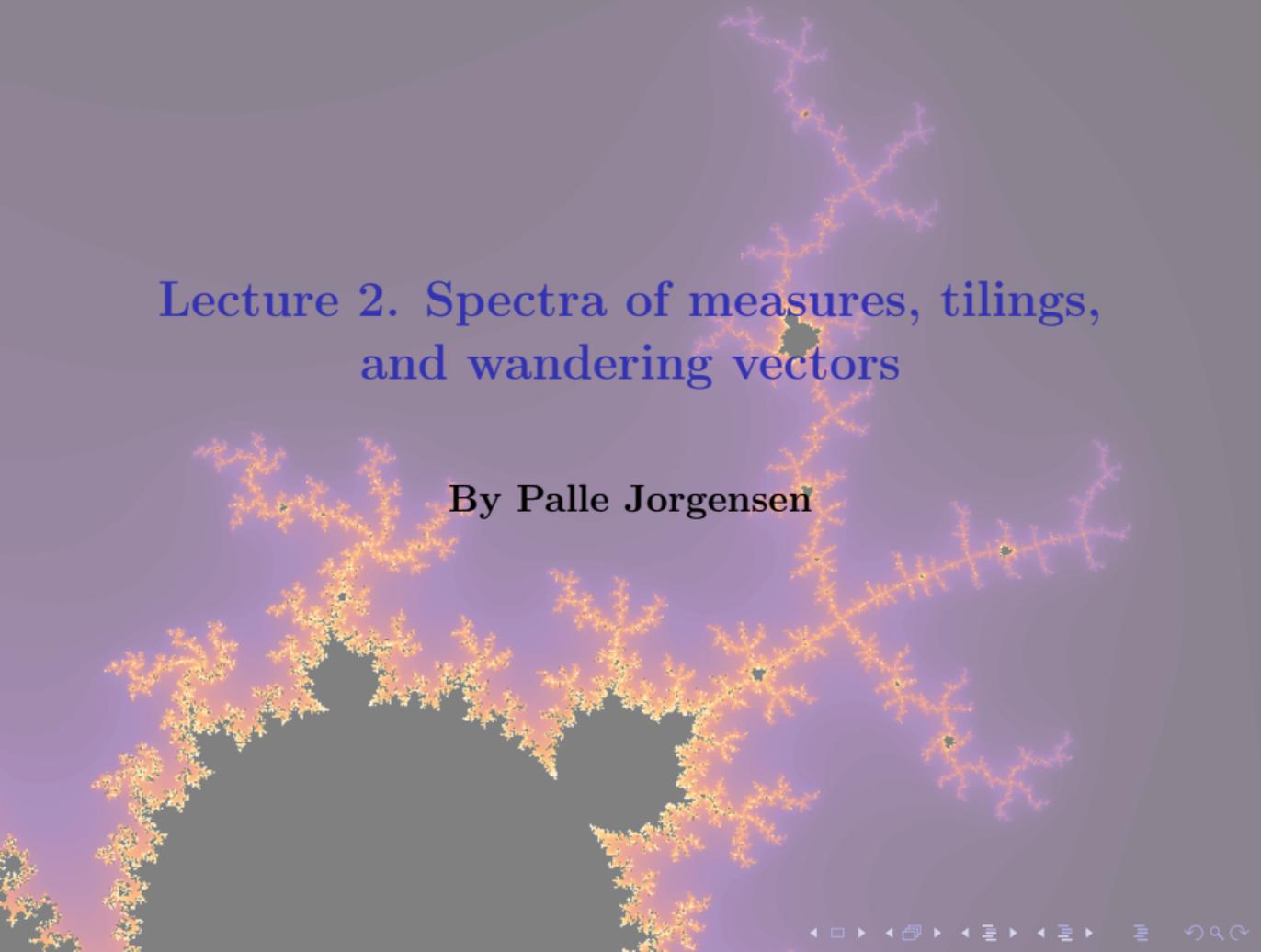
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Lecture 2. Spectra of measures, tilings, and wandering vectors

By Palle Jorgensen

Abstract

There is an intimate relations between systems of tiling by translations on the one hand, and orthogonal Fourier bases on the other. Representation theory makes a link between the two, but the tile-spectral question is deep and difficult; so far only partially resolved. One tool of inquiry is that of “wandering vectors” or wandering subspaces. The term “wandering” has its origin in the study of systems of isometries in Hilbert space. It has come to refer to certain actions in a Hilbert space which carries representations: When the action generates orthogonal vectors, we refer to them as wandering vectors; similarly for closed subspaces. In the case of representations of groups, this has proved a useful way of generating orthogonal Fourier bases; — when they exist. In the case of representations of the Cuntz algebras, the “wandering” idea has become a tool for generating nested and orthogonal subspaces. The latter includes multiresolution subspaces for wavelet systems and for signal/image processing algorithms.

1. Spectra of measures and wandering vectors

Wandering vectors

Definition 1.1. Let \mathcal{U} be a family of unitary operators acting on a Hilbert space \mathcal{H} . We say that a vector $v_0 \neq 0$ in \mathcal{H} is a *wandering vector* if $\{Uv_0 : U \in \mathcal{U}\}$ is an orthogonal family of vectors.

Definition 1.2. Let Γ be a locally compact abelian group and let $G = \widehat{\Gamma}$ be its dual group. For a point $\gamma \in \Gamma$, write

$$\langle \gamma, g \rangle = e_\gamma(g), \quad (g \in G). \quad (1.1)$$

We say that a subset S of Γ is a *spectrum* for a Borel probability measure μ_0 on G if the set $\{e_\gamma : \gamma \in S\}$ is an orthonormal basis for $L^2(\mu_0)$.

Fourier frames

Definition 1.3. Let $A, B > 0$ and let \mathcal{H} be a Hilbert space a family of vectors $\{e_i : i \in I\}$ in \mathcal{H} is called a *frame with bounds* A, B if

$$A\|f\|^2 \leq \sum_{i \in I} |\langle e_i, f \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

Definition 1.4. A subset S of Γ is a *frame spectrum* with bounds A, B for a Borel probability measure μ_0 on G if the set $\{e_\gamma : \gamma \in S\}$ is a frame with bounds A, B for $L^2(\mu_0)$.

Which sets appear as spectra of some measure?

Theorem 1.5 (Dutkay-Jor). Let $S \subset \Gamma$ be an arbitrary subset. Then the subset S is a spectrum/frame spectrum with bounds A, B for a Borel probability measure μ_0 on G if and only if there exists a triple (\mathcal{H}, v_0, U) where \mathcal{H} is a complex Hilbert space, $v_0 \in \mathcal{H}$, $\|v_0\| = 1$ and $U(\cdot)$ is a strongly continuous representation of Γ on \mathcal{H} such that $\{U(\gamma)v_0 : \gamma \in S\}$ is an orthonormal basis/frame with bounds A, B for \mathcal{H} .

Moreover, in this case μ_0 can be chosen such that

$$\langle v_0, U(\xi)v_0 \rangle_{\mathcal{H}} = \int_G e_{\xi}(g) d\mu_0(g) \text{ for all } \xi \in \Gamma \quad (1.2)$$

and there is an isometric isomorphism

$$W : L^2(G, \mu_0) \rightarrow \mathcal{H}$$

such that

$$We_{\gamma} = U(\gamma)v_0 \text{ for all } \gamma \in \Gamma. \quad (1.3)$$

2. Unitary groups and spectral sets

Definition 2.1. Let Ω be a bounded Borel subset of \mathbb{R} .

- ▶ A *unitary group of local translations* on Ω is a strongly continuous one parameter unitary group $U(t)$ on $L^2(\Omega)$ s.t. for all $f \in L^2(\Omega)$ and any $t \in \mathbb{R}$,

$$(U(t)f)(x) = f(x+t) \text{ for a.e } x \in \Omega \cap (\Omega - t) \quad (2.1)$$

- ▶ If Ω is spectral with spectrum Λ , we define The Fourier transform $\mathcal{F} : L^2(\Omega) \rightarrow l^2(\Lambda)$

$$\mathcal{F}f = \left(\left\langle f, \frac{1}{\sqrt{|\Omega|}} e_\lambda \right\rangle \right)_{\lambda \in \Lambda}, \quad (f \in L^2(\Omega)). \quad (2.2)$$

- ▶ The unitary group associated to Λ is

$$U_\Lambda(t) = \mathcal{F}^{-1} \hat{U}_\Lambda(t) \mathcal{F}. \quad (2.3)$$

Theorem 2.2 (Dutkay-Jor). Let Ω be a bounded Borel subset of \mathbb{R} . Assume that Ω is spectral with spectrum Λ . Let U_Λ be the associated unitary group as in (2.3). Then $U := U_\Lambda$ is a unitary group of local translations.

Theorem 2.3 (Dutkay-Jor). Let Ω be a bounded Borel subset of \mathbb{R} with $|\Omega| = 1$. Let $p \in \mathbb{N}$. Suppose Ω p -tiles \mathbb{R} by $\frac{1}{p}\mathbb{N}$.

► Then, for a.e. $x \in \mathbb{R}$ the set

$$\Omega_x := \left\{ k \in \mathbb{Z} : x + \frac{k}{p} \in \Omega \right\} \quad (2.4)$$

has exactly p elements

$$\Omega_x = \{k_0(x) < k_1(x) < \dots < k_{p-1}(x)\}. \quad (2.5)$$

For almost every $x \in \mathbb{R}$ there exist unique $y \in [0, \frac{1}{p})$ and $i \in \{0, \dots, p-1\}$ such that $y + \frac{k_i(y)}{p} = x$.

- ▶ The functions k_i have the following property

$$k_i\left(x + \frac{1}{p}\right) = k_i(x) - 1, \quad (x \in \mathbb{R}, i = 0, \dots, p-1). \quad (2.6)$$

- ▶ Consider the space of $\frac{1}{p}$ -periodic vector valued functions $L^2([0, \frac{1}{p}), \mathbb{C}^p)$. The operator $W : L^2(\Omega) \rightarrow L^2([0, \frac{1}{p}), \mathbb{C}^p)$ defined by

$$(Wf)(x) = \begin{pmatrix} f\left(x + \frac{k_0(x)}{p}\right) \\ \vdots \\ f\left(x + \frac{k_{p-1}(x)}{p}\right) \end{pmatrix}, \quad (x \in [0, \frac{1}{p}), f \in L^2(\Omega)), \quad (2.7)$$

is an isometric isomorphism with inverse

$$W^{-1} \begin{pmatrix} f_0 \\ \vdots \\ f_{p-1} \end{pmatrix} (x) = f_i(y), \text{ if } x = y + \frac{k_i(y)}{p}, \quad (2.8)$$

with $y \in [0, \frac{1}{p})$, $i \in \{0, \dots, p-1\}$.

- ▶ A set Λ of the form $\Lambda = \{0 = \lambda_0, \lambda_1, \dots, \lambda_{p-1}\} + p\mathbb{Z}$ is a spectrum for Ω if and only if $\{\lambda_0, \dots, \lambda_{p-1}\}$ is a spectrum for $\frac{1}{p}\Omega_x$ for a.e. $x \in [0, \frac{1}{p})$.
- ▶ The exponential functions are mapped by W as follows:

$$(W e_{\lambda_i + np})(x) = e_{\lambda_i + np}(x) \begin{pmatrix} e_{\lambda_i(\frac{k_0(x)}{p})} \\ \vdots \\ e_{\lambda_i(\frac{k_{p-1}(x)}{p})} \end{pmatrix} =: F_{i,n}(x), \quad (2.9)$$

($i = 0, \dots, p-1, n \in \mathbb{Z}, x \in [0, \frac{1}{p})$).

- For x in \mathbb{R} define the $p \times p$ unitary matrix \mathcal{M}_x which has column vectors

$$v_i(x) := \frac{1}{\sqrt{p}}(e_{\lambda_i}(\frac{k_0(x)}{p}), e_{\lambda_i}(\frac{k_1(x)}{p}), \dots, e_{\lambda_i}(\frac{k_{p-1}(x)}{p}))^t,$$

$$i = 0, \dots, p-1.$$

Let U_Λ be the group of local translations on Ω associated to a spectrum Λ . Consider the one-parameter unitary group \mathcal{U}_p on $L^2([0, \frac{1}{p}), \mathbb{C}^p)$ defined by

$$(\mathcal{U}_p(t)F)(x) = \mathcal{M}_x \mathcal{M}_{x+t}^* F(x+t). \quad (2.10)$$

Then W intertwines U_Λ and \mathcal{U}_p :

$$WU_\Lambda(t) = \mathcal{U}_p(t)W. \quad (2.11)$$

On dimension $k = 1$, a strongly continuous unitary representation U of the additive group \mathbb{R} exists, acting by local translations on $L^2(\Omega)$, if and only if (Ω, Λ) is a spectral pair where Λ is the spectrum of the unitary representation U .

Theorem 2.4 (Dutkay-Jor). The set $\Omega = \cup_{i=1}^n (\alpha_i, \beta_i)$ is spectral if and only if there exists a strongly continuous one parameter unitary group $(U(t))_{t \in \mathbb{R}}$ on $L^2(\Omega)$ with the property that, for all $t \in \mathbb{R}$ and $f \in L^2(\Omega)$:

$$(U(t)f)(x) = f(x + t), \quad (2.12)$$

for almost every $x \in \Omega \cap (\Omega - t)$.

3. Quasiperiodic Spectra and Orthogonality for Iterated Function System Measures

New spectra from old

Theorem 3.1 (Dutkay-Jor). Let A be a finite subset of \mathbb{Z}_+ with $0 \in A$. The following affirmations are equivalent:

1. $A + [0, 1]$ is a spectral set.
2. A is a spectral set.

In this case, any spectrum of $A + [0, 1]$ has the form $\mathbb{Z} + \Lambda_A$, where Λ_A is a spectrum for A . Moreover $A + [0, 1]$ has only finitely many spectra that contain 0.

Affine iterated function system

Definition 3.2. Let A be a $d \times d$ expansive integer matrix.

- ▶ We say that a matrix is *expansive* if all its eigenvalues have absolute value > 1 .
- ▶ Let B be a finite subset of \mathbb{R}^d . We call the family of maps $(\tau_b)_{b \in B}$,

$$\tau_b(x) = A^{-1}(x + b), \quad x \in \mathbb{R}^d, b \in B,$$

an *affine iterated function system (affine IFS)*.

Theorem 3.3 (Dutkay-Jor). Let μ_B be the invariant measure associated to the affine IFS $\tau_b(x) = A^{-1}(x + b)$, $x \in \mathbb{R}$, $b \in B$, where B is a finite set of integers, $0 \in B$, and $A \in \mathbb{Z}$, $A \geq 2$. Suppose there exists a set L of integers with $0 \in L$ such that $A^{-1}L$ is a spectrum for B . If $\#B < A$, then μ_B has infinitely many spectra.

4. Spectral Theory of Multiple Intervals

Multiple intervals

Definition 4.1. Fix $n > 2$, let

$$-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_n < \alpha_n < \infty$$

and let

$$\Omega := \mathbb{R} \setminus \left(\bigcup_{k=1}^n [\beta_k, \alpha_k] \right) = \bigcup_{k=0}^n J_k. \quad (4.1)$$

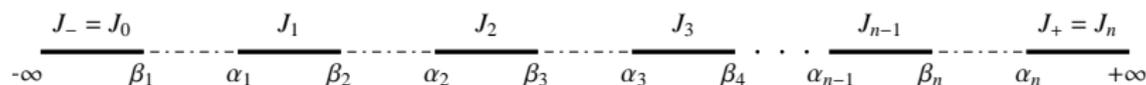


Figure 4.1: $\Omega = \bigcup_{k=0}^n J_k = \left(\bigcup_{k=1}^{n-1} J_k \right) \cup (J_- \cup J_+)$, i.e., Ω = the complement in \mathbb{R} of n finite and disjoint intervals.

Momentum operators

Definition 4.2. The *maximal momentum operator* is

$$P := \frac{1}{i2\pi} \frac{d}{dx} \quad (4.2)$$

acting in $L^2(\Omega)$, with domain $\mathcal{D}(P)$ equal to the set of absolutely continuous functions on Ω where both f and Pf are square-integrable. Recall,

$$\mathcal{D}(P_{min}) = \{f \in \mathcal{D}(P) ; f = 0 \text{ on } \partial\Omega\}.$$

Note. All the selfadjoint extension operators for P_{min} are indexed by $B \in U(n)$.

Theorem 4.3 (Jorgensen, Pedersen, Tian). Let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be a system of interval endpoints:

$$-\infty < \beta_1 < \alpha_1 < \beta_2 < \cdots < \beta_n < \alpha_n < \infty, \quad (4.3)$$

with $J_0 = J_- = (-\infty, \beta_1)$, $J_n = J_+ = (\alpha_n, \infty)$, and $J_i = (\alpha_i, \beta_{i+1})$, $i = 1, \dots, n-1$. Let $B \in U(n)$ be chosen non-degenerate (fixed), and let

$$\psi_\lambda(x) := \psi_\lambda^{(B)}(x) = \left(\sum_{i=0}^n \chi_i(x) A_i^{(B)}(\lambda) \right) e_\lambda(x), \quad (4.4)$$

be as the generalized eigenfunctions.

Theorem 4.3 cont.

For $f \in L^2(\Omega)$, setting

$$(V_B f)(\lambda) = \langle \psi_\lambda, f \rangle_\Omega = \int \overline{\psi_\lambda(y)} f(y) dy, \quad (4.5)$$

we then get the following orthogonal expansions:

$$f = \int_{\mathbb{R}} (V_B f)(\lambda) \psi_\lambda(\cdot) d\lambda \quad (4.6)$$

where the convergence in (4.6) is to be taken in the L^2 -sense via

$$\|f\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} |(V_B f)(\lambda)|^2 d\lambda, \quad f \in L^2(\Omega). \quad (4.7)$$

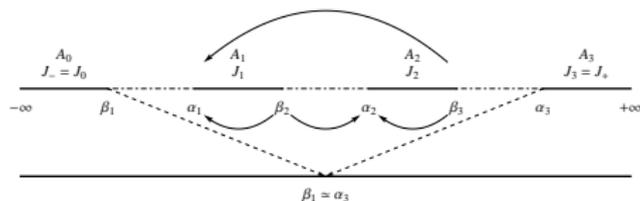
Scattering theory

Theorem 4.3 cont.

Moreover, we have

$$V_B U_B(t) = M_t V_B, \quad t \in \mathbb{R} \quad (4.8)$$

where $(M_t g)(\lambda) = e_{\lambda}(-t) g(\lambda)$, for all $t, \lambda \in \mathbb{R}$, and all $g \in L^2(\mathbb{R})$.



$$L^2(\Omega) = L^2(J_- \cup J_+) \oplus L^2(J_1 \cup J_2)$$

$$U_B(t) = U_B^{\text{cont}}(t) \oplus U_B^{\text{bound-state}}(t)$$

Figure 4.2: Transition between intervals in Ω .

Theorem 4.4 (Jorgensen, Pedersen, Tian). Set $J_0 = J_-$, $J_n = J_+$. Let B be determined

$$B = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ c & \mathbf{0} \end{pmatrix}, \quad c \in \mathbb{C}, |c| = 1. \quad (4.9)$$

Then the continuous spectrum of P_B is the real line and the discrete spectrum of P_B is $\bigcup_{k=1}^{n-1} \frac{1}{\ell_k} \mathbb{Z}$, where $\ell_k = \beta_{k+1} - \alpha_k$ is the length of the k th bounded interval. The multiplicity of each eigenvalue λ is $\#\{1 \leq k \leq n-1 \mid \ell_k \lambda \in \mathbb{Z}\}$. Hence, 0 is an eigenvalue with multiplicity $n-1$ and counting multiplicity the discrete spectrum has uniform density $\sum_{k=1}^{n-1} \ell_k$, in the sense that, for any a we have

$$\frac{\text{number of eigenvalues in } [a-n, a+n]}{2n} \xrightarrow{n \rightarrow \infty} \sum_{k=1}^{n-1} \ell_k.$$

5. Orthogonal exponentials, translations, and Bohr completions

Question. Let μ be a probability measure on \mathbb{R}^n . Suppose there is a (strongly continuous) group of unitary transformations $(U(t))_{t \in \mathbb{R}^n}$ on $L^2(\mu)$ such that for every measurable set O , and $t \in \mathbb{R}^n$ with $O, O+t \subset \text{supp}(\mu)$

$$U(t)\chi_{O+t} = \chi_O.$$

Is μ is a spectral measure?

Definition 5.1. Let A be a finite subset of \mathbb{R}^n . $N := \#A$. We say that A is a *spectral* set if the normalized counting measure δ_A on A is a spectral measure, $\delta_A := \frac{1}{N} \sum_{a \in A} \delta_a$. A spectrum for A is a spectrum for the measure δ_A . We denote by $L^2(A) := L^2(\delta_A)$.

Theorem 5.2 (Dutkay, Han, Jorgensen). Let A be a finite subset of \mathbb{R}^n . The following affirmations are equivalent:

1. The set A is spectral.
2. There exists a continuous group of unitary operators $(U(t))_{t \in \mathbb{R}^n}$ on $L^2(A)$, i.e., $U(t+s) = U(t)U(s)$, $t, s \in \mathbb{R}^n$ such that

$$U(a - a')\chi_a = \chi_{a'} \quad (a, a' \in A), \quad (5.1)$$

where $\chi_a(x) = \begin{cases} 1, & x = a \\ 0, & x \in A \setminus \{a\}. \end{cases}$

Definition 5.3. We shall use Pontryagin's duality for locally compact abelian groups H , i.e.,

$$\widehat{H} = \{ \chi : H \rightarrow \mathbb{T} \mid \chi \text{ continuous, } \chi(h_1 + h_2) = \chi(h_1)\chi(h_2), \quad (5.2) \\ \chi(-h) = \overline{\chi(h)}, h, h_1, h_2 \in H \}$$

- ▶ \widehat{H} is a locally compact abelian group under the operation $(\chi_1\chi_2)(h) := \chi_1(h)\chi_2(h)$, $h \in H$.
- ▶ $\widehat{\widehat{H}} \cong H$, i.e., the natural embedding $H \hookrightarrow \widehat{\widehat{H}}$ is onto.
- ▶ H is compact iff \widehat{H} is discrete.

We apply this to $(\mathbb{R}^n, +)$. When it is equipped with the discrete topology, it is denoted \mathbb{R}_{disc}^n .

- ▶ It follows that $G := \widehat{\mathbb{R}_{disc}^n}$ is a compact abelian group with normalized Haar measure μ_{Bohr} .
- ▶ Dualizing the natural mapping $\mathbb{R}_{disc}^n \hookrightarrow \mathbb{R}^n$ (continuous!) we get

$$\mathbb{R}^n = \widehat{\mathbb{R}^n} \hookrightarrow G; \quad (5.3)$$

i.e., \mathbb{R}^n is naturally embedded into G : hence the name “*Bohr compactification*”.

Bohr and almost periodic:

Definition 5.4. A continuous function f on \mathbb{R}^n is said to be *almost periodic* if for all $\epsilon > 0$ there exists $T \in \mathbb{R}_+$ such that for all $y \in \mathbb{R}^n$ there exists $p \in y + Q_T$ such that

$$|f(x) - f(x + p)| < \epsilon, \text{ for all } x \in \mathbb{R}^n. \quad (5.4)$$

Moreover, if f is almost periodic, then

$$\lim_{T \rightarrow \infty} \langle f \rangle_T = \langle f \rangle = \int_G f d\mu_{Bohr}, \quad (5.5)$$

where \mathbb{R}^n is embedded in G via (5.3). In particular, a continuous almost periodic function \mathbb{R}^n extends naturally to a continuous function on G .

Theorem 5.5 (Dutkay, Han, Jorgensen). Let $\xi : \mathbb{R}^n \rightarrow G$ denote Bohr's embedding (5.3), i.e.,

$$\langle \xi(x), \lambda \rangle = e_\lambda(x) =: e^{2\pi i \lambda \cdot x}, \quad (x, \lambda \in \mathbb{R}^n) \quad (5.6)$$

and set

$$\tilde{e}_\lambda(\chi) := \chi(\lambda), \quad (\lambda \in \mathbb{R}_{disc}^n, \chi \in G) \quad (5.7)$$

Let μ be a finite measure on \mathbb{R}^n and let $\Lambda \subset \mathbb{R}^n$ be the subset of \mathbb{R}^n . Then the set $E(\Lambda) := \{e_\lambda\}_{\lambda \in \Lambda}$ is orthonormal in $L^2(\mu)$ iff the embedding given by (5.6)-(5.7)

$$W_\Lambda : e_\lambda \mapsto \tilde{e}_\lambda \text{ where } \tilde{e}_\lambda(\chi) = \chi(\lambda), \chi \in G, \quad (5.8)$$

$$W_\Lambda : \mathcal{H}_\Lambda := \text{clspan}\{e_\lambda\} \hookrightarrow L^2(G),$$

is an isometric operator.

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Lecture 3. The universal tiling conjecture in dimension one and operator fractals

By Palle Jorgensen

Abstract

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum (has an orthogonal Fourier basis) if and only if it is possible to tile \mathbb{R}^d by a set of translates of Ω [Fug74]. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of \mathbb{R}^d and Iosevich et al. [IKT01] proved that no smooth symmetric convex body Ω with at least one point of nonvanishing Gaussian curvature can admit an orthogonal basis of exponentials.

Using complex Hadamard matrices of orders 6 and 12, Tao [Tao04] constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in \mathbb{R}^5 or \mathbb{R}^{11} . Tao's results were extended to lower dimensions, down to $d = 3$, but the problem is still open for $d = 1$ and $d = 2$.

Summary of some affirmative recent results: The conjecture has been proved in a great number of special cases (e.g., all convex planar bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling \mathbb{Z} indeed implies one side of Fuglede's conjecture [CM99]. Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group [Ami05].

1. On the universal tiling conjecture in dimension one

We formulate the following “Universal Tiling Conjecture” for a fixed number $p \in \mathbb{N}$:

Conjecture 1.1 ([UTC(p)]). Let $p \in \mathbb{N}$. Let $\Gamma := \{\lambda_0 = 0, \lambda_1, \dots, \lambda_{p-1}\}$ be a subset of \mathbb{R} with p elements. Assume Γ has a spectrum of the form $\frac{1}{p}A$ with $A \subset \mathbb{Z}$. Then for every finite family A_1, A_2, \dots, A_n of subsets of \mathbb{Z} such that $\frac{1}{p}A_i$ is a spectrum for Γ for all i , there exists a common tiling subset \mathcal{T} of \mathbb{Z} such that the set A_i tiles \mathbb{Z} by \mathcal{T} for all $i \in \{1, \dots, n\}$.

Theorem 1.2 (Dutkay-Jor). The following affirmations are equivalent.

1. The Universal Tiling Conjecture is true for all $p \in \mathbb{N}$.
2. Every bounded Lebesgue measurable spectral set tiles by translations.

Moreover, if these statements are true and if Ω , $|\Omega| = 1$, is a bounded Lebesgue measurable set which has a spectrum with period p , then Ω tiles by a subset \mathcal{T} of $\frac{1}{p}\mathbb{Z}$.

The spectral-tile implication in the Fuglede conjecture is equivalent to some formulations of this implication for some special classes of sets Ω :

Theorem 1.3 (Dutkay-Jor). The following affirmations are equivalent:

1. For every finite union of intervals with rational endpoints $\Omega = \cup_{i=1}^n (\alpha_i, \beta_i)$ with $|\Omega| = 1$, if Ω has a spectrum Λ with period p , then Ω tiles \mathbb{R} by a subset \mathcal{T} of $\frac{1}{p}\mathbb{Z}$.
2. For every finite union of intervals with integer endpoints $\Omega = \cup_{i=1}^n (\alpha_i, \beta_i)$, $|\Omega| = N$, if Ω has a spectrum Λ with minimal period $\frac{r}{N}$, $r \in \mathbb{Z}$, then $\frac{N}{r}$ is an integer and Ω tiles \mathbb{R} with a subset \mathcal{T} of $\frac{N}{r}\mathbb{Z}$.
3. Every bounded Lebesgue measurable spectral set tiles by translations.

2. Unitary Representations of Wavelet Groups and Encoding of Iterated Function Systems in Solenoids

Definition 2.1. On $L^2(\mathbb{R}^d)$ we denote by T_k the *translation operator* $(T_k f)(x) = f(x - k)$, $g \in L^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, and by U the *dilation operator* $(Uf)(x) = \frac{1}{\sqrt{|\det A|}} f(A^{-1}x)$. Their Fourier transform is

$$(\hat{T}_k h)(x) = e^{2\pi i k \cdot x} f(x) \quad (2.1)$$

$$(\hat{U}h)(x) = \sqrt{|\det A|} h(A^T x), \quad (2.2)$$

$(h \in L^2(\mathbb{R}^d), x \in \mathbb{R}^d, k \in \mathbb{R}^d)$. The operators $\{U, T_k\}$ (or $\{\hat{U}, \hat{T}_k\}$) define a representation of the group G_A on $L^2(\mathbb{R}^d)$.

Definition 2.2. We denote by $\mathcal{S}_A(1)$ the set of sequences $(z_n)_{n \in \mathbb{N}} \in \mathcal{S}_A$ such that $\lim_{n \rightarrow \infty} z_n = \mathbf{1}$. On $\mathcal{S}_A(1)$ consider the measure $\tilde{\mu}$ defined by

$$\int_{\mathcal{S}_A(1)} f d\tilde{\mu} = \int_{\mathbb{T}^d} \sum_{(z_n)_{n \in \mathbb{N}} \in \mathcal{S}_A(1), \theta_0((z_n)_{n \in \mathbb{N}}) = z} f((z_n)_{n \in \mathbb{N}}) d\mu(z), \quad (2.3)$$

where μ is the Haar measure on \mathbb{T}^d .

On $L^2(\mathcal{S}_A(1), \tilde{\mu})$ define the operators

$$(\tilde{T}_k f)(z_0, z_1, \dots) = z_0^k f(z_0, z_1, \dots), \quad (2.4)$$

$$(\tilde{U} f)(z_0, z_1, \dots) = \sqrt{|\det A|} f(\sigma_A(z_0, z_1, \dots)). \quad (2.5)$$

Theorem 2.3 (Dutkay-Jor).

- ▶ The measure $\tilde{\mu}$ satisfies the following invariance property

$$\int_{\mathcal{S}_A(1)} f \circ \sigma_A d\tilde{\mu} = \frac{1}{|\det A|} \int_{\mathcal{S}_A(1)} f d\tilde{\mu}, \quad (f \in L^1(\mathcal{S}_A(1), \tilde{\mu})). \quad (2.6)$$

- ▶ The operators \tilde{T}_k , $k \in \mathbb{Z}^d$ and \tilde{U} are unitary, and

$$\tilde{U}\tilde{T}_k\tilde{U}^{-1} = \tilde{T}_{Ak}, \quad (k \in \mathbb{Z}^d) \quad (2.7)$$

so $\{\tilde{U}, \tilde{T}_k\}$ generate a representation of the group G_A on $L^2(\mathcal{S}_A(1), \tilde{\mu})$.

Theorem 2.3 cont.

- ▶ The map \hat{i} is a measure preserving transformation between \mathbb{R}^d and $\mathcal{S}_A(1)$.
- ▶ The operator $\mathcal{W} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathcal{S}_A(1), \tilde{\mu})$ defined by

$$\mathcal{W}f = f \circ \hat{i}^{-1}$$

is an intertwining isometric isomorphism,

$$\mathcal{W}\hat{T}_k = \tilde{T}_k\mathcal{W}, \quad (k \in \mathbb{Z}^d), \quad \mathcal{W}\hat{U} = \tilde{U}\mathcal{W}. \quad (2.8)$$

3. Scalar spectral measures associated with an Operator-Fractal

Notation

Definition 3.1. Let $L^2(\mu_{\frac{1}{4}})$ be the Hilbert space with the associated spectrum, $\Gamma(\frac{1}{4})$, where

$$\Gamma\left(\frac{1}{4}\right) = \left\{ \sum_{i=0}^m a_i 4^i : a_i \in \{0, 1\} \right\} = \{0, 1, 4, 5, 16, 17, 20, \dots\}. \quad (3.1)$$

Thus, $L^2(\mu_{\frac{1}{4}})$ has an orthonormal basis (ONB):

$$E\left(\Gamma\left(\frac{1}{4}\right)\right) = \left\{ e_\gamma(t) = e^{2\pi i \gamma t} \mid \gamma \in \Gamma\left(\frac{1}{4}\right) \right\}. \quad (3.2)$$

Definition 3.2. Set

$$Ue_\gamma = e_{5\gamma}. \quad (3.3)$$

The operator $M = M_{e_1}$ is multiplication by the exponential e_1 :

$$M_{e_1}e_\gamma = e_{\gamma+1}. \quad (3.4)$$

A representation of \mathcal{O}_2 on $L^2(\mu_{\frac{1}{4}})$ is given by the operators

$$S_0e_\gamma = e_{4\gamma} \text{ and } S_1e_\gamma = e_{4\gamma+1}. \quad (3.5)$$

Connections to representations of the Cuntz-algebras

Definition 3.3. Let \mathcal{O}_N denote the Cuntz algebra with N generators, i.e., the C^* -algebra on symbols $\{s_i\}_{i=1}^N$, satisfying:

$$s_i^* s_j = \delta_{ij} \mathbb{1}, \quad \text{and} \quad \sum_{i=1}^N s_i s_i^* = \mathbb{1}, \quad (3.6)$$

where $\mathbb{1}$ denotes the unit element in \mathcal{O}_N . By a representation of \mathcal{O}_N we mean a function $s_i \mapsto S_i = \pi(s_i)$ such that

$$S_i^* S_j = \delta_{ij} I, \quad \text{and} \quad \sum_{i=1}^N S_i S_i^* = I. \quad (3.7)$$

We say that $\pi \in \text{Rep}(\mathcal{O}_N, \mathcal{H})$ if (3.7) holds.

Proposition 3.4. Fix a unit vector $v \in L^2(\mu_{\frac{1}{4}})$. Let m_v^U and m_v^{MU} be the spectral measures associated with U and $M_{e_1}U$ respectively. Then

$$m_v^U = m_{S_0^*v}^U + m_{S_1^*v}^{MU}. \quad (3.8)$$

Theorem 3.5 (Jorgensen, Kornelson, Shuman). Let $v \in L^2(\mu_{\frac{1}{4}})$. Then

$$m_v^U = |\langle v, e_0 \rangle|^2 \delta_1 + \sum_{k=0}^{\infty} m_{S_1^* S_0^{*k} v}^{MU}. \quad (3.9)$$

Theorem 3.6 (Jorgensen, Kornelson, Shuman). Fix a unit vector $v \in L^2(\mu_{\frac{1}{4}})$. Suppose $S \in B(L^2(\mu_{\frac{1}{4}}))$ commutes with both U and U^* , and suppose $\langle v \rangle_U$ is invariant under S . Then for any $\psi \in C(\mathbb{T})$,

$$S\pi_U(\psi)v = \sqrt{\frac{dm_{Sv}}{dm_v}}(U)v = \pi_U\left(\sqrt{\frac{dm_{Sv}}{dm_v}}\psi\right)v. \quad (3.10)$$

4. An Operator-Fractal

- ▶ Let $\lambda = \frac{1}{2^n}$ for a natural number n and consider the Bernoulli measure with scale factor λ . It is known that $L^2(\mu_\lambda)$ has a Fourier basis.
- ▶ When $L^2(\mu_\lambda)$ has more than one Fourier basis, there are natural unitary operators U , indexed by a subset of odd scaling factors p ; each U is defined by mapping one ONB to another. The unitary operator U can also be orthogonally decomposed according to the Cuntz relations. Moreover, this *operator-fractal* U exhibits its own self-similarity.

Bernoulli convolution measure

- ▶ Let $\{\tau_+, \tau_-\}$ be the IFS

$$\tau_+(x) = \lambda(x + 1), \quad \tau_-(x) = \lambda(x - 1), \quad (4.1)$$

where the scaling factor λ is in $(0, 1)$.

- ▶ The IFS $\{\tau_+, \tau_-\}$ generates a compact *attractor set*, X_λ , that satisfies

$$X_\lambda = \tau_+(X_\lambda) \cup \tau_-(X_\lambda).$$

The IFS also generates a measure μ_λ (the Bernoulli convolution), supported on X_λ , which satisfies a similar invariance equation:

$$\mu_\lambda = \frac{1}{2} \left(\mu_\lambda \circ \tau_+^{-1} \right) + \frac{1}{2} \left(\mu_\lambda \circ \tau_-^{-1} \right). \quad (4.2)$$

- ▶ $\Gamma = \Gamma(\lambda)$: orthonormal Fourier basis. When such an orthonormal Fourier basis exists, we say that μ_λ is a *spectral measure*.
- ▶ Canonical spectrum (Jorgensen, Pedersen):

$$\Gamma = \Gamma\left(\frac{1}{2n}\right) = \left\{ \sum_{i=0}^m a_i (2n)^i : a_i \in \left\{0, \frac{n}{2}\right\}, m \text{ finite} \right\}. \quad (4.3)$$

Definition 4.1. Given the Bernoulli measure μ with scale factor $\frac{1}{2n}$, let Γ be the canonical spectrum. We define S_0, S_1 on the canonical ONB $E(\Gamma)$:

$$S_0 e_\gamma = e_{2n\gamma}, \text{ and} \quad (4.4)$$

$$S_1 e_\gamma = e_{\frac{n}{2}+2n\gamma}. \quad (4.5)$$

Definition 4.2. Consider an arbitrary pair for $\lambda = \frac{1}{2n}$ where the canonical set $\Gamma = \Gamma(\frac{1}{2n})$ and the scaled set $p\Gamma$ are both spectra. Define the operator U on Γ by

$$U e_\gamma = e_{p\gamma}. \quad (4.6)$$

- ▶ Decomposition of the canonical spectrum Γ .

$$\Gamma_0 = \Gamma \setminus 2n\Gamma = \frac{n}{2} + 2n\Gamma$$

$$\Gamma_1 = 2n\Gamma \setminus (2n)^2\Gamma = 2n\left(\frac{n}{2} + 2n\Gamma\right)$$

$$\vdots \quad \vdots$$

$$\Gamma_k = (2n)^k\Gamma \setminus (2n)^{k+1}\Gamma = (2n)^k\left(\frac{n}{2} + 2n\Gamma\right).$$

- ▶ It is readily verified that the sets Γ_k are disjoint and form a partition of $\Gamma \setminus \{0\}$.

The matrix for U

Theorem 4.3 (Jorgensen, Kornelson, Shuman). The matrix representation of the operator U , for the given ordering of the index set Γ , has a block diagonal structure of the form

$$U \simeq \begin{array}{c|c|c|c|c|c|c} & 0 & \Gamma_0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & \cdots \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \hline \Gamma_0 & 0 & U|_{W_0} & 0 & 0 & 0 & \cdots \\ \hline \Gamma_1 & 0 & 0 & U|_{W_0} & 0 & 0 & \cdots \\ \hline \Gamma_2 & 0 & 0 & 0 & U|_{W_0} & 0 & \cdots \\ \hline \Gamma_3 & 0 & 0 & 0 & 0 & U|_{W_0} & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \quad (4.7)$$

In other words,

$$U \simeq P_{e_0} \oplus \bigoplus_{k=0}^{\infty} U|_{W_0},$$

Theorem 4.4 (Jorgensen, Kornelson, Shuman). Let M_{e_1} be the multiplication operator by the function e_1 on $L^2(\mu)$, and let P_{e_0} be the orthogonal projection onto $sp\{e_0\}$. Then U has the form

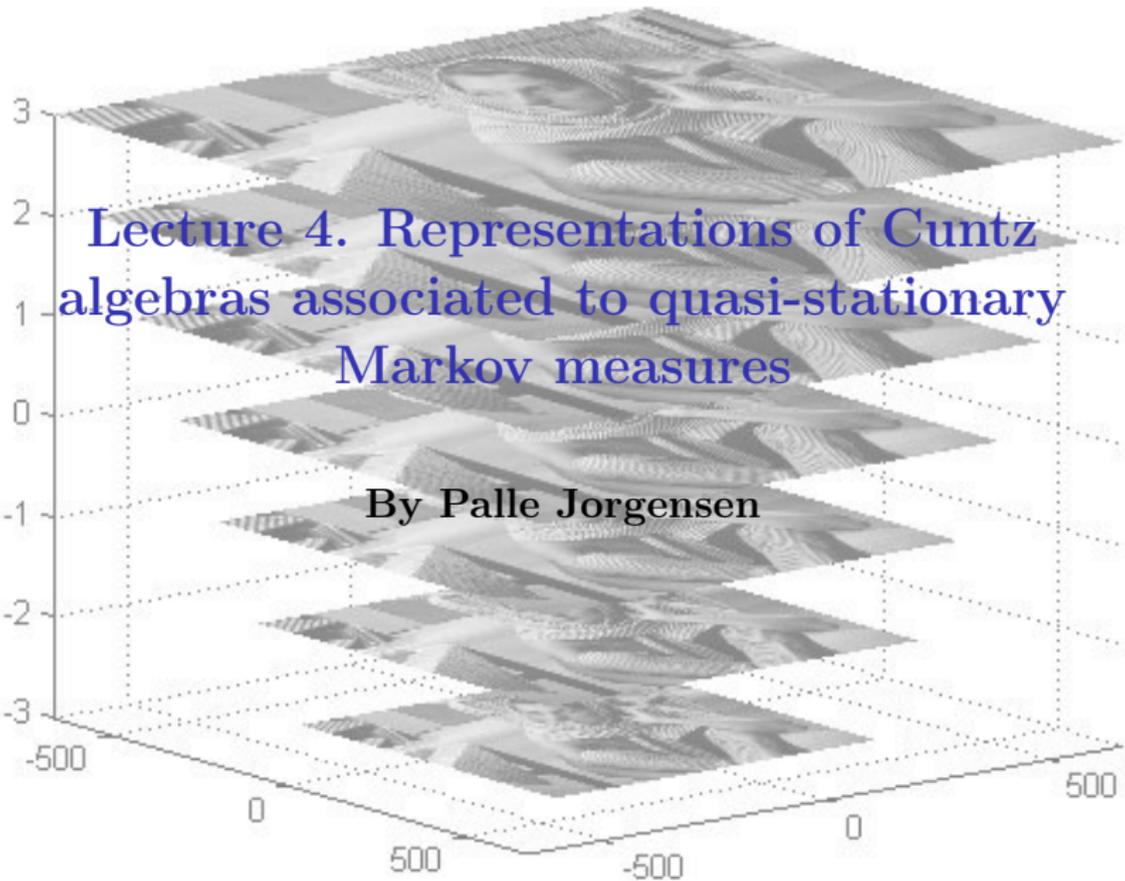
$$U \simeq P_{e_0} \oplus \bigoplus_{k=0}^{\infty} M_{e_1} U. \quad (4.8)$$

Theorem 4.5 (Jorgensen, Kornelson, Shuman). The matrix for $U|_{W_0}$ has the following block form, where the index set Γ_0 for the ONB of W_0 is ordered by $\{1\} \cup \bigsqcup \tilde{\Gamma}_k$. The notation $*$ indicates an entry or block that is not necessarily zero and 0 indicates an entry or block that contains all zeros.

$$U|_{W_0} \simeq \begin{array}{c|cccccc} & 1 & \tilde{\Gamma}_0 & \tilde{\Gamma}_1 & \tilde{\Gamma}_2 & \tilde{\Gamma}_3 & \cdots \\ \hline 1 & 0 & * & 0 & 0 & 0 & \cdots \\ \hline \tilde{\Gamma}_0 & * & 0 & * & * & * & \cdots \\ \hline \tilde{\Gamma}_1 & 0 & * & 0 & 0 & 0 & \cdots \\ \hline \tilde{\Gamma}_2 & 0 & * & 0 & 0 & 0 & \cdots \\ \hline \tilde{\Gamma}_3 & 0 & * & 0 & 0 & 0 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \quad (4.9)$$

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Lecture 4. Representations of Cuntz
algebras associated to quasi-stationary
Markov measures

By Palle Jorgensen

Abstract

Representations of Cuntz algebras that arise from the action of stochastic matrices on sequences from \mathbb{Z}_n are considered. This action gives rise to an invariant measure, which depending on the choice of stochastic matrices, may satisfy a finite tracial condition. If so, the measure is ergodic under the action of the shift on the sequence space, and thus yields a representation of a Cuntz algebra. The measure provides spectral information about the representation in that equivalent representations of the Cuntz algebras for different choices of stochastic matrices occur precisely when the measures satisfy a certain equivalence condition.

Recursive multiresolutions and basis constructions in Hilbert spaces are key tools in analysis of fractals and of iterated function systems in dynamics: Use of multiresolutions, selfsimilarity, and locality, yield much better pointwise approximations than is possible with traditional Fourier bases. The approach here will be via representations of the Cuntz algebras. It is motivated by applications to an analysis of frequency sub-bands in signal or image-processing, and associated multi-band filters: With the representations, one builds recursive subdivisions of signals into frequency bands.

1. Representations of Cuntz algebras associated to quasi-invariant Markov measures

quasi-stationary Markov measures

Definition 1.1. An $N \times N$ matrix T is called *stochastic* if the sum of the entries in each row is 1. (We use column vectors for multiplication on the right and row vectors for multiplication on the left).

Let $\lambda = (\lambda_0, \dots, \lambda_{N-1})$ be a *positive probability vector*, i.e., $\sum_{i \in \mathbb{Z}_N} \lambda_i = 1$ and let $\{T^{(n)}\}_{n \in \mathbb{N}}$ be a sequence of stochastic $N \times N$ matrices with positive entries such that

$$\lambda T^{(1)} = \lambda \tag{1.1}$$

$$\lambda_i > 0, T_{i,j}^{(n)} > 0, \quad (i, j \in \mathbb{Z}_N, n \in \mathbb{N}). \tag{1.2}$$

Definition 1.2. Define the Borel probability measure $\mu = \mu_{\lambda,T}$ on \mathcal{K}_N , first, on cylinder sets: for $I = i_1 \dots i_n$

$$\mu(\mathcal{C}(I)) = \lambda_{i_1} T_{i_1, i_2}^{(1)} T_{i_2, i_3}^{(2)} \dots T_{i_{n-1}, i_n}^{(n-1)}, \quad (1.3)$$

and then extending it to the Borel sigma-algebra, using the Kolmogorov extension theorem.

- ▶ We say that $\mu = \mu_{\lambda,T}$ is the *Markov measure associated to* λ and T .
- ▶ We say that the Markov measure μ is *quasi-stationary* if

$$\sum_{n=1}^{\infty} \left| \frac{T_{x_n, x_{n+1}}^{(n)}}{T_{x_n, x_{n+1}}^{(n+1)}} - 1 \right| < \infty, \text{ for all } x_1 x_2 \dots \in \mathcal{K}_N. \quad (1.4)$$

Theorem 1.3 (Dutkay-Jor). Let μ be a quasi-stationary Markov measure. Then $\mu \circ \sigma_j^{-1} \ll \mu$ for all $j \in \mathbb{Z}_N$ and

$$\frac{d(\mu \circ \sigma_j^{-1})}{d\mu}(x_1 x_2 \dots) = \delta_{j, x_1} \frac{\lambda_{x_2}}{\lambda_{x_1} T_{x_1, x_2}^{(1)}} F(x_2 x_3 \dots) \quad (1.5)$$

where

$$F(x_1 x_2 \dots) = \prod_{n=1}^{\infty} \frac{T_{x_n, x_{n+1}}^{(n)}}{T_{x_n, x_{n+1}}^{(n+1)}} \text{ and } 0 < F(x_1 x_2 \dots) < \infty. \quad (1.6)$$

Theorem 1.3 cont.

Also,

$$\frac{d(\mu \circ \sigma^{-1})}{d\mu}(x_1 x_2 \dots) = \frac{1}{F(x_1 x_2 \dots)}. \quad (1.7)$$

Set

$$f_j^2(x_1 x_2 \dots) := \delta_{j, x_1} \frac{\lambda_{x_2}}{\lambda_{x_1} T_{x_1, x_2}^{(1)}} F(x_2 x_3 \dots), \quad f_j \geq 0.$$

The operators S_j on $L^2(\mu)$ defined by

$$S_j f = f_j (f \circ \sigma), \quad (f \in L^2(\mu), j \in \mathbb{Z}_N) \quad (1.8)$$

form a nonnegative monic representation of \mathcal{O}_N .

Theorem 1.4 (Dutkay-Jor). Let μ be a quasi-stationary Markov measure associated to (λ, T) and let $(S_i)_{i \in \mathbb{Z}_N}$ the corresponding representation of \mathcal{O}_N . Then

1. The measure μ is ergodic with respect to the endomorphism σ .
2. The representation of \mathcal{O}_N is irreducible.

Theorem 1.5 (Dutkay-Jor). Let μ be a quasi-stationary Markov measure associated to (λ, T) . Let $T^{(\infty)}$ be the limit of the stochastic matrices $T^{(n)}$. Let λ_∞ be a positive probability row vector such that $\lambda_\infty T^{(\infty)} = \lambda_\infty$. (The existence of such a vector follows from the Perron-Frobenius theorem). Suppose the following condition is satisfied, for all $i, j \in \mathbb{Z}_N$:

$$\sum_{n=1}^{\infty} n |T_{i,j}^{(n)} - T_{i,j}^{(n+1)}| < \infty. \quad (1.9)$$

Let μ_∞ be the Markov measure associated to λ_∞ and the constant sequence with fixed matrix $T^{(\infty)}$. Then

1. The measures μ and μ_∞ are equivalent.
2. The monic representations of \mathcal{O}_N associated to μ and to μ_∞ are equivalent.

2. Monopoles, dipoles, and harmonic functions on Bratteli diagrams

Settings

- ▶ (G, c) : an electrical network supported by a locally finite connected graph $G = (V, E)$ together with a symmetric conductance function $c = c_{xy}$;
- ▶ the Laplace operator

$$(\Delta u)(x) = \sum_{y \sim x} c_{xy}(u(x) - u(y))$$

where $u : V \rightarrow \mathbb{R}$;

- ▶ $P = (p(x, y))$: the transition probabilities matrix whose entries determine a reversible random walk (X_n) on the vertex set V of G ;
- ▶ \mathcal{H}_E : the Hilbert space of functions on V of finite energy.

Theorem 2.1 (Bezuglyi-Jor). Let (V, E, c) be a transient electrical network with transition probabilities matrix P . Let $G(x, y)$ be the Green's function determined by P . Then, for any vertex $x \in V$, the function

$$w_x : a \mapsto w_x(a) = \frac{G(a, x)}{c(x)}, \quad a \in V,$$

is a monopole in \mathcal{H}_E , and

$$a \mapsto v_{x_1, x_2}(a) := \frac{G(a, x_1)}{c(x_1)} - \frac{G(a, x_2)}{c(x_2)}, \quad a \in V,$$

defines a dipole from \mathcal{H}_E where x_1, x_2 are any vertices from V .

Theorem 2.2 (Bezuglyi-Jor). Let $(B(V, E), c)$ be a weighted Bratteli diagram with associated sequences of matrices (\overleftarrow{P}_n) . Then a sequence of vectors $f_n \in \mathbb{R}^{|V_n|}$ represents a harmonic function $f = (f_n) : V \rightarrow \mathbb{R}$ if and only if for any $n \geq 1$

$$f_n - \overrightarrow{P}_{n-1} f_{n-1} = \overleftarrow{P}_n f_{n+1},$$

where $\overrightarrow{P}_n = \overleftarrow{P}_n^T$.

Theorem 2.3 (Bezuglyi-Jor). Let $f = (f_n) \geq 0$ be a function on V such that $\overleftarrow{P}_n f_{n+1} = f_n$. Then the sequence $(h_n(x))$ converges pointwise to a harmonic function $H(x)$. Moreover, for every $x \in V$, there exists $n(x)$ such that $h_i(x) = H(x), i \geq n(x)$. Equivalently, the sequence $(f_n \circ X_{\tau(V_n)})$ converges in $L^1(\Omega_x, \mathbb{P}_x)$.

3. Infinite-dimensional transfer operators, endomorphisms, and measurable partitions

Theorem 3.1 (Bezuglyi-Jor). Let $(X, \mathcal{B}, \mu, \sigma)$ be a measure preserving non-invertible dynamical system. Let $\mathcal{H} = L^2(\mu)$ and define

$$S : f \mapsto f \circ \sigma : \mathcal{H} \rightarrow \mathcal{H}.$$

Then S is an isometry. The adjoint of S is

$$S^* = \frac{(g d\mu) \circ \sigma^{-1}}{d\mu}, \quad g \in \mathcal{H}.$$

Transfer operators for piecewise monotone maps

Definition 3.2. An onto endomorphism α of $(0, 1)$ is called *piecewise monotone* if $(0, 1)$ can be partitioned into a finite or infinite family (J_k) of subintervals $J_k = (t_{k-1}, t_k)$ such that the restriction of α on each J_k is a continuous monotone one-to-one map onto $(0, 1)$ (in many examples, the map α is assumed to be differentiable on each J_k). For every k , there exists an inverse branch β_k of α such that β_k maps $(0, 1)$ onto J_k and satisfies the condition

$$\alpha \circ \beta_k(x) = x, \quad x \in (0, 1).$$

We will assume implicitly that the collection of disjoint subintervals (J_k) of $(0, 1)$ is countable.

Definition 3.3. Let α , $(\beta_k : k \in \mathbb{N})$, and J_k be as above. Suppose that $\pi = (p_k : k \in \mathbb{N})$ is a probability infinite-dimensional positive vector (probability distribution), i.e., $p_k > 0$ and $\sum_k p_k = 1$. Let a measure μ on $X = (0, 1)$ satisfy the property

$$\mu = \sum_{k=1}^{\infty} p_k \mu \circ \beta_k^{-1}. \quad (3.1)$$

Then μ is called an *iterated function systems measure (IFS measure)* for the iterated function system $(\beta_k : k \in \mathbb{N})$.

Theorem 3.4 (Bezuglyi-Jor). Let α be a piecewise monotone endomorphism of $(0,1)$, and let $(J_k : k \in \mathbb{N})$ be the corresponding collection of the disjoint intervals. Suppose that a measure μ is non-atomic and satisfies relation 3.1. Then, the entries (p_k) of the probability distribution $\pi = (p_k : k \in \mathbb{N})$ are determined by formula

$$p_k = \frac{\int_{J_k} \alpha(x) d\mu(x)}{\int_0^1 x d\mu(x)} = \mu(J_k).$$

Theorem 3.5 (Bezuglyi-Jor). Take an α -invariant measure μ on $(0, 1)$, $\mu \circ \alpha^{-1} = \mu$. Suppose that $R = R_\pi$ is the transfer operator acting on measurable functions such that

$$R(f)(x) = \sum_{k=1}^{\infty} p_k f(\beta_k(x)),$$

where the probability distribution $\pi = (p_k)$ is defined by

$$p_k := \frac{\int_{J_k} \alpha(x) d\mu(x)}{\int_0^1 x d\mu(x)} = \mu(\beta_k(0, 1)).$$

Then μ is R -invariant if and only if, for any $k, m \in \mathbb{N}$,

$$\left(\int_0^1 x d\mu(x) \right) \int_{J_k} \alpha(x)^m d\mu(x) = \left(\int_0^1 x^m d\mu(x) \right) \int_{J_k} \alpha(x) d\mu(x). \quad (3.2)$$

4. Representations of Cuntz-Krieger relations, dynamics on Bratteli diagrams, and path-space measures

Definition 4.1. Let (X, μ) be a probability measure space with non-atomic measure μ . We consider a finite family $\{\sigma_i : i \in \Lambda\}$ of one-to-one μ -measurable maps σ_i defined on a subset D_i of X and let $R_i = \sigma_i(D_i)$. The family $\{\sigma_i\}$ is called a *semibranching function system (s.f.s.)* if the following conditions hold:

- (i) $\mu(R_i \cap R_j) = 0$ for $i \neq j$ and $\mu(X \setminus \bigcup_{i \in \Lambda} R_i) = 0$;
- (ii) $\mu \circ \sigma_i \ll \mu$ and

$$\rho_\mu(x, \sigma_i) := \frac{d\mu \circ \sigma_i}{d\mu}(x) > 0 \text{ for } \mu\text{-a.e. } x \in D_i;$$

- (iii) there exists an endomorphism $\sigma : X \rightarrow X$ (called a *coding map*) such that $\sigma \circ \sigma_i(x) = x$ for μ -a.e. $x \in D_i$, $i \in \Lambda$.

If, additionally to properties (i) - (iii), we have $\bigcup_{i \in \Lambda} D_i = X$ (μ -a.e.), then the s.f.s. $\{\sigma_i : i \in \Lambda\}$ is called *saturated*.

Definition 4.2. We also say that a saturated s.f.s. satisfies *condition (C-K)* if for any $i \in \Lambda$ there exists a subset $\Lambda_i \subset \Lambda$ such that up to a set of measure zero

$$D_i = \bigcup_{j \in \Lambda_i} R_j.$$

In this case, condition (C-K) defines a 0-1 matrix \tilde{A} by the rule:

$$\tilde{a}_{i,j} = 1 \iff j \in \Lambda_i, \quad i \in \Lambda. \quad (4.1)$$

Then the matrix \tilde{A} is of the size $|\Lambda| \times |\Lambda|$.

Theorem 4.3 (Bezuglyi-Jor). Let $\{\sigma_i : i \in \Lambda\}$ and $\{\sigma'_i : i \in \Lambda\}$ be two isomorphic saturated s.f.s. defined on measure spaces (X, m) and (X', m') , respectively. Let also $\sigma : X \rightarrow X$ and $\sigma' : X' \rightarrow X'$ be the corresponding coding maps. Suppose that $\{T_i = T_i(m) : i \in \Lambda\}$ and $\{T'_i = T'_i(m') : i \in \Lambda\}$ are operators acting respectively on $L^2(X, m)$ and $L^2(X', m')$ according to the formulas:

Theorem 4.3 cont.

$$(T_i\psi)(x) = \chi_{R_i}(x)\rho_m(\sigma(x), \sigma_i)^{-1/2}\psi(\sigma(x)) \\ i \in \Lambda, \psi \in L^2(X, m); \quad (4.2)$$

$$(T'_i\xi)(x) = \chi_{R'_i}(x)\rho_{m'}(\sigma'(x), \sigma'_i)^{-1/2}\xi(\sigma'(x)), \quad (4.3) \\ i \in \Lambda, \xi \in L^2(X', m').$$

If \tilde{A} is the matrix defined by (4.1), then the representations π and π' of $\mathcal{O}_{\tilde{A}}$ determined by $\{T_i : i \in \Lambda\}$ and $\{T'_i : i \in \Lambda\}$, respectively, are unitarily equivalent.

Lemma 4.4. Let B and B' be given as above and let $m = m(P_n)$ and $m'm'(P'_n)$ be Markov measures defined on X_B and $X_{B'}$. Then for $\bar{\alpha} : X_B \rightarrow X_{B'}$, the measure $m' \circ \bar{\alpha}$ is equivalent to m if and only if for any $x = (x_i) \in X_B$

$$0 < \prod_{i=1}^{\infty} \frac{(p')_{s(\alpha(x_i)), \alpha(x_i)}^{(i)}}{p_{s(x_i), x_i}^{(i)}} < \infty. \quad (4.4)$$

Theorem 4.5 (Bezuglyi-Jor). Let B and B' be two 0-1 stationary simple Bratteli diagrams and let $\{\sigma_e : e \in E\}$ and $\{\sigma'_{e'} : e' \in E'\}$ be the corresponding s.f.s. defined on (X_B, m) and $(X_{B'}, m')$. Suppose that $\alpha : E \rightarrow E'$ is an admissible map and $\bar{\alpha} : X_B \rightarrow X_{B'}$ is the one-to-one transformation generated by α . Then $\bar{\alpha}$ implements the isomorphism of $\{\sigma_e : e \in E\}$ and $\{\sigma'_{\alpha(e)} : e \in E\}$ if and only if at least one of the following conditions hold:

Theorem 4.5 cont.

- ▶ $m = \mu, m' = \mu'$ where μ and μ' are the measures invariant with respect to \mathcal{R} and \mathcal{R}' respectively satisfying the invariance relation $\mu' \circ \bar{\alpha} = \mu$;
- ▶ $m = \nu, m' = \nu'$ where $\nu = \nu(P)$ and $\nu' = \nu(P')$ are stationary Markov measures satisfying condition (4.4);
- ▶ $m = m(P_n)$ and $m' = m'(P'_n)$ are Markov measures satisfying condition (4.4).

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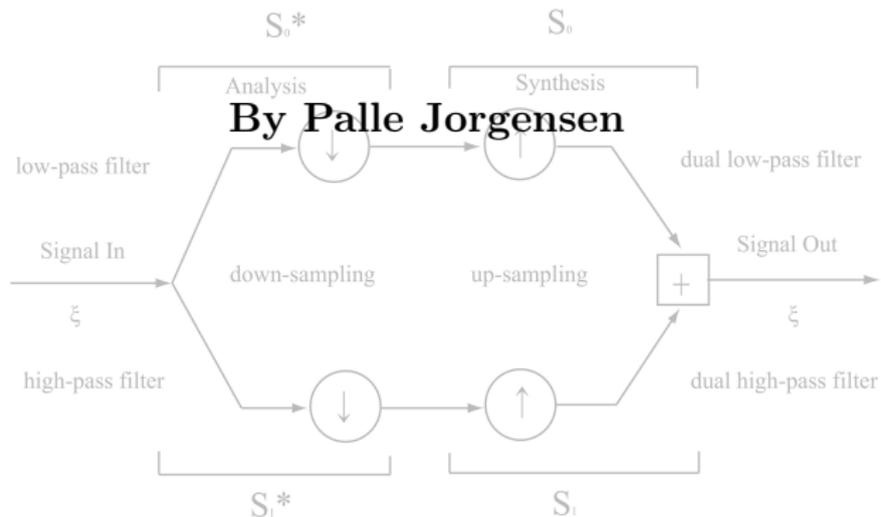


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Lecture 5. The Cuntz relations and kernel decompositions



Abstract

As demonstrated in earlier talks, the constructions of spectral measures often utilize “Cuntz isometries”, namely isometries that satisfy the Cuntz relations. This talk will discuss how understanding specific representations of the Cuntz algebras yields information concerning other spectra for a spectral measure. Conversely, beginning with a representation of a Cuntz algebra, a Markov measure can be associated to the representation which gives spectral information about the representation.

1. Atomic representations of Cuntz algebras

Definition 1.1. Fix an integer $N \geq 2$. The *Cantor group on N letters* is

$$\mathcal{K} = \mathcal{K}_N = \prod_1^{\infty} \mathbb{Z}_N = \{(\omega_1 \omega_2 \dots) : \omega_i \in \mathbb{Z}_N \text{ for all } i = 1, \dots\},$$

an infinite Cartesian product.

The elements of \mathcal{K} are infinite words. On the Cantor group, we consider the product topology. We denote by $\mathcal{B}(\mathcal{K})$ the sigma-algebra of Borel subsets of \mathcal{K} .

Definition 1.2. For a finite word $I = i_1 \dots i_k \in \mathbb{Z}_N^k$, we define the corresponding *cylinder set*

$$\mathcal{C}(I) = \{\omega \in \mathcal{K} : \omega_1 = i_1, \dots, \omega_k = i_k\}. \quad (1.1)$$

The Pontryagin dual group of \mathcal{K} is

$$\widehat{\mathcal{K}} = \{(\xi_1 \xi_2 \dots \xi_p 00 \dots) : \xi_1, \dots, \xi_p \in \mathbb{Z}_N, p \in \mathbb{N}\}.$$

The Fourier duality is given by

$$\langle \omega, \xi \rangle = \prod_{k=1}^{\infty} \langle \omega_k, \xi_k \rangle = \prod_{k=1}^{\infty} e^{2\pi i \frac{\omega_k \xi_k}{N}}, \quad (\omega \in \mathcal{K}, \xi \in \widehat{\mathcal{K}}). \quad (1.2)$$

Theorem 1.3 (Dutkay, Haussermann, Jorgensen). Let $(S_i)_{i \in \mathbb{Z}_N}$ be a representation of the Cuntz algebra \mathcal{O}_N on a Hilbert space \mathcal{H} . For every finite word I , define the projection

$$P(\mathcal{C}(I)) = S_I S_I^*. \quad (1.3)$$

Then P extends to a projection-valued measure on $\mathcal{B}(\widehat{\mathcal{K}})$.

For $\xi \in \widehat{\mathcal{K}}$, $\xi = \xi_1 \dots \xi_n 0 \dots$, define the operator

$$U(\xi) = \sum_{I \in \mathbb{Z}_N^n} \langle I, \xi \rangle S_I S_I^*. \quad (1.4)$$

Then U defines a unitary representation of $\widehat{\mathcal{K}}$ on \mathcal{H} .

Theorem 1.3, cont. Also the projection-valued measure P is the spectral decomposition (as defined by the Stone-Naimark-Ambrose-Godement theorem) of the unitary representation U , i.e.,

$$U(\xi) = \int_{\mathcal{K}} \langle \omega, \xi \rangle P(d\omega), \quad (\xi \in \widehat{\mathcal{K}}). \quad (1.5)$$

For $i \in \mathbb{Z}_N$ and $A \in \mathcal{B}(\mathcal{K})$

$$S_i P(A) S_i^* = P(\sigma_i(A)). \quad (1.6)$$

$$\sum_{i \in \mathbb{Z}_N} S_i P(\sigma_i^{-1}(A)) S_i^* = P(A), \quad (A \in \mathcal{B}(\mathcal{K})). \quad (1.7)$$

For a finite dimensional, cyclic invariant subspace M , there can be no non-periodic atoms and all cyclic atoms must be contained in M .

Theorem 1.4 (Dutkay, Haussermann, Jorgensen). Let $(S_i)_{i \in \mathbb{Z}_N}$ be a representation of \mathcal{O}_N on a Hilbert space \mathcal{H} and let P be the associated projection valued measure. Suppose M is a finite dimensional, cyclic invariant subspace for S_i^* , $i \in \mathbb{Z}_N$. Then for any non-periodic word $\omega \in \mathcal{K}$, $P(\omega) = 0$. For any cyclic word ω , $P(\omega)\mathcal{H} \subset M$.

Definition 1.5. A representation $(S_l)_{l \in \mathbb{Z}_N}$ of the Cuntz algebra \mathcal{O}_N on a Hilbert space \mathcal{H} is called *permutative* if there exists an orthonormal basis $\{e_i : i \in I\}$ such that for all $i \in I$, $j \in \mathbb{Z}_N$, the vector $S_j e_i$ is an element of this basis. This defines the *branching maps* $\sigma_j : I \rightarrow I$ by

$$S_j e_i = e_{\sigma_j(i)}, \quad (j \in \mathbb{Z}_N, i \in I). \quad (1.8)$$

Define the *coding map* $E : I \rightarrow \mathcal{K}_N$ by

$$E(i) = j_0 j_1 \dots, \text{ where } \sigma^k(i) \in \sigma_{j_k}(I), \text{ for all } k \in \mathbb{N}. \quad (1.9)$$

Theorem 1.6 (Dutkay, Haussermann, Jorgensen). Let $(S_i)_{i=0}^{N-1}$ be a permutative representation of the Cuntz algebra \mathcal{O}_N on a Hilbert space \mathcal{H} with orthonormal basis $\{e_\lambda : \lambda \in \Lambda\}$ and encoding mapping E . Then the representation is purely atomic and supported on $E(\Lambda)$. Moreover, for $\omega \in E(\Lambda)$, $P(\omega)$ is the projection onto the closed span of the vectors e_λ , $\lambda \in E^{-1}(\omega)$.

2. Characterizations of rectangular (para)-unitary rational Functions

Theorem 2.1 (Alpay, Jorgensen, Lewkowicz). Let $F(z)$ be a $p \times m$ -valued rational function with poles within the open unit disk (Schur stable).

1. Assume that $p \geq m$.

- ▶ $F(z)$ is in \mathcal{U} (=lossless) if and only if, it admits $(p+n) \times (m+n)$ minimal realization matrix

$$R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

satisfying

$$R^* \cdot \text{diag}\{I_n \quad I_p\} \cdot R = \text{diag}\{I_n \quad I_m\}. \quad (2.1)$$

Theorem 2.1 cont.

- ▶ If (2.1) holds, one can always find $\tilde{B} \in \mathbb{C}^{n \times (p-m)}$ and $\tilde{D} \in \mathbb{C}^{p \times (p-m)}$ so that the $(n+p) \times (n+p)$ augmented matrix

$$R_{n+p} := \left(\begin{array}{c|cc} A & B & \tilde{B} \\ \hline C & D & \tilde{D} \end{array} \right), \quad (2.2)$$

is unitary, i.e.

$$R_{n+p}^* R_{n+p} = I_{n+p} = R_{n+p} R_{n+p}^*. \quad (2.3)$$

- ▶ If (2.3) holds, one can always find, a constant isometry U_{iso} so that

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = R = R_{n+p} \cdot \left(\begin{array}{c|c} I_n & 0_{n \times m} \\ \hline 0_{p \times n} & U_{\text{iso}} \end{array} \right) \quad \begin{array}{l} U_{\text{iso}} \in \mathbb{C}^{p \times m} \\ U_{\text{iso}}^* U_{\text{iso}} = I_m \end{array} \quad (2.4)$$

Theorem 2.1 cont.

2. Assume that $m \geq p$.

- ▶ $F(z)$ is in \mathcal{U} (=lossless) if and only if, it admits $(p+n) \times (m+n)$ minimal realization matrix

$$R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \text{ satisfying}$$

$$R \cdot \text{diag}\{I_n \quad I_m\} \cdot R^* = \text{diag}\{I_n \quad I_p\}. \quad (2.5)$$

- ▶ If (2.5) holds, one can always find $\tilde{C} \in \mathbb{C}^{(m-p) \times n}$ and $\tilde{D} \in \mathbb{C}^{(m-p) \times m}$ so that the augmented matrix

$$R_{n+m} := \left(\begin{array}{c|c} A & B \\ \hline C & D \\ \tilde{C} & \tilde{D} \end{array} \right), \quad (2.6)$$

is unitary, i.e.

$$R_{n+m}^* R_{n+m} = I_{n+m} = R_{n+m} R_{n+m}^*. \quad (2.7)$$

Theorem 2.1 cont.

- ▶ If (2.7) holds, one can always find, a constant coimetry U_{coiso} so that

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = R = \left(\begin{array}{c|c} I_n & 0_{n \times m} \\ \hline 0_{p \times n} & U_{\text{coiso}} \end{array} \right) \cdot R_{n+m} \quad \begin{array}{l} U_{\text{coiso}} \in \mathbb{C}^{p \times m} \\ U_{\text{coiso}} U_{\text{coiso}}^* = I_p \end{array} \quad (2.8)$$

Theorem 2.2 (Alpay, Jorgensen, Lewkowicz). Let $F(z)$ be a $p \times m$ -valued rational function.

1. Assume that $p \geq m$. $F(z)$ is in \mathcal{U} if and only if, there exists in \mathcal{U} , a $p \times p$ -valued rational function $F_p(z)$, so that

$$F(z) = F_p(z)U_{\text{iso}} \quad \begin{array}{l} U_{\text{iso}} \in \mathbb{C}^{p \times m} \\ U_{\text{iso}}^* U_{\text{iso}} = I_m \end{array}$$

2. Assume that $m \geq p$. $F(z)$ is in \mathcal{U} if and only if, there exists in \mathcal{U} a $m \times m$ -valued rational function $F_m(z)$, so that

$$F(z) = U_{\text{coiso}}F_m(z) \quad \begin{array}{l} U_{\text{coiso}} \in \mathbb{C}^{p \times m} \\ U_{\text{coiso}}U_{\text{coiso}}^* = I_p \end{array} .$$

3. Pure states on \mathcal{O}_d

Definition 3.1. If $\hat{\omega}$ is any state on \mathcal{O}_d , let $(\mathcal{H}, \Omega, S_1, \dots, S_d)$ be the corresponding representation, and $(\mathcal{K}, \Omega, V_1, \dots, V_d)$ the corresponding Popescu system, and define the endomorphism σ of $\mathcal{B}(\mathcal{H})$ by $\sigma(\cdot) = \sum_{i=1}^d S_i S_i^*$, and the unital completely positive map σ of $\mathcal{B}(\mathcal{K})$ by $\sigma(\cdot) = \sum_{i=1}^d V_i V_i^*$.

Theorem 3.2 (Bratteli-Jor). If $\hat{\omega}$ is a state on \mathcal{O}_d , the following conditions are equivalent.

1. $\hat{\omega}$ is pure.
2. $\sigma(X) = X$ implies $X \in \mathbb{C}\mathbb{1}_{\mathcal{H}}$, $X \in \mathcal{B}(\mathcal{H})$.
3. $\sigma(Y) = Y$ implies $Y \in \mathbb{1}_{\mathcal{K}}$, $Y \in \mathcal{B}(\mathcal{K})$.
4. $\{V_i, V_i^*\}$ acts irreducibly on \mathcal{K} , and $P \in \pi(\mathcal{O}_d)''$.

Theorem 3.3 (Bratteli-Jor). Let \mathcal{K} be a Hilbert space, and let $V_1, \dots, V_d \in \mathcal{B}(\mathcal{K})$ be operators satisfying

$$\sum_{i \in \mathbb{Z}_d} V_i V_i^* = \mathbb{1}.$$

- ▶ Then \mathcal{K} can be embedded into a larger Hilbert space $\mathcal{H} = \mathcal{H}_V$ carrying a representation S_1, \dots, S_d of the Cuntz algebra \mathcal{O}_d such that if $P : \mathcal{H} \rightarrow \mathcal{K}$ is the projection onto \mathcal{K} we have

$$V_i^* = S_i^* P$$

(i.e., $S_i^* \mathcal{K} \subset \mathcal{K}$ and $S_i^* P = P S_i^* P = V_i^*$) and \mathcal{K} is cyclic for the representation.

Theorem 3.3, cont.

- ▶ The system $(\mathcal{H}, S_1, \dots, S_d, P)$ is unique up to unitary equivalence, and if $\sigma : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{K})$ is defined by

$$\sigma(A) = \sum_i V_i A V_i^*$$

then the commutant of the representation $\{S_1, \dots, S_d\}'$ is isometrically order isomorphic to the fixed point set $\mathcal{B}(\mathcal{K})^\sigma = \{A \in \mathcal{B}(\mathcal{K}) \mid \sigma(A) = A\}$ by the map $A' \mapsto P A' P$.

Theorem 3.3, cont.

- ▶ Moreover generally, if $W_1, \dots, W_d \in \mathcal{B}(\mathcal{K})$ is another set of operators satisfying

$$\sum_{i \in \mathbb{Z}_d} W_i W_i^* = \mathbb{1}$$

and T_1, \dots, T_d are the corresponding representations of s_1, \dots, s_d , then there is an isometric linear isomorphism between intertwiners $U : \mathcal{H}_V \rightarrow \mathcal{H}_W$, i.e., operators satisfying

$$US_i = T_i U,$$

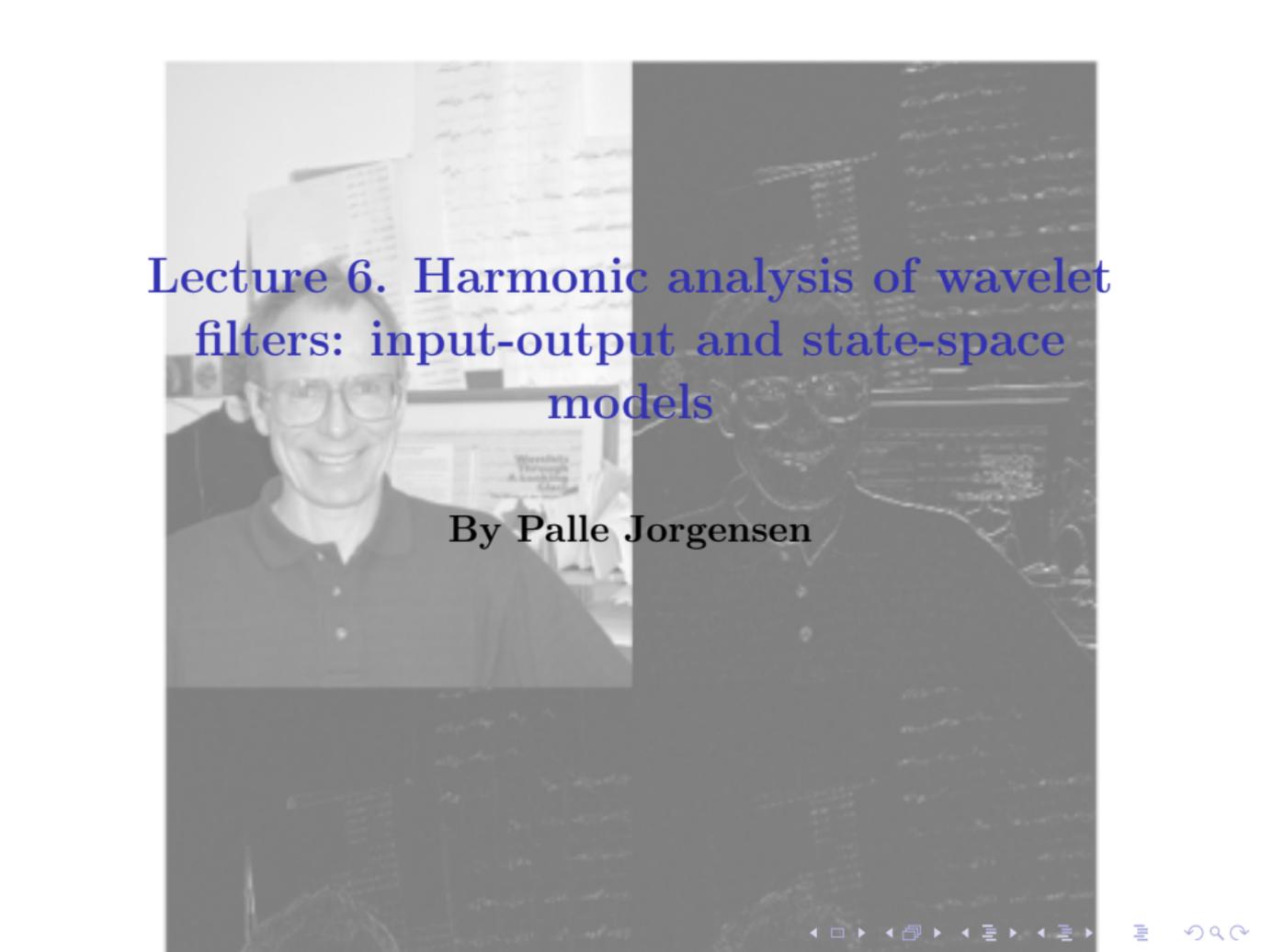
and operator $V \in \mathcal{B}(\mathcal{K})$ such that

$$\sum_{i \in \mathbb{Z}_d} W_i V W_i^* = V,$$

given by the map $U \mapsto V = PUP$.

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Lecture 6. Harmonic analysis of wavelet filters: input-output and state-space models

By Palle Jorgensen

Abstract

Wavelets have a dual existence between the discrete and continuous realms manifested in the discrete and continuous wavelet transforms. Wavelet filters give another bridge between the smooth and non-smooth domains in that the convergence of the cascade algorithm yields wavelets and wavelet transforms in a smooth setting, i.e. \mathbb{R}^d , and also the non-smooth setting such as the Cantor dust, depending on the parameters embedded in the choice of wavelet filters.

1. Wavelets on Fractals

Terminology for measures ν on \mathbb{T}

- ▶ If $f \in C(\mathbb{T})$, set

$$\nu(f) := \int_{\mathbb{T}} f d\nu = \int_{\mathbb{T}} f(z) d\nu(z).$$

- ▶ If $\tau: \mathbb{T} \rightarrow \mathbb{T}$ is a measurable transformation, set

$$\nu^{\tau^{-1}}(E) = \nu(\tau^{-1}(E))$$

for Borel sets $E \subset \mathbb{T}$.

- ▶ If $R: C(\mathbb{T}) \rightarrow C(\mathbb{T})$ is linear, set

$$(\nu R)(f) = \nu(Rf) = \int_{\mathbb{T}} (Rf) d\nu.$$

- ▶ We introduce

$$\mathcal{L}(m_0) := \{\nu \in M_1(\mathbb{T}) : \nu R_{m_0} = \nu\}. \quad (1.1)$$

We now return to the case of the middle-third Cantor set \mathbf{C} .

- ▶ Set $s := \log_3(2)$, and view $\chi_{\mathbf{C}}$ as an element in the Hilbert space $L^2(\mathbb{R}, (dx)^s)$.
- ▶ We recall the usual unitary operators $(Uf)(x) := \frac{1}{\sqrt{2}}f(\frac{x}{3})$, and $(T_k f)(x) = f(x - k)$, $k \in \mathbb{Z}$, and the relation

$$UT_k U^{-1} = T_{3k}, \quad k \in \mathbb{Z}. \quad (1.2)$$

- ▶ If $m_0(z) = \sum_k a_k z^k$, $m_0 \in L^\infty(\mathbb{T})$, is given, we define the cascade approximation operator $M = M_{m_0}$ as before

$$\begin{aligned} Mf(x) &= U^{-1}m_0(T)f(x) \\ &= \sqrt{2} \sum_{k \in \mathbb{Z}} a_k f(3x - k), \quad f \in L^2(\mathbb{R}, (dx)^s). \end{aligned}$$

- ▶ The condition

$$\frac{1}{3} \sum_{w^3=z} |m_0(w)|^2 = 1, \text{ a.e. } z \in \mathbb{T}, \quad (1.3)$$

will be a standing assumption on m_0 .

- ▶ Define the sequence $m_0^{(n)}(z) := m_0(z)m_0(z^3) \cdots m_0(z^{3^{n-1}})$, and we say that m_0 has *frequency localization* if the limit of an associated sequence of measures,

$$\lim_{n \rightarrow \infty} |m_0^{(n)}(z)|^2 d\mu(z)$$

exists; i.e., if there is a $\nu \in M_1(\mathbb{T})$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(z) |m_0^{(n)}(z)|^2 d\mu(z) = \int_{\mathbb{T}} f(z) d\nu(z), \quad \forall f \in C(\mathbb{T}). \quad (1.4)$$

- ▶ Recall that if $m_0 \in \text{Lip}(\mathbb{T})$ is assumed, and R_{m_0} has Perron-Frobenius spectrum, then it has frequency localization, and the limit measure ν satisfies

$$\lim_{n \rightarrow \infty} \text{dist}_{\text{Haus}}(d\nu, |m_0^{(n)}|^2 d\mu) = 0. \quad (1.5)$$

Theorem 1.1 (Dutkay-Jor). The Dichotomy Theorem.
 Let $m_0 \in L^\infty(\mathbb{T})$ be given, and suppose (1.3) holds, and let ν be the corresponding limit measure. Assume further that, for $k \in \mathbb{Z}_+$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} |m_0^{(n)}(z)|^2 A_{0,0}^{(k)}(z) d\mu(z) = \nu(A_{0,0}^{(k)}), \quad (1.6)$$

where A is the matrix function

$$A_{j,k}(z) := \langle m_k, m_j \rangle_N(z), \quad z \in \mathbb{T};$$

and $A^{(k)}(z) := A(z)A(z^3)\dots A(z^{3^k})$.

Let $M = M_{m_0}$ be the cascade approximation operator in $L^2(\mathbb{R}, (dx)^s)$, $s = \log_3(2)$.

Theorem 1.1, cont. Then the limit

$$\lim_{n \rightarrow \infty} M^n \chi_{\mathbf{C}} \text{ exists in } L^2(\mathbb{R}, (dx)^s) \quad (1.7)$$

if and only if there is a Borel subset $E \subset \mathbb{T}$ such that $\nu(E) = 1$ (i.e., E is a supporting set for ν), and $m_0(z) = \frac{1+z^2}{\sqrt{2}}$, for all $z \in E$. In the special case where A is further assumed continuous and m_0 has frequency localization, the condition (1.6) is automatically satisfied,

$$Mf(x) = (M_{\mathbf{C}}f)(x) = f(3x) + f(3x - 2) \quad (1.8)$$

and

$$M_{\mathbf{C}}\chi_{\mathbf{C}} = \chi_{\mathbf{C}}. \quad (1.9)$$

Low pass filters

We will consider low-pass filters m_0 with the following properties:

- (1) $m_0 \in \text{Lip}(\mathbb{T})$;
- (2) m_0 has a finite number of zeros;
- (3) $R_{m_0}(1) = 1$.

Theorem 1.2 (Dutkay-Jor). Suppose m_0 satisfies (1), (2), (3). Then exactly one of the following affirmations is true:

- (i) There exists an (m_0, N) -cycle. In this case, the invariant measures ν with

$$\nu(R_{m_0}(f)) = \nu(f), (f \in C(\mathbb{T}))$$

are atomic supported on the (m_0, N) -cycles. The wavelet representation associated to $(m_0, 1)$ is a direct sum of cyclic amplifications of $L^2(\mathbb{R})$.

Theorem 1.2, cont.

- (ii) There are no (m_0, N) -cycles. In this case there are no eigenvalues for $R_{m_0} |_{C(\mathbb{T})}$ with $|\lambda| = 1$ other than $\lambda = 1$; 1 is a simple eigenvalue. There exists a unique probability measure ν on \mathbb{T} which is invariant for R_{m_0} (i.e., $\nu(R_{m_0}(f)) = \nu(f)$ for $f \in C(\mathbb{T})$).

$$\lim_{n \rightarrow \infty} R_{m_0}^n(f) = \nu(f) \text{ uniformly } f \in C(\mathbb{T}). \quad (1.10)$$

2. Minimality of the data in wavelet filters

Theorem 2.1 (Jorgensen). Let $T^{(A)}$ be a wavelet representation of \mathcal{O}_N on $\mathcal{H} = L^2(\mathbb{T})$, and assume the genus of A is g . Let r_0 be as in (2.1),

$$r_0 = \left\lfloor \frac{gN - 1}{N - 1} \right\rfloor, \quad (2.1)$$

and let P be the projection onto $\mathcal{K} := \text{span} \{z^{-k}; 0 \leq k \leq r_0\}$. Suppose there is a second projection $E \in \mathcal{B}(\mathcal{K})$ such that $0 \neq E \neq P$, and E commutes with $T_i^{(A)*}P$ for all $i = 0, 1, \dots, N - 1$.

Theorem 2.1, cont.

Then it follows that E is diagonal with respect to the basis $\{z^{-k}; k = 0, 1, \dots, r_0\}$ in \mathcal{K} . Moreover, $A(z)$ has a matrix corner of the form

$$V \begin{pmatrix} z^{n_0} & 0 & \cdots & 0 \\ 0 & z^{n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{n_{M-1}} \end{pmatrix}, \quad (2.2)$$

where $V \in U_M(\mathbb{C})$, and where the exponents n_i of the diagonal corner in $A(z)$ satisfy $0 \leq n_i \leq g - 1$ for all $i = 0, 1, \dots, M - 1$.

Theorem 2.2 (Jorgensen).

- (a) Let $A \in \mathcal{P}(\mathbb{T}, U_N(\mathbb{C}))$ be given. Suppose it is purely non-diagonal, and let $T^{(A)}$ be the corresponding wavelet representation of \mathcal{O}_N on $L^2(\mathbb{T})$. Then it follows that $T^{(A)}$ is irreducible.
- (b) Let r_0 be as in (2.1). Let p, q , $0 \leq p \leq q \leq r_0$ be the optimal numbers such that

$$\mathcal{K}_{\text{red}} = \langle \{-p, -(p+1), \dots, -q\} \rangle \quad (2.3)$$

is $T_i^{(A)*}$ -invariant for all i , and further satisfies

$$\langle \{0, -1, \dots, -p+1, -q-1, \dots, -r_0\} \rangle \subset [\mathcal{O}_N \mathcal{K}_{\text{red}}]. \quad (2.4)$$

Theorem 2.2, cont.

Then the following three properties hold:

- (i) $T_i^{(A)*}(\mathcal{K}_{\text{red}}) \subset \mathcal{K}_{\text{red}}$ for all i ,
 - (ii) \mathcal{K}_{red} is cyclic (for $L^2(\mathbb{T})$),
 - (iii) \mathcal{K}_{red} is minimal with respect to properties (i)–(ii).
- (c) The minimal space \mathcal{K}_{red} from (b) is reduced from the right if $N - 1$ divides $gN - 1$, where g is the genus, and if not, it is $\langle \{-p, \dots, -r_0\} \rangle$; so it is only “truncated” at one end when $N - 1$ does not divide $gN - 1$.

3. Measures in wavelet decompositions

We now turn to the computation of the measures $\mu_f(\cdot) = \|E(\cdot)f\|^2$ in the special case when the representation of \mathcal{O}_N arises from a system of subband filters.

- ▶ A system of subband filters corresponding to N subbands is a set of L^∞ -functions m_0, m_1, \dots, m_{N-1} on \mathbb{T} such that the following matrix function on \mathbb{T} takes unitary values:

$$\frac{1}{\sqrt{N}} \left(m_j \left(z e^{i2\pi \frac{k}{n}} \right) \right)_{0 \leq j, k < N}. \quad (3.1)$$

Specifically, for a.e. $z \in \mathbb{T}$, the $N \times N$ matrix of (3.1) is assumed unitary.

Theorem 3.1 (Jorgensen). Let the functions $(m_j)_{0 \leq j < N}$ satisfy the condition (3.1) and let S_j be the corresponding wavelet representation of \mathcal{O}_N on the Hilbert space $L^2(\mathbb{T})$. Let $f \in L^2(\mathbb{T})$, $\|f\| = 1$, then

$$\mu_f(J_k(a)) = \sum_{n \in \mathbb{Z}} \left| (f \bar{m}_a)^\wedge(nN^k) \right|^2, \quad (3.2)$$

where $k \in \mathbb{Z}_+$, and $a = (a_1, \dots, a_k) \in \Gamma_N^k$.

Theorem 3.2 (Jorgensen). Let $N \in \mathbb{N}$, and let $\{A_k(a)\}_{k \in \mathbb{N}, a \in \Gamma_N^k}$ be an N -adic system of partitions of a compact metric space X . Let $(S_i)_{0 \leq i < N}$ be a representation of \mathcal{O}_N on a Hilbert space \mathcal{H} , and let $E^A(\cdot)$ be the corresponding projection-valued measure.

(a) Then there is a set f_1, f_2, \dots (possibly finite), $f_i \in \mathcal{H}$, $\|f_i\| = 1$, such that the measures

$$\mu_i(\cdot) := \|E^A(\cdot) f_i\|^2 \tag{3.3}$$

are mutually singular.

Theorem 3.2, cont.

(b) For each i , there is a unique isometry

$$V_i: L^2(X, \mu_i) \longrightarrow \mathcal{H} \quad (3.4)$$

satisfying the following three conditions:

$$V_i \chi_{A_k(a)} = S_a S_a^* f_i \quad \text{for } k \in \mathbb{N}, a \in \Gamma_N^k, \quad (3.5)$$

$$V_i^* S_a S_a^* V_i = M_{\chi_{A_k(a)}}, \quad (3.6)$$

and

$$V_i(L^2(X, \mu_i)) = \mathcal{H}_{f_i}. \quad (3.7)$$

(c) Moreover, $\mathcal{H} = \sum_i^\oplus \mathcal{H}_{f_i}$, where $\mathcal{H}_{f_i} := [\mathcal{C}f_i]$.

4. Fourier series on fractals: a parallel with wavelet theory

Definition 4.1. Let A be a $d \times d$ integer matrix. Let B, L be two finite subsets of \mathbb{Z}^d of the same cardinality $|B| = |L| =: N$. Then (A, B, L) is called a *Hadamard triple* if the matrix

$$\frac{1}{\sqrt{N}} \left(e^{2\pi i A^{-1} b \cdot l} \right)_{b \in B, l \in L} \quad (4.1)$$

is unitary. (This is called a complex Hadamard matrix.)

Assumption 4.2. There is set $L \subset \mathbb{Z}^d$ of the same cardinality as B , $|B| = |L| =: N$ such that (A, B, L) is a Hadamard triple. Moreover we will assume that $0 \in B$ and $0 \in L$.

Proposition 4.3 (LW02). Let A be a $d \times d$ expansive integer matrix, $B, L \subset \mathbb{Z}^d$ with $|B| = |L| = N$. Then (A, B, L) is a Hadamard triple if and only if

$$\sum_{l \in L} |m_B((A^T)^{-1}(x + l))|^2 = 1, \quad (x \in \mathbb{R}^d) \quad (4.2)$$

where m_B is the function defined in

$$m_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x}, \quad (x \in \mathbb{R}^d). \quad (4.3)$$

It is clear that we have now a QMF condition for our fractal setting.

Definition 4.4. We see that m_0 satisfies the *quadrature mirror filter (QMF)* condition iff (Def.)

$$\sum_{l \in \mathcal{L}} |m_0((A^T)^{-1}(x + l))|^2 = 1, \quad (x \in \mathbb{R}^d), \quad (4.4)$$

where \mathcal{L} is a complete set of representatives for $\mathbb{Z}^d / A^T \mathbb{Z}^d$.

Definition 4.5. We say that the iterated function system

$$\sigma_l(x) = (A^T)^{-1}(x + l), \quad (x \in \mathbb{R}^d, l \in L)$$

is *dual* to the IFS $(\tau_b)_{b \in B}$ if (A, B, L) is a Hadamard triple.

With the Hadamard triple we have a first candidate for a spectrum of μ_B :

$$\Lambda_0 := \left\{ \sum_{k=0}^n (A^T)^k l_k \mid l_k \in L, n \in \mathbb{N} \right\}. \quad (4.5)$$

Lawton's theorem and transfer operators

Theorem 4.6 (Lawton). Let m_0 be a Lipschitz low-pass filter with $m_0(0) = 1$ and satisfying the QMF condition. Then the scaling function φ is an orthonormal scaling function (i.e., its translates are orthonormal) if and only if the only continuous \mathbb{Z}^d -periodic functions h that satisfy

$$\sum_{l \in \mathcal{L}} |m_0((A^T)^{-1}(x + l))|^2 h((A^T)^{-1}(x + l)) = h(x), \quad (x \in \mathbb{R}^d) \quad (4.6)$$

are the constants. (Recall that \mathcal{L} is a complete set of representatives for $\mathbb{Z}^d/A^T\mathbb{Z}^d$.)

Lawton's theorem brings a new character into play: the transfer operator.

Definition 4.7. The operator defined on \mathbb{Z}^d -periodic functions f on \mathbb{R}^d by

$$(R_{m_0}f)(x) = \sum_{l \in \mathcal{L}} |m_0((A^T)^{-1}(x+l))|^2 f((A^T)^{-1}(x+l)), \quad (x \in \mathbb{R}^d) \quad (4.7)$$

is called the *transfer operator*.

In the fractal setting, one uses the function m_B for the “weight” in place of m_0 , and the dual IFS $(\sigma_l)_{l \in L}$ to replace the inverse branches $x \mapsto (A^T)^{-1}(x+l)$. Thus the transfer operator is defined by

$$(R_{m_B}f)(x) = \sum_{l \in L} |m_B(\sigma_l(x))|^2 f(\sigma_l(x)), \quad (x \in \mathbb{R}^d). \quad (4.8)$$

Proposition 4.8 (Dutkay-Jor). The following affirmations hold:

1. In the wavelet context, with h_φ defined in (4.9),

$$h_\varphi(x) := \sum_{k \in \mathbb{Z}} |\hat{\varphi}(x+k)|^2 = 1 \quad (x \in \mathbb{Z}^d). \quad (4.9)$$

Then $R_{m_0} h_\varphi = h_\varphi$.

2. In the fractal context, if Λ_0 is the candidate for a spectrum, defined in (4.5), and h_{Λ_0} is the function defined in (4.10),

$$h_\Lambda(x) := \sum_{\lambda \in \Lambda} |\hat{\mu}_B(x+\lambda)|^2 = 1 \quad (x \in \mathbb{R}^d); \quad (4.10)$$

then one has $R_{m_B} h_{\Lambda_0} = h_{\Lambda_0}$.

5. W-Markov measures, transfer operators, wavelets and multiresolutions

Axiom 5.1.

(i) Let $\mathcal{G}_n \subset \mathcal{F}$ be the smallest sigma-algebra for which the variables T_0, \dots, T_n are measurable. We have that $V_n^* V_{n+1}$ does not depend on n , and

$$\mathbb{E}(f \circ T_{n+1} | \mathcal{G}_n) = \mathbb{E}(f \circ T_{n+1} | \mathcal{F}_n) = R(f) \circ T_n, \quad n = 0, 1, \dots \quad (5.1)$$

(ii) The measures μ_0 and μ_1 are equivalent.

- ▶ We refer to (5.1) as the Markov property in the present setting. If in the expression $R_{A,B}f = V_A^* V_B f$, to $A = T_n$ and $B = T_{n+1}$ and if moreover R_{T_{n+1}, T_n} is independent of n we get

$$\mathbb{E}(f \circ T_{n+1} | \mathcal{F}_n) = (R(f)) \circ T_n.$$

Iterating we get

$$\mathbb{E}(f \circ T_{n+k} | \mathcal{F}_n) = \left(R^k(f) \right) \circ T_n.$$

Theorem 5.2 (Alpay, Jorgensen, Lewkowicz). Let $(\Omega, \mathcal{F}, \mathbb{P}, (T_n)_{n \in \mathbb{N}_0})$ satisfy axioms (i) and (ii) above. Then there is a probability measure \mathbb{P}^\times on the Cartesian product $\prod_0^\infty X$, and an isomorphism \widehat{T} between $(\Omega, \mathcal{F}, \mathbb{P}, (T_n)_{n \in \mathbb{N}_0})$ and $(\prod_{n=0}^\infty X, \mathcal{C}, \mathbb{P}^\times, (\pi_n)_{n \in \mathbb{N}_0})$, meaning that

$$\pi_n \circ \widehat{T} = T_n, \quad n = 0, 1, \dots \quad (5.2)$$

Homogeneous Markov chains (HMC)

Notation. Let (X, \mathcal{B}_X) be a set with a fixed sigma-algebra \mathcal{B}_X . We consider another measure-space (Y, \mathcal{D}) .

- ▶ Let ψ_0, ψ_1, \dots be a sequence of independent identically distributed (i.i.d.) Y -valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with probability distribution ν . Such a sequence is called a white noise or a driving sequence.
- ▶ One way to construct such a sequence is as follows. One takes

$$\Omega_Y = \prod_{n=0}^{\infty} Y \quad (= Y^{\mathbb{N}_0})$$

endowed with the cylinder sigma-algebra \mathcal{C} , and the infinite product measure $\nu_{\infty} = \nu \times \nu \cdots$, and set

$$\psi_n(y_0, y_1, \dots) = y_n$$

► Thus

$$\nu(D) = \nu_\infty(\psi_n^{-1}(D)), \quad D \in \mathcal{D}, \quad \text{and } n = 0, 1, \dots \quad (5.3)$$

In particular,

$$\int_{\Omega} F(\cdot, \psi_0(\omega)) d\nu_\infty(\omega) = \int_Y F(\cdot, y) d\nu(y) \quad (5.4)$$

► We now consider a measurable map F from $X \times Y$ into X , where Y is another measure-space. We define

$$(R_F f)(x) = \int_Y f(F(x, y)) d\nu(y), \quad f \in \mathcal{M}(X, \mathcal{B}). \quad (5.5)$$

- Let $\Omega_Y = \prod_{n=0}^{\infty} Y$, and for $\omega = (y_0, y_1, \dots) \in \Omega_Y$, we define (with $F_y = F(\cdot, y)$)

$$\begin{aligned}\omega|n &= (y_0, \dots, y_n) \\ F_{\omega|n} &= F_{y_n} F_{y_{n-1}} \cdots F_{y_1} F_{y_0}.\end{aligned}\tag{5.6}$$

We assume that

$$\bigcap_{n=1}^{\infty} F_{\omega|n}(X) = \{x_{\omega}\}\tag{5.7}$$

is a singleton. We then set

$$V(\omega) = x_{\omega}.\tag{5.8}$$

Theorem 5.3 (Alpay, Jorgensen, Lewkowicz). Let ν be a probability measure on (Y, \mathcal{D}) , and let $\times_{n=0}^{\infty} \nu$ be the corresponding infinite product measure on Ω . Assume that (5.7) is in force, and let V be defined by (5.8). The formula

$$\mu(B) = (\times_{n=0}^{\infty} \nu)(V^{-1}(B)), \quad B \in \mathcal{B}_X, \quad (5.9)$$

then defines a measure on (X, \mathcal{B}) which satisfies

$$\mu R_F = \mu, \quad (5.10)$$

that is

$$\iint_{X \times Y} f(F(x, y)) d\nu(y) d\mu(x) = \int_X f(x) d\mu(x) \quad (5.11)$$

holds $\forall f \in \mathcal{M}(X, \mathcal{B}_X)$.

6. Filters and Matrix Factorization

The 2×2 case: Polynomials

Theorem 6.1 (Sweldens). Let $A \in SL_2(\text{pol})$, then there are $l, p \in \mathbb{Z}_+$, $K \in \mathbb{C} \setminus \{0\}$ and polynomial functions U_1, \dots, U_p , L_1, \dots, L_p such that

(6.1)

The assertion that Theorem 6.1 holds with small modifications in the 3×3 case.

- ▶ In the definition of $A \in SL_3(\text{pol})$, it is understood that $A(z)$ has $\det A(z) \equiv 1$ and that the entries of the inverse matrix $A(z)^{-1}$ are again polynomials.
- ▶ Note that if L, M, U and V are polynomials, then the four matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ L & 1 & 0 \\ 0 & M & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ L & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & U & 0 \\ 0 & 1 & V \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & U \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.2}$$

are in $SL_3(\text{pol})$.

► Note that

$$\begin{pmatrix} 1 & 0 & 0 \\ L & 1 & 0 \\ 0 & M & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -L & 1 & 0 \\ LM & -M & 1 \end{pmatrix} \quad \text{and} \quad (6.3)$$

$$\begin{pmatrix} 1 & U & 0 \\ 0 & 1 & V \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -U & UV \\ 0 & 1 & -V \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.4)$$

Theorem 6.2 (Jor-Song). Let $A \in SL_3(\text{pol})$; then the conclusion in Theorem 6.1 carries over with the modification that the alternating upper and lower triangular matrix-functions now have the form (6.2) or (6.3)-(6.4) where the functions L_j, M_j, U_j and $V_j, j = 1, 2, \dots$ are polynomials.

The $N \times N$ case

Theorem 6.3 (Jor-Song). Let $N \in \mathbb{Z}_+$, $N > 1$, be given and fixed. Let $A \in SL_N(\text{pol})$; then the alternative factors in the product are upper and lower triangular matrix-functions in $SL_N(\text{pol})$. We may take the lower triangular matrix-factors $\mathcal{L} = (L_{i,j})_{i,j \in \mathbb{Z}_N}$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_p & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{p+1} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{N-1} & 0 & 1 \end{pmatrix}.$$

Theorem 6.3, cont.

- ▶ The polynomial entries are given by

$$\begin{cases} L_{i,i} \equiv 1, \\ L_{i,j}(z) = \delta_{i-j,p} L_i(z); \end{cases} \quad (6.5)$$

- ▶ The upper triangular factors of the form $\mathcal{U} = (U_{i,j})_{i,j \in \mathbb{Z}_N}$ with

$$\begin{cases} U_{i,i} \equiv 1, \\ L_{i,j}(z) = \delta_{i-j,p} U_i(z). \end{cases} \quad (6.6)$$

7. Orthogonal Exponentials for Bernoulli Iterated Function Systems

Orthogonal exponentials with respect to $\nu_{\frac{3}{4}}$

Theorem 7.1 (Jorgensen, Kornelson, Shuman). There exist infinitely many infinite sets Λ such that $\{e_\ell : \ell \in \Lambda\}$ is a mutually orthogonal set of functions with respect to the measure $\nu_{\frac{3}{4}}$.

Theorem 7.2 (Jorgensen, Kornelson, Shuman). Define the set Γ_k for each $k \in \mathbb{N}$ as follows:

$$\Gamma_k = \left\{ \sum_{j=k-1}^p \frac{a_j 4^j}{3^k} : p \text{ finite, } a_j \in \{0, 1\} \right\} \cup \{0\}$$

Each set $\{e_\gamma : \gamma \in \Gamma_k\}$ is an orthonormal family in $L^2(X_{\frac{3}{4}}, \nu_{\frac{3}{4}})$.

Rational values of λ

Theorem 7.3 (Jorgensen, Kornelson, Shuman). Let $\lambda \in \mathbb{Q} \cap (0, 1)$ and let $\lambda = \frac{a}{b}$ be in reduced form. If b is odd, then any collection of pairwise orthogonal exponential functions in the Hilbert space $L^2(X_\lambda, \nu_\lambda)$ can have only finitely many elements. If b is even, then there exists a countably infinite collection of orthogonal exponentials in $L^2(X_\lambda, \nu_\lambda)$.

8. Harmonic analysis of iterated function systems with overlap

Definition 8.1. Let (X, \mathcal{B}, μ) be a finite measure space, and let $(\tau_i)_{i=1}^N$ be a finite system of measurable endomorphisms, $\tau_i: X \rightarrow X$, $i = 1, \dots, N$; and suppose μ is some normalized equilibrium measure. We then say that the system has *essential overlap* if

$$\sum_{i \neq j} \mu(\tau_i(X) \cap \tau_j(X)) > 0. \quad (8.1)$$

Theorem 8.2 (Jorgensen, Kornelson, Shuman). Let (X, \mathcal{B}, μ) and $(\tau_i)_{i=1}^N$ be as in Definition 8.1; in particular we assume that μ is some (τ_i) -equilibrium measure. We assume further that each τ_i is of finite type. Let $\mathcal{H} = L^2(\mu)$,

$$F_i: f \mapsto \frac{1}{\sqrt{N}} f \circ \tau_i,$$

and let $\mathbb{F} = (F_i)$ be the corresponding column isometry. Then \mathbb{F} maps onto $\bigoplus_1^N L^2(\mu)$ if and only if (τ_i) has zero μ -essential overlap.

The general case

Theorem 8.3 (Jorgensen, Kornelson, Shuman). Let $N \in \mathbb{N}$, $N \geq 2$, be given, and let $(\tau_i)_{i \in \mathbb{Z}_N}$ be a contractive IFS in a complete metric space. Let (X, μ) be the Hutchinson data. Let $P (= P_{1/N})$ be the Bernoulli measure on $\Omega = \prod_1^\infty \mathbb{Z}_N = \mathbb{Z}_N^{\mathbb{N}}$. Let $\pi: \Omega \rightarrow X$ be the encoding mapping. Set

$$F_i f := \frac{1}{\sqrt{N}} f \circ \tau_i \quad \text{for } f \in L^2(X, \mu), \text{ and} \quad (8.2)$$

$$S_i^* \psi := \frac{1}{\sqrt{N}} \psi \circ \sigma_i \quad \text{for } \psi \in L^2(\Omega, P), \quad (8.3)$$

where σ_i denotes the shift map.

Theorem 8.3, cont.

(a) Then the operator $V: L^2(X, \mu) \rightarrow L^2(\Omega, P)$ given by

$$Vf = f \circ \pi \tag{8.4}$$

is isometric.

(b) The following intertwining relations hold:

$$VF_i = S_i^*V, \quad i \in \mathbb{Z}_N. \tag{8.5}$$

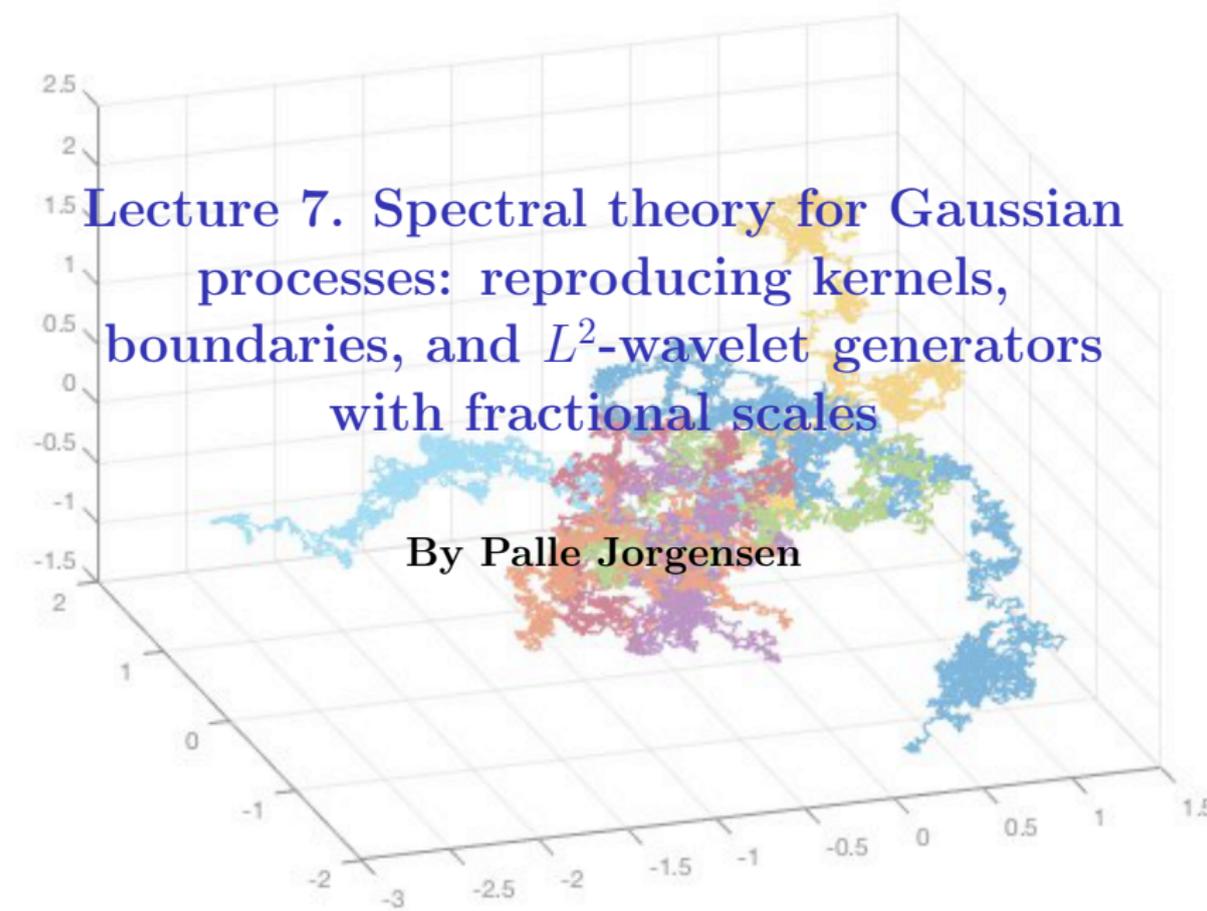
(c) The isometric extension $L^2(X, \mu) \hookrightarrow L^2(\Omega, P)$ of the (F_i) -relations is minimal in the sense that $L^2(\Omega, P)$ is the closure of

$$\bigcup_n \bigcup_{i_1 i_2 \dots i_n} S_{i_1} S_{i_2} \cdots S_{i_n} V L^2(X, \mu). \tag{8.6}$$

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Lecture 7. Spectral theory for Gaussian processes: reproducing kernels, boundaries, and L^2 -wavelet generators with fractional scales

By Palle Jorgensen

Abstract

Gaussian processes for whose spectral (meaning generating) measure is spectral (meaning possesses orthogonal Fourier bases) are considered. These Gaussian processes admit an Itô-like stochastic integration as well as harmonic and wavelet analyses of related Reproducing Kernel Hilbert Spaces.

1. Spectral theory and generators for Gaussian processes

Definition 1.1. A Gaussian (noise) stochastic process indexed by (M, \mathcal{B}, μ) consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

- ▶ Ω is a set (sample space), \mathcal{F} is a sigma-algebra of subsets (events) of Ω , and \mathbb{P} is a probability measure defined on \mathcal{F} .

We assume that, for all $A \in \mathcal{B}$ such that $\mu(A) < \infty$, there is a Gaussian random variable

$$W_A = W_A^{(\mu)} : \Omega \longrightarrow \mathbb{R} \quad (1.1)$$

with zero mean and variance $\mu(A)$ (that is, $W_A \sim N(0, \mu(A))$), the Gaussian with zero mean and variance $\mu(A)$).

Definition 1.2. Let (M, \mathcal{B}) be fixed, and let \mathcal{H} denote the corresponding Hilbert space of sigma-function. For $\mu \in (M, \mathcal{B})$ we set

$$\mathcal{H}(\mu) = \left\{ f\sqrt{d\mu} \mid f \in \mathbf{L}^2(d\mu) \right\}, \quad (1.2)$$

and

$$\mathcal{H}_1(\mu) = \left\{ f\sqrt{d\mu} \mid f \in \mathbf{L}^2(d\mu), |f| \leq 1 \mu \text{ a.e.} \right\}. \quad (1.3)$$

Theorem 1.3 (Alpay-Jor). Let \mathcal{H} be the sigma-Hilbert space. Let (M, \mathcal{B}) be a sigma-finite measure space, and μ a positive measure on \mathcal{B} . Then, the map

$$f\sqrt{d\mu} \longrightarrow W^{(\mu)}(f), \quad (1.4)$$

defined for every $\mu \in \mathcal{M}(M, \mathcal{B})$ and $f \in \mathbf{L}^2(d\mu)$, extends to an isometry $F \mapsto \widetilde{W}(F)$ from \mathcal{H} into $\mathbf{L}^2(\Omega_s, Q)$. Furthermore, $\{\widetilde{W}(F)\}_{F \in \mathcal{H}}$ is a Gaussian \mathcal{H} -process, i.e.,

$$\mathbb{E}_Q(\widetilde{W}(F_1)\widetilde{W}(F_2)) = \langle F_1, F_2 \rangle_{\mathcal{H}}, \quad \forall F_1, F_2 \in \mathcal{H}. \quad (1.5)$$

Representation of $W^{(\mu)}$ in an arbitrary probability space (Ω, \mathcal{F}, P)

Theorem 1.4 (Alpay-Jor). Let $W^{(\mu)}$ be represented in a probability space $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\varphi_j\}_{j \in \mathbb{N}}$ be an orthonormal basis in $\mathbf{L}^2(\mu)$. Then, there is a system $\{Z_j\}_{j \in \mathbb{N}}$ of i.i.d. $N(0, 1)$ random variables such that

$$W_A^{(\mu)} = \sum_{j=1}^{\infty} \left(\int_A \varphi_j(t) d\mu(t) \right) Z_j(\cdot) \quad (1.6)$$

holds almost everywhere on Ω with respect to \mathbb{P} .

Theorem 1.5 (Alpay-Jor). Let (M, \mathcal{B}, μ) be fixed, and let the associated process be realized in $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Suppose

$$\mathcal{F} = \sigma - \text{alg.} \left\{ W_A^{(\mu)} \mid A \in \mathcal{B} \right\}. \quad (1.7)$$

Then, the following assertions hold:

- (i) For all $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ and $a, b \in \mathbb{R}$ (with $a < b$) we have

$$\mathbb{P} \left(\left\{ \omega \in \Omega \mid a < W_A^{(\mu)}(\omega) \leq b \right\} \right) = \gamma_1 \left(\left(\frac{a}{\sqrt{\mu(A)}}, \frac{b}{\sqrt{\mu(A)}} \right] \right), \quad (1.8)$$

where γ_1 is the standard $N(0, 1)$ -Gaussian.

Theorem 1.5, cont.

(ii) There is a measure isomorphism

$$\Psi : \Omega \longrightarrow \times_{\mathbb{N}} \mathbb{R}$$

such that

$$\mathbb{P} \circ \Psi^{-1} = Q_K \quad \text{and} \quad W^{(Q_K, \mu)} \circ \Psi = W^{(\mu)} \quad (1.9)$$

hold almost everywhere on Ω , and where $W^{(Q, \mu)}$ denotes the realization of $W^{(\mu)}$ on $(\times_{\mathbb{N}} \mathbb{R}, Q_K)$.

2. Realizations of infinite products, Ruelle operators and wavelet filters

Notation 2.1. We characterized wavelet filters as functions of the form

$$M(z) = QU(z^N)\Delta(z)V \quad (2.1)$$

where

- ▶ $V = \frac{1}{\sqrt{N}} \left(\epsilon_N^{-\ell j} \right)_{\ell, j=0, \dots, N-1}$, $\epsilon_N := e^{i\frac{2\pi}{N}}$, is (up to scaling) the usual discrete Fourier transform matrix;
- ▶ U is a rational $(N \times N)$ -valued function which takes unitary values on the unit circle, with no poles outside the closed unit disk, and
- ▶ Q is an arbitrary (constant) unitary matrix.

One can explicitly write (2.1) as

$$M(z) = \frac{1}{\sqrt{N}} \begin{pmatrix} m_0(z) & m_0(\epsilon_N z) & \cdots & m_0(\epsilon_N^{N-1} z) \\ m_1(z) & m_1(\epsilon_N z) & \cdots & m_1(\epsilon_N^{N-1} z) \\ \vdots & & & \\ m_{N-1}(z) & m_{N-1}(\epsilon_N z) & \cdots & m_{N-1}(\epsilon_N^{N-1} z) \end{pmatrix}. \quad (2.2)$$

Note that $M(z)$ in (2.1) is unitary on \mathbb{T} .

- Following (2.2) one can write,

$$\begin{pmatrix} m_0(z) \\ m_1(z) \\ \vdots \\ m_{N-1}(z) \end{pmatrix} = QU(z^N) \begin{pmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(N-1)} \end{pmatrix}. \quad (2.3)$$

- In the sequel, by choosing in (2.1)

$$Q = (U(1)\Delta(1)V)^* = (U(1)V)^* \quad (2.4)$$

we shall normalize the filters so that in (2.1), $M(1) = I_N$.

- For $m_0(z)$ in (2.2) we set:

$$m(z) := m_0(z). \quad (2.5)$$

The wavelet father function $\varphi(w)$ is given by its Fourier transform

$$\widehat{\varphi}(w) = \prod_{k=1}^{\infty} m(e^{\frac{2\pi iw}{N^k}}). \quad (2.6)$$

Definition 2.2. Let $M(z)$ is a matrix-valued rational function, analytic at infinity, with realization

$$M(z) = D + C(zI - A)^{-1}B. \quad (2.7)$$

We assume that $1 \geq \|M(z)\|$ for all $z \in \mathbb{T}$.

Theorem 2.3 (Alpay, Jorgensen, Lewkowicz). Given a square $M(z)$ in (2.7).

(i) Assume that

$$\lim_{k \rightarrow \infty} \|D^k\|^{1/k} < 1. \quad (2.8)$$

Then, the operators \mathcal{A} , \mathcal{B} , \mathcal{C} are bounded, where

$$\mathcal{A} = \begin{pmatrix} A & BC & BDC & BD^2C & \cdots \\ 0 & A & BC & BDC & \cdots \\ 0 & 0 & A & BC & \cdots \end{pmatrix}, \quad (2.9)$$
$$\ell_2(\mathbb{N}) \otimes \mathbb{C}^m \implies \ell_2(\mathbb{N}) \otimes \mathbb{C}^m;$$

Theorem 2.3, cont.

$$\mathcal{B} = \begin{pmatrix} \vdots \\ BD^2 \\ BD \\ B \end{pmatrix}, \quad \mathbb{C} \longrightarrow \ell_2(\mathbb{N}) \otimes \mathbb{C}^m, \quad (2.10)$$

$$\mathcal{C} = (C \quad DC \quad D^2C \quad \dots), \quad \ell_2(\mathbb{N}) \otimes \mathbb{C}^m \longrightarrow \mathbb{C}^m. \quad (2.11)$$

(ii) \mathcal{A} in (2.9) is the block-Toeplitz operator with symbol

$$A + zB(I - zD)^{-1}C.$$

Theorem 2.4 (Alpay, Jorgensen, Lewkowicz). Assume that $M(z)$ in (2.7) has no singularity at the point $z = 1$ and that $M(1) = I$, and let $(z_k)_{k \in \mathbb{N}_0}$ be a sequence of complex numbers which are not poles of M and such that

$$\sum_{k=0}^{\infty} |1 - z_k| < \infty. \quad (2.12)$$

Then it holds that

$$\prod_{k=1}^{\infty} M(z_k) = \mathcal{C}(\Lambda(z) - \mathcal{A})^{-1} \mathcal{B} \quad (2.13)$$

where \mathcal{A} , \mathcal{B} and \mathcal{C} are defined by (2.9)-(2.11), and where

$$\Lambda(z) = \begin{pmatrix} z_1 I_n & & \\ & z_2 I_n & \\ & & \ddots \end{pmatrix}.$$

3. Infinite-Dimensional Measure Spaces and Frame Analysis

Definition 3.1. A Borel probability measure $\mu \in \mathcal{P}$ is a *probabilistic frame* if there exists $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \int_{\mathbb{R}^N} |\langle x, y \rangle|^2 d\mu(y) \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{R}^N. \quad (3.1)$$

- ▶ The constants A and B are called lower and upper probabilistic frame bounds, respectively.
- ▶ When $A = B$, μ is called a tight probabilistic frame.

Markov measures from frames

Proposition 3.2. Let \mathcal{H} be a Hilbert space, and let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a frame in \mathcal{H} with frame bounds α, β , $0 < \alpha \leq \beta < \infty$, i.e.,

$$\alpha \|x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, \varphi_n \rangle|^2 \leq \beta \|x\|^2 \quad (3.2)$$

holds for all $x \in \mathcal{H}$.

We then get a system of transition probabilities

$$p_{x,y} = \frac{|\langle x, y \rangle|^2}{c(x)} \quad (3.3)$$

with

$$c(x) := \sum_{n \in \mathbb{N}} |\langle x, \varphi_n \rangle|^2. \quad (3.4)$$

Prop. 3.2, cont.

Moreover, for $x, y \in \mathcal{H} \setminus \{0\}$, we have:

1. Reversible:

$$c(x)p_{x,y} = c(y)p_{y,x}$$

2. Markov-rules: $p_{x,\varphi_n} \leq 1$ for $n \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} p_{x,\varphi_n} = 1.$$

3. Normalization:

$$p_{x,y} \leq \|y\|^2 / \alpha$$

where α is the lower frame bound from (3.2).

Translation

Theorem 3.3 (Jor-Song). We can define μ^x

$$\mu^x(E) := \mu(E - x), \quad x \in \mathcal{H}, \quad E \subset S' \quad \text{Borel}$$

and the Radon-Nikodym derivative is

$$\frac{d\mu^x}{d\mu} \in L_+^1(S', \mu)$$

$$\frac{d\mu^x}{d\mu}(\omega) = e^{(T_\mu x)(\omega) - \frac{1}{2}\|x\|^2}, \quad \omega \in S'. \quad (3.5)$$

Theorem 3.4 (Jor-Song). Let \mathcal{H} and μ be as above, $\dim \mathcal{H} = \aleph_0$; and let $x, y \in \mathcal{H}$. Set

$$\mathcal{E}_\mu(x)(\cdot) = e^{(T_\mu x)(\cdot)} e^{-\frac{1}{2}\|x\|^2} \quad \text{on } S', \quad (3.6)$$

see (3.5). Then

$$\int_{S'} \mathcal{E}_\mu(x)(\omega) \langle y, \omega \rangle^2 d\mu(\omega) = \langle x, y \rangle^2 + \|y\|^2; \quad (3.7)$$

and the following co-cycle property holds:

$$\mathcal{E}_\mu(x_1)(\omega) \mathcal{E}_\mu(x_2)(\omega) = e^{-\langle x_1, x_2 \rangle_{\mathcal{H}}} \mathcal{E}_\mu(x_1 + x_2)(\omega), \quad (3.8)$$

for all $x_1, x_2 \in \mathcal{H}$, and $\omega \in S'$.

4. Decomposition of wavelet representations and Martin boundaries

Theorem 4.1 (Dutkay, Jorgensen, Silvestrov). Suppose $r : (X, \mu) \rightarrow (X, \mu)$ is ergodic. Assume $|m_0|$ is not constant 1 μ -a.e., non-singular, i.e., $\mu(m_0(x) = 0) = 0$, and $\log |m_0|^2$ is in $L^1(X)$. Then the wavelet representation $(\mathcal{H}, U, \pi, \varphi)$ is reducible.

Theorem 4.2 (Dutkay, Jorgensen, Silvestrov). In the hypotheses of Theorem 4.1, there exist a fundamental domain \mathcal{F} . The wavelet representation associated to m_0 has the following direct integral decomposition:

$$[\mathcal{H}, U, \pi] = \int_{\mathcal{F}}^{\oplus} [\mathcal{H}_z, U_z, \pi_z] d\mu_{\infty}(z),$$

where the component representations $[\mathcal{H}_z, U_z, \pi_z]$ in the decomposition are irreducible for a.e., z in \mathcal{F} , relative to μ_{∞} .

Martin boundary

Definition 4.3. A point $x_0 \in X$ is called *regular* if the following two conditions are satisfied:

- (i) The sets $r^{-n}(x_0)$, $n \in \mathbb{N}$ are mutually disjoint.
 - (ii) None of the sets $r^{-n}(x_0)$, $n \geq 0$ intersect the set of zeroes of W .
- For a point $x_0 \in X$, define the set $\mathcal{T}(x_0) := \cup_{n \geq 0} r^{-n}(x_0)$. We call this the *tree with root at x_0* . If x_0 is regular and $x \in \mathcal{T}(x_0)$, define $n(x_0)$ to be the unique non-negative integer such that $r^{n(x_0)}(x) = x_0$.

Definition 4.4. For $x, y \in \mathcal{T}(x_0)$ define the transition probabilities $p(x, y)$ as follows:

$$p(x, y) := \begin{cases} W(y), & \text{if } r(y) = x, \\ 0, & \text{otherwise.} \end{cases} \quad (4.1)$$

A function u on $\mathcal{T}(x_0)$ is called *p-harmonic* if

$$u(x) = \sum_{y \in \mathcal{T}(x_0)} p(x, y)u(y), \quad (x \in \mathcal{T}(x_0)). \quad (4.2)$$

Theorem 4.5 (Dutkay, Jorgensen, Silvestrov). For any p -harmonic function $u \geq 0$, there exists a measure ν on Ω_{x_0} such that

$$u(x) = \frac{1}{W(x)W(r(x)) \dots W(r^{n(x)-1}(x))} \nu(V_{r^{n(x)}(x), r^{n(x)-1}(x), \dots, x}), \quad (4.3)$$

$$(x \in \mathcal{T}(x_0)).$$

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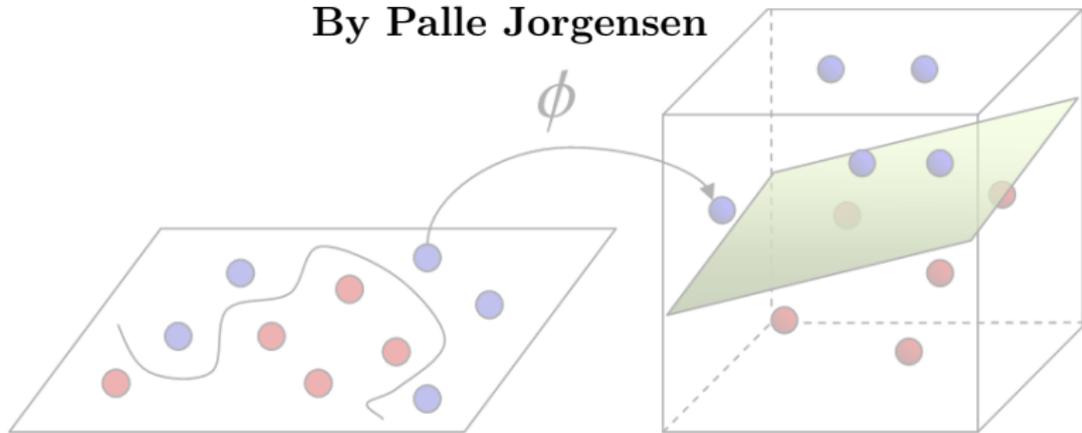
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Lecture 8. Reproducing Kernel Hilbert Spaces arising from groups

By Palle Jorgensen



Abstract

Reproducing Kernel Hilbert Spaces appear in the study of spectral measures. Spectral measures give rise to positive definite functions via the Fourier transform. Reversing this process will be the focus of the ninth talk. This talk will set the stage by discussing Reproducing Kernel Hilbert Spaces that appear in the context of positive definite functions, and the harmonic analysis of those Reproducing Kernel Hilbert Spaces.

1. Nonuniform sampling, reproducing kernels, and the associated Hilbert spaces

Symmetric pair

Definition 1.1. Let $k : V \times V \longrightarrow \mathbb{C}$ (or \mathbb{R}) be a positive definite kernel, and \mathcal{H} be the corresponding RKHS as above.

- ▶ The RKHS $\mathcal{H} = \mathcal{H}(k)$ is said to have the *discrete mass* property (\mathcal{H} is called a *discrete RKHS*), if $\delta_x \in \mathcal{H}$, for all $x \in V$. Here, $\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$, i.e., the Dirac mass at $x \in V$. If \mathcal{H} has the finite-mass property, then we get a dual pair of operators as follows:

Definition 1.2. $A : l^2(V) \longrightarrow \mathcal{H} (= \mathcal{H}(k))$,
 $\mathcal{D}(A) = \text{span} \{ \delta_x \}$ dense in $l^2(V)$, with

$$A\delta_x = \delta_x \in \mathcal{H};$$

$B : \mathcal{H} \longrightarrow l^2(V)$, $\mathcal{D}(B) = \text{span} \{ k_x \}$ dense in \mathcal{H} , where

Case 1. RKHS:

$$Bk_x = \delta_x \tag{1.1}$$

Case 2. Relative RKHS:

$$Bk_x = \delta_x - \delta_o. \tag{1.2}$$

Proposition 1.3 (Jor-Tian). The system (A, B) from Definition 1.1 is a symmetric pair, i.e.,

$$\langle Au, v \rangle_{\mathcal{H}} = \langle u, Bv \rangle_{l^2}, \quad \forall u \in \mathcal{D}(A), \forall v \in \mathcal{D}(B). \tag{1.3}$$

2. Reproducing kernels and choices of associated feature spaces, in the form of L^2 -spaces

Theorem 2.1 (Jor-Tian). Let (V, \mathcal{B}) be a measure space, μ a σ -finite measure on \mathcal{B} , and set $\mathcal{B}_{fin} = \{A \in \mathcal{B} ; \mu(A) < \infty\}$. Let K be a p.d. kernel on $\mathcal{B}_{fin} \times \mathcal{B}_{fin}$, and let $\mathcal{H}(K)$ be the corresponding RKHS. Suppose $\mathcal{H}(K)$ consists of signed measures; and set

$$\mathcal{H}(\mu) := \left\{ F \in \mathcal{H}(K) ; dF \ll d\mu, \text{ and } \frac{dF}{d\mu} \in L^2(\mu) \right\}. \quad (2.1)$$

Then

$$\mathcal{H}(\mu) \subseteq \mathcal{H}(K^{(\mu)}), \quad (2.2)$$

where

$$K^{(\mu)}(A, B) := \mu(A \cap B), \quad \forall A, B \in \mathcal{B}_{fin}; \quad (2.3)$$

and therefore $\exists c(\mu) < \infty$ such that $c(\mu) K^{(\mu)} - K$ is positive definite.

Definition 2.2. We shall say that $L^2(\mu)$ is a *feature space* if there is a function $r : U \rightarrow L^2(\mu)$ such that

$$K(x, y) = \int_S r_x(s) r_y(s) d\mu(s), \quad \forall x, y \in U. \quad (2.4)$$

(In the complex case, the RHS in 2.4 will instead be $\int_S r_x(s) \overline{r_y(s)} d\mu(s)$.)

Theorem 2.3 (Jor-Tian). Let $K : U \times U \longrightarrow \mathbb{R}$ be a positive definite (p.d.) kernel, and let (S, \mathcal{B}, μ, r) be a feature space; in particular, we have $r_x \in L^2(\mu)$, $\forall x \in U$, and $K(x, y) = \int_S r_x(s) r_y(s) d\mu(s)$.

1. Set

$$R(x, A) = \int_A r_x(s) d\mu(s), \quad x \in U, \quad A \in \mathcal{B}_{fin}.$$

Then R is a measure in the second variable, and it is measurable in $x (\in U)$.

2. For all $F \in \mathcal{H}(K)$, we have

$$\|F\|_{\mathcal{H}(K)}^2 = \int_S \left| \langle F(\cdot), R(\cdot, ds) \rangle_{\mathcal{H}(K)} \right|^2, \quad \text{and} \quad (2.5)$$

$$F(x) = \int_S r_x(s) \langle F(\cdot), R(\cdot, ds) \rangle_{\mathcal{H}(K)}. \quad (2.6)$$

3. Fourier duality for fractal measures with affine scales

Definition 3.1. Let a $d \times d$ matrix R be given. Assume $R \in M_d(\mathbb{Z})$ and that R is expansive. Let $B, L \subset \mathbb{R}^d$ be such that $0 \in B, 0 \in L, N = \#B = \#L$ and assume

$$R^k b \cdot l \in \mathbb{Z}, \text{ for all } b \in B, l \in L, k \in \mathbb{Z}, k \geq 0 \quad (3.1)$$

and further that the matrix

$$\frac{1}{\sqrt{N}} \left(e^{2\pi i R^{-1} b \cdot l} \right)_{b \in B, l \in L} \quad (3.2)$$

is unitary. Set

$$\Gamma(B) := \left\{ \sum_{k=0}^n R^k b_k : b_k \in B, n \in \mathbb{Z}_+ \right\}, \text{ and} \quad (3.3)$$

$$\Gamma(L) := \left\{ \sum_{k=0}^n (R^T)^k l_k : l_k \in L, n \in \mathbb{Z}_+ \right\} \quad (3.4)$$

Theorem 3.2 (Dutkay-Jor). Let R, B, L be as stated in Definition 3.1. Assume further that $R^T = R$. If there is a $G \in \text{GL}_d(\mathbb{R})$ such that $G = G^T$, $GR = RG$, $G(B) = L$. Then the two spectral functions

$$\sigma_{\Gamma(L)}^{(B)}(t) = \sum_{\gamma \in \Gamma(L)} |\widehat{\mu}_B(t + \gamma)|^2, \quad (3.5)$$

$$\sigma_{\Gamma(B)}^{(L)}(t) = \sum_{\xi \in \Gamma(B)} |\widehat{\mu}_L(t + \xi)|^2, \quad (3.6)$$

satisfy

$$G\Gamma(B) = \Gamma(L) \quad (3.7)$$

and

$$\sigma_{\Gamma(B)}^{(L)}(t) = \sigma_{\Gamma(L)}^{(B)}(Gt) \text{ for all } t \in \mathbb{R}^d. \quad (3.8)$$

Finite cycles

Definition 3.3. Let $n \in \mathbb{Z}_+$ be given, and set $R = 2n$, $B = \{0, 2\}$ and let $\mu = \mu_{1/2n} = \mu_B$ be the fractal measure. We say that $p \in v_+$ odd is *admissible* if and only if there are non-trivial B -extreme τ_L -cycles. Here $L = L_{n,p} = \{0, np/2\}$. (Are there admissible values of p not divisible by $2n - 1$?)

Theorem 3.4 (Dutkay-Jor). Let $n \in \mathbb{Z}_+$ be given. Set

$$p = \sum_{i=0}^{2n-1} (2n)^i. \quad (3.9)$$

Then p is admissible and not divisible by $2n - 1$. There are associated B -extreme cycles of length $2n$.

4. On common fundamental domains

Theorem 4.1 (Dutkay, Han, Jorgensen, Picioroaga).

Consider two commuting measure-preserving actions of some countable (possibly finite) discrete groups Γ and Λ on the same measure space (M, \mathcal{B}, m) . Assume in addition that both actions have fundamental domains of finite positive measures, X for Γ and Y for Λ , and $m(X) \geq m(Y)$. Then the following affirmations are equivalent:

1. The two actions have a common tiling system.
2. For all sets $A \in \mathcal{B}$ which are invariant for both Γ and Λ , the following equality holds

$$m(A \cap X) = \frac{m(X)}{m(Y)} \cdot m(A \cap Y). \quad (4.1)$$

Theorem 4.2 (Dutkay, Han, Jorgensen, Picioroaga).

Let G be a locally compact group of polynomial growth with Haar measure m . Suppose Γ and Λ are two uniform lattices in G . Consider the action of Γ on G on the left and the action of Λ on G on the right. If $\text{cov}_G(\Gamma) \geq \text{cov}_G(\Lambda)$, then the two actions have a common tiling system.

5. Families of spectral sets for Bernoulli convolutions

Definition 5.1. Given S a subset of \mathbb{R} , we will use the notation $E(S)$ to be the set of exponential functions

$$E(S) := \{e_s(x) := \exp(2\pi isx) \mid s \in S\}. \quad (5.1)$$

Theorem 5.2 (Jorgensen, Kornelson, Shuman). The set $E(3\Gamma(\frac{1}{8}))$ forms an ONB for $L^2(\mu_{\frac{1}{8}})$.

Theorem 5.3 (Jorgensen, Kornelson, Shuman). If $p \in 2\mathbb{N} + 1$ such that $p < \frac{2(2n-1)}{\pi}$, then $(\mu_{\frac{1}{2n}}, p\Gamma(\frac{1}{2n}))$ is a spectral pair for $L^2(\mu_{\frac{1}{2n}})$.

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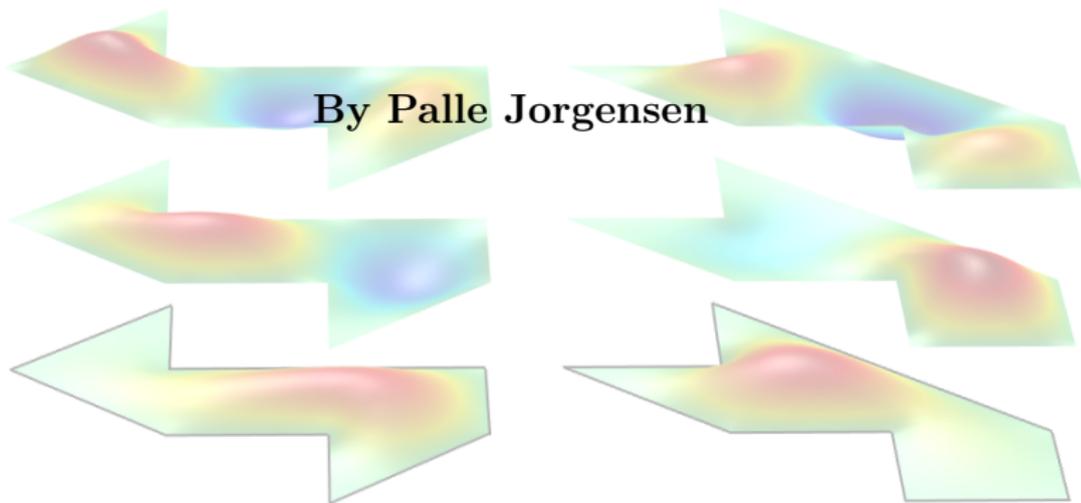
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Lecture 9. Extensions of positive definite functions

By Palle Jorgensen



Abstract

We consider the question of spectral measures from the perspective of positive definite functions. Since the measures are spectral, the corresponding positive definite functions have special properties in terms of their zero sets. This correspondence leads to the natural question of whether this process can be reversed. Bochner's theorem implies that positive definite functions are the Fourier transform of measures, but whether those measures are spectral becomes a subtle problem. Thus, by considering certain functions on appropriate subsets, the question of spectrality can be formulated as whether the function can be extended to a positive definite function. The answer is sometimes yes, using the harmonic analysis of Reproducing Kernel Hilbert Spaces.

1. Extensions of Positive Definite Functions: Applications and Their Harmonic Analysis

Two Extension Problems

1. The study of extensions of *locally defined continuous and positive definite (p.d.) functions* F on groups on the one hand, and, on the other;
2. The question of extensions for an associated *system of unbounded Hermitian operators with dense domain* in a reproducing kernel Hilbert space (RKHS) \mathcal{H}_F associated to F .

Definition 1.1. We say that $(U, \mathcal{K}, k_0) \in \text{Ext}(F)$ iff

1. U is a strongly continuous unitary representation of G in the Hilbert space \mathcal{K} , containing the RKHS \mathcal{H}_F ; and
2. there exists $k_0 \in \mathcal{K}$ such that

$$F(g) = \langle k_0, U(g)k_0 \rangle_{\mathcal{K}}, \quad \forall g \in \Omega^{-1} \cdot \Omega. \quad (1.1)$$

Definition 1.2. Let $\text{Ext}_1(F) \subset \text{Ext}(F)$ consisting of (U, \mathcal{H}_F, F_e) with

$$F(g) = \langle F_e, U(g)F_e \rangle_{\mathcal{H}_F}, \quad \forall g \in \Omega^{-1} \cdot \Omega; \quad (1.2)$$

where $F_e \in \mathcal{H}_F$ satisfies $\langle F_e, \xi \rangle_{\mathcal{H}_F} = \xi(e)$, $\forall \xi \in \mathcal{H}_F$, and e denotes the neutral (unit) element in G , i.e., $eg = g$, $\forall g \in G$.

Definition 1.3. Let $Ext_2(F) := Ext(F) \setminus Ext_1(F)$, consisting of the solutions to problem (1.1) for which $\mathcal{K} \not\supseteq \mathcal{H}_F$, i.e., unitary representations realized in an enlargement Hilbert space.

Theorem 1.4 (Jorgensen, Pedersen, Tian). The following two conditions are equivalent:

1. F is extendable to a continuous p.d. function \tilde{F} defined on \mathbb{R} , i.e., \tilde{F} is a continuous p.d. function defined on \mathbb{R} and $F(x) = \tilde{F}(x)$ for all x in $\Omega - \Omega$.
2. There is a Hilbert space \mathcal{K} , an isometry $W : \mathcal{H}_F \rightarrow \mathcal{K}$, and a strongly continuous unitary group $U_t : \mathcal{K} \rightarrow \mathcal{K}$, $t \in \mathbb{R}$, such that if A is the skew-adjoint generator of U_t , we have

$$WF_\varphi \in dom(A), \quad \text{and} \quad (1.3)$$

$$AWF_\varphi = WF_{\varphi'}. \quad (1.4)$$

$$\begin{array}{ccc}
 (F, |t| < a) & \xrightarrow{\text{p.d. extension}} & \tilde{F}(t) \underset{t \in \mathbb{R}}{=} \langle F_0, U(t) F_0 \rangle_{\mathcal{H}_F} \\
 \downarrow & & \uparrow \\
 (\mathcal{H}_F, D^{(F)}) & & U(t) = \int_{\mathbb{R}} e^{it\lambda} P_U(d\lambda) \\
 \downarrow & & \uparrow \text{spectral resolution } P_U \\
 A^{(F)} \supset D^{(F)} & & \\
 (A^{(F)})^* = -A^{(F)} & \xrightarrow{\text{operator extension}} & U(t) = e^{tA^{(F)}}, t \in \mathbb{R}
 \end{array}$$

Figure 1.1: Extension correspondence. From locally defined p.d F to $D^{(F)}$, to a skew-adjoint extension, and the unitary one-parameter group, to spectral resolution, and finally to an associated element in $Ext(F)$.

A Model of All Deficiency Index-(1, 1) Operators

Theorem 1.5 (Jorgensen, Pedersen, Tian). Let \mathcal{H} be a separable Hilbert space, and let S be a Hermitian operator with dense domain in \mathcal{H} . Suppose the deficiency indices of S are (d, d) ; and suppose one of the selfadjoint extensions of S has simple spectrum.

Then the following two conditions are equivalent:

1. $d = 1$;
2. for each of the selfadjoint extensions T of S , we have a unitary equivalence between (S, \mathcal{H}) on the one hand, and a system $(S_\mu, L^2(\mathbb{R}, \mu))$ on the other, where μ is a Borel probability measure on \mathbb{R} .

Theorem 1.5, cont.

- ▶ Moreover,

$$\begin{aligned} \text{dom}(S_\mu) = \left\{ f \in L^2(\mu) \mid \lambda f(\cdot) \in L^2(\mu), \right. \\ \left. \text{and } \int_{\mathbb{R}} (\lambda + i) f(\lambda) d\mu(\lambda) = 0 \right\}, \end{aligned} \quad (1.5)$$

and

$$(S_\mu f)(\lambda) = \lambda f(\lambda), \quad \forall f \in \text{dom}(S_\mu), \forall \lambda \in \mathbb{R}. \quad (1.6)$$

- ▶ In case μ satisfies condition (1.6), then the constant function $\mathbf{1}$ (in $L^2(\mathbb{R}, \mu)$) is in the domain of S_μ^* , and

$$S_\mu^* \mathbf{1} = i \mathbf{1} \quad (1.7)$$

i.e., $(S_\mu^* \mathbf{1})(\lambda) = i$, a.a. λ w.r.t. $d\mu$.

2. Harmonic analysis of a class of reproducing kernel Hilbert spaces arising from groups

Definition 2.1.

- ▶ G : a given locally compact abelian group, write the operation in G additively;
- ▶ dx : denotes the Haar measure of G (unique up to a scalar multiple.)
- ▶ \widehat{G} : the dual group, i.e., \widehat{G} consists of all continuous homomorphisms: $\lambda : G \rightarrow \mathbb{T}$, $\lambda(x + y) = \lambda(x) \lambda(y)$, $\forall x, y \in G$; $\lambda(-x) = \overline{\lambda(x)}$, $\forall x \in G$. Occasionally, we shall write $\langle \lambda, x \rangle$ for $\lambda(x)$. Note that \widehat{G} also has its Haar measure.

Theorem 2.2 (Jorgensen, Pedersen, Tian).

1. Let F and \mathcal{H}_F be as above; and let $\mu \in \mathcal{M}(\widehat{G})$; then there is a positive Borel function h on \widehat{G} s.t. $h^{-1} \in L^\infty(\widehat{G})$, and $hd\mu \in \text{Ext}(F)$, if and only if $\exists K_\mu < \infty$ such that

$$\int_{\widehat{G}} |\widehat{\varphi}(\lambda)|^2 d\mu(\lambda) \leq K_\mu \int_{\Omega} \int_{\Omega} \overline{\varphi(y_1)} \varphi(y_2) F(y_1 - y_2) dy_1 dy_2. \quad (2.1)$$

2. Assume $\mu \in \text{Ext}(F)$, then

$$(fd\mu)^\vee \in \mathcal{H}_F, \quad \forall f \in L^2(\widehat{G}, \mu). \quad (2.2)$$

Theorem 2.3 (Jorgensen, Pedersen, Tian). Let $F : \Omega - \Omega \rightarrow \mathbb{C}$ be continuous, and positive definite on $\Omega - \Omega$; and assume $Ext(F) \neq \emptyset$. Let $\mu \in Ext(F)$, and let $T_\mu(F_\phi) := \widehat{\varphi}$, defined initially only for $\varphi \in C_c(\Omega)$, be the isometry $T_\mu : \mathcal{H}_F \rightarrow L^2(\mu) = L^2(\widehat{G}, \mu)$. Then $Q_\mu := T_\mu T_\mu^*$ is a projection in $L^2(\mu)$ with $K_\Omega(\cdot)$ as kernel:

$$(Q_\mu f)(\lambda) = \int_{\widehat{G}} K_\Omega(\lambda - \xi) f(\xi) d\mu(\xi), \quad \forall f \in L^2(\widehat{G}, \mu), \forall \lambda \in \widehat{G}. \quad (2.3)$$

Theorem 2.4 (Jorgensen, Pedersen, Tian). Let $\Omega \subset \mathbb{R}^n$ be given, $\Omega \neq \emptyset$, open and connected. Suppose F is given p.d. and continuous on $\Omega - \Omega$, and assume $Ext(F) \neq \emptyset$. Let U be the corresponding unitary representations of $G = \mathbb{R}^n$, and let $P_U(\cdot)$ be its associated PVM acting on \mathcal{H}_F (= the RKHS of F .) Then \mathcal{H}_F splits up as an orthogonal sum of three closed and $U(\cdot)$ invariant subspaces

$$\mathcal{H}_F = \mathcal{H}_F^{(atom)} \oplus \mathcal{H}_F^{(ac)} \oplus \mathcal{H}_F^{(sing)} \quad (2.4)$$

with these subspaces characterized as follows:

1. The PVM $P_U(\cdot)$ restricted to $\mathcal{H}_F^{(atom)}$ is purely atomic;
2. $P_U(\cdot)$ restricted to $\mathcal{H}_F^{(ac)}$ is absolutely continuous with respect to the Lebesgue measure $d\lambda = d\lambda_1 \cdots d\lambda_n$ on \mathbb{R}^n ; and
3. $P_U(\cdot)$ is continuous, purely singular, when restricted to $\mathcal{H}_F^{(sing)}$.

Theorem 2.4, cont.

- ▶ **Case $\mathcal{H}_F^{(atom)}$.** If $\lambda \in \mathbb{R}^n$ is an atom in $P_U(\cdot)$, i.e., $P_U(\{\lambda\}) \neq 0$, where $\{\lambda\}$ denotes the singleton with λ fixed; then $P_U(\{\lambda\})\mathcal{H}_F$ is one-dimensional, and the function $e_\lambda(x) := e^{i\lambda \cdot x}$, (complex exponential) restricted to Ω , is in \mathcal{H}_F . We have:

$$P_U(\{\lambda\})\mathcal{H}_F = \mathbb{C}e_\lambda \Big|_{\Omega}. \quad (2.5)$$

- ▶ **Case $\mathcal{H}_F^{(sing)}$.** Vectors $\xi \in \mathcal{H}_F^{(sing)}$ are characterized by the following property:

The measure

$$d\mu_\xi(\cdot) := \|P_U(\cdot)\xi\|_{\mathcal{H}_F}^2 \quad (2.6)$$

is continuous and purely singular.

Theorem 2.4, cont.

- **Case $\mathcal{H}_F^{(ac)}$.** If $\xi \in \mathcal{H}_F^{(ac)}$, then it is represented as a continuous function on Ω , and

$$\langle \xi, F_\varphi \rangle_{\mathcal{H}_F} = \int_{\Omega} \overline{\xi(x)} \varphi(x) dx_{(\text{Lebesgue meas.})}, \quad \forall \varphi \in C_c(\Omega). \quad (2.7)$$

Moreover, there is a $f \in L^2(\mathbb{R}^n, \mu)$ such that

$$\int_{\Omega} \overline{\xi(x)} \varphi(x) dx = \int_{\mathbb{R}^n} \overline{f(\lambda)} \widehat{\varphi}(\lambda) d\mu(\lambda), \quad \forall \varphi \in C_c(\Omega); \quad (2.8)$$

and

$$\xi = (fd\mu)^\vee \Big|_{\Omega}. \quad (2.9)$$

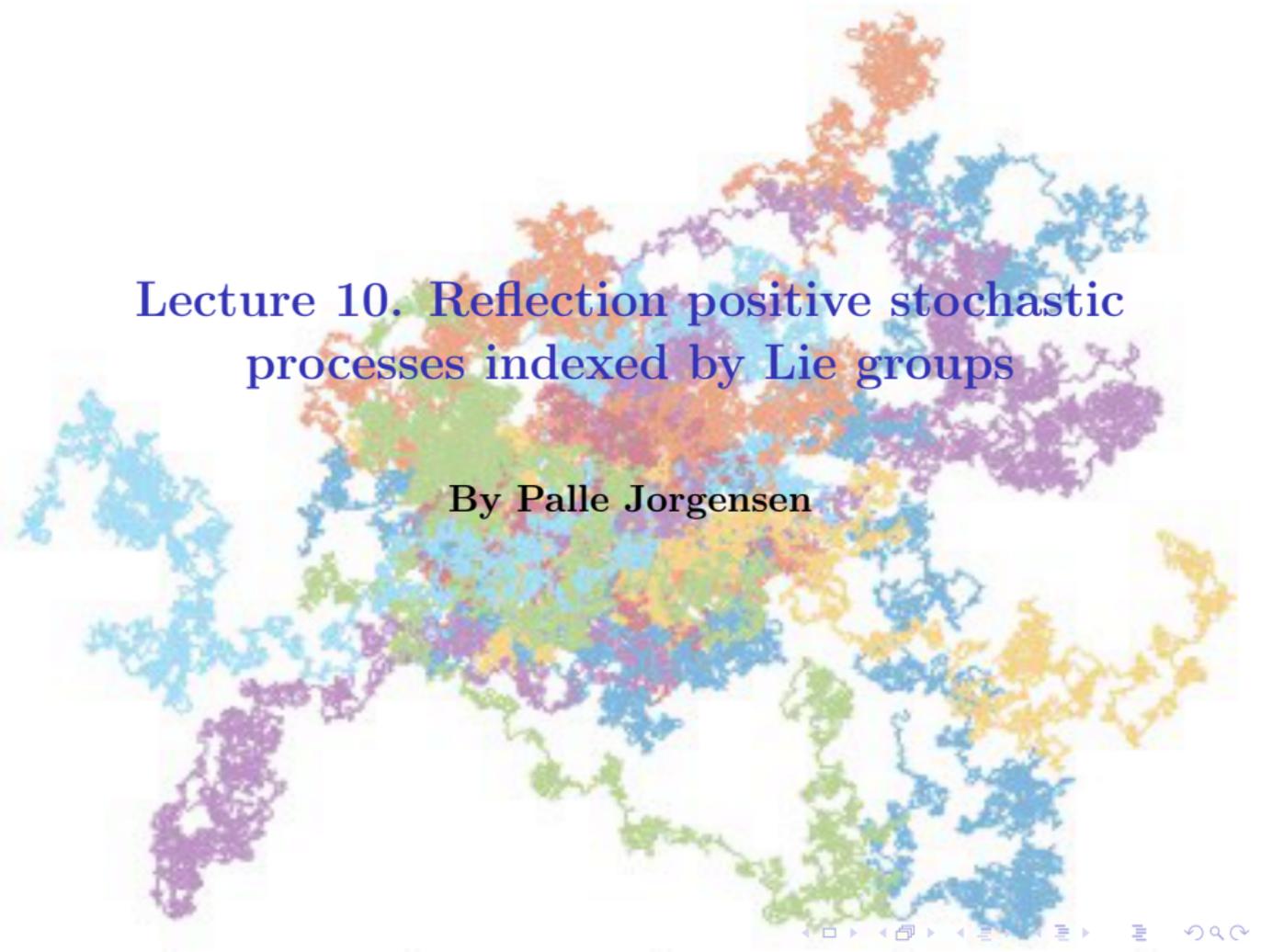
(We say that $(fd\mu)^\vee$ is the μ -extension of ξ .)

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Lecture 10. Reflection positive stochastic processes indexed by Lie groups

By Palle Jorgensen

Abstract

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of reflection positivity has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

1. Unitary Representations of Lie Groups with Reflection Symmetry

We consider a class of unitary representations of a Lie group G which possess a certain reflection symmetry defined as follows.

Notation. If π is a representation of G in some Hilbert space \mathbb{H} , we introduce the following three structures:

1. $\tau \in \text{Aut}(G)$ of period 2;
2. $J : \mathbb{H} \rightarrow \mathbb{H}$ is a unitary operator of period 2 such that $J\pi(g)J^* = \pi(\tau(g))$, $g \in G$ (this will hold if π is of the form $\pi_+ \oplus \pi_-$ with π_+ and $\pi_- \circ \tau$ unitarily equivalent); it will further be assumed that there is a closed subspace $K_0 \subset \mathbb{H}$ which is invariant under $\pi(H)$, $H = G^\tau$, or more generally, an open subgroup of G^τ ;
3. positivity is assumed in the sense that $\langle v, J(v) \rangle \geq 0$, $v \in K_0$.

Definition 1.1. A unitary representation π acting on a Hilbert space $H(\pi)$ is said to be *reflection symmetric* if there is a unitary operator $J : H(\pi) \rightarrow H(\pi)$ such that

1. $J^2 = id$.
2. $J\pi(g) = \pi(\tau(g))J, \quad g \in G$.

Note. If (1) holds then π and $\pi \circ \tau$ are equivalent.

Furthermore, generally from (2) we have $J^2\pi(g) = \pi(g)J^2$.

Thus, if π is irreducible, then we can always renormalize J such that (1) holds. Let $H = G^\tau = \{g \in G \mid \tau(g) = g\}$ and let \mathfrak{h} be the Lie algebra of H . Then $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. Define $\mathfrak{q} = \{Y \in \mathfrak{g} \mid \tau(Y) = -Y\}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

Reflection Symmetry for Complementary Series

Theorem 1.2 (Jor-Ólafsson). Assume that G/H is non-compactly causal and such that there exists a $w \in K$ such that $Ad(w)|_{\mathfrak{a}} = -1$. Let π_ν be a complementary series such that $\nu \leq L_{\text{pos}}$. Let C be the minimal H -invariant cone in \mathfrak{q} such that $S(C)$ is contained in the contraction semigroup of HP_{max} in G/P_{max} . Let Ω be the bounded realization of $H/H \cap K$ in $\bar{\mathfrak{n}}$. Let $J(f)(x) := f(\tau(x)w^{-1})$. Let K_0 be the closure of $\mathcal{C}_c^\infty(\Omega)$ in H_ν . Then the following holds:

Theorem 1.2, cont.

1. $(G, \tau, \pi_\nu, C, J, K_0)$ satisfies the positivity conditions (PR1)–(PR2).
2. π_ν defines a contractive representation $\tilde{\pi}_\nu$ of $S(C)$ on K such that $\tilde{\pi}_\nu(\gamma)^* = \tilde{\pi}_\nu(\tau(\gamma)^{-1})$.
3. There exists a unitary representation $\tilde{\pi}_\nu^c$ of G^c such that
 - (i) $d\tilde{\pi}_\nu^c(X) = d\tilde{\pi}_\nu(X) \quad \forall X \in \mathfrak{h}$.
 - (ii) $d\tilde{\pi}_\nu^c(iY) = i d\tilde{\pi}_\nu(Y) \quad \forall Y \in C$.

2. Osterwalder-Schrader Axioms-Wightman Axioms

Definition 2.1. A closed convex cone $C \subset \mathfrak{q}$ is *hyperbolic* if $C^\circ \neq \emptyset$ and if $\text{ad } X$ is semisimple with real eigenvalues for every $X \in C^\circ$.

We will assume the following for (G, π, τ, J) :

- (PR1)** π is reflection symmetric with reflection J ;
- (PR2)** there is an H -invariant hyperbolic cone $C \subset \mathfrak{q}$ such that $S(C) = H \exp C$ is a closed semigroup and $S(C)^\circ = H \exp C^\circ$ is diffeomorphic to $H \times C^\circ$;
- (PR3)** there is a subspace $0 \neq \mathbf{K}_0 \subset \mathbf{H}(\pi)$ invariant under $S(C)$ satisfying the positivity condition

$$\langle v, v \rangle_J := \langle v, J(v) \rangle \geq 0, \quad \forall v \in \mathbf{K}_0.$$

Theorem 2.2 (Jor-Ólafsson). Assume that (π, C, \mathbf{H}, J) satisfies (PR1)–(PR3). Then the following hold:

1. $S(C)$ acts via $s \mapsto \tilde{\pi}(s)$ by contractions on \mathbf{K} .
2. Let G^c be the simply connected Lie group with Lie algebra \mathfrak{g}^c . Then there exists a unitary representation $\tilde{\pi}^c$ of G^c such that $d\tilde{\pi}^c(X) = d\tilde{\pi}(X)$ for $X \in \mathfrak{h}$ and $i d\tilde{\pi}^c(Y) = d\tilde{\pi}(iY)$ for $Y \in C$.
3. The representation $\tilde{\pi}^c$ is irreducible if and only if $\tilde{\pi}$ is irreducible.

Definition 2.3. Let W be a G -invariant cone in \mathfrak{g} . We denote the set of all unitary representations π of G with $W \subset W(\pi)$ by $\mathcal{A}(W)$. A unitary representation π is called W -admissible if $\pi \in \mathcal{A}(W)$.

It turns out that the irreducible representations in $\mathcal{A}(W)$ are *highest weight representations*. A (\mathfrak{g}^c, K^c) -module is a complex vector space \mathbf{V} such that

- 1) \mathbf{V} is a \mathfrak{g}^c -module.
- 2) \mathbf{V} carries a representation of K^c , and the span of $K^c \cdot v$ is finite-dimensional for every $v \in \mathbf{V}$.
- 3) For $v \in \mathbf{V}$ and $X \in \mathfrak{k}^c$ we have

$$X \cdot v = \lim_{t \rightarrow 0} \frac{\exp(tX) \cdot v - v}{t}.$$

- 4) For $Y \in \mathfrak{g}^c$ and $k \in K^c$ the following holds for every $v \in \mathbf{V}$:

$$k \cdot (X \cdot v) = (\text{Ad}(k)X) \cdot [k \cdot v].$$

Definition 2.4. Let \mathbf{V} be a (\mathfrak{g}^c, K^c) -module. Then \mathbf{V} is a highest-weight module if there exists a nonzero element $v \in \mathbf{V}$ and a $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ such that

- 1) $X \cdot v = \lambda(X)v$ for all $X \in \mathfrak{t}$.
- 2) There exists a positive system Δ^+ in Δ such that $\mathfrak{g}_{\mathbb{C}}^c(\Delta^+) \cdot v = 0$.
- 3) $\mathbf{V} = U(\mathfrak{g}^c) \cdot v$.

The element v is called a primitive element of weight λ .

Theorem 2.5 (Jor-Ólafsson). Let $\rho \in \mathcal{A}(W)$ be irreducible. Then the corresponding (\mathfrak{g}^c, K^c) -module is a highest-weight module and equals $U(\mathfrak{p}^-)\mathbf{W}^\lambda$. In particular, every weight of \mathbf{V}_{K^c} is of the form

$$\nu - \sum_{\alpha \in \Delta(\mathfrak{p}^+, \mathfrak{t}_{\mathbb{C}})} n_\alpha \alpha.$$

Furthermore, $\langle \nu, H_\alpha \rangle \leq 0$ for all $\alpha \in \Delta_p^+$.

3. Reflection Positive Stochastic Processes Indexed by Lie Groups

Definition 3.1. A *reflection positive Hilbert space* is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where \mathcal{E} is a Hilbert space, θ a unitary involution and \mathcal{E}_+ a closed subspace which is θ -positive in the sense that the hermitian form $\langle v, w \rangle_\theta := \langle \theta v, w \rangle$ is positive semidefinite on \mathcal{E}_+ . For a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$, let

$$\mathcal{N} := \{u \in \mathcal{E}_+ : \langle \theta u, u \rangle = 0\}$$

and let $\hat{\mathcal{E}}$ be the completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\theta$. Let $q: \mathcal{E}_+ \rightarrow \hat{\mathcal{E}}, v \mapsto q(v) = \hat{v}$ be the canonical map. Then

$$\mathcal{E}_+^\theta := \{v \in \mathcal{E}_+ : \theta v = v\}$$

is the maximal subspace of \mathcal{E}_+ on which q is isometric.

Definition 3.2. Let $(\mathcal{E}, \mathcal{E}_+, \theta)$ be a reflection positive Hilbert space. If $\mathcal{E}_0 \subseteq \mathcal{E}_+$ is a closed subspace, $\mathcal{E}_- := \theta(\mathcal{E}_+)$, and E_0, E_{\pm} the orthogonal projections onto \mathcal{E}_0 and \mathcal{E}_{\pm} , then we say that $(\mathcal{E}, \mathcal{E}_0, \mathcal{E}_+, \theta)$ is of *Markov type* if

$$E_+ E_0 E_- = E_+ E_- . \tag{3.1}$$

Proposition 3.3. Let $(U_g)_{g \in G}$ be a reflection positive unitary representation of (G, S, τ) on $(\mathcal{E}, \mathcal{E}_+, \theta)$, let $\mathcal{E}_0 \subseteq (\mathcal{E}_+)^{\theta}$ be a subspace and $\Gamma = q|_{\mathcal{E}_0} : \mathcal{E}_0 \rightarrow \widehat{\mathcal{E}}$. If $(\mathcal{E}, \mathcal{E}_0, \mathcal{E}_+, \theta)$ is of Markov type, then the following assertions hold:

- (i) The reflection positive function $\varphi : G \rightarrow B(\mathcal{E}_0)$,
 $\varphi(g) := E_0 U_g E_0$, is multiplicative on S .
- (ii) $\varphi(s) = \Gamma^* \hat{U}_s \Gamma$ for $s \in S$, i.e., Γ intertwines $\varphi|_S$ with \hat{U} .

Reconstruction Theorem

Theorem 3.4 (Jorgensen, Neeb, Ólafsson). Let (G, τ) be a symmetric Lie group and $S \subseteq G$ be a \sharp -invariant subsemigroup satisfying $G = S \cup S^{-1}$. Then every positive semigroup structure for (G, S, τ) is associated to some (G, S, τ) -probability space $((Q, \Sigma, \mu), \Sigma_0, U, \theta)$.

Standard path space structures for locally compact groups

Theorem 3.5 (Jorgensen, Neeb, Ólafsson). Suppose that Q is a second countable locally compact group. Let μ be the measure on $Q^{\mathbb{R}}$ corresponding to the symmetric convolution semigroup $(\mu_t)_{t \geq 0}$ of probability measures on Q and the measure ν on Q for which the operators $P_t f = f * \mu_t$ define a positive semigroup structure on $L^2(Q, \nu)$. Then the translation action $(U_t \omega)(s) := \omega(s - t)$ on $P(Q) = Q^{\mathbb{R}}$ is measure preserving and μ is invariant under $(\theta \omega)(t) := \omega(-t)$.

Theorem 3.5, cont.

We thus obtain a reflection positive one-parameter group of Markov type on $\mathcal{E} := L^2(P(Q), \mathfrak{B}^{\mathbb{R}}, \mu)$ with respect to

- ▶ $\mathcal{E}_+ := L^2(P(Q), \mathfrak{B}^{\mathbb{R}_+}, \mu)$, for which
- ▶ $\mathcal{E}_0 := \text{ev}_0^*(L^2(Q, \nu)) \cong L^2(Q, \nu)$ and
- ▶ $\hat{\mathcal{E}} \cong L^2(Q, \nu)$ with $q(F) = E_0 F$ for $F \in \mathcal{E}_+$.

We further have

$$E_0 U_t E_0 = P_t \quad \text{holds for} \quad P_t f = f * \mu_t,$$

so that the U -cyclic subrepresentation generated by \mathcal{E}_0 is a unitary dilation of the hermitian one-parameter semigroup $(P_t)_{t \geq 0}$ on $L^2(Q, \nu)$.

4. Reflection Positivity and Spectral Theory

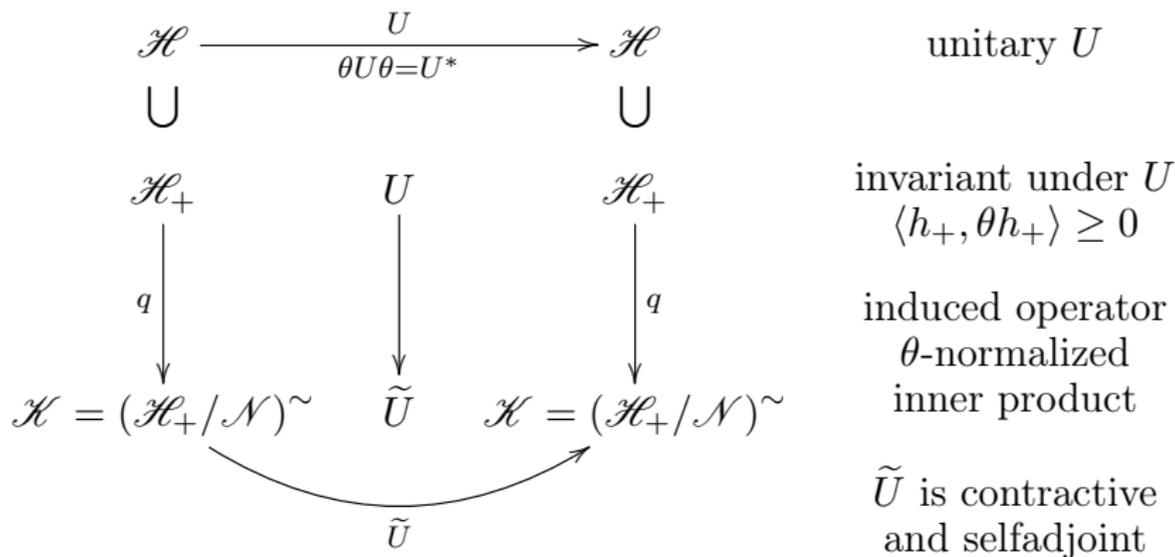


Figure 4.1: Reflection positivity. A unitary operator U transforms into a selfadjoint contraction \tilde{U} .

Maximal Reflections

Definition 4.1. Let \mathcal{H} be a Hilbert space and θ a reflection on \mathcal{H} . Let $P = \text{proj} \{x \in \mathcal{H} ; \theta x = x\}$, so that $\theta = 2P - I_{\mathcal{H}}$. Set

$$\text{Sub}_{OS}(\theta) = \{E_+ ; E_+ \text{ is a projection in } \mathcal{H} \text{ s.t. } E_+ \theta E_+ \geq 0\}. \quad (4.1)$$

Note. As usual properties for projections have equivalent formulation for closed subspaces: In this case, we may identify elements in $\text{Sub}_{OS}(\theta)$ with closed subspaces \mathcal{H}_+ such that

$$\langle h_+, \theta h_+ \rangle \geq 0, \text{ for } \forall h_+ \in \mathcal{H}_+. \quad (4.2)$$

Set $\mathcal{H}_+ := E_+ \mathcal{H}$.

Theorem 4.2 (Jor-Tian). Let \mathcal{H} , θ , and P be as stated, and consider the corresponding $Sub_{OS}(\theta)$ as in (4.1), or equivalently (4.2).

Then $Sub_{OS}(\theta)$ is an ordered lattice of projections, and it has the following family of maximal elements: Let

$C : P\mathcal{H} \rightarrow P^\perp\mathcal{H}$ be a contractive operator, and set

$$\mathcal{H}_+(P, C) := \{x + Cx ; x \in P\mathcal{H}\}. \quad (4.3)$$

Then $\mathcal{H}_+(P, C)$ is maximal in $Sub_{OS}(\theta)$, and every maximal element in $Sub_{OS}(\theta)$ has this form for some contraction $C : P\mathcal{H} \rightarrow P^\perp\mathcal{H}$.

Theorem 4.3 (Jor-Tian). Let \mathcal{H} , \mathcal{H}_\pm , θ , and U be as above, i.e., we are assuming O.S.-positivity; and further that U satisfies

$$\theta U \theta = U^*; \text{ and} \tag{4.4}$$

$$U \mathcal{H}_+ \subseteq \mathcal{H}_+ \text{ (equivalently, } E_+ U E_+ = U E_+ \text{.)} \tag{4.5}$$

Let P be the projection onto $\{h \in \mathcal{H} ; \theta h = h\}$, i.e., we have $\theta = 2P - I_{\mathcal{H}}$.

Theorem 4.3, cont.

1. Then

$$PUE_+ = PU^*\theta E_+. \quad (4.6)$$

2. If $C : P\mathcal{H} \rightarrow P^\perp\mathcal{H}$ denotes the corresponding contraction, then there is a unique operator $U_P : P\mathcal{H} \rightarrow P\mathcal{H}$ such that $U_P = PUP$; and, if $h_+ = x + Cx$, $x \in P\mathcal{H}$, then

$$\left\| \tilde{U}q(h_+) \right\|_{\mathcal{H}}^2 = \|U_P x\|_{\mathcal{H}}^2 - \|CU_P x\|_{\mathcal{H}}^2. \quad (4.7)$$

3. In particular, we have

$$\|U_P x\|_{\mathcal{H}}^2 - \|CU_P x\|_{\mathcal{H}}^2 \leq \|x\|_{\mathcal{H}}^2 - \|Cx\|_{\mathcal{H}}^2, \quad \forall x \in P\mathcal{H}.$$

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Appendix, with some of Jorgensen's recollections of conversations with Irving Segal, Marshall Stone, Bent Fuglede, and David Shale *

While this paper emphasizes orthogonality, stressing fractal measures, many of the problems have a history beginning with the case when the measure μ under consideration is a restriction of Lebesgue measure on \mathbb{R}^d . The original problem was motivated by von Neumann's desire to use his Spectral Theorem on partial differential operators (PDOs), much like the Fourier methods had been used boundary value problems in ordinary differential equations.

What follows is an oral history, tracing back some key influences in the subject, step-by-step, three or four mathematical generations. The reader is cautioned that it is a subjective account. And it should be added that analysis of fractal measures derived inspiration from a diverse variety of sources: potential theory, geometric measure theory, the study of singularities of solutions to partial differential equations (PDEs), and dynamics. Here the focus is on operator theory and harmonic analysis, subjects which in turn derived their inspiration from much the same sources, and from quantum physics. A central character in our account below is *projection valued measures* as envisioned by John von Neumann [vN32] and Marshall Stone [Sto32].

In what follows we trace influences from operator theory and mathematical physics, with projection valued measures playing a key role. Much of it was passed onto us from generations back.

We begin the history of what we now call the Fuglede problem [Fug74] with conversations the second named author had with Irving Segal, Marshall Stone, Bent Fuglede, and David Shale. Since this never found its way into print, in this appendix, we draw up connections to early parts of operator theory.

Starting in the early fifties, only a few years after the end of WWII, and still in the shadow of the war, mathematical physicists returned to the center of attraction around Niels Bohr in Copenhagen. Both Niels Bohr, and his mathematician brother Harald were still active; and still inspired members of the next generation. This is at the same time the generation of mathematicians who inspired me, along with a large number of other researchers in functional analysis, in operator theory, in partial differential equations, in harmonic analysis, and in mathematical physics.

Included in the flow of scientists who visited Copenhagen around the time was some leading analysts from the University of Chicago, and elsewhere in the US, Marshall Stone, Irving Segal, Richard Kadison, George Mackey, and Arthur Wightman to mention only a few. What this group of analysts has in common is that the members were all much inspired by John von Neumann's view on mathematical physics and operator theory. From conversations with Segal, I sense that von Neumann was viewed as a sort of Demi-God.

The forties and the fifties was the Golden Age of functional analysis, the period when the spectral theorem, and the spectral representation theorem, were being applied to problems in

*For details, see [DJ08].

harmonic analysis, and in quantum field theory. Here are some examples: (1) Irving Segal’s Plancherel theorem for unitary representations of locally compact unimodular groups [Seg50]; (2) the Gelfand–Naimark–Segal construction of representations of C^* -algebras from quantum mechanical states [Nm56, Nm59]; (3) Stone’s theorem for unitary one-parameter groups; and its generalization [Sto32], (4) the Stone–Naimark–Ambrose–Godement theorem for representations of locally compact abelian groups; (5) the rigorous formulation of the Stone–von Neumann uniqueness theorem [vN32, vN68]; and (6) its generalization into what became the Mackey machine for the study of unitary representations of semi-direct products of continuous groups, including (7) the imprimitivity theorem [Mac52, Mac53] for induced representations of unitary representations of locally compact groups. See also [SG65, GrW54, H55, Sha62].

In this context, we find much work involving unbounded operators in Hilbert space, typically non-commuting operators, for example axiomatic formulations of the momentum and position operators, and in a context where the dense domain of the operators must be taken very seriously. The simplest such operators are the momentum and position operators Q and P that form toy examples of quantum fields; i.e., the operators of Heisenberg’s uncertainty inequality [GrW54].

While the spectral theorem had already played a major role in mathematical physics, at the same time Lars Hörmander [H55] and Lars Gårding [GrW54] in Sweden were using functional analytic methods in the study of elliptic problems in PDE theory.

In his first visit to Copenhagen in the early fifties, Irving Segal brought with him one of his graduate students, David Shale, who was supposed to have written a thesis on commuting selfadjoint extensions of partial differential operators, and apparently achieved some initial results; never published in any form.

A baby example: To understand the derivative operator d/dx in a finite interval, we know that von Neumann’s deficiency indices help. Von Neumann’s deficiency indices is a pair of numbers (n, m) which serve as obstructions for formally hermitian operators with dense domain in Hilbert space to be selfadjoint.

For a single formally hermitian operator in Hilbert space \mathcal{H} , it is known that selfadjoint extensions exist if and only if $n = m$. The size of n ($= m$) is a measure of the cardinality of the set of selfadjoint extensions: The different selfadjoint extensions are parameterized by partial isometries between two subspaces in \mathcal{H} each of dimension n .

The property of *selfadjointness* is essential, as the spectral theorem doesn’t apply to formally hermitian operators. If the usual derivative operator d/dx is viewed as an operator in $L^2(0, 1)$ with dense domain \mathcal{D} consisting of differentiable functions vanishing at the end-points, then one checks that it is formally skew-hermitian, and that its deficiency indices are $(1, 1)$. As a result, we see that the derivative operator with zero-endpoint conditions has a one-parameter family of selfadjoint extensions; the family being indexed by the circle. In fact, the family has a simple geometric realization as follows: Thinking of functions in $L^2(0, 1)$ as wave functions, there is then clearly only one degree of freedom in choosing selfadjoint extensions. As we translate the wave function to the right, and hit the boundary point, a free phase must be assigned. This amounts to the ‘closing up’ the wave functions at the endpoints of the unit interval. What goes out at $x = 1$ must return at $x = 0$, possibly with a fixed phase correction. With this in mind, it was natural to ask for an analogue of von Neumann’s deficiency indices for several operators. In fact this is a dream that was never realized, but one which still serves to inspire me.

In the context of \mathbb{R}^d , i.e., higher dimensions, it is very natural to ask the similar extension question for bounded open domains Ω in \mathbb{R}^d . And it was clear what to expect.

Going to d dimensions: If functions in $L^2(\Omega)$ are translated locally in the d different coordinate directions, we will expect that the issue of selfadjoint extension operators should be related to the matching of phases on the boundary of Ω , and therefore related to the *tiling* of \mathbb{R}^d by translations of Ω ; i.e., with translations of Ω which cover \mathbb{R}^d , and which do not overlap on

sets of positive Lebesgue measure. The global motion by continuous translation in the d coordinate directions will be determined uniquely by the spectral theorem if we can find commuting selfadjoint extensions of the d partial derivative operators

$$i \frac{\partial}{\partial x_j}, \quad j = 1, \dots, d,$$

defined on the dense domain \mathcal{D} of differentiable functions on Ω which vanish on the boundary. We can take $\mathcal{D} = C_c^\infty(\Omega)$. These d operators are commuting and formally hermitian, but not selfadjoint. In fact when $d > 1$, each of the operators has deficiency indices (∞, ∞) . So in each of the d coordinate directions, $i \partial/\partial x_j|_{\mathcal{D}}$ has an infinite variety of selfadjoint extensions. But experimentation with examples shows that ‘most’ choices of Ω will yield non-commuting selfadjoint extensions. Each operator individually does have selfadjoint extensions, and the question is if they can be chosen to be mutually commuting. By this we mean that the corresponding projection-valued spectral measures commute.

The spectral representation for every selfadjoint extension $H_j \supset i \partial/\partial x_j|_{\mathcal{D}}$ has the form $H_j = \int_{\mathbb{R}} \lambda E_j(d\lambda)$, $j = 1, \dots, d$, where $E_j: \mathcal{B}(\mathbb{R}) \rightarrow \text{Projections}(L^2(\Omega))$ denotes the Borel subsets of \mathbb{R} . We say that a family of d selfadjoint extensions H_1, \dots, H_d of the respective $i \partial/\partial x_j$ operators is commuting if $E_j(A_j) E_k(A_k) = E_k(A_k) E_j(A_j)$ for all $A_j, A_k \in \mathcal{B}(\mathbb{R})$, and $j \neq k$.

When commuting extensions exist, we form the product measure

$$E = E_1 \times \dots \times E_d$$

on $\mathcal{B}(\mathbb{R}^d)$, and set

$$U(t) = \int_{\mathbb{R}^d} e^{i\lambda \cdot t} E(d\lambda),$$

where $t, \lambda \in \mathbb{R}^d$ and $\lambda \cdot t = \sum_{j=1}^d \lambda_j t_j$. Then clearly

$$U(t) U(t') = U(t + t'), \quad t, t' \in \mathbb{R}^d,$$

i.e., U is a unitary representation of \mathbb{R}^d acting on $\mathcal{H} = L^2(\Omega)$.

Theorem 0.1. (Fuglede–Jorgensen–Pedersen [Fug74, Jr82, Ped87]) *Let Ω be a non-empty connected open and bounded set in \mathbb{R}^d , and suppose that the d operators*

$$i \frac{\partial}{\partial x_j} \Big|_{C_c^\infty(\Omega)}, \quad j = 1, \dots, d,$$

have commuting extensions in $L^2(\Omega)$. Then there is a subset $S \subset \mathbb{R}^d$ such that

$$\{ e^{is \cdot x} \mid s \in S \}$$

is an orthogonal basis in $L^2(\Omega)$.

Remark 0.2. It might be natural to expect that an open spectral set Ω will have its connected components spectral, or at least have features predicted by the spectrum of the bigger set. This is not so as the following example (due to Steen Pedersen) shows. Details below!

If the set Ω in Theorem 0.1 is not assumed connected, the conclusion in Theorem 0.1 would be false. If a spectral set Ω is disconnected, then properties of the connected components are not immediately discerned from knowing that there is a spectrum for Ω .

An example showing this can be constructed by taking $\Omega = \Omega(p)$ to the following set obtained from a unit square, dividing it into two triangles along the main diagonal, followed by a translation of the triangle under the diagonal.

Details: Start with the following two open triangles making up a fixed unit-square, divided along the main diagonal. Now make a translation of the triangle under the diagonal by a non-zero integer amount p in the x -direction; i.e., by the vector $(p, 0)$, leaving the upper triangle alone .

Further, let $\Omega(p)$ be the union of the resulting two p -separated triangles. Hence the two connected components in $\Omega(p)$ will be two triangles; the second obtained from the first by a mirror image and a translation. Neither of these two disjoint open triangles is spectral; see [Fug74]. Nonetheless, as spectrum for $\Omega(p)$ we may take the unit-lattice \mathbb{Z}^2 . To see this, one may use a simple translation argument in $L^2(\Omega(p))$, coupled with the fact that $e_\lambda(p) = 1$ for all $\lambda \in \mathbb{Z}^2$.

There are two interesting open connected sets in the plane that are known [Fug74] not to be spectral. They are the open disk and the triangle. Of those two non-spectral sets, the triangles may serve as building blocks for spectral sets. Not the disks!

Remark 0.3. It is enough to assume that $\Omega \subset \mathbb{R}^d$ is open and has positive finite Lebesgue measure.

If the set Ω in the theorem is disconnected, then it is not necessarily true that there is a single set S , such that (Ω, S) is a spectral pair.

To see this, take $d = 1$ and $\Omega = (0, 1) \cup (2, 4)$, i.e., two open intervals as specified. The reason that there cannot be a set S such that $\{e_s \mid s \in S\}$ is an ONB in $L^2(\Omega)$ is that the polynomial $z^4 - z^2 + z - 1$ does not have enough zeros to make (Ω, S) a spectral pair.

Proof. Fuglede proved, under more stringent assumptions, that the spectrum of E is atomic, i.e., as a measure, E is supported on a discrete subset $S \subset \mathbb{R}^d$. Moreover, each $E(\{s\}) L^2(\Omega)$ for $s \in S$ is one-dimensional, and $E(\{s\}) L^2(\Omega) = \mathbb{C} e^{is \cdot x}$, where $e_s(x) = e^{is \cdot x}$ is the restriction to Ω of the complex exponential function corresponding to vector frequency $s = (s_1, \dots, s_d) \in S$. Since a spectral measure is *orthogonal*, i.e., $E(A) E(A') = 0$ if $A \cap A' = \emptyset$, it follows that

$$\{e_s \mid s \in S\}$$

is an orthogonal basis for $L^2(\Omega)$; in other words, we say that (Ω, S) is a *spectral pair*. \square

By the reasoning from the $d = 1$ example, by analogy, one would expect that the existence of commuting selfadjoint extensions will force Ω to tile \mathbb{R}^d by translations, at least if Ω is also connected. And in any case, one would expect that issues of spectrum and tile for bounded sets Ω in \mathbb{R}^d are related.

Segal suggested to his U. of Chicago Ph.D. student David Shale (originally from New Zealand) that he should take a closer look at when an open bounded set Ω in \mathbb{R}^d has commuting selfadjoint extensions for the d formally hermitian, and formally commuting partial derivatives. Apparently the research group around Irving Segal was very active, and everyone talked to one another. At the time, Bent Fuglede was a junior professor in Copenhagen, and he was thinking about related matters. Moreover, he quickly showed that if you add an assumption, then the results for problems of selfadjoint extensions do follow closely the simple case of the unit interval, i.e., $d = 1$.

Actually Fuglede had initially complained to Segal that the problem was hard, and Segal suggested to add an assumption, assuming in addition that the local action by translations in Ω is multiplicative. With this, Fuglede showed that for special configuration of sets Ω , there are selfadjoint extensions, and that they are associated in a natural way with lattices L in \mathbb{R}^d . The

spectrum of the representation U is a lattice L . By a *lattice* we mean a rank- d discrete additive subgroup of \mathbb{R}^d .

Suppose now that d commuting selfadjoint extensions exist for some bounded open domain Ω in \mathbb{R}^d ; and suppose in addition the multiplicative condition is satisfied. When the spectral theorem of Stone–Naimark–Ambrose–Godement (the SNAG theorem) is applied to a particular choice of d associated commuting unitary one-parameter groups, Fuglede showed that Ω must then be a fundamental domain (also called a tile for translations) for the lattice L^* which is dual to L . If L is a lattice in \mathbb{R}^d , the *dual lattice* L^* is

$$L^* = \{ \lambda \in \mathbb{R}^d \mid \lambda \cdot s \in 2\pi\mathbb{Z} \text{ for all } s \in L \}.$$

But more importantly, Fuglede pointed out that the several-variable variant of the spectral theorem (the SNAG theorem), and some potential theory, shown that if in addition Ω is assumed connected, and has a 'regular' boundary, then the existence of commuting selfadjoint extensions implies that $L^2(\Omega)$ has an orthonormal basis of complex exponentials

$$\{ \exp(is \cdot x) \mid s \text{ in some discrete subset } S \subset \mathbb{R}^d \}.$$

In a later paper [Jr82], Jorgensen suggested that a pair of sets (Ω, S) be called a *spectral pair*, and that S called a spectrum of Ω . With this terminology, we can state Fuglede's conjecture as follows: A measurable subset Ω of \mathbb{R}^d with finite positive Lebesgue measure has a spectrum, i.e., is the first part of a spectral pair, if and only if Ω tiles \mathbb{R}^d with some set of translation vectors in \mathbb{R}^d .

Apparently Fuglede's work was all done around 1954, and was in many ways motivated by von Neumann's thinking about unbounded operators.

In the mean time, David Shale left the problem, and he ended up writing a very influential Ph.D. thesis on representations of the canonical anticommutation relations in the case of an infinite number of degrees of freedom. See [Sha62].

Irving Segal explained to me in 1976 how his suggestions from the early fifties had been motivated by von Neumann. Von Neumann had hoped that all the work he and Marshal Stone had put into the axiomatic formulation of the spectral representation theorem would pay off on the central questions on the theory of PDEs. Initially, Fuglede wasn't satisfied with what he had proved, and he didn't get around to publishing his paper until 1974, see [Fug74], and only after being prompted by Segal, who had in the mean time become the founding editor of the *Journal of Functional Analysis*.

Fuglede had apparently felt that we needed to first understand non-trivial examples that arise from tilings by translations with vectors that can have irregular configurations, and aren't related in any direct way to a lattice. Natural examples for Ω for $d = 2$ that come to mind are the open interior of the triangle or of the disk. But Fuglede proved in [Fug74] that these two planar sets do *not* have spectra, i.e., they do not have the basis property for any subset $S \subset \mathbb{R}^2$. Specifically, in either case, there is no $S \subset \mathbb{R}^2$ such that $\{ e_s|_{\Omega} \mid s \in S \}$ is an orthogonal basis for $L^2(\Omega)$. This of course is consistent with the spectrum-tile conjecture.

Very importantly, in his 1974 paper, Fuglede calculated a number of instructive examples that showed the significance of combinatorics and of finite cyclic groups in our understanding of spectrum and tilings; and he made precise what is now referred to as the Fuglede conjecture ([JP94, JP00, PW01, Ped04, Jr82]).

Fuglede's question about equivalence of the two properties (existence of orthogonal Fourier frequencies for a given measurable subset Ω in \mathbb{R}^d) and the existence of a subset which makes Ω tile \mathbb{R}^d by translations, was for $d = 1, 2$, and perhaps 3. But of course the question is intriguing for any value of the dimension d . In recent years, a number of researchers, starting with Tao,

have now produced examples in higher dimensions giving negative answers: By increasing the dimension, it is possible to construct geometric obstructions to tiling which do not have spectral theoretic counterparts; and vice versa. For some of these examples we refer the reader to [FMM06, Tao04, LW95, LW96b, LW96c, LW96a, IKT03, Mat05, KL96].

In the talks, joint papers are cited, involving Jorgensen and co-authors. The list of the most frequently cited papers in the present lectures includes (alphabetic by last name): Alpay, Daniel; Bezuglyĭ, S. I.; Bratteli, Ola; Dutkay, Dorin Ervin; Herr, John; Kornelson, Keri A.; Neeb, Karl-Hermann; Ólafsson, Gestur; Packer, Judith A.; Pedersen, Steen; Shuman, Karen L.; Song, Myung-Sin; Sun, Qiyu; Tian, Feng; Weber, Eric.

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