Chapter 3

The Lattice of Theories

Let $\mathcal{S}$ be algebraizable and K the unique **quasivariety** equivalent to $\mathcal{S}$. Because of the intimate connection between a deductive system and its theory lattice it is to be expected that $\mathbf{Th}\mathcal{S}$ is isomorphic to the lattice $\mathbf{Th}\,\mathsf{K}$ of equational theories of K (this lattice is defined below), and this is indeed the case as is shown in Lemma 3.5 below. Moreover the isomorphism persists even when we enrich both $\mathbf{Th}\mathcal{S}$ and $\mathbf{Th}\,\mathsf{K}$ by the system of join-continuous operators induced by the substitutions in the common underlying language $\mathcal{L}$. What is more interesting however is that the converse also holds: the isomorphism of the enriched lattices $\mathbf{Th}\mathcal{S}$ and $\mathbf{Th}\,\mathsf{K}$ guarantees the equivalence of $\mathcal{S}$ and K. This is the main result of the chapter and is given in Theorem 3.7.

We begin by looking more closely at the equational consequence relation $\models_{\mathsf{K}}$ for an arbitrary class K of algebras.

Any set Γ of equations closed under K-consequence (i.e., $\Gamma \models_{\mathsf{K}} \varphi \approx \psi$ implies $\varphi \approx \psi \in \Gamma$) is called an *equational theory of* K (a K-*theory* for short). By an *equational theory* (without reference to K) we mean a K-theory where K is the class of all algebras of the given type. We denote the set of K-theories by $Th\,\mathsf{K}$. The basic properties of K-theories closely parallel those of $\mathcal{S}$-theories as one would expect. For any $\Gamma \subseteq Eq$ we define

$$Cn_{\mathsf{K}}\Gamma = \{\varphi \approx \psi \in Eq : \Gamma \models_{\mathsf{K}} \varphi \approx \psi\},$$

the smallest K-theory including Γ. The notion of a set of *generators* of a K-theory is defined in the obvious way. $Th\,\mathsf{K}$ is closed under arbitrary intersection and thus forms a complete lattice $\mathbf{Th}\,\mathsf{K} = \langle Th\,\mathsf{K}, \cap, \vee^{\mathsf{K}}\rangle$. The largest theory is Eq, and the smallest is the set of identities of K. The equational consequence relation $\models_{\mathsf{K}}$ is completely determined by the operator Cn_{K}, and also by the theory lattice $\mathbf{Th}\,\mathsf{K}$.

Not every equational consequence relation is finitary. The finitary ones can be characterized in terms of their theory lattices. The proof of the following lemma is similar to that of Lemma 1.1 and will also be omitted.

Lemma 3.1 *Let* K *be any class of algebras. The following three conditions are equivalent.*

(i) $\models_{\mathsf{K}}$ *is finitary;*

(ii) *The compact elements of* $\mathbf{Th}\,\mathsf{K}$ *coincide with the finitely generated* K-*theories;*

(iii) $Th\,\mathsf{K}$ *is closed under unions of directed sets.*

Each of the conditions (i)–(iii) implies

(iv) *the lattice* $\mathbf{Th}\,\mathsf{K}$ *is algebraic.* ∎

Let σ be an arbitrary substitution. We take $\sigma(\Gamma) = \{\sigma\varphi \approx \sigma\psi : \varphi \approx \psi \in \Gamma\}$ and $\sigma^{-1}(\Gamma) = \{\varphi \approx \psi : \sigma\varphi \approx \sigma\psi \in \Gamma\}$. If Θ is a K-theory, then so is $\sigma^{-1}(\Theta)$ because $\models_{\mathsf{K}}$ is always stuctural. In general $\sigma(\Theta)$ is not a theory. We define

$$\sigma_{\mathsf{K}}(\Theta) = Cn_{\mathsf{K}}\sigma(\Theta).$$

The analogue of Lemma 1.2 holds and the proof is similar.

Lemma 3.2 *Let* K *be any class of algebras.*

(i) $Th\,\mathsf{K}$ *is closed under inverse substitution.*

(ii) $\sigma_{\mathsf{K}}(Cn_{\mathsf{K}}\Gamma) = Cn_{\mathsf{K}}\sigma(\Gamma)$ *for every* $\Gamma \subseteq Eq$ *and substitution* σ.

(iii) σ_{K} *is a join-continuous mapping of* $\mathbf{Th}\,\mathsf{K}$ *into itself.* ∎

In the following discussion $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is a deductive system, and K is an algebraic semantics for $\mathcal{S}$ with defining equations $\delta \approx \epsilon$. ($\mathcal{S}$ is not assumed to be algebraizable.) We also assume K is a quasivariety, or at least that $\models_{\mathsf{K}}$ is finitary. We define two functions H_{K} from $Th\,\mathsf{K}$ into $Th\,\mathcal{S}$, and Ω_{K} from $Th\,\mathcal{S}$ back to $Th\,\mathsf{K}$. For every $\Theta \in Th\,\mathsf{K}$ let

$$H_{\mathsf{K}}\Theta = \{\varphi \in Fm : \delta(\varphi) \approx \epsilon(\varphi) \in \Theta\}.$$

$H_{\mathsf{K}}\Theta$ is a $\mathcal{S}$-theory. To see this assume $H_{\mathsf{K}}\Theta \vdash_{\mathcal{S}} \psi$. Since K is an algebraic semantics for $\mathcal{S}$, $\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in H_{\mathsf{K}}\Theta\} \models_{\mathsf{K}} \delta(\psi) \approx \epsilon(\psi)$. But the left hand side is included in Θ, so $\delta(\psi) \approx \epsilon(\psi) \in \Theta$, and hence $\psi \in H_{\mathsf{K}}\Theta$. It is easily established that H_{K} is *meet-continuous* in the sense that $H_{\mathsf{K}}(\bigcap_{i\in I}\Theta_i) = \bigcap_{i\in I} H_{\mathsf{K}}\Theta_i$ for every system of K-theories. In general H_{K} is not join-continuous.

For each $T \in Th\,\mathcal{S}$ define

$$\Omega_{\mathsf{K}}T = Cn_{\mathsf{K}}(\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in T\}). \tag{1}$$

Both H_{K} and Ω_{K} depend on the defining equations $\delta \approx \epsilon$, but note that, if $\mathcal{S}$ is algebraizable, and K is the unique equivalent semantics for $\mathcal{S}$, then every system of defining equations gives the same H_{K} and Ω_{K} by 2.15

H_{K} and Ω_{K} are clearly order-preserving.

Lemma 3.3 (i) $\Omega_{\mathsf{K}} Cn_{\mathcal{S}}\Gamma = Cn_{\mathsf{K}}\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \Gamma\}$ *for every* $\Gamma \subseteq Fm$.

(ii) Ω_{K} *is a join-continuous map from* $\mathbf{Th}\mathcal{S}$ *into* $\mathbf{Th}\,\mathsf{K}$.

(iii) Ω_{K} *preserves unions of directed sets of theories. I. e.,*

$$\Omega_{\mathsf{K}}(\bigcup_{i\in I} T_i) = \bigcup_{i\in I} \Omega_{\mathsf{K}} T_i$$

for every system T_i, $i \in I$, *of* $\mathcal{S}$*-theories that is directed by inclusion.*

Proof. (i) It is only necessary to prove the inclusion from left to right; the reverse inclusion is obvious. Suppose $\varphi \approx \psi \in \Omega_{\mathsf{K}} Cn_{\mathcal{S}}\Gamma$. Then

$$\{\delta(\chi) \approx \epsilon(\chi) : \chi \in Cn_{\mathcal{S}}\Gamma\} \models_{\mathsf{K}} \varphi \approx \psi. \tag{2}$$

$\chi \in Cn_{\mathcal{S}}\Gamma$ means $\Gamma \vdash_{\mathcal{S}} \chi$, and hence

$$\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \Gamma\} \models_{\mathsf{K}} \{\delta(\chi) \approx \epsilon(\chi) : \chi \in Cn_{\mathcal{S}}\Gamma\}.$$

Combined with (2) this gives $\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \psi$, i. e., $\varphi \approx \psi \in Cn_{\mathsf{K}}\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \Gamma\}$.

(ii) Let T_i, $i \in I$, be any system of $\mathcal{S}$-theories. Also let $\Gamma = \bigcup_{i\in I} T_i$ so that $\bigvee^{\mathcal{S}}_{i\in I} T_i = Cn_{\mathcal{S}}\Gamma$. Then using part (i) we get

$$\begin{aligned}
\Omega_{\mathsf{K}}\Big(\bigvee_{i\in I}^{\mathcal{S}} T_i\Big) &= \Omega_{\mathsf{K}} Cn_{\mathcal{S}}\Gamma \;=\; Cn_{\mathsf{K}}\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \Gamma\}\\
&= Cn_{\mathsf{K}}\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \bigcup_{i\in I} T_i\}\\
&= Cn_{\mathsf{K}}\Big(\bigcup_{i\in I}\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in T_i\}\Big)\\
&= Cn_{\mathsf{K}}\Big(\bigcup_{i\in I} Cn_{\mathsf{K}}\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in T_i\}\Big)\\
&= Cn_{\mathsf{K}}\Big(\bigcup_{i\in I} \Omega_{\mathsf{K}} T_i\Big)\\
&= \bigvee_{i\in I}^{\mathcal{S}} \Omega_{\mathsf{K}} T_i.
\end{aligned}$$

(iii) By (ii) we have $\Omega_{\mathsf{K}}(\bigcup_{i\in I} T_i) = \Omega_{\mathsf{K}}(\bigvee^{\mathcal{S}}_{i\in I} T_i) = \bigvee^{\mathsf{K}}_{i\in I} \Omega_{\mathsf{K}} T_i$. But $\Omega_{\mathsf{K}} T_i$, $i \in I$, is also a directed system since Ω_{K} is order-preserving. Thus, by the assumption that $\models_{\mathsf{K}}$ is finitary, $\bigvee^{\mathsf{K}}_{i\in I} \Omega_{\mathsf{K}} T_i = \bigcup_{i\in I} \Omega_{\mathsf{K}} T_i$. ∎

Lemma 3.4 (i) $H_{\mathsf{K}}\Omega_{\mathsf{K}} T = T$ *for every* $T \in Th\,\mathcal{S}$.

(ii) $\Omega_{\mathsf{K}} H_{\mathsf{K}}\Theta \subseteq \Theta$ *for every* $\Theta \in Th\,\mathsf{K}$, *and* $\Omega_{\mathsf{K}} H_{\mathsf{K}}\Theta = \Theta$ *just in case* $\Theta \in \Omega_{\mathsf{K}}(Th\,\mathcal{S})$, *i.e.,* Θ *is the image of some* $\mathcal{S}$*-theory under* Ω_{K}.

Proof. (i) Clearly $H_{\mathsf{K}}\Omega_{\mathsf{K}}T \supseteq T$. Suppose $\psi \in H_{\mathsf{K}}\Omega_{\mathsf{K}}T$. Unraveling the definitions of H_{K} and Ω_{K} we get

$$\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in T\} \models_{\mathsf{K}} \delta(\psi) \approx \epsilon(\psi).$$

Since K is an algebraic semantics for $\mathcal{S}$ with defining equations $\delta \approx \epsilon$, we immediately get $T \vdash_{\mathcal{S}} \psi$, and hence $\psi \in T$.

(ii) $\Omega_{\mathsf{K}}H_{\mathsf{K}}\Theta = Cn_{\mathsf{K}}\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in H_{\mathsf{K}}\Theta\} = Cn_{\mathsf{K}}\{\delta(\varphi) \approx \epsilon(\varphi) : \delta(\varphi) \approx \epsilon(\varphi) \in \Theta\} \subseteq Cn_{\mathsf{K}}\Theta = \Theta$. If $\Theta = \Omega_{\mathsf{K}}T$ for some $T \in Th\mathcal{S}$, then $\Omega_{\mathsf{K}}H_{\mathsf{K}}\Theta = \Omega_{\mathsf{K}}H_{\mathsf{K}}\Omega_{\mathsf{K}}T = \Omega_{\mathsf{K}}T = \Theta$ by part (i). ∎

It follows from 3.4(i) that Ω_{K} is a bijection between $Th\mathcal{S}$ and $\Omega_{\mathsf{K}}(Th\mathcal{S}) \subseteq Th\,\mathsf{K}$. Since Ω_{K} is order preserving, $\Omega_{\mathsf{K}}(Th\mathcal{S})$ forms a complete lattice under the ordering relation (actually set-theoretic inclusion) it inherits from $\mathbf{Th}\,\mathsf{K}$. We denote this lattice by $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$; it is isomorphic to $\mathbf{Th}\mathcal{S}$ via Ω_{K}. In general $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$ is not a sublattice of $\mathbf{Th}\,\mathsf{K}$ since $\Omega_{\mathsf{K}}(Th\mathcal{S})$ need not be closed under intersection. However, the join operations of the two lattices coincide as we see in the next lemma.

Let $\mathbf{L}$ and $\mathbf{M}$ be complete lattices with $L \subseteq M$. $\mathbf{L}$ is called a *join-complete subsemilattice* of $\mathbf{M}$ if $\bigvee^{\mathbf{L}}_{i\in I} a_i = \bigvee^{\mathbf{M}}_{i\in I} a_i$ for any system $a_i, i \in I$, of elements of L. $\mathbf{L}$ is *compact* in $\mathbf{M}$ if the compact elements of $\mathbf{L}$ coincide with the compact elements of $\mathbf{M}$ that lie in $\mathbf{L}$.

Lemma 3.5 *Assume* K *is an algebraic semantics for* $\mathcal{S}$ *with defining equations* $\delta \approx \epsilon$.

(i) Ω_{K} *maps* $\mathbf{Th}\mathcal{S}$ *isomorphically onto a compact and join-complete subsemilattice of* $\mathbf{Th}\,\mathsf{K}$.

(ii) K *is equivalent to* $\mathcal{S}$ *with defining equations* $\delta \approx \epsilon$ *iff* Ω_{K} *maps* $\mathbf{Th}\mathcal{S}$ *isomorphically onto all of* $\mathbf{Th}\,\mathsf{K}$.

Proof. (i) We must show that $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$ is a compact, join-complete subsemilattice of $\mathbf{Th}\,\mathsf{K}$. Let $\Theta_i, i \in I$, be an arbitrary system of K-theories in $\Omega_{\mathsf{K}}(Th\mathcal{S})$. Let $\Theta_i = \Omega_{\mathsf{K}}T_i$ with $T_i \in Th\mathcal{S}$. Since Ω_{K} is join-continuous,

$$\bigvee^{\mathsf{K}}_{i\in I} \Theta_i = \bigvee^{\mathsf{K}}_{i\in I} \Omega_{\mathsf{K}}T_i = \Omega_{\mathsf{K}}\left(\bigvee^{\mathcal{S}}_{i\in I} T_i\right) = \bigvee^{\mathbf{L}}_{i\in I} \Omega_{\mathsf{K}}T_i = \bigvee^{\mathbf{L}}_{i\in I} \Theta_i,$$

where $\mathbf{L} = \Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$. Thus $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$ is a join-complete subsemilattice of $\mathbf{Th}\,\mathsf{K}$. This implies at once that every element of $\Omega_{\mathsf{K}}(Th\mathcal{S})$ that is compact in $\mathbf{Th}\mathcal{S}$ must also be compact in $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$. For the converse suppose Θ is compact in $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$. Then $H_{\mathsf{K}}\Theta$ is compact in $\mathbf{Th}\mathcal{S}$ since $\Omega_{\mathsf{K}}H_{\mathsf{K}}\Theta = \Theta$ and Ω_{K} is an isomorphism between $\mathbf{Th}\mathcal{S}$ and $\Omega_{\mathsf{K}}(\mathbf{Th}\mathcal{S})$. Thus $H_{\mathsf{K}}\Theta$ is finitely generated. Let Γ be a finite set of generators of $H_{\mathsf{K}}\Theta$ so that $H_{\mathsf{K}}\Theta = Cn_{\mathcal{S}}\Gamma$.

Then $\Theta = \Omega_K H_K \Theta = \Omega_K Cn_S \Gamma = Cn_K\{\delta(\vartheta) \approx \epsilon(\vartheta) : \vartheta \in \Gamma\}$; the last equality follows by 3.3(i). Thus Θ is a finitely generated K-theory, and hence compact in **Th** K since $\models_K$ is assumed to be finitary.

(ii) Assume K is equivalent to $\mathcal{S}$, and let Δ be a system of equivalence formulas for K. In view of 3.4(ii) it suffices to show that $\Omega_K H_K \Theta \supseteq \Theta$ for each $\Theta \in Th\,K$. Let $\varphi \approx \psi \in \Theta$. By 2.8(ii), $\delta(\varphi \,\Delta\, \psi) \approx \epsilon(\varphi \,\Delta\, \psi) \in \Theta$. Thus $\varphi \,\Delta\, \psi \in H_K\Theta$ by definition of H_K, and hence $\delta(\varphi \,\Delta\, \psi) \approx \epsilon(\varphi \,\Delta\, \psi) \in \Omega_K H_K \Theta$. So by 2.8(ii) again we get $\varphi \approx \psi \in \Omega_K H_K \Theta$

Assume conversely that $\Omega_K(\mathbf{Th}\mathcal{S}) = \mathbf{Th}\,K$. Then by 3.4, H_K is an isomorphism from **Th** K onto $\mathbf{Th}\mathcal{S}$ with inverse Ω_K. Let p, q be any pair of fixed variables. (For instance we can take them to be the first two variables in the sequence $\mathrm{p}_0, \mathrm{p}_1, \ldots$.) Let $\Theta = Cn_K\{p \approx q\}$. Θ is compact in **Th** K since it is finitely generated. So $H_K\Theta$ is compact in $\mathbf{Th}\mathcal{S}$ and hence finitely generated. Let $\varphi_j(p, q, r_0, \ldots r_{k-1})$, $j < m$, be a finite set of generators for $H_K\Theta$ where the $r_0, \ldots, r_{k-1}$ include each variable distinct from p, q that occurs in at least one of the φ_j. Then by 3.4(ii) and 3.3(i),

$$\Theta = \Omega_K H_K \Theta = Cn_K\{\delta(\varphi_j) \approx \epsilon(\varphi_j) : j < m\}.$$

So $\{\delta(\varphi_j) \approx \epsilon(\varphi_j) : j < m\} \models_K p \approx q$. Let σ be a substitution that leaves p, q fixed and maps each r_i to p. Then, since $\models_K$ is structural, we get $\{\delta(\sigma\varphi_j) \approx \epsilon(\sigma\varphi_j) : j < m\} \models_K p \approx q$. So $\Theta \subseteq \Omega_K(Cn_S\{\sigma\varphi_j : j < m\})$. Applying H_K to both sides, we get $H_K\Theta \subseteq Cn_S\{\sigma\varphi_j : j < m\}$.

For the inclusion in the opposite direction observe that $\delta(\varphi_j) \approx \epsilon(\varphi_j) \in \Omega_K H_K \Theta = \Theta$ for each $j < m$. Thus $p \approx q \models_K \delta(\varphi_j) \approx \epsilon(\varphi_j)$. Applying σ we get $p \approx q \models_K \delta(\sigma\varphi_j) \approx \epsilon(\sigma\varphi_j)$, i.e., $\delta(\sigma\varphi_j) \approx \epsilon(\sigma\varphi_j) \in \Theta$ for all $j < m$. Thus by definition of H_K, $\sigma\varphi_j \in H_K\Theta$ for each $j < m$. So $H_K\Theta = Cn_S\{\sigma\varphi_j : j < m\}$.

Define $\Delta_j(p, q) = \sigma\varphi_j \;(= \varphi_j(p, q, p, \ldots, p))$ for each j. The mutual consequence relation $p \approx q \dashv\models_K \delta(p \,\Delta\, q) \approx \epsilon(p \,\Delta\, q)$ follows without difficulty from the equalities $\Omega_K H_K \Theta = \Theta$ and $H_K\Theta = Cn_S\{p \,\Delta\, q\}$; the structurality of $\models_K$ then gives 2.8(ii). ∎

The fact that Ω_K maps $\mathbf{Th}\mathcal{S}$ isomorphically onto a compact join-complete subsemilattice of **Th** K is not enough to characterize K as an algebraic semantics for $\mathcal{S}$. Such a characterization requires in addition some restriction on how Ω_K behaves with regard to substitutions. The fundamental property of Ω_K needed for this purpose is commutativity with substitution.

Lemma 3.6 *Let σ be any substitution. Then for all $T \in Th\mathcal{S}$, $\Omega_K\sigma_S(T) = \sigma_K(\Omega_K T)$.*

Proof. We observe first of all that, for every $\Gamma \subseteq \boldsymbol{Eq}$,

$$Cn_{\mathsf{K}}\sigma(Cn_{\mathsf{K}}\Gamma) = Cn_{\mathsf{K}}\sigma(\Gamma). \tag{3}$$

In fact $Cn_{\mathsf{K}}\sigma(Cn_{\mathsf{K}}\Gamma) = \sigma_{\mathsf{K}}(Cn_{\mathsf{K}}\Gamma) = Cn_{\mathsf{K}}\sigma(\Gamma)$; Lemma 3.2(ii) is used twice here.

$$\begin{aligned}
\Omega_{\mathsf{K}}\sigma_{\mathcal{S}}(T) &= \Omega_{\mathsf{K}}Cn_{\mathcal{S}}\sigma(T) \\
&= Cn_{\mathsf{K}}(\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in \sigma(T)\}) \qquad \text{by 3.3(i)} \\
&= Cn_{\mathsf{K}}\sigma(\{\delta(\psi) \approx \epsilon(\psi) : \psi \in T\}) \\
&= Cn_{\mathsf{K}}\sigma(Cn_{\mathsf{K}}\{\delta(\psi) \approx \epsilon(\psi) : \psi \in T\}) \qquad \text{by (3)} \\
&= \sigma_{\mathsf{K}}(Cn_{\mathsf{K}}\{\delta(\psi) \approx \epsilon(\psi) : \psi \in T\}) \\
&= \sigma_{\mathsf{K}}(\Omega_{\mathsf{K}}T). \blacksquare
\end{aligned}$$

We are now ready for our promised characterization of equivalent algebraic semantics by means of the theory lattices.

A substitution σ is called *surjective* if for each $\varphi \in Fm$ there exists a $\varphi' \in Fm$ such that $\sigma\varphi' = \varphi$. Clearly σ is surjective iff for each variable p there exists a variable p' such that $\sigma p' = p$.

Theorem 3.7 *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a deductive system, and let* K *be a quasivariety, or, more generally, any class of algebras such that $\models_{\mathsf{K}}$ is finitary.*

(i) K *is an algebraic semantics for $\mathcal{S}$ iff there exists an isomorphism from* $\mathbf{Th}\mathcal{S}$ *onto a compact, join-complete subsemilattice of* $\mathbf{Th}\,\mathsf{K}$ *that commutes with substitutions.*

(ii) K *is equivalent to $\mathcal{S}$ iff there exists an isomorphism from* $\mathbf{Th}\mathcal{S}$ *onto the whole of* $\mathbf{Th}\,\mathsf{K}$ *that commutes with substitutions.*

(iii) *Statements* (i) *and* (ii) *remain true when "substitutions" is replaced by "surjective substitutions".*

Proof. (i) That the condition is necessary follows from Lemmas 3.5(i) and 3.6. To prove it is sufficient suppose Ξ is an isomorphism from $\mathbf{Th}\mathcal{S}$ onto a compact and join-complete subsemilattice of $\mathbf{Th}\,\mathsf{K}$ that commutes with substitutions. Let p be a fixed but arbitrary variable, and let $T = Cn_{\mathcal{S}}\{p\}$ and $\Theta = \Xi T$. Θ is compact in $\mathbf{Th}\,\mathsf{K}$ since it is the image of a compact element of $\mathbf{Th}\mathcal{S}$, and $\Xi(\mathbf{Th}\mathcal{S})$ is compact in $\mathbf{Th}\,\mathsf{K}$ by assumption. Thus Θ is finitely generated by 3.1(ii). Let $\kappa_i(p, r_0, \dots, r_{k-1}) \approx \lambda_i(p, r_0, \dots, r_{k-1})$, for $i < n$, be a finite system of generators for Θ. Let σ be any substitution such that $\sigma p = p$ and $\sigma r_j = p$ for $j < k$; notice that σ can be taken to be surjective. Notice also that $\sigma_{\mathcal{S}}(T) = \sigma_{\mathcal{S}}(Cn_{\mathcal{S}}\{p\}) = Cn_{\mathcal{S}}\{\sigma p\} = T$. Thus $\Theta = \Xi T = \Xi\sigma_{\mathcal{S}}(T) = \sigma_{\mathsf{K}}(\Xi T) = \sigma_{\mathsf{K}}(Cn_{\mathsf{K}}\{\kappa(p, \bar{r}) \approx \lambda(p, \bar{r})\}) = Cn_{\mathsf{K}}\sigma(\{\kappa(p, \bar{r}) \approx \lambda(p, \bar{r})\}) = Cn_{\mathsf{K}}\{\kappa(p, p, \dots, p) \approx \lambda(p, p, \dots, p)\}$. Set

$$\delta_i(p) = \kappa_i(p, p, \dots, p)\ ,\ \epsilon_i(p) = \lambda_i(p, p, \dots, p) \tag{4}$$

for each $i < n$. Then $\delta(p) \approx \epsilon(p)$ is also a set of generators for Θ.

Let φ be any formula and let σ be any substitution that takes p to φ; notice again that a surjective σ can be chosen. Then $\Xi Cn_{\mathcal{S}}\{\varphi\} = \Xi Cn_{\mathcal{S}}\{\sigma p\} = \Xi\sigma_{\mathcal{S}} Cn_{\mathcal{S}}\{p\} = \sigma_{\mathsf{K}}\Xi Cn_{\mathcal{S}}\{p\} = \sigma_{\mathsf{K}}\Theta = \sigma_{\mathsf{K}} Cn_{\mathsf{K}}\{\delta(p) \approx \epsilon(p)\} = Cn_{\mathsf{K}}\sigma\{\delta(p) \approx \epsilon(p)\} = Cn_{\mathsf{K}}\{\delta(\varphi) \approx \epsilon(\varphi)\}$. For any $\Gamma \subseteq Fm$ we have

$$\begin{aligned} \Xi Cn_{\mathcal{S}}\Gamma &= \Xi\Big(\bigvee_{\varphi\in\Gamma}^{\mathcal{S}} Cn_{\mathcal{S}}\{\varphi\} \Big) = \bigvee_{\varphi\in\Gamma}^{\mathsf{K}} \Xi Cn_{\mathcal{S}}\{\varphi\} \\ &= \bigvee_{\varphi\in\Gamma}^{\mathsf{K}} Cn_{\mathsf{K}}\{\delta(\varphi) \approx \epsilon(\varphi)\} = Cn_{\mathsf{K}}\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in \Gamma\}. \end{aligned}$$

Thus

$$\begin{aligned} \Gamma \vdash_{\mathcal{S}} \psi &\Leftrightarrow Cn_{\mathcal{S}}\{\psi\} \subseteq Cn_{\mathcal{S}}\Gamma \\ &\Leftrightarrow \Xi Cn_{\mathcal{S}}\{\psi\} \subseteq \Xi Cn_{\mathcal{S}}\Gamma \\ &\Leftrightarrow Cn_{\mathsf{K}}\{\delta(\psi) \approx \epsilon(\psi)\} \subseteq Cn_{\mathsf{K}}\{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in \Gamma\} \\ &\Leftrightarrow \{\delta(\varphi) \approx \epsilon(\varphi) : \varphi \in \Gamma\} \models_{\mathsf{K}} \delta(\psi) \approx \epsilon(\psi). \end{aligned}$$

Hence K is an algebraic semantics for $\mathcal{S}$ with defining equations $\delta \approx \epsilon$. Observe that the isomorphism Ξ coincides with the function Ω_{K} defined in (1) using the defining equations described in (4).

(ii) is an straightforward consequence of part (i) and 3.5(ii). To see that (iii) holds we need only recall the observations made in the proof of (i) that the two substitutions used there could be taken to be surjective. ∎

It follows from the above proof that, if $\mathcal{S}$ is algebraizable and K is its equivalent algebraic semantics, then Ω_{K} defined in (1) is the *unique* isomorphism from **Th**$\mathcal{S}$ to **Th** K that commutes with substitution. In the algebraizable case Ω_{K} has a particularly simple characterization in terms of the equivalence formulas associated with K. The proof of the following lemma is straightforward and will be omitted.

Lemma 3.8 *Let $\mathcal{S}$ be algebraizable and* K *the equivalent algebraic semantics for $\mathcal{S}$. Let Δ be a system of equivalence formulas for* K. *Then, for every $T \in Th\mathcal{S}$, $\Omega_{\mathsf{K}}T = \{\varphi \approx \psi : \varphi\,\Delta\,\psi \in T\}$.* ∎

We shall have occasion to use this characterization in the next chapter.