A Different Look at the Optimal Control of the Brockett Integrator

Domenico D'Alessandro* and Zhifei Zhu†

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Abstract

We propose a method to analyze the three dimensional nonholonomic system known as the *Brockett integrator* and to derive the (energy) optimal trajectories between two given points. Our method uses symmetry reduction and an analysis of the quotient space associated with the action of a (symmetry) group on $\mathbb{R}^3$. By lifting the Riemannian geodesics with respect to an appropriate metric from the quotient space back to the original space $\mathbb{R}^3$, we derive the optimal trajectories of the original problem.

1 Introduction

This paper explores a different method to derive the optimal trajectories for the first order controllable system proposed by R. Brockett in [4], which is often referred to as the *Brockett integrator*. The system has the state in $\mathbb{R}^3$ and evolves as

$$\frac{d}{dt}\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -y \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} u_2, \tag{1}$$

where $\vec{x} = (x, y, z)^T$ and u_1, u_2 are the control functions. It is a (local) canonical form for systems with non-holonomic constraints and it models a variety of robotic and mobile systems. We refer to Chapters 7,8 of the book [17] for applications and generalizations of this system. Several issues concerning this model have been explored in the literature. These include stabilization to the origin (see, e.g., [5] and [21]) and optimal control (see, e.g., [7], [17], [22]), which is our focus here. In particular, we are interested in the *optimal steering problem* with minimum energy, that is, in finding the control to steer the state $\vec{x}$ from $\vec{x}(0) = (0, 0, 0)^T$ to $\vec{x}(1) = X_f$ for some arbitrary $\vec{x}_f$ in $\mathbb{R}^3$ while minimizing the energy-like cost

$$J := \int_0^1 (u_1^2 + u_2^2) dt. \tag{2}$$

We emphasize that the solution to this problem is known and our goal here is to give a derivation from a different perspective. In fact, the optimal control problem was investigated in [4]. Utilizing the Euler-Lagrange Equations, which give necessary conditions of optimality, R. Brockett obtained a cost minimizing control in the form

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\lambda t) & -\sin(\lambda t) \\ \sin(\lambda t) & \cos(\lambda t) \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix},$$

*Department of Mathematics, Iowa State University, daless@iastate.edu
†Corresponding author, Department of Mathematics, Iowa State University, zhifeiz@iastate.edu

where $\lambda \in \mathbb{R}$. He also demostrated that the optimal path from $\vec{x}(0) = (0,0,0)$ to $\vec{x}(1) = (0,0,a)$ is obtained when $\lambda = -2\pi$ and $u_1(0)^2 + u_2(0)^2 = \frac{a}{2\pi}$. In [7], D. D'Alessandro and A. Ferrante obtained this result for a more general model than (1) using *sufficient* conditions of optimality. In [22], S. Sinha proved the existence of the optimal solution and obtained a general algorithm for the optimal trajectories. The sinusoidal form of the optimal control laws inspired sub-optimal steering strategies for more general systems with non-holonomic constraints [18].

It is well known (see, e.g., [1], [16]) that, for systems with non-holonomic constraints such as (1), there is a relation between the optimal steering trajectories with minimal energy, the minimum time problem with bounded norm for the control and the problem of finding sub-Riemannian geodesics for a given sub-Riemannian metric. We shall briefly review these facts in section 2 and derive the appropriate sub-Riemannian metric for our system. Therefore, the problem is transformed into a sub-Riemannian problem. We then observe that there exists a group (G) action on the state space (R^3) which transforms optimal trajectories into optimal trajectories. Therefore, once one found an optimal trajectory, one has in fact found a *family* of optimal trajectories where one can be transformed into the other by an element of the symmetry group G. This suggests considering the problem in the *quotient space* R^3/G. We describe the group action and the structure of the quotient space in section 3. Care must be taken because the quotient space R^3/G does not have the structure of a manifold but it is only a *stratified space* [3] [20]. Nevertheless, we can put a Riemannian structure on a dense and open subset of the quotient R^3/G, called the *regular part*, so that the sub-Riemannian length of an admissible (horizontal) curve in R^3 coincides with the Riemannian length of its projection onto the regular part in R^3/G. We do this in section 4. The idea is then to calculate the sub-Riemannian geodesics, that is, the optimal trajectories, by using Riemannian geometry on the quotient space and then 'lifting back' the trajectories on R^3. So our approach, can be summarized in the diagram 1.

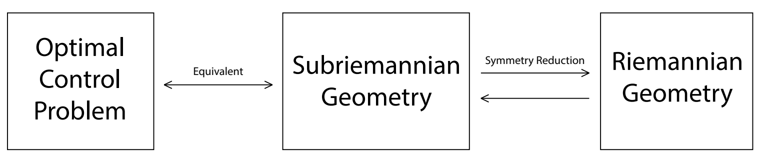

Figure 1: Schematic description of the approach to compute optimal trajectories for the optimal steering problem of the Brockett's integrator

In our case, we will see that the Riemannian geodesics can be calculated explicitly since the geodesic equations can be integrated. This will be shown in section 5. In section 6, we illustrate how to obtain the sub-Riemannian geodesics by the lifting back process. In section 7, we give some concluding remarks on the method presented and its potential application to more general models.

2 Minimum energy control and sub-Riemannian geometry

2.1 Sub-Riemannian geometry

We review some basic facts on sub-Riemannian geometry and in the next subsections we will make the connection with the optimal control problem of interest here. More datails can be found in standard textbooks such as [1], [16].

A **sub-Riemannian structure** on a manifold M, (M, Δ, g), is defined by a sub-bundle Δ of the tangent bundle TM with the canonical projection $\pi_\Delta : \Delta \to M$. For each $p \in M$, the fibre $\Delta_p = \pi_\Delta^{-1}(p)$ is a subspace of the tangent space T_pM, which is often assumed to be of constant dimension, i.e., $\dim(\Delta_p)$ independent of $p \in M$. The sub-Riemannian structure is usually obtained by restricting a Riemannian metric g defined on T_pM to Δ_p. This gives a non-degenerate inner product on Δ_p for each p, $g_p(\cdot, \cdot)$.

A curve $\gamma : [0, T] \to M$ is said to be **horizontal** if $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for almost every $t \in [0, T]$. Some regularity assumptions are made on horizontal curves. In particular, they are assumed to be Lipschitz continuous and therefore differentiable almost everywhere with $\dot{\gamma}$ essentially bounded. This means that there exists a smooth map $h : [0, T] \to TM$, with $h(t) \in T_{\gamma(t)}M$ and uniformly bounded for every t (in the given Riemannian metric) and such that $\dot{\gamma} = h(t)$ almost everywhere. Moreover $\dot{\gamma}$ is assumed to be non-zero almost everywhere. A horizontal curve is said to be *parametrized by constant speed* if $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))$ is constant almost everywhere. It is said to be *parametrized by arclength* if $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = 1$ almost everywhere.

The length of a horizontal curve γ is defined as

$$\text{length}(\gamma) := \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt. \tag{3}$$

It is important to notice that the length of a horizontal curve does not change after *reparametrization*. A reparametrization is a Lipschitz continuous, monotone and surjective map $\phi : [0, T^{'}] \to [0, T]$, (set $t = \phi(s)$) and a reparametrization of a curve γ is the curve $\gamma_\phi : [0, T^{'}] \to M$, defined as

$$\gamma_\phi := \gamma \circ \phi.$$

The curve γ_ϕ is horizontal if γ is horizontal since the tangent vector $\frac{d\gamma_\phi}{ds}$ is, by the chain rule, proportional to $\frac{d\gamma}{dt}$. The length of γ_ϕ can be computed by the following:

$$\text{length}(\gamma_\phi) = \int_0^{T^{'}} \sqrt{g_{\gamma_\phi}(s)\left(\frac{d\gamma_\phi}{ds}, \frac{d\gamma_\phi}{ds}\right)}ds = \int_0^{T^{'}} \sqrt{g_{\gamma_\phi(s)}\left(\frac{d\gamma_\phi}{dt}, \frac{d\gamma_\phi}{dt}\right)}\left|\frac{d\phi}{ds}\right| ds$$
$$= \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)}dt = \text{length}(\gamma).$$

Furthermore, let $L := \text{length}(\gamma)$ and assume that we use as the reparametrization $t = \phi(s)$, where $\phi : [0, \alpha L] \to [0, T]$, is the inverse of

$$s = \alpha \int_0^t \sqrt{g_{\gamma(\tau)}(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}d\tau.$$

We have, by the chain rule,

$$g_{\gamma_\phi(s)}\left(\frac{d\gamma_\phi}{ds}, \frac{d\gamma_\phi}{ds}\right) = g_{\gamma_{\phi(s)}}\left(\frac{d\gamma}{dt}(\phi(s)), \frac{d\gamma}{dt}(\phi(s))\right)\left(\frac{d\phi}{ds}\right)^2 = \frac{1}{\alpha^2}.$$

Therefore, the reparametrization γ_ϕ is parametrized by constant speed $\frac{1}{\alpha}$. We have (cf. also [1] Lemma 3.16).

Lemma 1. *For any horizontal curve γ joining two points in M, there exists a horizontal curve reparametrization of γ which has the same length and is parametrized by constant speed.*

The *sub-Riemannian distance* between two points p and q in M is defined as the infimum among the lengths of the horizontal curves joining p and q, that is,

$$\text{dist}(p, q) = \inf_\gamma \text{length}(\gamma). \tag{4}$$

A horizontal curve which realizes such an infimum is called a **sub-Riemannian geodesic**. From the Lemma 1 we know that, in looking for sub-Riemmanian geodesics, we can restrict ourselves to curves parametrized by constant speed.

Often, especially in the context of control theory, sub-Riemannian structures are specified by giving a set of vector fields $\{X_1, ..., X_m\}$ which are, at every point p, a basis for the subspace $\Delta_p \subseteq T_pM$. A common assumption that such a set of vector fields is *bracket generating*, that is, the Lie algebra of vector fields generated by $\{X_1, ..., X_m\}$ is such that, at every point p, it spans the whole T_pM. Under such an assumption the **Chow-Raschevskii theorem** (see, e.g., [1], [16]) says that, for any two points in M, (assumed connected) there exists a minimizing sub-Riemannian geodesic, that is, the inf in (4) is, in fact, attained.

2.2 Sub-Riemannian geometry and minimum energy control

Assume now that the vector fields $\{X_1, ..., X_m\}$ describing the sub-Riemannian structure are *orthonormal* with respect to the sub-Riemannian metric, that is, at every point $p \in M$, $g_p(X_j(p), X_k(p)) = \delta_{j,k}$. Any horizontal curve γ with $\gamma(0) = p$ satisfies

$$\dot{\gamma} = \sum_{j=1}^{m} u_j(t) X_j(\gamma(t)), \qquad \gamma(0) = p, \tag{5}$$

for certain essentially bounded 'control functions' $u_j = u_j(t)$, $j = 1, ..., m$, and viceversa, given a set of control functions $u_j = u_j(t)$, $j = 1, ..., m$, equation (5) determines a horizontal curve γ starting at p. The control functions u_j are related to the curve γ by

$$u_j(t) = g_{\gamma(t)} \left(X_j(\gamma(t)), \dot{\gamma}(t) \right),$$

and the energy (cost) integral $\int_0^T \|u(t)\|^2 dt$ can be expressed in terms of the curve γ as

$$\int_0^T \|u(t)\|^2 dt = \int_0^T g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt := J(\gamma). \tag{6}$$

Therefore, the steering problem with minimum energy from a point p to a point q in M, for system (5), is equivalent to finding the horizontal curves connecting p and q which minimize $J(\gamma)$ in (6).

The following fact (see, e.g., [1] Lemma 3.64) gives the relation between the minimum energy problem and the minimum sub-Riemannian length problem, that is, between the problem of minimizing length(γ) in (3) and $J = J(\gamma)$ in (6).

Lemma 2. *A horizontal curve γ connecting p and q in M minimizes $J = J(\gamma)$ if and only if it minimizes $L :=$ length(γ) and is parametrized by constant speed.*

Proof. From the Cauchy-Schwartz inequality

$$\left(\int_0^T f(t) b(t) dt \right)^2 \leq \left(\int_0^T f(t)^2 dt \right) \left(\int_0^T b(t)^2 dt \right)$$

applied to the functions $f = \sqrt{g_\gamma(\dot{\gamma}, \dot{\gamma})}$, $b \equiv 1$, we obtain

$$(L(\gamma))^2 \leq T J(\gamma), \tag{7}$$

with equality if and only if $g_\gamma(\dot{\gamma}, \dot{\gamma})$ is proportional to 1, that is, if γ is parametrized by constant speed.

Now assume that γ minimizes L and is parametrized by constant speed. We have, using (7) with equality $(L(\gamma))^2 = TJ(\gamma)$. If there was another horizontal curve γ_1 with $J(\gamma_1) < J(\gamma)$, we would have, again using (7),

$$(L(\gamma_1))^2 \leq TJ(\gamma_1) < TJ(\gamma) = (L(\gamma))^2,$$

which contradicts the minimality of γ (for L).

Viceversa, assume γ minimizes J but it does not minimize L. Then there exists a γ_1 with $L(\gamma_1) =$ length$(\gamma_1) < L(\gamma) =$ length(γ). According to Lemma 1, we can take γ_1 parametrized by constant speed so that

$$TJ(\gamma_1) = (L(\gamma_1))^2 < (L(\gamma))^2 \leq TJ(\gamma),$$

which contradicts the minimality of γ for J. Therefore γ must minimize L as well. For the γ's that minimize L (7) always holds with the equality only if γ is parametrized by constant speed. Therefore, the minimizing γ for J must also be parametrized by constant speed. □

In view of the above lemma we can solve a minimum energy problem by finding a sub-Riemannian geodesic for an appropriate metric which is such that the vector fields determining the sub-Riemannian structure are orthonormal. If necessary, we can then reparametrize the geodesic so that it is parametrized by constant speed.[1]

Our focus in this paper is on the *steering with minimum energy problem.* However there is also a connection of sub-Riemannian geometry with the *minimum time problem with bounded control* which, for completeness, we report here. A proof can be found for example in [2] Theorem 1.

Proposition 3. *For system (5) the following two facts are equivalent:*

1. $\gamma : [0,T] \to M$ *is a minimizing sub-Riemannian geodesic joining* p *and* q *parametrized by constant speed* L, *i.e.,* $g_\gamma(\dot\gamma, \dot\gamma) = L^2$, *a.e.*
2. $\gamma : [0,T] \to M$ *is a minimum time trajectory for system (5) subject to* $\gamma(0) = p$, $\gamma(T) = q$ *and the constraint* $\|u\| \leq L$, *a.e..*

2.3 Application to the Brockett integrator

We now turn to the application of the above described concepts to the system (1) which is the focus of this paper. For the system (1), the underlying manifold M is $\mathbb{R}^3$ and the sub-Riemannian structure is determined by the vector fields (cf. (1))

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}. \tag{8}$$

We consider as the (sub-)Riemannian metric on $\mathbb{R}^3$ in the Cartesian coordinates (x, y, z) any metric determined by the matrix

$$\tilde g = \begin{pmatrix} 1 - \frac{y^2}{k} & \frac{xy}{k} & 0 \\ \frac{xy}{k} & 1 - \frac{x^2}{k} & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix}, \tag{9}$$

where $k = k(x, y)$ is any function such that $k(x, y) > x^2+y^2$. In fact, with this condition, application of Sylvester criterion shows that the matrix $\tilde g$ in (9) is positive definite, at every point in $\mathbb{R}^3$. Furthermore we have $g(X_j, X_k) = \delta_{j,k}$, $j, k = 1, 2$. For example

$$\begin{pmatrix} 1 & 0 & -y \end{pmatrix} \begin{pmatrix} 1 - \frac{y^2}{k} & \frac{xy}{k} & 0 \\ \frac{xy}{k} & 1 - \frac{x^2}{k} & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} = 0.$$

In the following calculations, we shall use $k = x^2 + y^2 + 1$.

3 Symmetries for the Brockett Integrator

3.1 Symmetries in sub-Riemannian problems

Symmetry reduction has a long and successful history in control theory (see, e.g., [11], [12], [13], [15], [19]). The idea is to use symmetries to reduce the complexity of the optimal control problem. In the context of sub-Riemannian geometry which interests us in this paper, for a sub-Riemannian manifold (M, Δ, g), and a problem with initial condition $p \in M$, we define a **symmetry group** G as a Lie transformation group acting on M by Φ_h, $h \in G$, which satisfies the following conditions

[1] We shall see in the proof of Lemma 11 below that the geodesics we find in our case are already parametrized by constant length.

C1

$$\Phi_h(p) = p, \qquad \forall h \in G,$$

C2 For every $h \in G$ and at every $q \in M$

$$\Phi_{h*}\Delta_q \subseteq \Delta_{\Phi_h(q)}$$

C3 For every $h \in G$, Φ_h is an isometry on Δ. This means that, for every $q \in M$ and for every two tangent vectors V_1 and V_2, in Δ_q we have

$$g_{\Phi_h(q)}(\Phi_{h*}V_1, \Phi_{h*}V_2) = g_q(V_1, V_2). \tag{10}$$

From the previous conditions, if γ is a horizontal curve connecting p to q in M, for every $h \in G$, $\Phi_h \circ \gamma$ is a horizontal curve connecting p to $\Phi_h(q)$. Furthermore, because of the **C3** condition above, the length of γ is the same as the length of $\Phi_h \circ \gamma$. In particular, if γ is a sub-Riemannian geodesic from p to q, $\Phi_h \circ \gamma$ is a sub-Riemannian geodesic from p to $\Phi_h(q)$. It is therefore natural to consider the length minimizing problem not on the manifold M but on on the quotient space M/G. In the following we denote by π the natural projection $\pi : M \to M/G$ which maps $q \in M$ to the equivalence class, i.e. the *orbit*, of elements q_1 such that $q_1 = \Phi_h(q)$ for some $h \in G$.

Standard results of Lie group transformation theory (see, e.g., [3]) show that M/G has the structure of a stratified space. The stratification is obtained by 'orbit type' as follows. If H is a subgroup of G which is the isotropy group of certain point $q \in M$, one considers the *isotropy type* (H), which is the equivalence class of subgroups that are conjugate by an element of G to H. These groups are themselves isotropy groups of elements in M. In particular H is the isotropy group of q, that is, $\Phi_h(q) = q \leftrightarrow h \in H$ if and only if xHx^{-1} is the isotropy group of $\Phi_x(q)$.

We notice that because of assumption **C1**, the isotropy type of the initial point p is (G), that is, the whole group G. A *partial ordering* can be defined on the set of isotropy types by saying that $(H_1) \leq (H_2)$, if the class (H_2) contains a group which has H_1 as one of its subgroups. With this definition (under assumption **C1**), (G) is a *maximal* isotropy type, as it is $\geq$ than every isotropy type. A standard result in Lie group transformation theory says that there also exists a *minimum* isotropy type, (H_{min}), that is, a type (K) such that $(K) \leq (H)$ for every type (H).

Let $M_{(H)}$ be the set in M of points whose isotropy group is in (H). Clearly, $(H_1) \leq (H_2)$ implies $M_{(H_2)} \subseteq M_{(H_1)}$. If two points q_1 and q_2 belong to the same orbit, then their isotropy groups belong to the same isotropy type[2] and therefore they belong to the same $M_{(H)}$. It makes sense therefore to consider the sets $M_{(H)}/G$. These are called the *orbit types* in M/G and form a stratification of M/G [3]. Of particular interest is the orbit type associated with the minimal isotropy type (H_{min}), $M_{(H_{min})}/G$. This can be proven to be a smooth manifold which is *connected, open and dense* in M/G. It is called the **regular part** of M/G. Its preimage, under the natural projection, $M_{(H_{min})}$, is also called the regular part in M. The set $M/G - M_{(H_{min})}/G$ is called the **singular part**.

In the following we shall follow the strategy that was advocated in [2], [8]. For the minimal sub-Riemannian length problem with symmetry, we define a Riemannian metric on M/G so that the length of the projection of any horizontal curve on M coincides with the sub-Riemannian length in M. However since M/G is not a manifold (and in fact because of property **C1** the initial point p is in the singular part of the manifold M) we will only (in the next section) put a Riemannian metric on the regular part $M_{(H_{min})}/G$ (which is however open and dense in M/G) and from this infer properties of and calculate the sub-Riemannian geodesics in M.

3.2 Application to the Brockett integrator

We now see how the above picture applies to the Brockett integrator system for which we assume that the initial point is a point on the z-axis in $\mathbb{R}^3$, that is, $p = (0, 0, a)^T, a \in \mathbb{R}$. We take as symmetry

[2]Assume $Hq_1 = q_1$ and $x^{-1}q_2 = q_1$. This gives $Hx^{-1}q_2 = Hq_1 = q_1$, and therefore $xHx^{-1} = xq_1 = q_2$. This implies that the isotropy group of q_2 contains xHx^{-1}. Exchanging the role of q_1 and q_2, we show that the isotropy group of q_2 is a subset of xHx^{-1}.

group on the underlying manifold $M = \mathbb{R}^3$, the Lie group $SO(2)$ realized by 3×3 matrices of the form

$$h := \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \theta \in \mathbb{R}, \tag{11}$$

with the action given by standard matrix-vector multiplication. With this choice, condition **C1** is verified. Furthermore, for a given $h \in G$ given by (11), we have using (8)

$$\Phi_{h*} X_1(q) = \cos(\theta) \frac{\partial}{\partial x} - \sin(\theta) \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \tag{12}$$

$$\Phi_{h*} X_2(q) = \sin(\theta) \frac{\partial}{\partial x} + \cos(\theta) \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}. \tag{13}$$

A direct verification shows that

$$\cos(\theta)\Phi_{h*}X_1(q) + \sin(\theta)\Phi_{h*}X_2(q) = X_1(\Phi_h(q)),$$

$$-\sin(\theta)\Phi_{h*}X_1(q) + \cos(\theta)\Phi_{h*}X_2(q) = X_2(\Phi_h(q)).$$

This shows property **C2** above and also property **C3** because according to the chosen metric X_1 and X_2 are orthonormal everywhere and therefore so are $\Phi_{h*}X_1$ and $\Phi_{h*}X_2$. Alternatively, we see for property **C3** that multiplication by an orthogonal matrix does not change the norm of the control in (6).

4 Riemannian metric on the quotient space

With the above defined action of $G \cong SO(2)$ on $M := \mathbb{R}^3$, there are only two isotropy types (H_{min}) given by the trivial group $H_{min} = \{\mathbf{1}\}$ containing only the identity and (G) which contains the whole group. The group H_{min} is the isotropy group of every point in $\mathbb{R}^3$ except for the points on the z-axis. This is $M_{(H_{min})}$, the regular part of M. An orbit in M is parametrized by two coordinates, $r := x^2 + y^2$ and the coordinate z. Therefore we can write the natural projection as $\pi : M \to M/G; (x, y, z) \mapsto (x^2 + y^2, z) := (r, z)$, where the orbit space M/G is realized by the half-plane $\{(r, z) | r \geq 0\}$. We denote the regular part of the quotient space as $\hat{M} := M_{(H_{min})}/G$. This can be taken as the open half-plane $\hat{M} = \{(r, z) | r > 0\}$.

We restrict the natural projection $\pi : M \to M/G$ to the regular part $M_{(H_{min})}$, i.e., $\pi : M_{(H_{min})} \to \hat{M}$. We shall define a Riemannian metric on $\hat{M}$ such that for any $q \in M_{(H_{min})}$,

$$\pi_*|_{\Delta_q} : \Delta_q \subset T_q\mathbb{R}^3 \to T_{\pi(q)}\hat{M}$$

is a linear isometry between Δ_q and $T_{\pi(q)}\hat{M}$ (in particular an isomorphism), where $\pi_*|_{\Delta_q}$ is the restriction of π_* to Δ_q.

Theorem 4. *The matrix*

$$\hat{g} = \begin{pmatrix} \frac{1}{4r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \tag{14}$$

defines a Riemannian metric g_Q on $\hat{M}$ in the coordinate (r, z), such that $\pi_*|_{\Delta_q} : \Delta_q \to T_{\pi(q)}\hat{M}$ is a linear isometry.

The metric (14) is similar to the metric of the Poincare' half plane model (see, e.g., [10] Example 3.10). We shall see that the behavior of the geodesics will also be similar.

Proof. For two tangent vectors in Δ_q with $q = (x, y, z)$, written as $Y_1 := aX_1 + bX_2$ and $Y_2 = cX_1 + dX_2$ with X_1 and X_2 defined in (8), using the metric g defined in (9) and the fact that, with respect to this metric, X_1 and X_2 are an orthonormal set, we have

$$g(aX_1 + bX_2, cX_1 + dX_2) = ac + bd. \tag{15}$$

By writing

$$aX_1 + bX_2 := a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + (bx - ay)\frac{\partial}{\partial z},$$

a direct calculation gives

$$\pi_*(aX_1 + bX_2) = 2(ax + by)\frac{\partial}{\partial r} + (bx - ay)\frac{\partial}{\partial z}. \tag{16}$$

Notice that the Kernel of π_* is zero since $x^2 + y^2 > 0$ on the regular part of $M = \mathbb{R}^3$. Using (16), a direct calculation with (14) gives

$$g_Q\left(\pi_*(aX_1 + bX_2), \pi_*(cX_1 + dX_2)\right) = ac + bd,$$

which coincides with the value found in (15). □

Theorem 4 implies that any horizontal curve γ is such that its projection $\pi(\gamma)$ onto $\hat{M}$ has the same length, with the given metric defined by (14). Viceversa, for any curve Γ in $\hat{M}$, its **lift** in $M_{(H_{min})}$ has the same length as Γ. The lift of a curve $\Gamma : [0, T] \to \hat{M}$ is any curve $\gamma : [0, T] \to M_{(H_{min})}$ such that $\pi(\gamma) = \Gamma$. If Γ is differentiable, denoting by $\dot{\Gamma} = \dot{\Gamma}(t)$ its tangent vector at time t, the lift through the point $\gamma(0) = \gamma_0$ satisfies the differential **lifting equation**

$$\dot{\gamma} = \pi_*^{-1}|_{\gamma(t)}\dot{\Gamma}, \qquad \gamma(0) = \gamma_0, \tag{17}$$

where, with some abuse of notation, what we have denoted by $\pi_*|_\gamma$ is, in fact, the restriction of $\pi_*|_\gamma$ to Δ_γ which according to theorem 4 is an isomorphism. By existence and uniqueness of solutions of initial value problems, a lift exists at least locally. The situation is similar to what described for *Riemannian submersions* (see, e.g., [10] pg. 185) but, in our case, the horizontal lift is not the same as the horizontal lift in that case because the orbits are in general not perpendicular to the distribution Δ_p. For Riemannian submersions the horizontal lift of a curve is not always defined globally, that is, on the entire interval [0,T] (cf., e.g., Proposition 4.28 in [14]). In our case, and for our specific example of the Brockett integrator, we can however say that every C^1 curve in $\hat{M}$ has a lift. To this aim, from (16), we obtain the specific expression for π_* in the basis $\{X_1, X_2\}$ in $T_pM_{(H_{min})}$ and $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\}$ in $T_{\pi(p)}\hat{M}$, which is given by

$$\pi_*|_{(x,y,z)} = \begin{pmatrix} 2x & 2y \\ -y & x \end{pmatrix}, \qquad \pi_*^{-1}|_{(x,y,z)} = \frac{1}{2(x^2 + y^2)}\begin{pmatrix} x & -2y \\ y & 2x \end{pmatrix} = \frac{1}{2r}\begin{pmatrix} x & -2y \\ y & 2x \end{pmatrix}.$$

This gives for the lifting equation (17), beside the obvious $\dot{z}$ equal to the second component of $\dot{\Gamma}$,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2r}\begin{pmatrix} \dot{r} & -2\dot{z} \\ 2\dot{z} & \dot{r} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{18}$$

which, once r in $[0, T]$ is bounded away from zero and $\dot{r}$ and $\dot{z}$ are continuous, is a linear equation. From the global existence of solutions of linear differential equations, we have:

Proposition 5. *Every C^1 curve in $\hat{M}$ has a lift satisfying (17), through any given point in $M_{(H_{min})}$.*

With this proposition, we can establish the correspondence between the sub-Riemannian geodesics in $(M, \Delta, g) = (\mathbb{R}^3, \Delta, g)$ and the Riemannian geodesics in $(\hat{M}, g_Q)$.

Theorem 6. *In the following statements 1 implies 2.*

1. *The curve* $\gamma : [0,T] \to \mathbb{R}^3$ *is a sub-Riemannian geodesic in* $\mathbb{R}^3$ *with initial condition* $p = (0,0,z_0)^T$ *and final condition* q *with* $\gamma(t) \in M_{H_{min}}$ *for each* $t \in (0,T)$.

2. $\pi \circ \gamma = \pi \circ \gamma(t)$ *is a Riemannian length minimizing geodesic in* $(\hat{M}, g_Q)$ *for* $t \in (0,T)$, *and* $\lim_{t\to 0^+} \pi \circ \gamma(t) = \pi\left((0,0,z_0)^T\right)$, $\lim_{t\to T^-} \pi \circ \gamma(t) = \pi(q)$.

Proof. Assume that there exist t_s and t_f with $0 < t_s < t_f < T$ such that $\pi \circ \gamma$ is *not* the minimizing Riemannian geodesic connecting $q_s := \pi(\gamma((t_s)))$ and $q_f := \pi(\gamma(t_f))$. Then there exists another Riemannian minimizing geodesic connecting q_s and q_f, call it Γ. Since Γ is a geodesic, it is C^∞ and therefore we can apply proposition 5 and it has a horizontal lift. Call it $\gamma_1 : [t_s, t_f]$ with $\gamma_1(t_s) = \gamma(t_s)$. The horizontal continuous curve equal to γ in $[0, t_s]$, and γ_1 in $[t_s, t_f]$, call it $\tilde{\gamma}$ is such that $\pi(\tilde{\gamma}(t_f)) = \pi(\gamma(t_f))$ has length less than γ and this contradicts the optimality of γ between $\gamma(0)$ and $\gamma(t_f)$. By continuity we also get the two limits at the endpoints. □

Therefore, according to Theorem 6, assuming that a sub-Riemannian geodesic γ, $[0,T] \to \mathbb{R}^3$, is such that $\gamma(t) \in M_{(H_{min})}$ for every $t \in (0,T)$, such a geodesic has to be found as the *lift* of a Riemannian geodesic in $\hat{M}$. In our case, the singular part of the state space $\mathbb{R}^3$ is the z-axis. If a sub-Riemannian geodesic γ is in the singular part for an interval of non-zero length $[t_1, t_2]$, this means $x(t) \equiv y(t) \equiv 0$, for $t \in [t_1, t_2]$ which implies $\dot{x}(t) \equiv \dot{y}(t) \equiv 0$ in the same interval and therefore $u_1(t) = u_2(t) \equiv 0$. According to (1), we have z is constant. These are curves that stop in a given point for a non-zero amount of time which we have excluded. Therefore the sub-Riemannian geodesics only intersect the singular part of the space in isolated points and in between these points we can obtain them as prescribed by the theorem as the lift of Riemannian geodesics in $\hat{M}$. We will carry out this program in the section 6 after having computed the Riemannian geodesics in section 5.

5 Geodesics in the Quotient Space

In this section, we compute the Riemannian geodesics in $(\hat{M}, g_Q)$ by solving the geodesic equations. It turns out that these equations can be explicitly integrated in this case.

By computing the Christoffel symbols in the (r,z) coordinates with the metric g_Q given in (14) (cf, e.g., formula (10) Chapter 2 in [10]) we obtain that they are all zero except

$$\Gamma^1_{11} = -\frac{1}{2r}, \quad \Gamma^1_{22} = \frac{2}{r}, \quad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{2r}.$$

Plugging these in the general geodesic equations (cf, e.g., formula (1) in Chapter 3 of [10]), we obtain the equations.

$$\frac{4(z')^2 - (r')^2}{2r} + r'' = 0 \quad \text{and} \quad z'' - \frac{r'z'}{r} = 0. \tag{19}$$

Theorem 7. *All geodesics* $\gamma_Q(t) = (r(t), z(t))$ *of* $(\hat{M}, g_Q)$, *with* $t \in (0,T)$, *and such that* $\lim_{t\to 0^+} r(t) = 0$ *and* z' *is not identically zero, can be written as*

$$r = a\sin^2(ct) \qquad z = \frac{a}{4}\left(2ct - \sin(2ct)\right) + b, \tag{20}$$

for some some parameters a, b, c *in* $\mathbb{R}$.

The form of the geodesics in the (r,z) plane starting from the origin is described in figure 2. Notice the symmetry about the r-axis. The geodesics starting from a different point on the z-axis have the same form but vertically shifted.

Proof. Integrating the second equation in (19) gives

$$z' = c_1 r,$$

for some $c_1 \in \mathbb{R}$. If z' is not identically zero $c_1 \neq 0$. Let $u = \frac{r'}{r}$. Substituting into the first equation and dividing by r gives

$$0 = \frac{r''}{r} + \frac{4(c_1 r)^2 - (r')^2}{2r^2} = \frac{2r''r - 2(r')^2}{2r^2} + \frac{(r')^2}{2r^2} + 2c_1^2 = u' + \frac{u^2}{2} + 2c_1^2,$$

which has solutions

$$u = 2c_1 \tan(c_2 - c_1 t),$$

for some $c_2 \in \mathbb{R}$. Integrating gives

$$r(t) = c_3 \cos^2(c_2 - c_1 t), \tag{21}$$

for some $c_3 \in \mathbb{R}$. Finally, integrating the equation $z' = c_1 r$, we obtain that

$$z(t) = -\frac{c_3}{4}\Big(2(c_2 - c_1 t) + \sin(2(c_2 - c_1 t))\Big) + c_4, \tag{22}$$

for parameters c_1, c_2, c_3, c_4.

Now, we assume that the geodesic is taken in the interval $(0, T)$. Imposing in (21) that $\lim_{t\to 0^+} r(t) = 0$, we obtain that $c_2 = \frac{(2k+1)\pi}{2}$. Using this into (21), we obtain

$$r(t) = c_3 \sin^2(c_1 t). \tag{23}$$

Plugging $c_2 = \frac{(2k+1)\pi}{2}$ into (22), we obtain

$$z(t) = c_4 - \frac{c_3(2k+1)\pi)}{4} + \frac{c_3 c_1}{2} t - \frac{c_3}{4}\sin(2c_1 t). \tag{24}$$

If we set $a = c_3$, $c = c_1$ and $b = c_4 - \frac{c_3(2k+1)\pi}{4}$, we obtain formulas (20).

□

For the case $z' \equiv 0$ we have the following

Theorem 8. *A class of geodesics $\gamma_Q(t) = (r(t), z(t))$ of $(\hat{M}, g_Q)$, with $t \in (0, T)$ and such that $\lim_{t\to 0^+} r(t) = \lim_{t\to 0^+} r^{'}(t) = 0$, and $z^{'} \equiv 0$ are*

$$r = Nt^2 \qquad z \equiv b, \tag{25}$$

for some some parameters $N > 0, b$ in $\mathbb{R}$.

Proof. Simply verify that for every N and b the functions (25) satisfy (19). □

6 Derivation of the Optimal Trajectories

We now lift the trajectories (20) in the quotient space to horizontal curves in $M = \mathbb{R}^3$.

In an interval $(0, T)$, using (20), we calculate the lifting equations (18) for x and y, while z is simply given by z in (20). After the application of elementary trigonometric identities, the equations (18) become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = c \begin{pmatrix} \frac{\cos(ct)}{\sin(ct)} & -1 \\ 1 & \frac{\cos(ct)}{\sin(ct)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

This equation can be explicitly integrated in an interval $[t_1, t_2] \subseteq (0, T)$ by first defining

$$\vec{y} := e^{-\int_{t_1}^{t} c\frac{\cos(c\tau)}{\sin(c\tau)} d\tau} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

which gives a linear system of differential equations with constant coefficients for $\vec{y}$. This method of solution gives for x and y the following form

$$\begin{aligned} x(t) &= N \sin(2ct + \phi) + k_x \\ y(t) &= -N \cos(2ct + \phi) + k_y, \end{aligned} \tag{26}$$

for parameters N, ϕ, k_x, k_y, c, which depend on the initial condition at t_1. Such parameters also have to satisfy

$$N \sin(\phi) + k_x = 0, \qquad -N \cos(\phi) + k_y = 0, \tag{27}$$

since $\lim_{t\to 0^+} x(t) = \lim_{t\to 0^+} y(t) = 0$. Therefore an optimal trajectory between two points of the z axis has the form

$$\begin{aligned} x(t) &= N \sin(2ct + \phi) - N \sin(\phi) \\ y(t) &= -N \cos(2ct + \phi) + N \cos(\phi) \end{aligned} \tag{28}$$

Remark 9. *The form of the optimal trajectories we found coincides with the sinusoidal trajectories in [4], [7], [17], where the optimal controls were found to be of the form*

$$u_1 = M \cos(\omega t + \psi), \qquad u_2 = M \sin(\omega t + \psi), \tag{29}$$

for parameters ω, M, and ψ, and x and y were obtained by integrating u_1 and u_2 above. Contrary to these references however, we have not used necessary and-or sufficient conditions from optimal control or calculus of variations but a symmetry reduction argument and Riemannian geometry.

Remark 10. *The above treatment is the lifting of the geodesics (20) described in Theorem 7. For the geodesics (25) we obtain the lifts $x(t) = at$, $y(t) = bt$, $z(t) \equiv z_0$, for parameters a and b and z_0, by using the lifting equations (18) with (25). This again coincides with some of the optimal trajectories obtained in [4], [7], [17], which are determined by the controls in (29) setting $\omega = 0$.*

In our treatment, the expression of the optimal trajectories (28) assumes that the entire optimal trajectories except (possibly) at the endpoints is in the regular part ($r > 0$). In general, under the assumption that the optimal geodesics are analytic one can prove that geodesics cease to be optimal when they touch the singular part of the space. This was shown in [2] (Corollary 3.6). The information about the analyticity of the sub-Riemannian geodesics can be obtained from application of the Pontryagin Maximum Principle.[3] If we want to avoid using such tools (in the spirit of this paper) we can verify this directly for the system of interest here. We do this in detail for the case of Theorem 7 (where the final z coordinate is different from the initial one) which is the more involved one. We state two lemmas and then conclude in Theorem 13.

Lemma 11. *The minimizing sub-Riemmanian geodesics from a point $(0, 0, z_0)$ to a point $(0, 0, z_1)$, $\gamma = \gamma(t)$, for $t \in [0, 1]$ is such that $\gamma(t) \in M_{(H_{min})}$ for every $t \in (0, 1)$.*

Proof. Using the explicit expressions for the geodesics in the quotient space (20), we can calculate explicitly the length of one 'branch' connecting a point $(r, z) = (0, \hat{z}_0)$ to a point $(r, z) = (0, \hat{z}_1)$ in $\hat{M}$. In particular with $\dot{\gamma}_Q = \dot{r}\frac{\partial}{\partial r} + \dot{z}\frac{\partial}{\partial z}$ we obtain with (14)

$$g_Q(\dot{\gamma}_Q, \dot{\gamma}_Q) = \frac{1}{4r}\dot{r}^2 + \frac{1}{r}\dot{z}^2 = \frac{1}{4r}a^2c^2\sin^2(2ct) + \frac{1}{r}a^2c^2\sin^4(ct) =$$

[3]In general determining smoothness properties of sub-Riemannian geodesics is one of the most important open problems in sub-Riemannian geometry (see, e.g., [24]).

$$= \frac{4a^2c^2\sin^2(ct)\cos^2(ct)}{4a\sin^2(ct)} + \frac{a^2c^2\sin^4(ct)}{a\sin^2(ct)} = ac^2,$$

which shows that the geodesics are parametrized by constant length. The length of the curve is $\int_0^T \sqrt{g_Q(\dot{\gamma}_Q(\tau), \dot{\gamma}_Q(\tau)}d\tau = \sqrt{a}|c|T$. Now setting $r = 0$ means from (20) that $|c|T = \pi$. Therefore, the total length is $L(\gamma_Q) = \sqrt{a}\pi$. Using $|c|T = \pi$ in the second one of (20) we get

$$|z_1 - z_0| = \frac{a|c|T}{2} = \frac{a\pi}{2}, \tag{30}$$

therefore the length for the corresponding sub-Riemannian geodesic is $L(\gamma) = \sqrt{2\pi}\sqrt{|z_1 - z_0|}$.

Now assume that we go from z_0 to z_1 through an intermediate point $\bar{z}$, that is, we use two branches. Then the total cost of the two branches γ_1 and γ_2 is

$$L(\gamma_1 + \gamma_2) = \sqrt{2\pi}\sqrt{|z_1 - \bar{z}|} + \sqrt{2\pi}\sqrt{|\bar{z} - z_0|} > \sqrt{2\pi}\sqrt{|z_1 - z_0|} = L(\gamma), \tag{31}$$

which shows that the trajectory with no internal point in the singular part is optimal.

□

The Riemannian geodesics (20) in the quotient space $\hat{M}$ have the form in figure 2. Equation

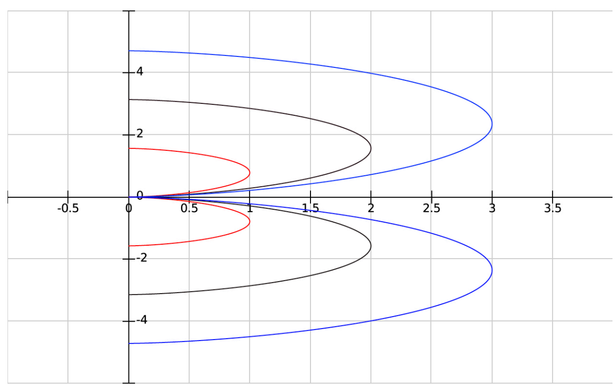

Figure 2: Riemannian geodesics in the $r - z$ plane (r is the horizontal axis and z is the vertical axis) starting from the point $(0, 0)$. Notice the symmetry with respect to the r axis obtained by changing c to $-c$ in (20). Geodesics from a point on the z axis different from the origin are obtained by shifting up or down the curves shown in this plot.

(30) in the proof shows that the parameter a is uniquely determined by the gap in z and therefore *uniqueness* of the geodesic is also verified.

The inequality (31) in the proof of Lemma 11 describes a triangle inequality saying that the path along one branch to a point in the z-axis (the singular part) is always shorter than the path through two or more branches.

Lemma 12. *Set the initial condition equal to $(r, z) = (0, z_0)$ in $\hat{M}$. Then, for any final condition (r_1, z_1), with $z_1 - z_0 \neq 0$ there exists a curve $\gamma_Q = \gamma_Q(t)$ of the form (20) with $t \in [0, T]$ and $\gamma(0) = (0, z_0)$ and $\gamma(T) = (r_1, z_1)$. In other terms, every point in $\hat{M}$ can be reached with just one branch of the geodesics (20) without intersecting the z axis.*

Proof. We assume that $\Delta z := z_1 - z_0 > 0$ and we take $c > 0$. The situation is perfectly analogous if $\Delta z := z_1 - z_0 < 0$ and we take $c < 0$. By defining $\alpha := ct \in (0, \pi)$, we can rewrite (20) as

$$r = a\sin^2(\alpha), \qquad \Delta z = \frac{a}{2}\left(\alpha - \sin(\alpha)\cos(\alpha)\right). \tag{32}$$

By setting $r = r_1$ in the first one and solving for a and replacing in the second one, we get $\Delta z = \Delta z(\alpha)$ given by the following function

$$\Delta z = \frac{r_1}{2\sin^2(\alpha)}\left(\alpha - \sin(\alpha)\cos(\alpha)\right).$$

This is a continuous function in $(0, \pi)$ and such that

$$\lim_{\alpha\to 0^+} \Delta z(\alpha) = 0, \qquad \lim_{\alpha\to\pi^-} \Delta z(\alpha) = +\infty,$$

and therefore it attains all positive values. Choosing α which satisfies $\Delta z = \Delta z(\alpha)$ for the given $\Delta z = z_1 - z_0$, we can then choose a so that $a = \frac{r_1}{\sin^2(\alpha)}$. □

In Lemma 11, we have seen that the shortest path to connect two points on the z axis is with a single geodesic (cf. the equation (30) showing uniqueness) never touching the z axis if not at the initial and final point. Now we want to prove that the same thing is true for a point in the regular part. From Lemma 12, we know that for a point in $(r_1, z_1) \in \hat{M}$ there exists a curve connecting to it. This curve will terminate on the z axis and be optimal. By the optimality principle, it will have to be optimal until (r_1, z_1) as well. Therefore, we can conclude with the following theorem.

Theorem 13. *Every optimal geodesics, connecting two points with different values of the z coordinate, only intersects the singular part (the z-axis) at most at the initial and final point.*

7 Concluding Remarks

The idea of treating a sub-Riemannian problem as a Riemannian problem via symmetry reduction has been successfully used in several instances in the control of quantum systems on Lie groups (see e.g., [2], [8]) and we have tested it here for a classical sub-Riemannian system, the Brockett integrator. A convenient feature of this model is that the resulting geodesic equations can be explicitly integrated. It is worth mentioning that such a symmetry reduction technique can be used not only for optimal control problems but also for *steering problems* between two points without the requirement of optimality. In this context, one prescribes a trajectory in the (lower dimensional) quotient space, and then, by lifting, one finds the trajectory in the original space and the corresponding control. This was done for example for quantum systems in [9]. The idea of prescribing a trajectory a priori and determining the control which induces it is particularly appealing in applications to robotics where one would like to prescribe trajectories which avoid given obstacles or are confined within a given space.

The Brockett integrator (1) is a canonical form for a class of systems in $\mathbb{R}^3$ with non-holonomy degree 1. These models can *locally* be written as in (1) (cf. [4] and Appendix A of [7]). Such class generalizes to $\mathbb{R}^n$ as (cf. [18])

$$\begin{aligned} \dot{x}_i &= u_i, \qquad i = 1, ..., m \\ \dot{z}_{ij} &= x_j u_i, \qquad i < j, \end{aligned} \tag{33}$$

where it is assumed that $n = m + \binom{m}{2}$. It is natural to ask whether a symmetry group exists for the more general class (33). We can more in general consider, for a vector $\vec{x} \in \mathbb{R}^m$ and a vector $\vec{z} \in \mathbb{R}^{n-m}$, the system

$$\begin{aligned} \dot{\vec{x}} &= \vec{u}, \\ \dot{z}_i &= \vec{x}^T M_i \vec{u}, \qquad i = 1, ..., n-m, \end{aligned} \tag{34}$$

for $m \times m$ matrices M_i, $i = 1, ..., n-m$, which includes (33) as a special case. By using the change of coordinates $z_i \to z_i - \frac{1}{2}\vec{x}^T M_i \vec{x}$, we can assume without loss of generality that the M_i's in (34) are skew-symmetric. Given a quadratic cost of the form $\int_0^T \|\vec{u}(\tau)\|^2 d\tau$ (cf. (2), we naturally look for a symmetry group which is a subgroup of $SO(m)$. This will be a symmetry group in the sense described in subsection 3.2 if the group commutes with the matrices M_i in (34). In the case of Brockett original generalization of the three-dimensional Brockett integrator (1), the corresponding matrices M_i form a basis of $so(m)$. There is no Lie subgroup of $SO(m)$ commuting with a basis of $so(m)$ except for the case $m = 2$ where $so(2)$ is Abelian, which is the case treated here. Therefore, except for the case $m = 2$, $n = 3$, treated in this paper, symmetry reduction, at least in the form advocated here, does not apply to the generalized Brockett integrators (33). There are however systems of the form (34) which still generalize the Brockett three-dimensional integrator for which the symmetry group is a nontrivial Lie subgroup of $SO(m)$. For these systems the methods applied in this paper are suitable.

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