Chapter 4

Two Intrinsic Characterizations

The characterization of algebraizable deductive systems given in Theorem 3.7 is not intrinsic since it depends on the *a priori* existence of an algebraic semantics satisfying certain special conditions. In this chapter we show that algebraizability is in fact an intrinsic property of deductive systems. The key to this result is the observation (Theorem 4.1) that, for every algebraizable system $\mathcal{S}$ with equivalent algebraic semantics K, the unique isomorphism Ω_{K} between $\mathbf{Th}\mathcal{S}$ and $\mathbf{Th}\,\mathsf{K}$ coincides with the Leibniz equivalence operator (see Chapter 1.4). In the main result of the chapter and of the paper, (Theorem 4.2), we show that $\mathcal{S}$ is algebraizable iff the Leibniz function satisfies two simple and natural conditions when restricted to the lattice of $\mathcal{S}$-theories. This result is then used to obtain a second characterization of algebraizability in terms of intrinsic properties of the equivalence formulas Δ and defining equations $\delta \approx \epsilon$ (Theorem 4.7). This latter characterization proves to be the most useful in establishing the algebraizability of specific deductive systems.

4.1 The Leibniz Operator

We begin by describing the connection between equational theories and congruence relations on the formula algebra. For each equational theory Θ define $\widehat{\Theta} = \{\langle\varphi,\psi\rangle : \varphi \approx \psi \in \Theta\}$. $\widehat{\Theta}$ is always a congruence on $\mathbf{Fm}$, and every congruence on the formula algebra is obtained in this way from an equational theory. Consequently, for every $\mathcal{S}$-theory T, the Leibniz relation $\boldsymbol{\Omega}T$ is associated with a unique equational theory; see Chapter 1.4 for the definition of $\boldsymbol{\Omega}T$. Recall that $\boldsymbol{\Omega}T$ is the largest congruence on Fm compatible with T (Theorem 1.5).

Theorem 4.1 *Let $\mathcal{S}$ be an algebraizable deductive system and* K *its equivalent algebraic semantics. Let Ω_{K} be the unique isomorphism between* $\mathbf{Th}\mathcal{S}$ *and* $\mathbf{Th}\,\mathsf{K}$. *Then for every $\mathcal{S}$-theory T we have $(\Omega_{\mathsf{K}}T)^{\widehat{}} = \boldsymbol{\Omega}T$.*

Proof. By Theorem 1.6 it suffices to show that $(\Omega_{\mathsf{K}}T)^{\widehat{}}$ is a congruence on $\mathbf{Fm}$ that is compatible with T and elementarily definable over the formula matrix $\langle \mathbf{Fm}, T\rangle$. $(\Omega_{\mathsf{K}}T\widehat{)}$ is a congruence since the transform of every equational theory is. That it is elementarily definable follows immediately from the characterization

$$\Omega_{\mathsf{K}}T = \{\varphi \approx \psi : \varphi \,\Delta\, \psi \in T\}$$

given in Lemma 3.8 (Δ is a system of equivalence formulas for K). This can also be used to show $(\Omega_{\mathsf{K}}T\widehat{)}$ is compatible with T. Suppose $\langle \varphi, \psi\rangle \in (\Omega_{\mathsf{K}}T\widehat{)}$, (i.e., $\varphi \approx \psi \in \Omega_{\mathsf{K}}T$) and $\varphi \in T$. Then $\varphi \,\Delta\, \psi \in T$ and hence $\psi \in T$ by the detachment property for Δ (Lemma 2.14). ∎

In the sequel we shall not bother to distinguish between the Leibniz relation $\boldsymbol{\Omega} T$ and the unique equational theory Θ such that $\widehat{\Theta} = \boldsymbol{\Omega} T$.

In every algebraizable system $\mathcal{S}$ the Leibniz function $\boldsymbol{\Omega}$ restricted to $\mathcal{S}$-theories is both injective and order-preserving ($T \subseteq S$ implies $\boldsymbol{\Omega} T \subseteq \boldsymbol{\Omega} S$); this is an immediate consequence of 4.1. According to the next theorem this is almost enough to guarantee that $\mathcal{S}$ is algebraizable.

Theorem 4.2 *A deductive system $\mathcal{S}$ is algebraizable iff the Leibniz equality function satisfies the following two conditions.*

(i) $\boldsymbol{\Omega}$ *is injective and order-preserving on* $\mathbf{Th}\mathcal{S}$*;*

(ii) $\boldsymbol{\Omega}$ *preserves unions of directed subsets of* $Th\mathcal{S}$.

In applying this theorem to show a deductive system $\mathcal{S}$ is algebraizable it suffices to verify that $\boldsymbol{\Omega}$ is injective and preserves unions since the latter condition implies $\mathcal{S}$ is order-preserving.

The condition that $\boldsymbol{\Omega}$ be order-preserving on $\mathbf{Th}\mathcal{S}$ defines the class of protoalgebraic logics; see Chapter 1.4.1. In Appendix B we present an example due to H. Andréka and I. Németi that shows condition 4.2(i) alone is not sufficient for algebraizability.

Assume $\mathcal{S}$ is algebraizable, and let K be its equivalent algebraic semantics. As we have already observed, $\boldsymbol{\Omega}$ is injective and order-preserving since it coincides with Ω_{K} on $Th\mathcal{S}$. It preserves unions of directed sets by 3.3(iii). Hence one of the two implications of the theorem holds. To get the one in the opposite direction we use Theorem 3.7. In order to be able to apply 3.7 we have to construct the class K that will eventually be the equivalent semantics for $\mathcal{S}$. We begin with a technical result about surjective substitutions.

Lemma 4.3 *Assume σ is a surjective substitution. Then, for every $\vartheta \in Fm$ and every variable p occurring in ϑ, there exists a $\vartheta' \in Fm$ and a variable q such that $\sigma(\vartheta'[\varphi/q]) = \vartheta[\sigma\varphi/p]$ for every $\varphi \in Fm$.*

Proof. An inverse image of a variable under any substitution must also be a variable. Thus since σ is surjective there exists for each variable r another variable r' such that $\sigma r' = r$. Let ϑ' be obtained from ϑ by simultaneously replacing each variable r different from p by r', and p by any variable q different from all the r'. It is easy to see that $\sigma(\vartheta'[\varphi/q]) = \vartheta[\sigma\varphi/p]$ for every $\varphi \in Fm$. ∎

For the rest of the chapter we assume that $\mathcal{S}$ is an arbitrary deductive system, and and that the Leibniz function $\boldsymbol{\Omega}$ is restricted to $Th\mathcal{S}$. We take

$$\boldsymbol{\Omega}(Th\mathcal{S}) = \{\boldsymbol{\Omega}T : T \in Th\mathcal{S}\}.$$

Lemma 4.4 *Assume $\boldsymbol{\Omega}$ is order-preserving.*

(i) $\boldsymbol{\Omega}(\bigcap_{i\in I} T_i) = \bigcap_{i\in I} \boldsymbol{\Omega}T_i$ *for every system T_i, $i \in I$, of $\mathcal{S}$-theories. Hence $\boldsymbol{\Omega}(Th\mathcal{S})$ is closed under arbitrary intersection.*

(ii) $\sigma^{-1}(\boldsymbol{\Omega}T) = \boldsymbol{\Omega}\sigma^{-1}(T)$ *for every $T \in Th\mathcal{S}$ and every surjective substitution σ. Hence $\boldsymbol{\Omega}(Th\mathcal{S})$ is closed under inverse surjective substitution.*

Proof. (i) Since $\boldsymbol{\Omega}$ is order-preserving, $\boldsymbol{\Omega}(\bigcap_{i\in I} T_i) \subseteq \bigcap_{i\in I} \boldsymbol{\Omega}T_i$. The reverse inclusion also holds since $\bigcap_{i\in I} \boldsymbol{\Omega}T_i$ is clearly compatible with $\bigcap_{i\in I} T_i$.

(ii) Let $T \in Th\mathcal{S}$ and σ be a surjective substitution. It is easy to check that $\sigma^{-1}(\boldsymbol{\Omega}T)$ is always a congruence. Let $\varphi \approx \psi \in \sigma^{-1}(\boldsymbol{\Omega}T)$ and $\varphi \in \sigma^{-1}(T)$. Then $\sigma\varphi \approx \sigma\psi \in \boldsymbol{\Omega}T$ and $\sigma\varphi \in T$, whence $\sigma\psi \in T$ as well since $\boldsymbol{\Omega}T$ is compatible with T. Thus $\psi \in \sigma^{-1}(T)$. This shows that $\sigma^{-1}(\boldsymbol{\Omega}T)$ is compatible with T, and hence that $\sigma^{-1}(\boldsymbol{\Omega}T) \subseteq \boldsymbol{\Omega}\sigma^{-1}(T)$ since $\boldsymbol{\Omega}\sigma^{-1}(T)$ is the largest congruence compatible with $\sigma^{-1}(T)$. For the reverse inclusion, suppose $\varphi \approx \psi \in \boldsymbol{\Omega}\sigma^{-1}(T)$. We have to show $\sigma\varphi \approx \sigma\psi \in \boldsymbol{\Omega}T$. Suppose not. Then there is a formula ϑ and a variable p occurring in ϑ such that $\vartheta[\sigma\varphi] \in T$ and $\vartheta[\sigma\psi] \notin T$, or vice versa. (This follows from the definition of $\boldsymbol{\Omega}T$; see Chapter 1.4.1). Because σ is surjective, there exists by 4.3 a formula ϑ' and a variable q such that $\sigma(\vartheta'[\varphi/q]) = \vartheta[\sigma\varphi/p]$ and $\sigma(\vartheta'[\psi/q]) = \vartheta[\sigma\psi/p]$. So $\vartheta'[\varphi/q] \in \sigma^{-1}(T)$ and $\vartheta'[\psi/q] \notin \sigma^{-1}(T)$. Hence $\varphi \approx \psi \notin \boldsymbol{\Omega}\sigma^{-1}(T)$, a contradiction. Thus $\boldsymbol{\Omega}\sigma^{-1}(T) \subseteq \sigma^{-1}(\boldsymbol{\Omega}T)$. ∎

Since $\boldsymbol{\Omega}(Th\mathcal{S})$ is closed under arbitrary intersections, it forms a complete lattice that we shall denote by $\boldsymbol{\Omega}(\mathbf{Th}\mathcal{S})$, and if $\boldsymbol{\Omega}$ is injective it is an isomorphism from $\mathbf{Th}\mathcal{S}$ onto $\boldsymbol{\Omega}(\mathbf{Th}\mathcal{S})$. In order to be able to apply 3.7 we must show that $\boldsymbol{\Omega}(\mathbf{Th}\mathcal{S})$ coincides with $\mathbf{Th}\,\mathsf{K}$ for some class K of algebras. (To be precise we shall show that $\mathbf{Th}\,\mathsf{K}$ and $\boldsymbol{\Omega}(\mathbf{Th}\mathcal{S})$ are isomorphic under the mapping $\Theta \mapsto \widehat{\Theta}$.) For this purpose we need some elementary results of universal algebra.

Recall that the formula algebra $\mathbf{Fm}$ has the set of formulas as universe, and, for each primitive connective ω of rank n, an operation $\omega^{\mathbf{Fm}}$ of the same rank defined by $\omega^{\mathbf{Fm}}(\varphi_0, \ldots, \varphi_{n-1}) = \omega\varphi_0 \ldots \varphi_{n-1}$. Any mapping of the variables

$\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots$ into an algebra **A** can be uniquely extended to a homomorphism of **Fm** into **A**. If $\varphi(p_0, \ldots, p_{n-1})$ is a formula and $a_0, \ldots, a_{n-1}$ elements of **A**, then the interpretation $\varphi^{\mathbf{A}}(a_0, \ldots, a_{n-1})$ of φ in **A** can be viewed as the image $h\varphi$ of φ under any homomorphism h from **Fm** into **A** that takes each variable $\mathbf{p}_i$ to the corresponding a_i. Thus for any class K of algebras we have that $\Gamma \models_{\mathsf{K}} \varphi \approx \psi$ iff $h\varphi = h\psi$ for every homomorphism of **Fm** into a member of K such that $h\xi = h\eta$ for every $\xi \approx \eta \in \Gamma$.

For every congruence (i.e., equational theory) Θ the quotient algebra $\mathbf{Fm}/\Theta$ is defined in the usual way. The elements are equivalence classes of Θ, and its algebraic structure is the one induced by that of **Fm**. The natural mapping from Fm onto Fm/Θ is a homomorphism. For each homomorphism of **Fm** into **A** the set of equations $\{\varphi \approx \psi : h\varphi = h\psi\}$ is a congruence called the *relation-kernel* of h. The relation-kernel of the natural mapping of **Fm** onto $\mathbf{Fm}/\Theta$ is Θ itself.

For any homomorphism h of **Fm** into a member of K the relation-kernel Θ of h is a K-theory. To see this suppose $\varphi \approx \psi \notin \Theta$. Then $h\varphi \neq h\psi$, but $h\xi = h\eta$ for every $\xi \approx \eta \in \Theta$. Hence $\Theta \not\models_{\mathsf{K}} \varphi \approx \psi$. More generally, for any $\Gamma \subseteq Eq$, the K-theory $Cn_{\mathsf{K}}\Gamma$ generated by Γ can be characterized as the intersection of the relation-kernels Θ of all homomorphisms of **Fm** into members of K such that $h\xi = h\eta$ for all $\xi \approx \eta \in \Gamma$.

Lemma 4.5 *Assume* $\mathbf{\Omega}$ *preserves unions of directed sets. Let*

$$\mathsf{K} = \{\mathbf{Fm}/\Theta : \Theta \in \mathbf{\Omega}(Th\mathcal{S})\}.$$

Then $\mathbf{\Omega}(Th\mathcal{S}) = Th\,\mathsf{K}$.

Proof. The hypothesis that $\mathbf{\Omega}$ preserves unions of directed sets implies that $\mathbf{\Omega}(Th\mathcal{S})$ is closed under directed unions. It also implies $\mathbf{\Omega}$ is order-preserving, and hence, by 4.4, that $\mathbf{\Omega}(Th\mathcal{S})$ is closed under arbitrary intersection and inverse surjective substitution.

Observe first of all that $\mathbf{\Omega}(Th\mathcal{S}) \subseteq Th\,\mathsf{K}$; this holds since each $\Theta \in \mathbf{\Omega}(Th\mathcal{S})$ is the relation-kernel of the natural map of **Fm** onto $\mathbf{Fm}/\Theta \in \mathsf{K}$.

We begin the proof of the reverse inclusion with the following sublemma:

> Let h be a homomorphism of **Fm** into a member $\mathbf{Fm}/\Theta$ of K with the property that each element of $\mathbf{Fm}/\Theta$ is the image of an infinite number of variables. Then the relation-kernel of h is of the form $\sigma^{-1}(\Theta)$ for some surjective substitution σ.

Let σ be any substitution such that $\sigma\mathbf{p}_i \in h\mathbf{p}_i$ for $i = 1, 2, 3, \ldots$, and, furthermore, such that each $\mathbf{p}_i$ is the image under σ of some $\mathbf{p}_j$; such a σ exists because of the assumption that each element of $\mathbf{Fm}/\Theta$ is the image of an infinite number of variables. Observe that σ is surjective, and that $h\mathbf{p}_i = \sigma\mathbf{p}_i/\Theta$ for each i.

Let f be the natural map from $\mathbf{Fm}$ onto $\mathbf{Fm}/\Theta$. Then $(f\circ\sigma)\mathbf{p}_i = f(\sigma\mathbf{p}_i) = \sigma\mathbf{p}_i/\Theta = h\mathbf{p}_i$ for every i. Thus $f\circ\sigma = h$ since they are homomorphisms that agree on the generators of $\mathbf{Fm}$. Let Ψ be the relation-kernel of h. For any $\varphi \approx \psi \in Eq$ we have $\varphi \approx \psi \in \Psi$ iff $h\varphi = h\psi$ iff $f(\sigma\varphi) = f(\sigma\psi)$ iff $\sigma\varphi \approx \sigma\psi \in \Theta$ iff $\varphi \approx \psi \in \sigma^{-1}(\Theta)$. Thus $\Psi = \sigma^{-1}(\Theta)$. This proves the sublemma.

Let $\Phi \in Th\,\mathsf{K}$. Assume for the time being that Φ is finitely generated, say $\Phi = Cn_{\mathsf{K}}\Gamma$ where Γ is finite. Φ, like any K-theory, can be expressed as the intersection of relation-kernels of homomorphisms of $\mathbf{Fm}$ into members of K. We will show that the fact that Φ is finitely generated guarantees these homomorphisms can all be taken to satisfy the special property of the homomorphism h of the sublemma.

Suppose $\varphi \approx \psi \notin \Phi$, i.e., $\Gamma \not\models_{\mathsf{K}} \varphi \approx \psi$. Then there exists a homomorphism h of $\mathbf{Fm}$ into $\mathbf{Fm}/\Theta$ for some $\Theta \in \mathbf{\Omega}(Th\mathcal{S})$ such that $h\xi = h\eta$ for each $\xi \approx \eta \in \Gamma$, but $h\varphi \neq h\psi$. We assume without loss of generality that h has the property of the hypothesis of the sublemma, for, if not, we can replace it by one that does since Γ and $\varphi \approx \psi$ together contain only finitely many variables. Let Ψ be the relation-kernel of h. Then $\varphi \approx \psi \notin \Psi$, but $\Gamma \subseteq \Psi$ and hence $\Phi = Cn_{\mathsf{K}}\Gamma \subseteq \Psi$ since Ψ is a K-theory. Thus Φ can be written as the intersection of a family of such Ψ, one for each $\varphi \approx \psi \notin \Phi$. By the sublemma each Ψ is of the form $\sigma^{-1}(\Theta)$ for some $\Theta \in \mathbf{\Omega}(Th\mathcal{S})$ and some surjective substitution σ. So $\Psi \in \mathbf{\Omega}(Th\mathcal{S})$ by 4.4(i), and hence $\Phi \in \mathbf{\Omega}(Th\mathcal{S})$ since $\mathbf{\Omega}(Th\mathcal{S})$ is closed under intersection.

Finally, assume Φ is an arbitrary K-theory. Let $\mathcal{C} = \{Cn_{\mathsf{K}}\Gamma : \Gamma \subseteq \Phi,\ \Gamma \text{ finite}\}$. Then $\Phi = \bigcup\mathcal{C}$. But by what we have just proved $\mathcal{C}$ is a subset of $Th\,\mathsf{K}$, and it is clearly directed by inclusion. So $\Phi \in \mathbf{\Omega}(Th\mathcal{S})$ by hypothesis. This proves that $Th\,\mathsf{K} \subseteq \mathbf{\Omega}(Th\mathcal{S})$. ∎

Thus if $\mathbf{\Omega}$ preserves unions of directed sets and is injective, then it is an isomorphism between the lattices $\mathbf{Th}\mathcal{S}$ and $\mathbf{Th}\,\mathsf{K}$ where $\mathsf{K} = \{\mathbf{Fm}/\Theta : \Theta \in \mathbf{\Omega}(Th\mathcal{S})\}$. The only thing that remains to show before we can apply 3.7 (ii),(iii) is that $\mathbf{\Omega}$ commutes with surjective substitutions.

Lemma 4.6 *Assume $\mathbf{\Omega}$ is injective and preserves unions of directed sets. Then $\mathbf{\Omega}$ commutes with surjective substitutions.*

Proof. Let σ be a surjective substitution and T a $\mathcal{S}$-theory. Let $\mathsf{K} = \{\mathbf{Fm}/\Theta \in \mathbf{\Omega}(Th\mathcal{S})\}$. We first show that $\sigma_{\mathsf{K}}(\mathbf{\Omega}T) \subseteq \mathbf{\Omega}\sigma_{\mathcal{S}}(T)$.

Let $\varphi \approx \psi \in \sigma(\mathbf{\Omega}T)$, say $\varphi = \sigma\varphi'$ and $\psi = \sigma\psi'$ where $\varphi' \approx \psi' \in \mathbf{\Omega}T$. Suppose $\varphi \approx \psi \notin \mathbf{\Omega}\sigma_{\mathcal{S}}(T)$, say $\vartheta[\varphi/p] \in \sigma_{\mathcal{S}}(T)$ and $\vartheta[\psi/p] \notin \sigma_{\mathcal{S}}(T)$. Since σ is surjective, by Lemma 4.3 there is a formula ϑ' and variable q such that

$\sigma(\vartheta'[\varphi'/q]) = \vartheta[\varphi/p]$ and $\sigma(\vartheta'[\psi'/q]) = \vartheta[\psi/p]$. Thus $\vartheta'[\varphi'/q] \in \sigma^{-1}(\sigma_S(T))$ and $\vartheta'[\psi'/q] \notin \sigma^{-1}(\sigma_S(T))$. So $\varphi' \approx \psi' \notin \boldsymbol{\Omega}(\sigma^{-1}\sigma_S(T))$. But, since $\boldsymbol{\Omega}$ is order-preserving by hypothesis, and $T \subseteq \sigma^{-1}(\sigma_S(T))$, we can conclude that $\varphi' \approx \psi' \notin \boldsymbol{\Omega}(T)$. This is a contradiction. Hence $\sigma(\boldsymbol{\Omega}T) \subseteq \boldsymbol{\Omega}\sigma_S(T)$, and thus $\sigma_{\mathsf{K}}(\boldsymbol{\Omega}T) \subseteq \boldsymbol{\Omega}\sigma_S(T)$.

To obtain the reverse inclusion we note that $\sigma_{\mathsf{K}}(\boldsymbol{\Omega}T)$ is a K-theory, and hence is a member of $\boldsymbol{\Omega}(Th\,\mathcal{S})$ by the previous lemma. So $\sigma_{\mathsf{K}}(\boldsymbol{\Omega}T) = \boldsymbol{\Omega}S$ for some $S \in Th\mathcal{S}$. Then $\boldsymbol{\Omega}T \subseteq \sigma^{-1}(\sigma_{\mathsf{K}}(\boldsymbol{\Omega}T)) = \sigma^{-1}(\boldsymbol{\Omega}S) = \boldsymbol{\Omega}\sigma^{-1}(S)$; we get the last equality from 4.4(ii). We can conclude now that $T \subseteq \sigma^{-1}(S)$. For suppose otherwise. Then $T \cap \sigma^{-1}(S) \neq T$. Since $\boldsymbol{\Omega}$ preserves intersection by 4.4(i), $\boldsymbol{\Omega}(T \cap \sigma^{-1}(S)) = \boldsymbol{\Omega}T \cap \boldsymbol{\Omega}\sigma^{-1}(S) = \boldsymbol{\Omega}T$, contradicting the premise $\boldsymbol{\Omega}$ is injective

Thus $T \subseteq \sigma^{-1}(S)$, and hence $\sigma T \subseteq \sigma(\sigma^{-1}(S)) = S$ since σ is surjective. So $\sigma_S(T) \subseteq S$, and therefore $\boldsymbol{\Omega}\sigma_S(T) \subseteq \boldsymbol{\Omega}S = \sigma_{\mathsf{K}}(\boldsymbol{\Omega}T)$. ∎

We can now apply 3.7(ii),(iii) to conclude that, under the hypothesis of 4.2, $\mathsf{K} = \{\mathbf{Fm}/\Theta : \Theta \in \boldsymbol{\Omega}(Th\,\mathcal{S})\}$ is an equivalent algebraic semantics for $\mathcal{S}$. This completes the proof of Theorem 4.2.

4.2 A Second Intrinsic Characterization

Although Theorem 4.2 gives an intrinsic and conceptually simpler characterization of algebraizable logics than 3.7, it is still not very useful for applications. The reason for this is that the definition of the Leibniz relation $\boldsymbol{\Omega}T$ given in Chapter 1.4, and also its alternative characterization as the largest congruence compatible with T, are difficult to work with. We give another intrinsic characterization that has proved useful in practice.

Theorem 4.7 *A deductive system $\mathcal{S}$ is algebraizable iff there exist a system Δ of formulas in two variables and a system $\delta \approx \epsilon$ of equations in a single variable such that the following conditions (i)–(v) hold for all $\varphi, \psi, \vartheta \in Fm$:*

(i) $\vdash_{\mathcal{S}} \varphi \,\Delta\, \varphi$;

(ii) $\varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \psi \,\Delta\, \varphi$;

(iii) $\varphi \,\Delta\, \psi,\ \psi \,\Delta\, \vartheta \vdash_{\mathcal{S}} \varphi \,\Delta\, \vartheta$;

For every primitive connective ω and all $\varphi_0, \ldots, \varphi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in Fm$ where n is the rank of ω

(iv) $\varphi_0 \,\Delta\, \psi_0, \ldots, \varphi_{n-1} \,\Delta\, \psi_{n-1} \vdash_{\mathcal{S}} \omega\varphi_0 \ldots \varphi_{n-1} \,\Delta\, \omega\psi_0 \ldots \psi_{n-1}$.

Finally, for all $\vartheta \in Fm$

(v) $\vartheta \dashv\vdash_{\mathcal{S}} \delta(\vartheta) \,\Delta\, \epsilon(\vartheta)$.

In this event Δ and $\delta \approx \epsilon$ are systems of equivalence formulas and defining equations for $\mathcal{S}$.

Proof. Suppose $\mathcal{S}$ is algebraizable, and let Δ and $\delta \approx \epsilon$ respectively be systems of equivalence formulas and defining equations for the equivalent algebraic semantics for $\mathcal{S}$. Then conditions (i)–(v) correspond to 2.13(i)–(iv) and 2.9(ii), respectively.

Suppose now that (i)–(v) hold. For each $T \in Th\mathcal{S}$ we define

$$\Omega_\Delta T = \{\varphi \approx \psi : \varphi \,\Delta\, \psi \in T\}.$$

Conditions (i)–(iii) say that $\Omega_\Delta T$ is a congruence. Suppose $\Omega_\Delta T = \Omega_\Delta S$. Let $\varphi \in T$. Then $\delta(\varphi) \,\Delta\, \epsilon(\varphi) \in T$ by (v), and hence $\delta(\varphi) \approx \epsilon(\varphi) \in \Omega_\Delta T$. Thus $\delta(\varphi) \approx \epsilon(\varphi) \in \Omega_\Delta S$, and so $\delta(\varphi) \,\Delta\, \epsilon(\varphi) \in S$ and $\varphi \in S$ by (v) again. This shows $T \subseteq S$, and by symmetry $S \subseteq T$. Thus Ω_Δ is injective.

Let T_i, $i \in I$, be a system of $\mathcal{S}$-theories directed by inclusion. $\varphi \approx \psi \in \Omega_\Delta(\bigcup_{i\in I} T_i)$ iff $\varphi \,\Delta\, \psi \in \bigcup_{i\in I} T_i$ iff $\varphi \,\Delta\, \psi \in T_i$ for some i iff $\varphi \approx \psi \in \Omega_\Delta T_i$ for some i iff $\varphi \approx \psi \in \bigcup_{i\in I} \Omega_\Delta T_i$. Thus Ω_Δ preserves unions of directed sets.

We can apply 4.2 as soon as we prove $\Omega_\Delta T = \boldsymbol{\Omega} T$ for each $T \in Th\mathcal{S}$. We begin by proving that Δ has the detachment property, i.e.,

$$\varphi\,,\, \varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \psi. \tag{1}$$

Let $T = Cn_{\mathcal{S}}\{\varphi, \varphi \,\Delta\, \psi\}$. Then, since $\varphi \vdash_{\mathcal{S}} \delta(\varphi) \,\Delta\, \epsilon(\varphi)$ by (v), we have

$$\delta(\varphi) \approx \epsilon(\varphi) \in \Omega_\Delta T. \tag{2}$$

But also $\varphi \approx \psi \in \Omega_\Delta T$. Thus φ can be replaced by ψ in (2), and hence $\delta(\psi) \approx \epsilon(\psi) \in \Omega_\Delta T$. Therefore, $\varphi, \varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \delta(\psi) \,\Delta\, \epsilon(\psi)$. We now get (1) by applying (v) again.

The detachment property implies that $\Omega_\Delta T$ is compatible with T, and hence $\Omega_\Delta \subseteq \boldsymbol{\Omega} T$. For the reverse inclusion, note that, since $\varphi \,\Delta\, \varphi \in T$, $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega} T$ yields $\varphi \,\Delta\, \psi \in T$, and hence $\langle \varphi, \psi \rangle \in \Omega_\Delta$. Alternatively, we can invoke Theorem 1.2 to show that $\Omega_\Delta T = \boldsymbol{\Omega} T$, as $\Omega_\Delta T$ is elementarily definable over the formula matrix $\langle \mathbf{Fm}, T \rangle$.

We have shown that $\Omega_\Delta = \boldsymbol{\Omega}$. We now apply 4.2 to conclude that $\mathcal{S}$ is algebraizable. ∎

Deductive systems satisfying conditions 4.7(i)–(iv) were first considered by Prucnal and Wroński [35] and have been extensively studied by Czelakowski [11].

Corollary 4.8 *A sufficient condition for a deductive system $\mathcal{S}$ to be algebraizable is that there exists a system Δ of formulas in two variables satisfying (i)–(iv) of Theorem 4.7 together with the following two conditions.*

(v) $\varphi,\ \varphi\ \Delta\ \psi \vdash_{\mathcal{S}} \psi$ *(detachment);*

(vi) $\varphi, \psi \vdash_{\mathcal{S}} \varphi \Delta \psi$ *(G-rule).*

In this event Δ and $p \approx p\ \Delta\ p$ are the equivalence formulas and defining equations for $\mathcal{S}$.

Proof. Let $\delta(p) = p$ and $\epsilon(p) = p\ \Delta\ p$. Then, for every $\vartheta \in Fm$,

$$\delta(\vartheta)\ \Delta\ \epsilon(\vartheta) = \vartheta\ \Delta\ (\vartheta\ \Delta\ \vartheta).$$

By (vi), $\vartheta,\ \vartheta\ \Delta\ \vartheta \vdash_{\mathcal{S}} \delta(\vartheta)\ \Delta\ \epsilon(\vartheta)$. Thus $\vartheta \vdash_{\mathcal{S}} \delta(\vartheta)\ \Delta\ \epsilon(\vartheta)$. since $\vartheta\ \Delta\ \vartheta$ is a $\mathcal{S}$-theorem. On the other hand, $\delta(\vartheta)\ \Delta\ \epsilon(\vartheta) \vdash_{\mathcal{S}} \vartheta$ by detachment. So 4.7(v) holds. ∎

The derived inference rule 4.8(vi) is called the *G-rule* by Suszko [40, p.34], [41, pp.92f.]. The G-rule holds in an algebraizable deductive system $\mathcal{S}$ iff the members of an arbitrary $\mathcal{S}$-theory are all identified under the Leibniz relation $\mathbf{\Omega}T$ (i.e., T is an equivalence class of $\mathbf{\Omega}T$). All the deductive systems algebraizable in the classical sense have the G-rule. However there are algebraizable systems that fail to have it, for instance **R**; see Chapter 5.2.2 below.

If conditions 4.7(i)–(v) hold in a deductive system, they continue to hold in every extension over the same language. This gives

Corollary 4.9 *Any extension of an algebraizable deductive system is itself algebraizable with the same equivalence formulas and defining equations.* ∎

In the course of the proof of 4.7 we established the following result.

Theorem 4.10 *Let $\mathcal{S}$, Δ, and $\delta \approx \epsilon$ be as in Theorem 4.7, and assume conditions 4.7(i)–(v) hold. Let*

$$\mathsf{K} = \{\mathbf{Fm}/\Omega_\Delta T : T \in Th\mathcal{S}\}.$$

Then K^{Q} is the unique equivalent quasivariety semantics for $\mathcal{S}$, and Δ and $\delta \approx \epsilon$ are respectively the equivalence formulas and defining equations for K^{Q}. ∎