Generalized Walsh bases and some applications to digital signal processing

Smooth and Non Smooth Harmonic Analysis Seminar
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Classic Walsh Basis

Rademacher functions:

$$\phi_n(x) = (-1)^k, \quad \frac{k}{2^n} \leq x < \frac{k+1}{2^n}, \quad k = 0, 1, \ldots, 2^n - 1$$

Orthogonal but not complete in $L^2[0,1]$.

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Walsh functions: all possible finite products of Rademacher's.

Paley- ordered Walsh functions :

$$w_n = \phi_{n_1}\phi_{n_2} \cdot \cdots \cdot \phi_{n_p}, \quad n = 2^{n_1} + 2^{n_2} + \ldots 2^{n_p}$$

$(w_n)_n$ *is orthogonal and complete in* $L^2[0,1]$

Generalized Walsh Bases

Walsh, Fourier and more bases are captured by the following construction:

Unitary $N \times N$ matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1} \end{pmatrix}$$

N filters m_i either polynomials of degree N with coefficients a_{ij} or

$$m_i(x) = \sqrt{N} \sum_{j=0}^{N-1} a_{ij} \chi_{[j/N,(j+1)/N)}(x), \quad x \in [0,1], i = 0, \ldots, N-1$$

Generalized Walsh Bases

Rademacher's played by

$$m_i(r^j(x)), \quad \textit{where } r(x) = Nx \textit{ mod } 1$$

Theorem

(Dutkay, P, Song)
The set $(w_n)_{n\geq 0}$ *forms an orthonormal basis in* $L^2[0,1]$.

$$w_n(x) = m_{i_0}(x)m_{i_1}(rx)m_{i_2}(r^2x)...m_{i_{p-1}}(r^{p-1}x)$$

where $n = i_0 + i_1 \cdot N + i_2N^2 + ... + i_{p-1}N^{p-1}$ *is the base N expansion of n*

Some properties

Theorem

(Dutkay, Harding, P)

- $\sum_{n=0}^{\infty} \langle f \, , \, w_n \rangle \, w_n$ *converges to* f *in* L^2

- *'Gap' sums* $\sum_{k=0}^{N^p} \langle f \, , \, w_k \rangle \, w_k$ *converge pointwise to* $f \in L^1$*, uniformly if* f *continuous*

- *if* f *piecewise constant the assignment*

$$A \longrightarrow (\langle f \, , \, w_n \rangle)_{n \geq 0}$$

is continuous.

- *'Filter' operators* $S_i : L^2 \to L^2$ *satisfy*

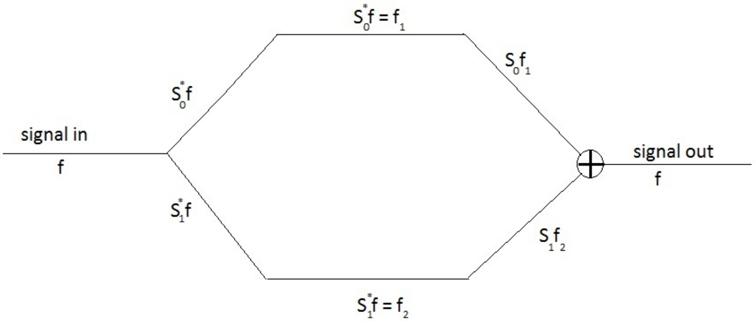
$$S_i f = m_i f(r)$$

$$S_i^* S_j = \delta_{i,j} I, \qquad \sum_{i=0}^{N-1} S_i S_i^* = I$$

Cuntz Relations

Definition

$$\text{Algebra } \mathcal{O}_N : \qquad S_i^* S_j = \delta_{i,j} I, \qquad \sum_{i=0}^{N-1} S_i S_i^* = I$$



Restricting to finite dimensions

Generalized Walsh transform $T : L^2 \to l^2$:

$$Tf = \langle f \, , \, w_n \rangle_{n \geq 0}$$

Discrete (and normalized) transform:
Signal space $\mathbf{R}^{N^p}$ *or* $\mathbf{C}^{N^p}$

$$Tf := A^{\otimes p} f$$

Expensive to implement $A \otimes B \otimes \ldots$ *but there is a 'fast' way for*

$$\overline{A}^{\otimes p} = \prod_{k=0}^{p-1} I_{N^k} \otimes \overline{A} \otimes I_{N^{p-1-k}}$$

Similar to FFT, fast Walsh: $O(M \log M)$ *operations,* $M = N^p$.

The Variance Criterion

$T : \mathbb{R}^n \to \mathbb{R}^n, \quad T^*T = I$

1. *Fix* $M < n$
2. $X = (x_1, x_2, \ldots, x_n),\ Y = TX = (y_1, y_2, \ldots, y_n)^t$
3. $cov(Y) := YY^t$
   Pick highest M *variances* $\sigma_{i_1,i_1}, \sigma_{i_2,i_2}, \ldots, \sigma_{i_M,i_M}$.
   Compression: $\tilde{Y} := (0, .., 0, y_{i_1}, 0, .., y_{i_2}, \ldots, 0, y_{i_M}, 0, ..0)$.
4. $\tilde{X} := T^{-1}(\tilde{Y})$ *approximates* X

Compare the (normalized) variance distributions:

$$k \to \frac{\sigma_{i_k,i_k}}{trace}$$

Compression Experiments

Two generalized Walsh based on 3×3 matrices

$$A := \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \qquad B := \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -0.2 & -0.58 & 0.78 \\ -0.79 & 0.57 & 0.22 \end{pmatrix} \tag{1}$$

Compression Experiments

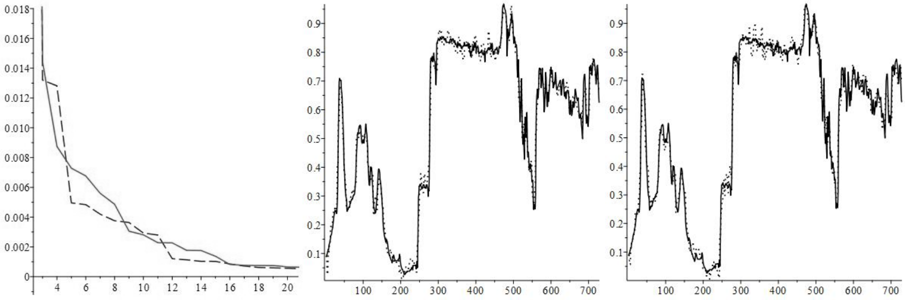


Figure: Variance distribution (left) for T_1-solid curve packs more energy than T_2 -dash. Approximation of a signal with T_1 (middle) and T_2(right), 90% components removed. Errors 0.84 and 0.95

Compression Experiments

Classic Walsh

$$A := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{2}$$

and a generalized Walsh based on 4×4*:*

$$B := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{3}$$

Compression Experiments

High variation signal, 2^8 length

$$X(i) := \begin{cases} \frac{i}{3i+1}, & \text{if } i|9 \\ \frac{i}{i+1}, & \text{otherwise} \end{cases}$$

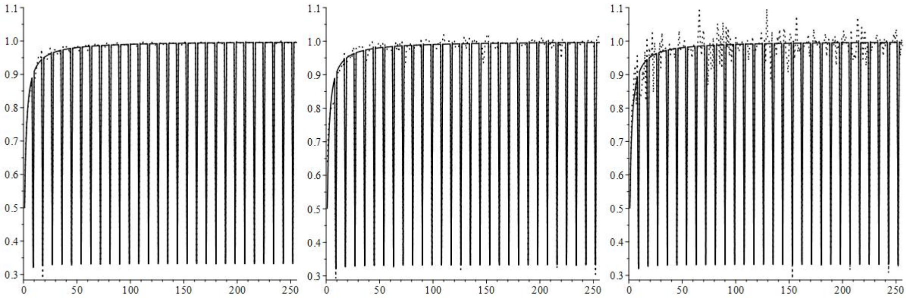


Figure: Approximation with DCT(left), classic Walsh(middle) and generalized 4 by 4 Walsh (right) with 55% components removed

Compression Experiments

Same signal, extended to 3^6 components. Apply now previous 3×3 generalized Walsh.

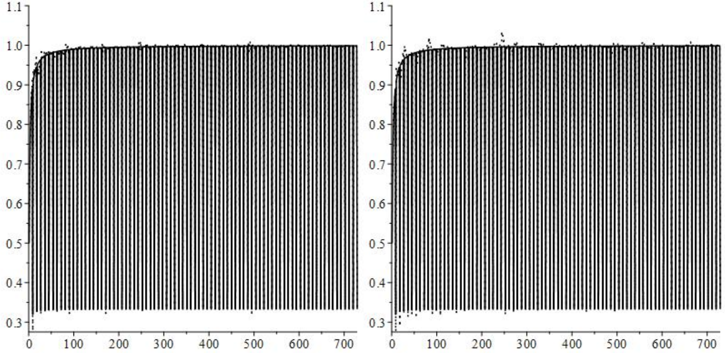


Figure: Same signal extended to dimension 3^6 with previous two generalized Walsh. 90% components removed. Recovery errors 0.02 and 0.08

Compression Experiments

Figure: original Lena pic, 256 by 256

Compression Experiments

Figure: classic Walsh Lena , 60% components removed, 256 by 256

Compression Experiments

Figure: generalized Walsh Lena, 60% components removed, 243 by 243

Compression Experiments

Figure: classic Walsh Lena , 88% components removed, 256 by 256

Compression Experiments

Figure: generalized Walsh Lena, 88% components removed, 243 by 243

Compression Experiments

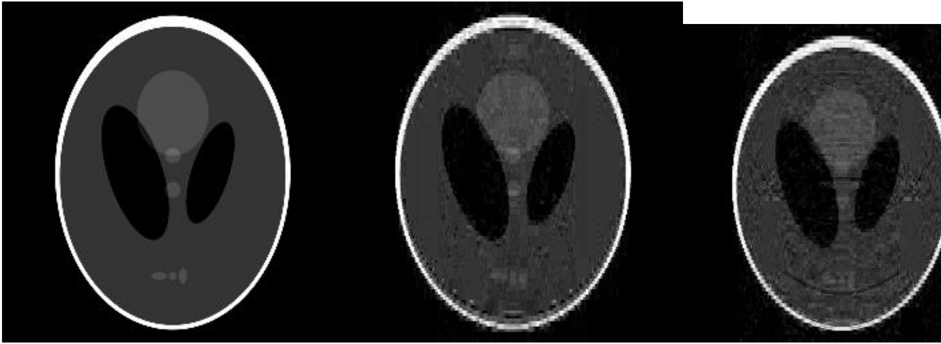


Figure: original phantom, classic Walsh, generalized Walsh, 88% compression

Compression Experiments

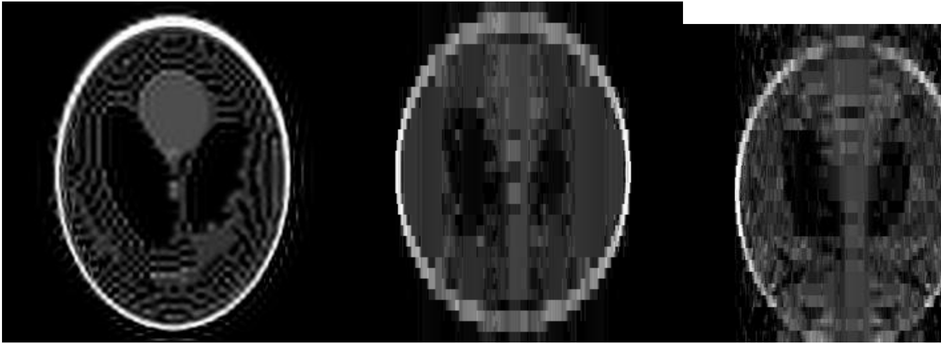
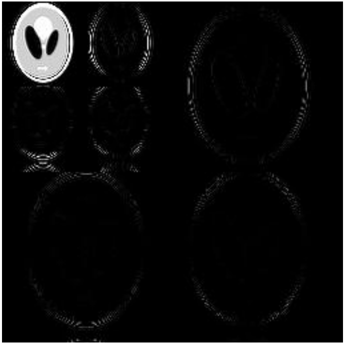


Figure: Wavelet, classic Walsh, generalized Walsh, 98% compression

Compression Experiments

The Discrete Wavelet transforms removes pixel values close to zero from this pic:



Uncertainty Principle

Uncertainty Principle for discrete signals (Donohoe, Stark '89)
$f \in \mathbf{C}^m$, $f \neq 0$, $\hat{f} \in \mathbf{C}^m$ *its discrete Fourier transform. Then*

$$|supp(f)| \cdot |supp(\hat{f})| \geq m \tag{4}$$

Theorem

(Dutkay, P, Silvestrov)
Let T *the generalized Walsh associated to* A, $f \in \mathbf{C}^{N^p}$, $f \neq 0$. *If* $\alpha > 0$ *satisfies* $max|a_{ij}| \leq (\frac{1}{\sqrt{N}})^{\alpha}$ *then :*
i)

$$|supp(f)| \cdot |supp(Tf)| \geq N^{p\alpha} \tag{5}$$

ii) $\alpha \leq 1$, *and* $= 1$ *if and only if* $\sqrt{N} \cdot A$ *is Hadamard matrix.*

Uncertainty Principle

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We want to recover $f \in \mathbf{C}^{N^p}$ *and know :*
i) $N_f = |supp(f)|$.
ii) a subset $B \subset supp(Tf)$ *of observed 'frequencies' with which form the signal*

$$\tilde{f}(k) := \begin{cases} Tf(k), & \text{if } k \in B \\ 0, & \text{otherwise} \end{cases}$$

iii) the number of unobserved 'frequencies' $N_w = |B^c|$ *satisfies*

$$2N_f N_w < N^{p\alpha} \tag{6}$$

Then f can be uniquely reconstructed.

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If $\tilde{f} = \tilde{g}$, $N_f = N_g$, *then signal* $h = f - g \neq 0$ *would violate the uncertainty principle.*

THANK YOU!